Simulations

- Physically-Based Animation
- ODE Solvers
- PDE Solvers

What natural phenomenon can we simulate?

Flocking Simulation

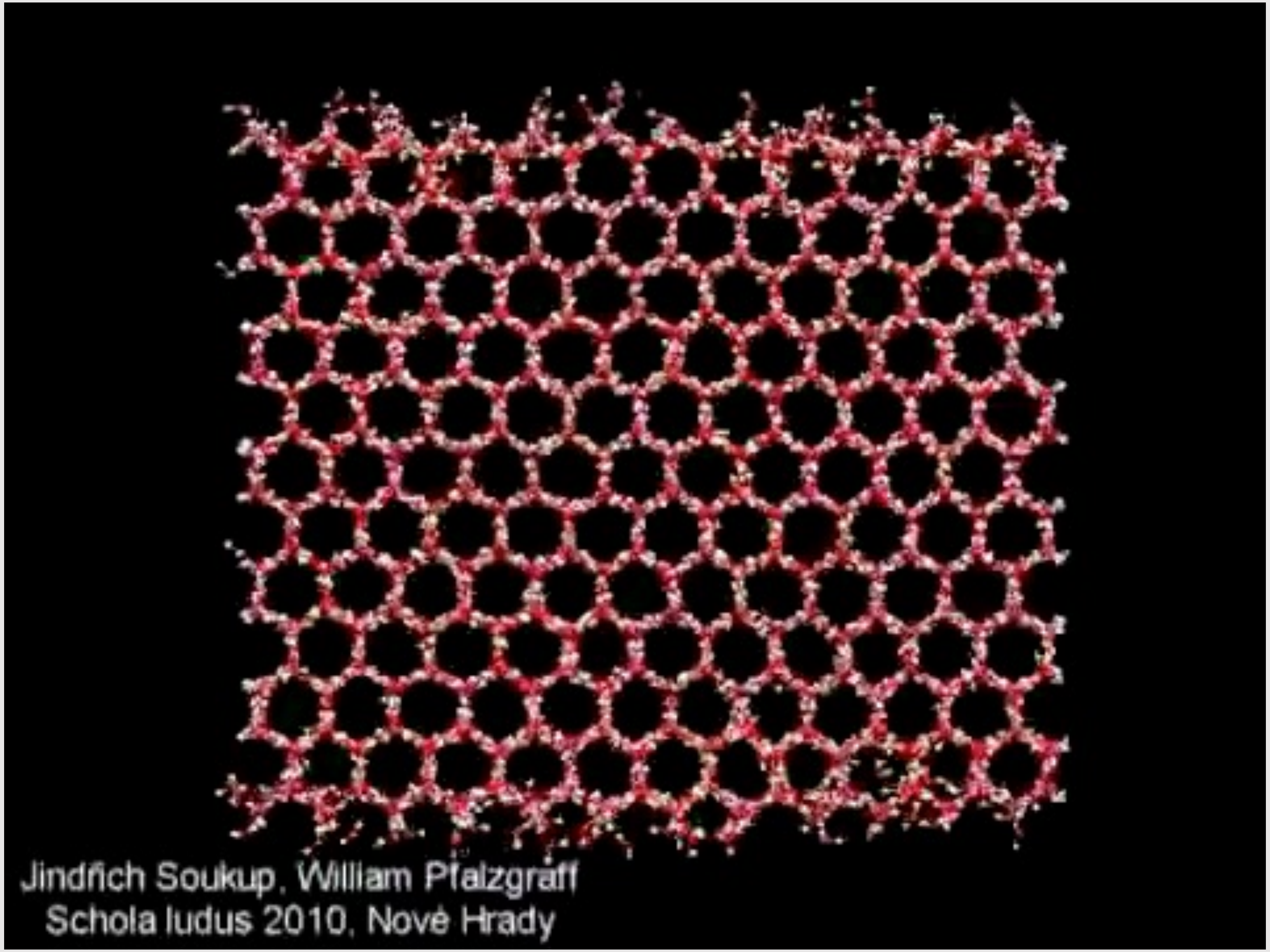
Crowd Simulation

Crowd Simulation

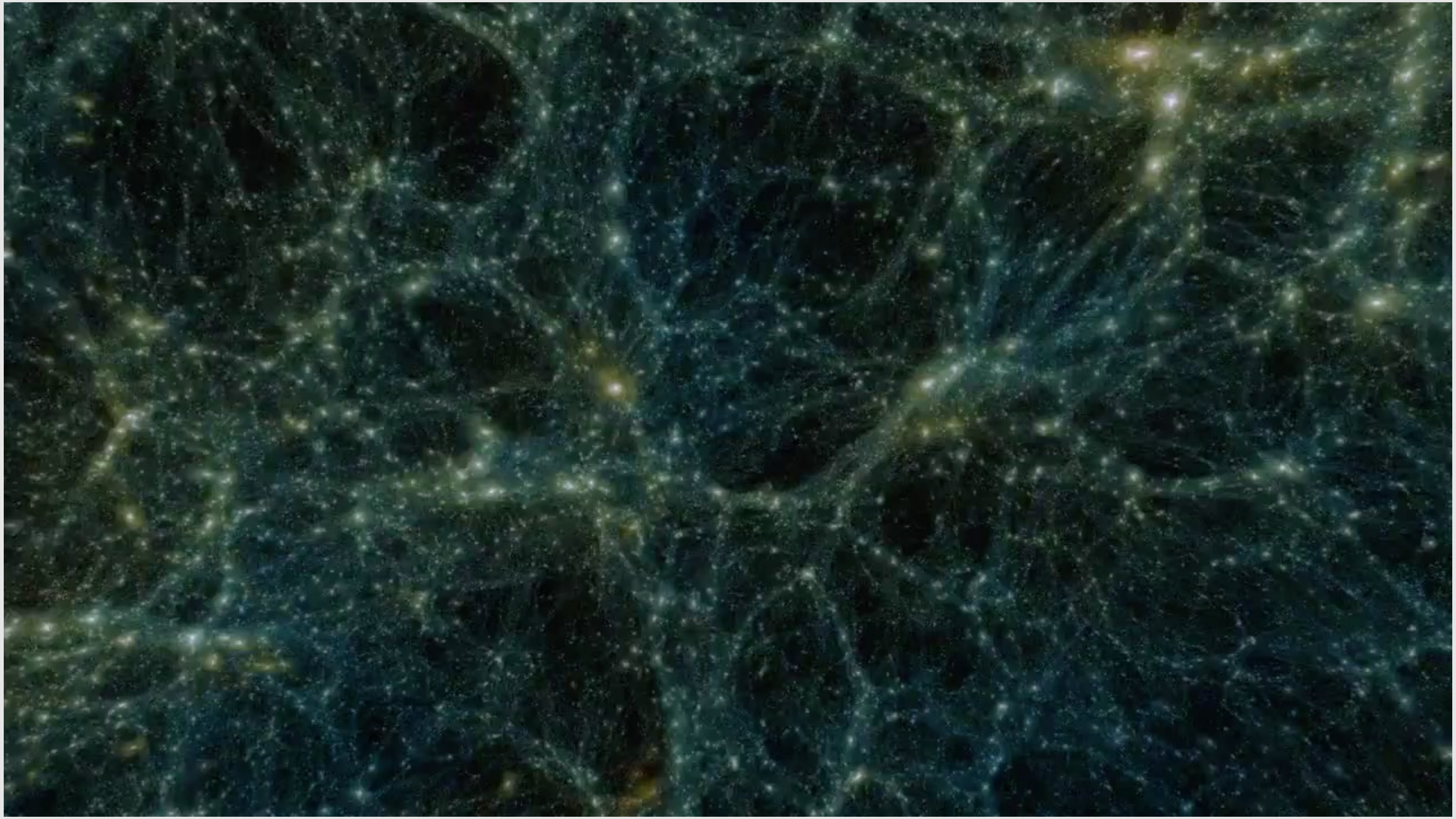
Fluid Simulation

Granular Material Simulation

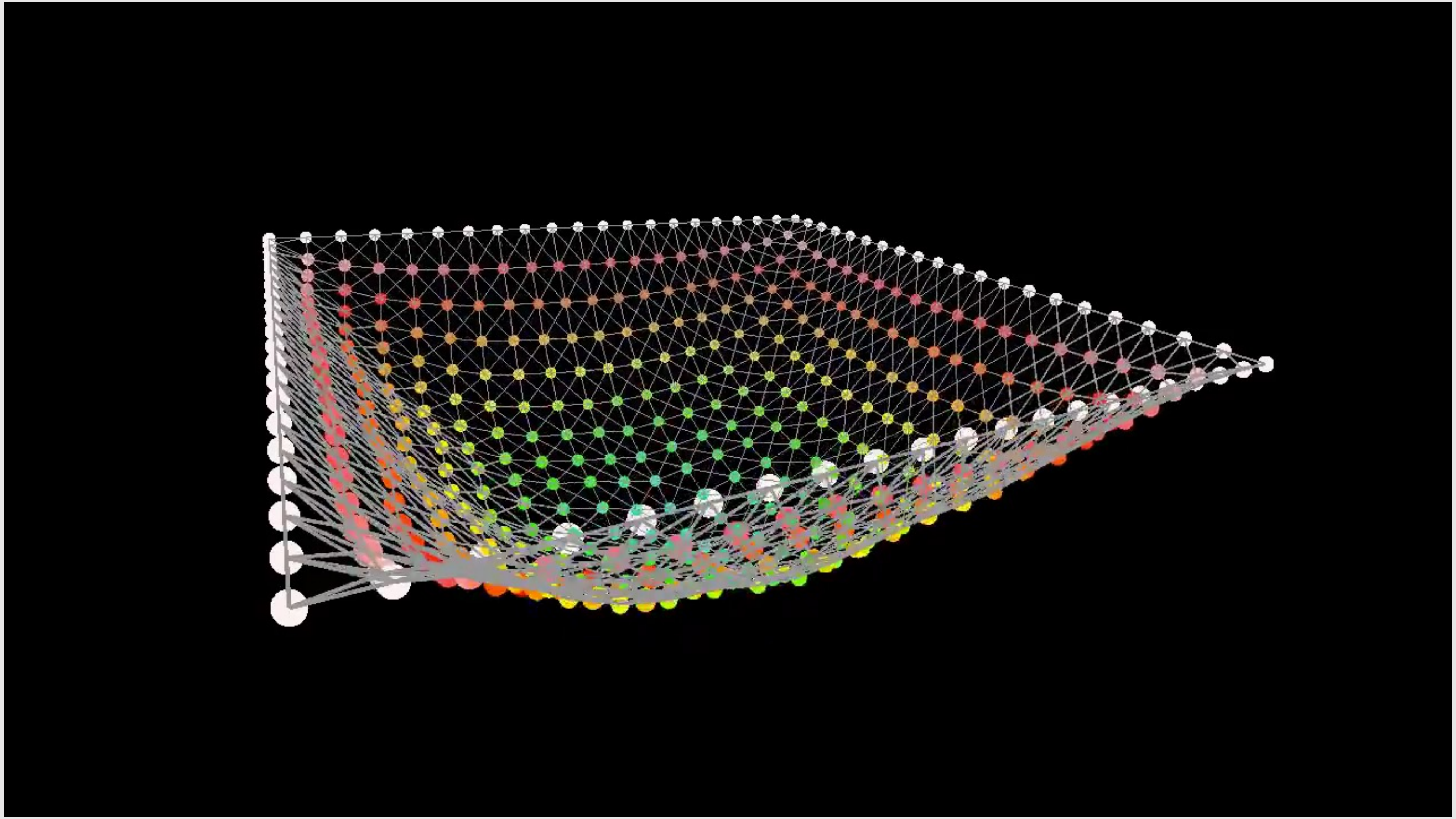
Molecular Dynamics Simulation



Cosmological Simulation

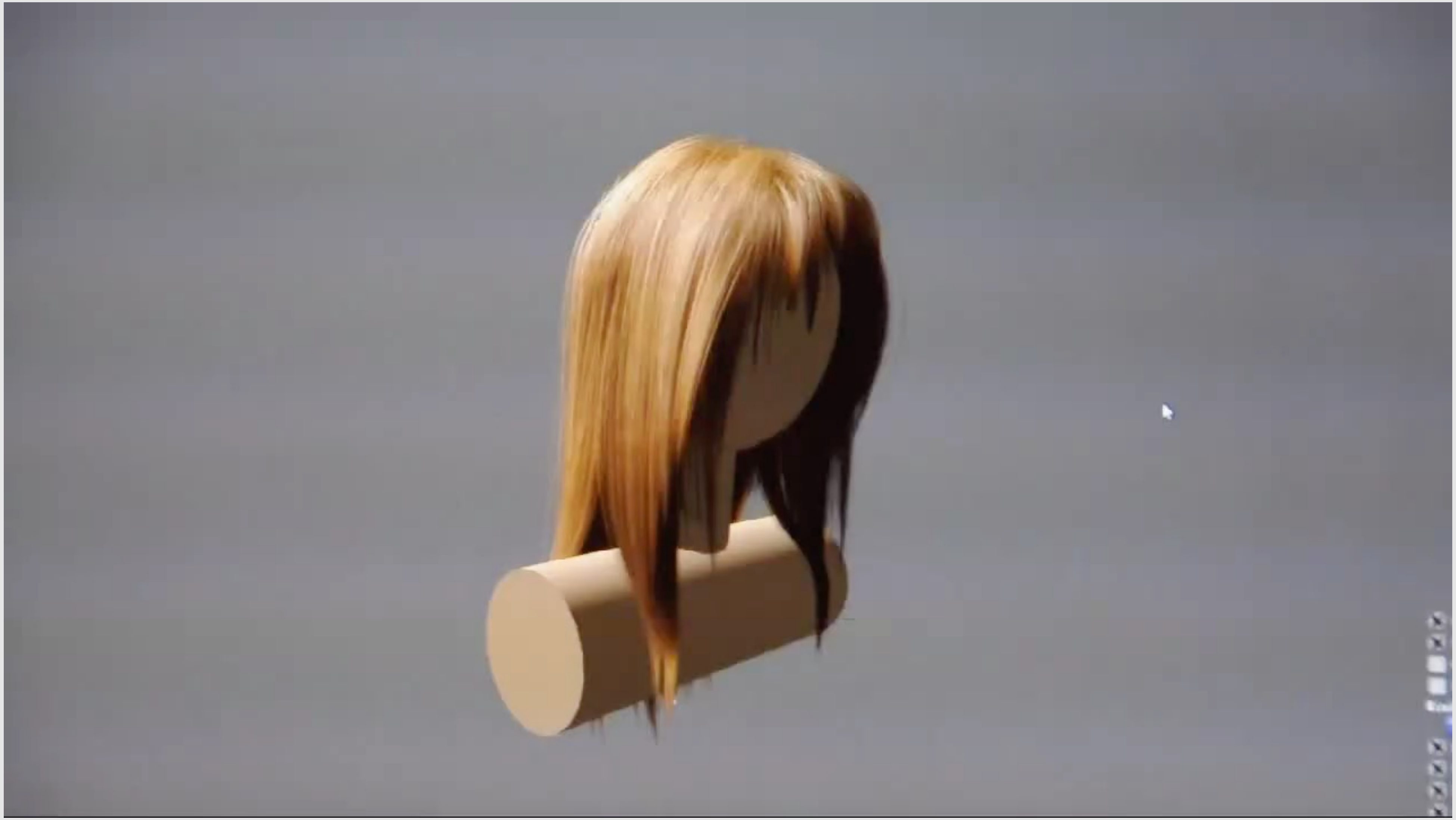


Mass-Spring Simulation

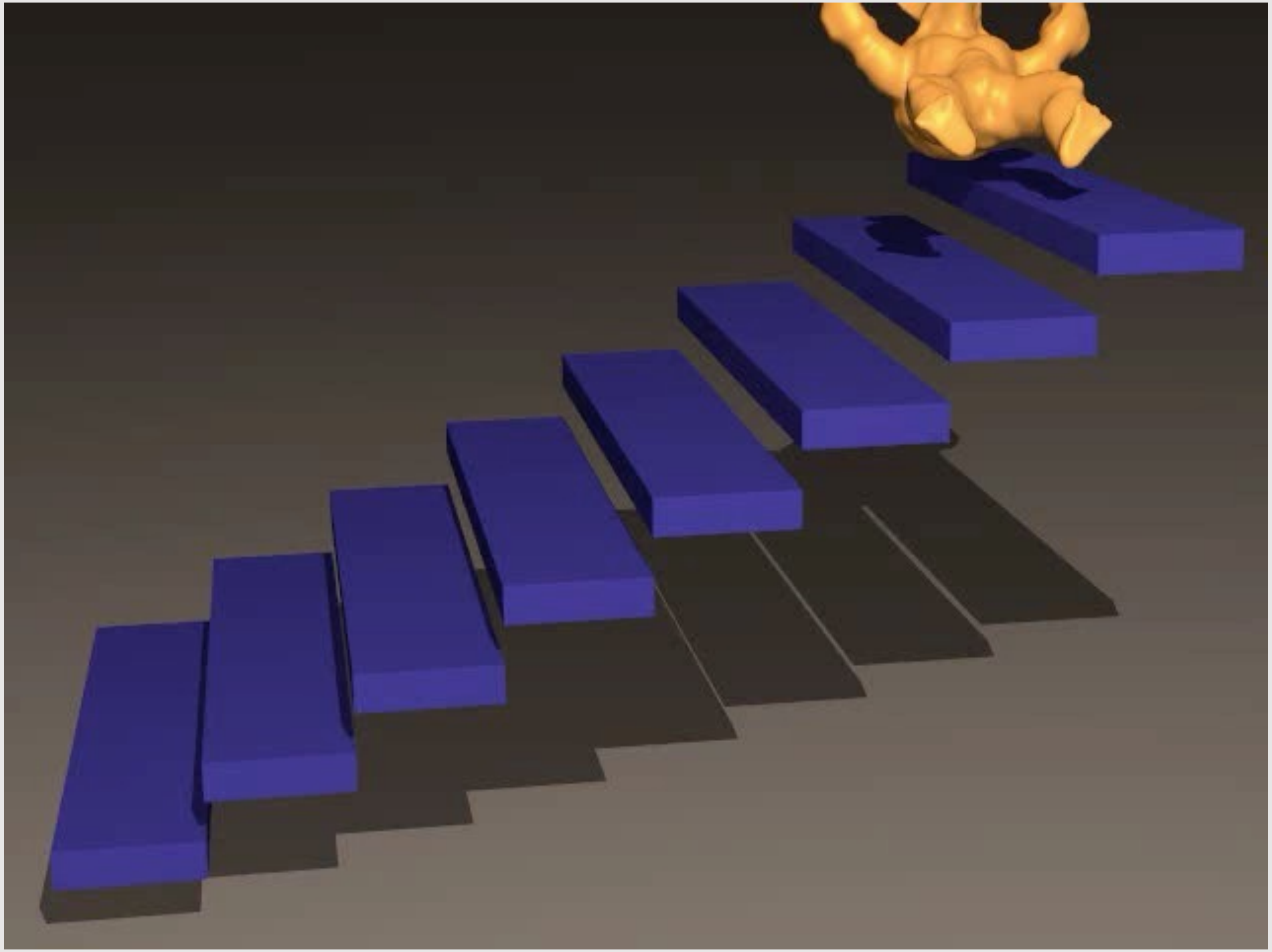


Cloth Simulation

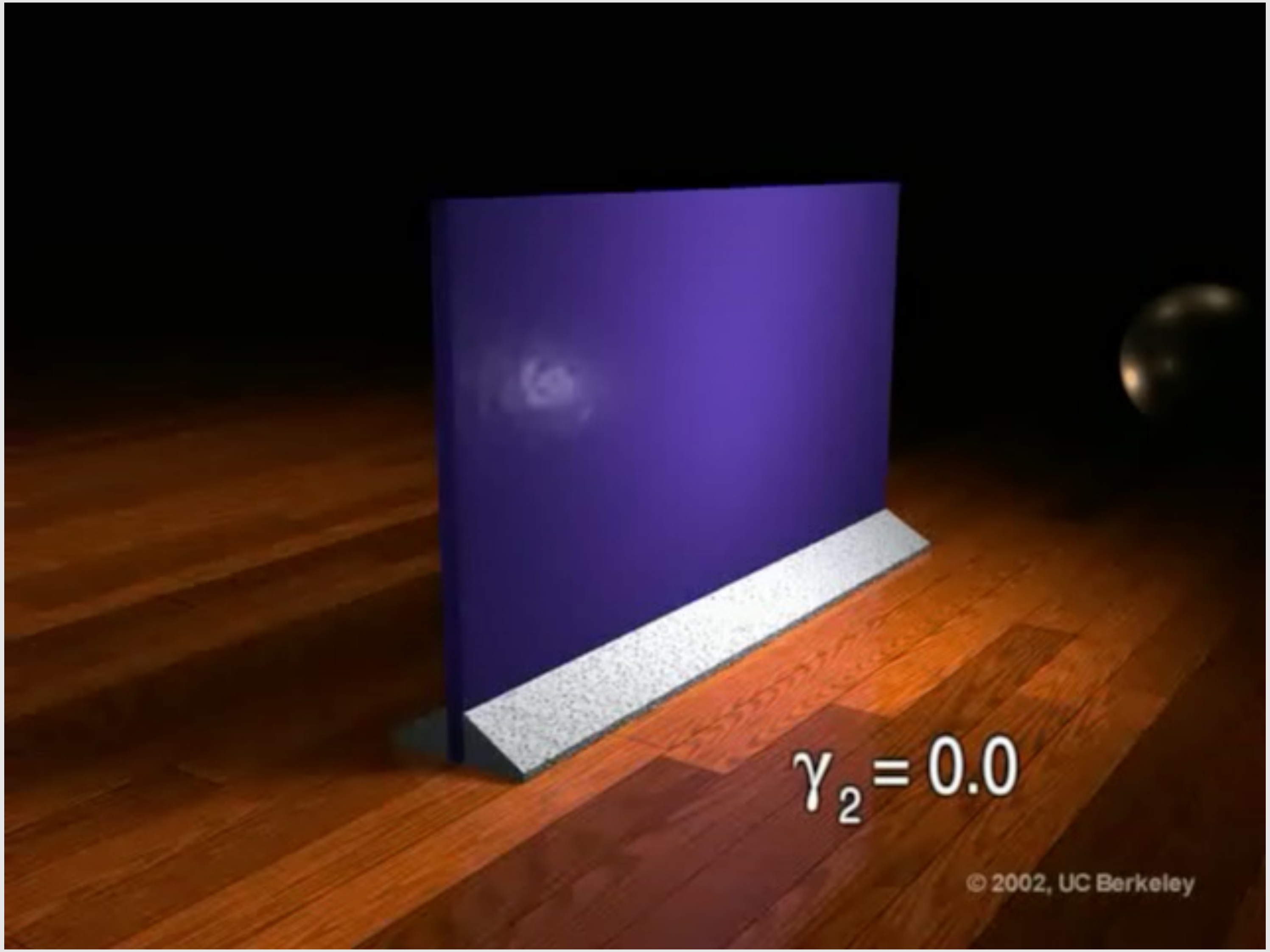
Hair Simulation



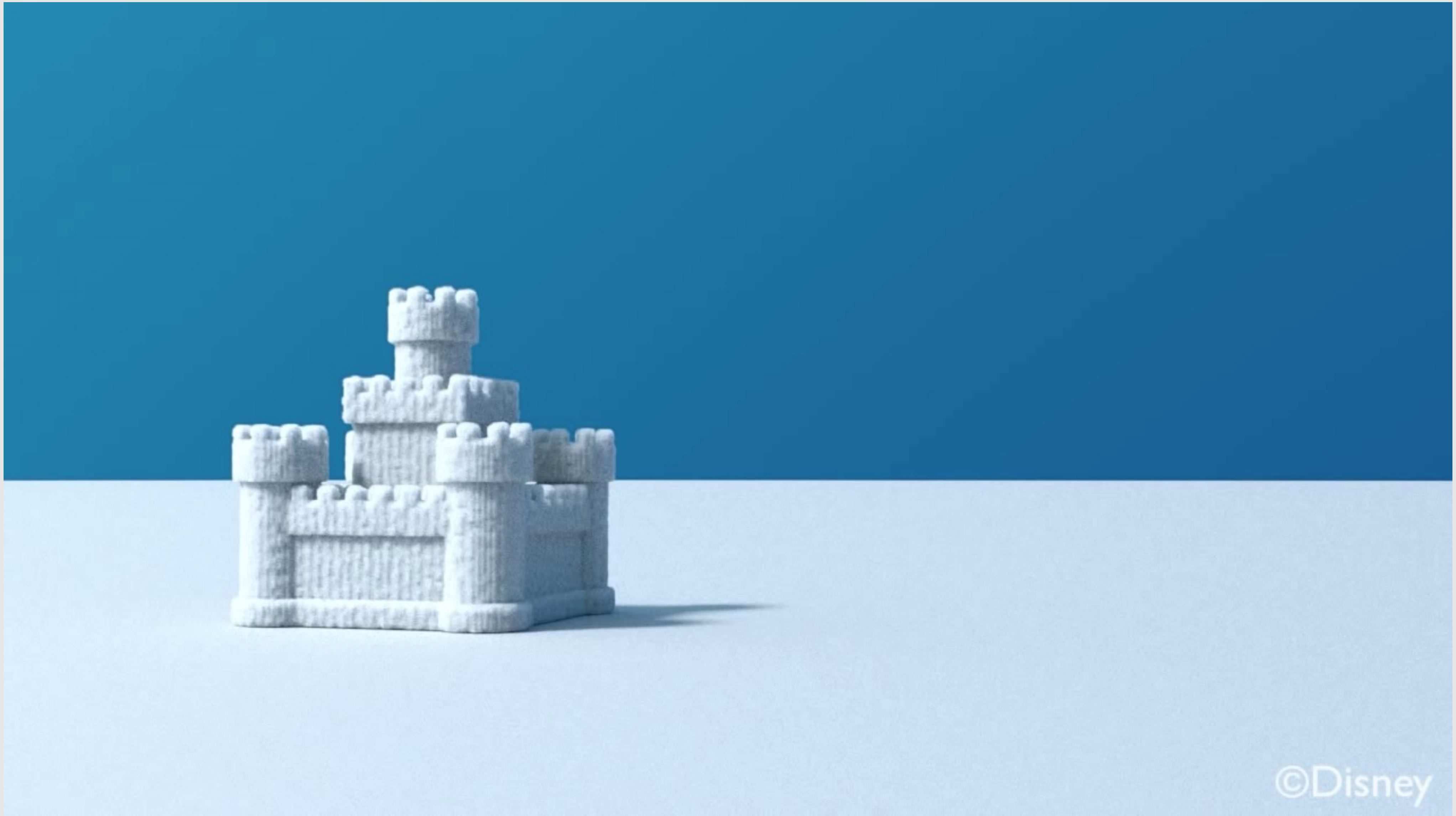
Elasticity Simulation



Fracture Simulation



Snow Simulation



Ok, simulation is cool,
How can we solve them analytically?

- ~~Physically-Based Animation~~
- ODE Solvers
- PDE Solvers

Ordinary Differential Equations

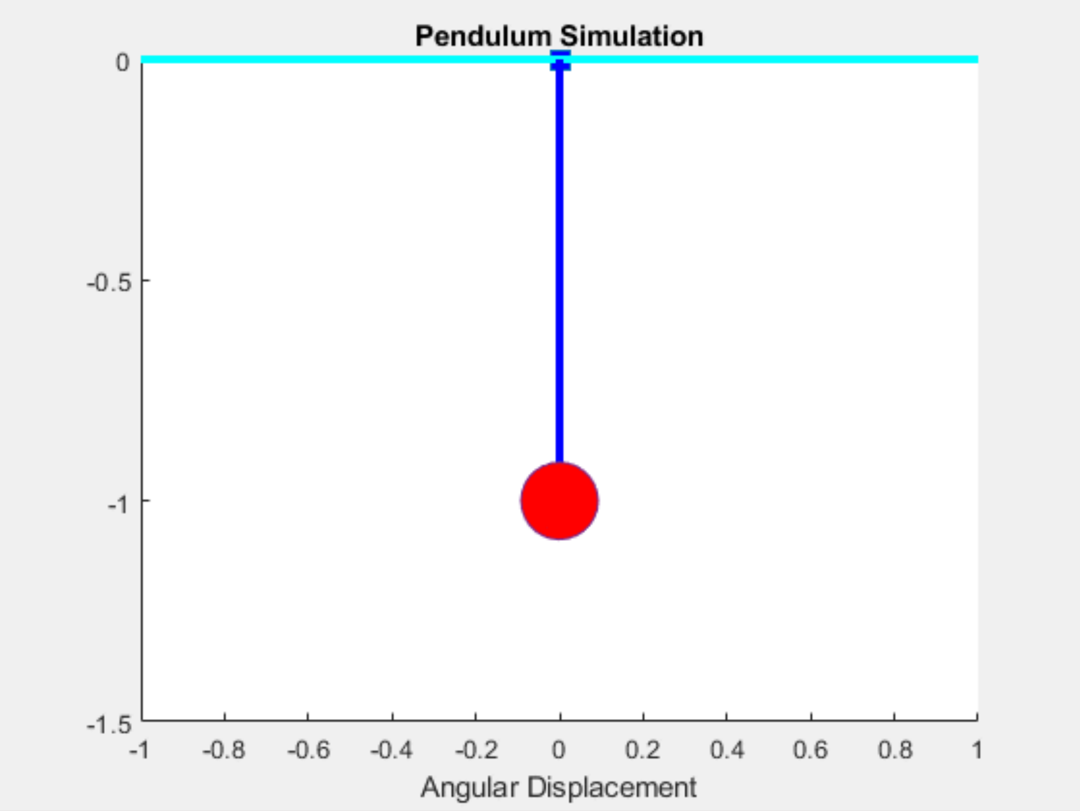
- **Ordinary Differential Equations (ODEs)** have a derivative with respect to one other variable
  - Ordinary - involves derivatives in time but not space
- Many dynamical systems can be described via an ODE in generalized coordinates:

$$\frac{d}{dt}q = f(q, \dot{q}, t)$$

- ODEs can also be used to model rates of growth proportional to some original value:

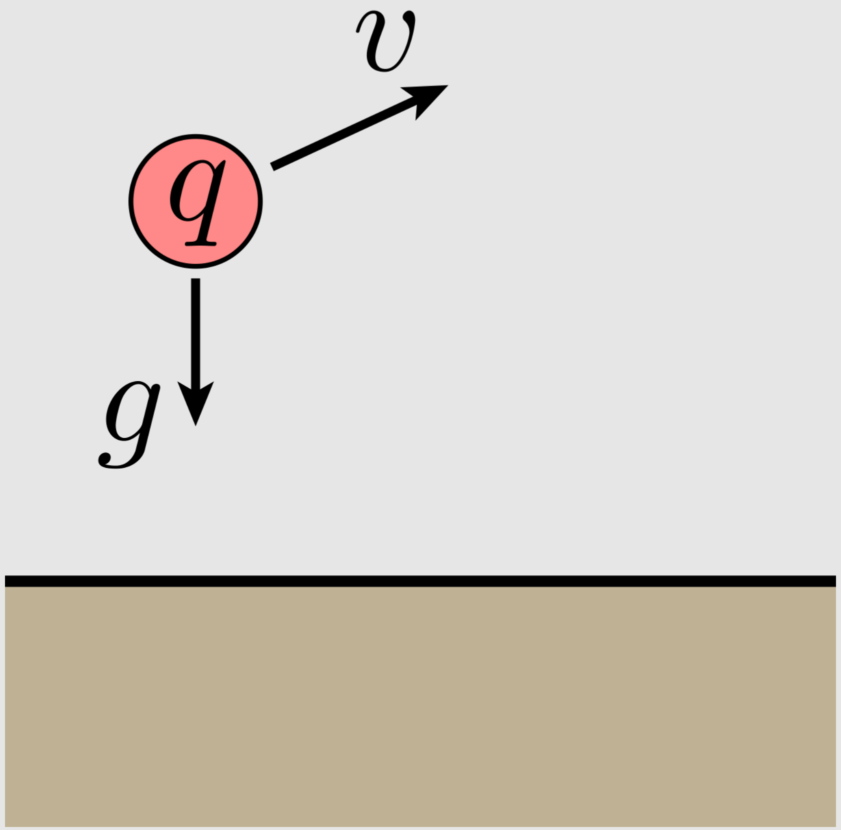
$$\frac{d}{dt}u(t) = au$$

  - **Solution:** $u(t) = be^{at}$
  - Describes exponential decay (a < 1), or stock (a > 1)



Simulation using second order ODE in MATLAB

Example: Throwing A Rock



- Consider a rock** of mass *m* tossed under force of gravity *g*
  - Easy to write dynamical equations, since only force is gravity:

$$\ddot{q} = g/m$$

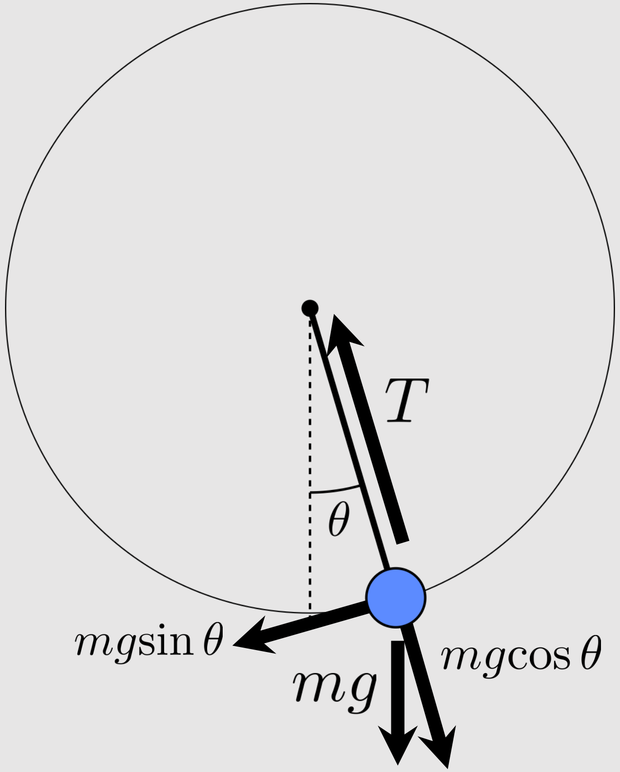
$$v(t) = v_0 + \frac{t}{m} g$$

$$q(t) = q_0 + t v_0 + \frac{t^2}{2m} g$$

Easy! We don't need a computer for simulation!

** Yes, the rock is spherical and has uniform density

Example: Pendulum



- Mass on end of a bar, swinging under gravity
- What are the equations of motion?
  - Same as “rock” problem, but constrained
  - Response tension $T(q)$ now varies based on configuration q
- Could use a “force diagram”
  - You probably did this for many hours in high school/college

Ok, maybe bring back the computer…

Lagrangian Mechanics

- Beautifully simple recipe:
  - Write down kinetic energy K
  - Write down potential energy U
  - Write down **Lagrangian**

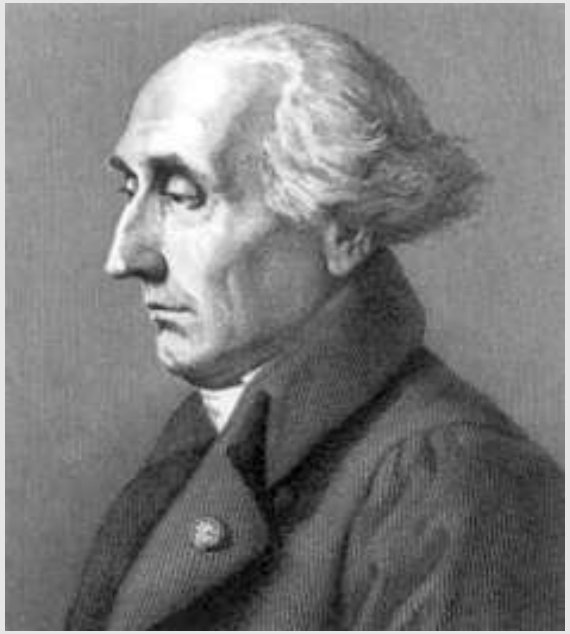
$$\mathcal{L} := K - U$$

  - Dynamics then given by **Euler-Lagrange equation**

mass times acceleration

$$\underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{\text{mass times acceleration}} = \underbrace{\frac{\partial \mathcal{L}}{\partial q}}_{\text{force}}$$

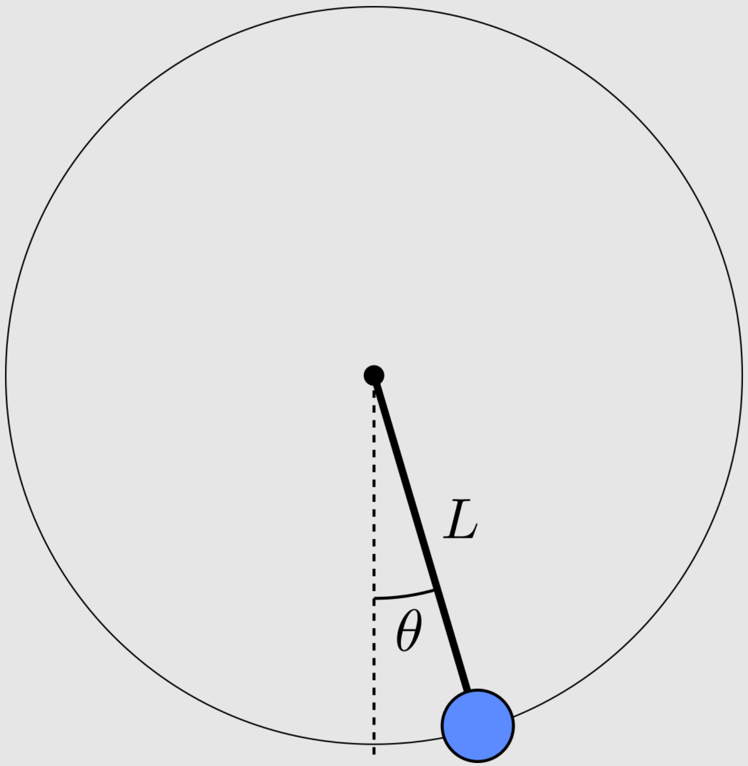
force



Joseph-Louis Langrange (1736 - 1813)

- Often easier to come up with (scalar) energies than forces
  - Very general, works in any kind of generalized coordinates
  - Helps develop nice class of numerical integrators (symplectic)

Lagrangian Mechanics: Pendulum



Simple configuration parameterization:

$$q = \theta$$

Kinetic energy:

$$K = \tfrac{1}{2} I\omega^2 = \tfrac{1}{2} mL^2\dot{\theta}^2$$

Potential energy:

$$U = mgh = -mgL\cos\theta$$

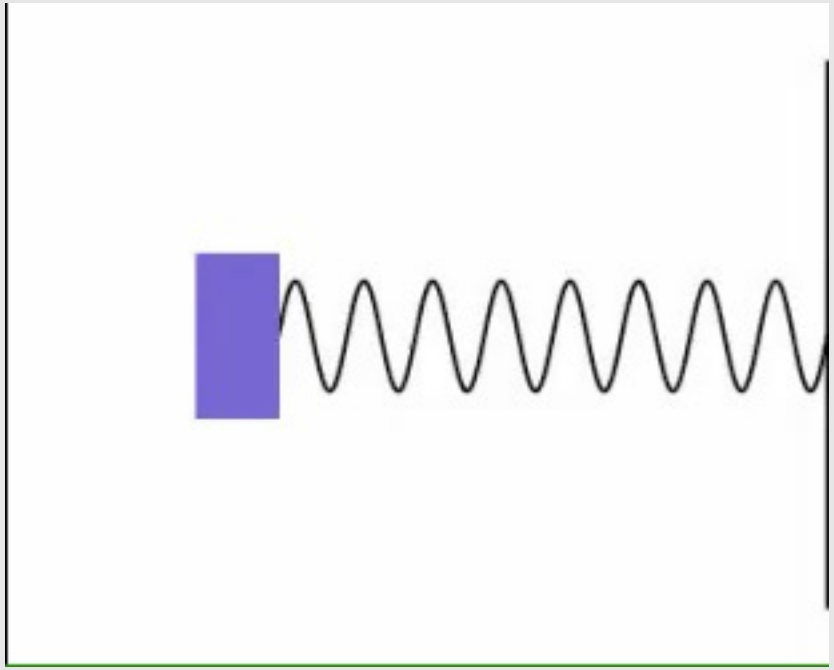
Euler-Lagrange equations:

$$\mathcal{L} = K - U = m(\tfrac{1}{2}L^2\dot{\theta}^2 + gL\cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{L}}{\partial \theta} = -mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \quad \Rightarrow \quad \boxed{\ddot{\theta} = -\tfrac{g}{L}\sin\theta}$$

Solving The Pendulum



[harmonic oscillation]

Simple equation for the pendulum:

$$\ddot{\theta} = -\tfrac{g}{L}\sin\theta$$

For small angles (e.g., clock pendulum) can approximate as:

$$\ddot{\theta} = -\tfrac{g}{L}\theta \;\Rightarrow \theta(t) = a\cos(t\sqrt{g/L} + b)$$

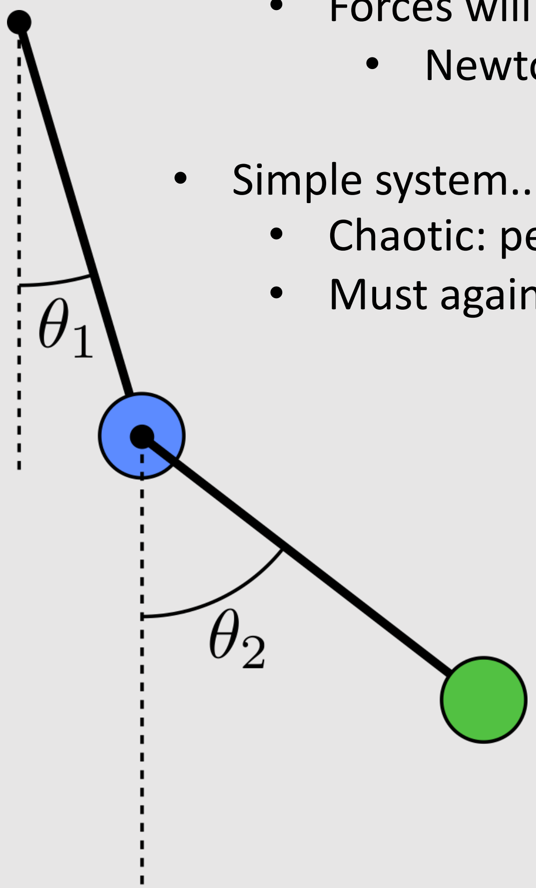
$\boldsymbol{sin\theta = \theta}$ **for small angles**

$$\frac{d^2}{d\theta^2}cos\theta = -cos\theta$$

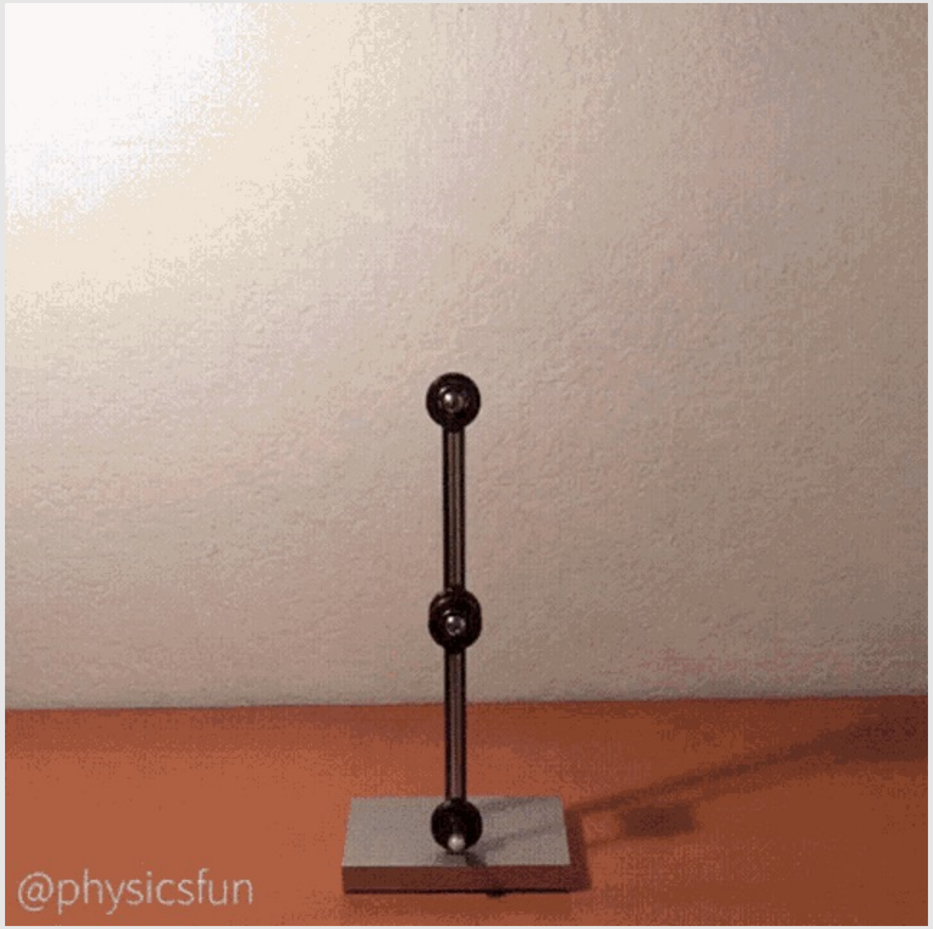
In general, there is no closed form solution!
Hence, we must use a numerical approximation

And pendulums are supposed to be easy to simulate!

Harder: Double Pendulum

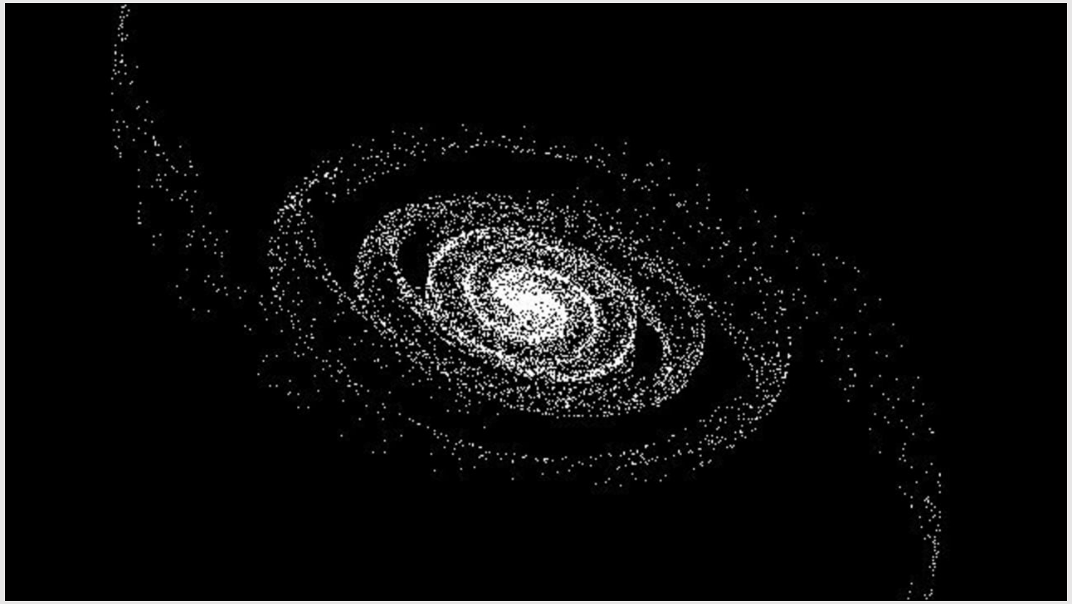


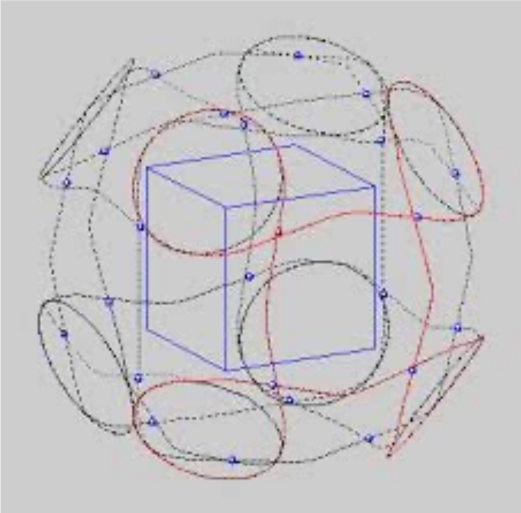
- Blue ball swings from pendulum
  - Green ball swings from blue ball
  - Forces will act on each other
    - Newton's 3rd law
- Simple system...not-so-simple motion
  - Chaotic: perturb input, wild changes to output
  - Must again use numerical approximation

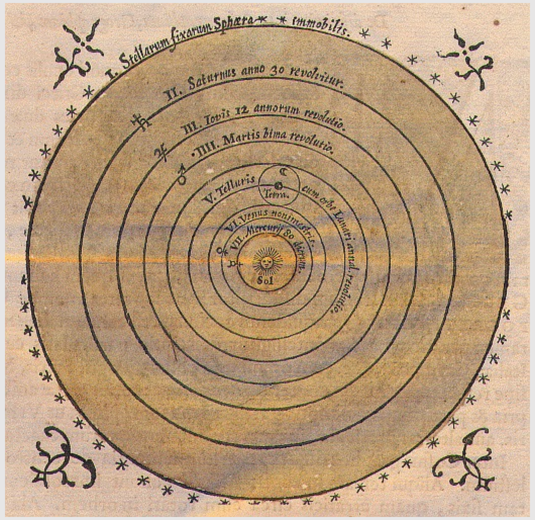


Even Harder: N-Body Problem

- Consider the Earth, moon, and sun
  - Where do they go?
  - Solution is trivial for two bodies
    - Assume one is fixed, solve for the other
- As soon as $n \geq 3$, gets chaotic
  - No closed form solution
- **Fun Fact:** this is a 15-418 homework assignment
  - Glad you aren't taking 15-418…







closed-form solution

Ok, so solving solutions analytically is hard,
How can we solve them numerically?

guess-and-check

Numerical Integration

- **Key idea:** replace derivatives with differences
  - With ODEs, only need to worry about derivative in **time**

- Replace time-continuous configuration function $q(t)$ with samples q_k in time

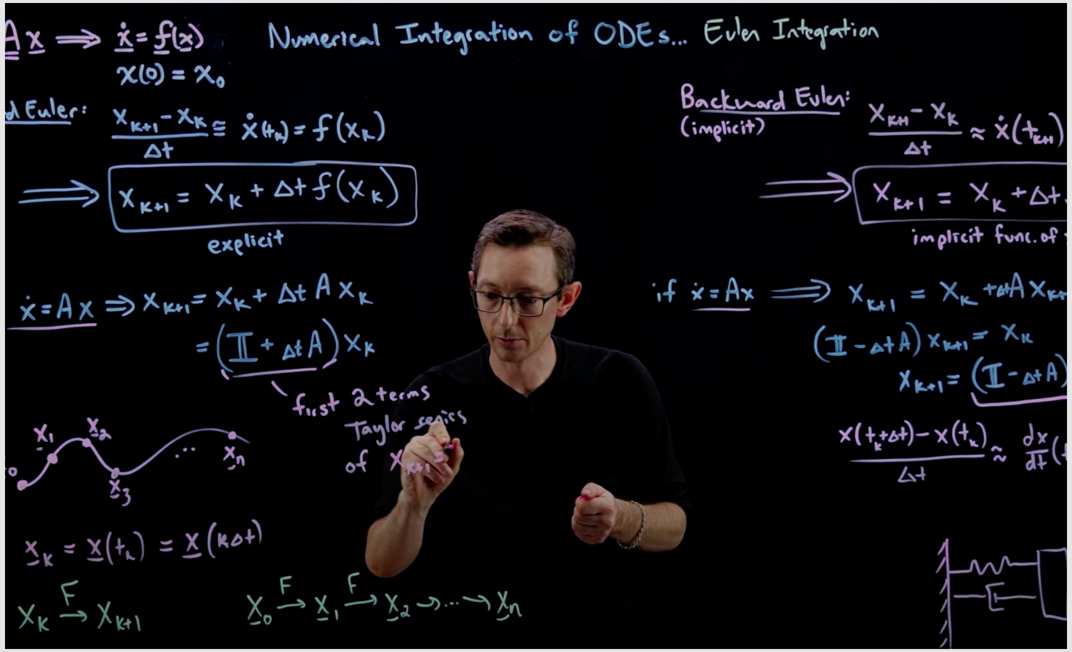
which q do we use?

new config (need to solve for)

current config

$$v(q(t)) = \frac{d}{dt}q(t) = \frac{q_{k+1} + q_k}{\tau}$$

time step



Deriving Forward & Backward Euler (2022) Steve Brunton

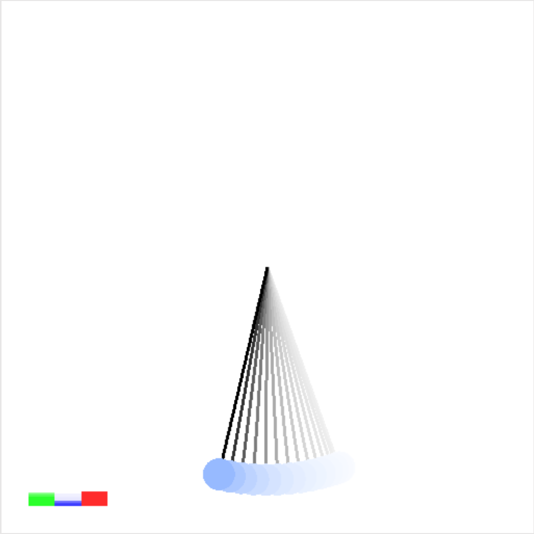
Forward Euler

- **Idea:** evaluate velocity at current configuration

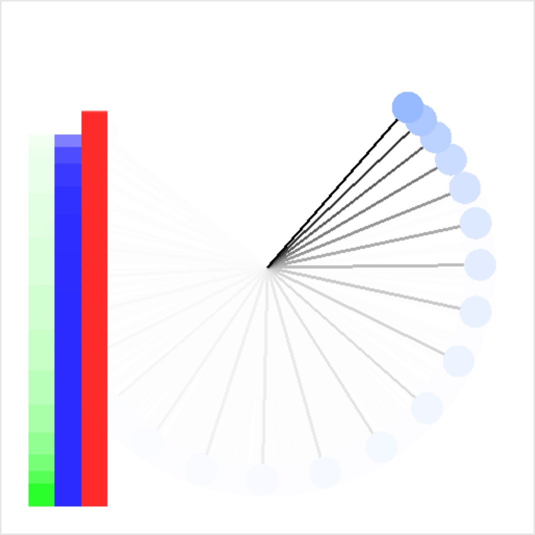
- New configuration can then be written explicitly in terms of known data:

$$q_{k+1} = q_k + \tau * v(q_k)$$

- Very intuitive: walk a tiny bit in the direction of the velocity



starts slow



gradually moves faster

Where did all this energy come from?

Forward Euler Analysis

Let's consider behavior of forward Euler for a simple linear ODE:

$$\dot{q} = -aq, \qquad a > 0$$

q should decay over time (loss of energy to global system).
Forward Euler approximation is:

$$q_{k+1} = q_k - \tau a q_k$$

$$q_{k+1} = (1 - \tau a) q_k$$

Which means after n steps, we have:

$$q_n = (1 - \tau a)^n q_0$$

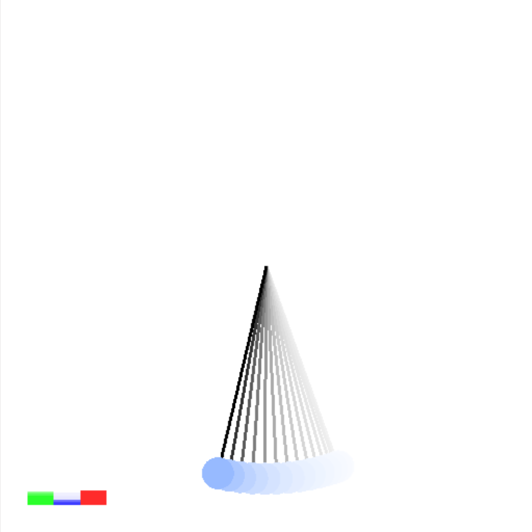
Decays only if $|1 - \tau a| < 1$, or equivalently, if $\tau < 2/a$
In practice: need very small time steps if a is large

Backward Euler

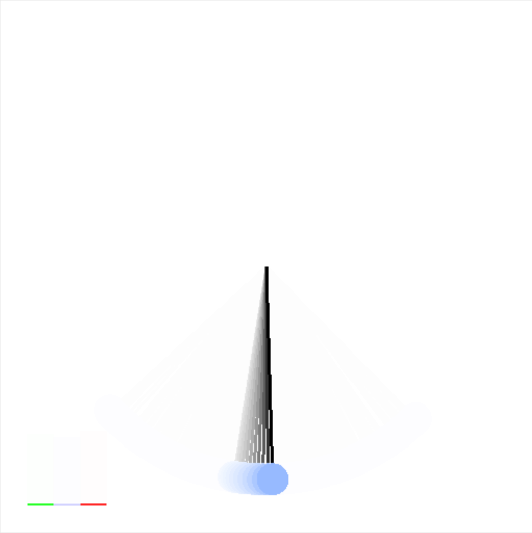
- **Idea:** evaluate velocity at next configuration
- New configuration implicit, output depends on input:

$$q_{k+1} = q_k + \tau * v(q_{k+1})$$

- Much harder to solve, since in general v can be very nonlinear!
  - More of a constraint problem: find a q_{k+1} that satisfies the above equation
  - Generally expensive to solve



starts slow



gradually slows down

Where did all this energy go?

Backward Euler Analysis

Again, let's consider a simple linear ODE:

$$\dot{q} = -aq, \qquad a > 0$$

q should decay over time (loss of energy to global system).
Backward Euler approximation is:

$$(q_{k+1} - q_k)/\tau = -aq_{k+1}$$

$$\frac{q_{k+1}}{\tau} + aq_{k+1} = \frac{q_k}{\tau}$$

$$(1 + \tau a)q_{k+1} = q_k$$

$$q_{k+1} = \frac{1}{1+\tau a} q_k$$

Which means after n steps, we have:

$$q_n = \left(\frac{1}{1+\tau a}\right)^n q_0$$

Decays only if $|1 + \tau a| > 1$, which is always true!
Backwards Euler is **unconditionally stable** for linear ODEs!

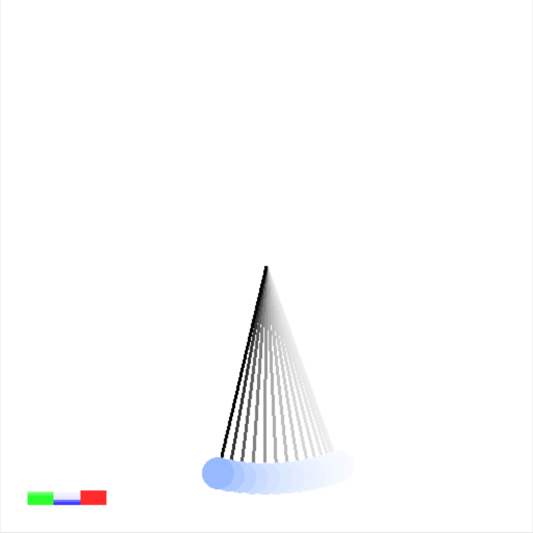
Symplectic Euler

- Nice alternative is Symplectic Euler
  - Update velocity using current configuration q_k
  - Update configuration using new velocity v_{k+1}

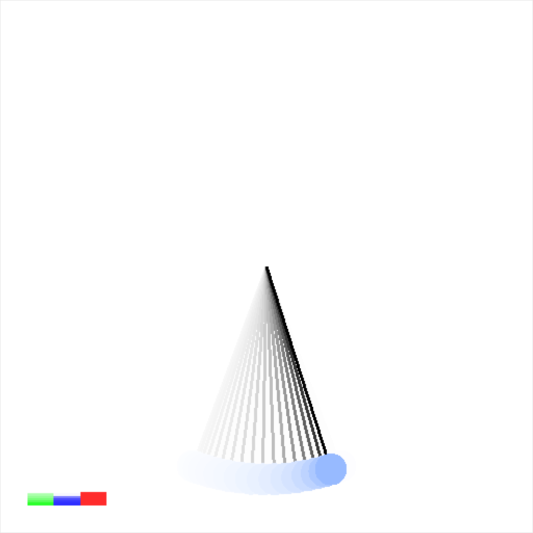
$$v_{k+1} = v_k + \tau * a(q_k)$$

$$q_{k+1} = q_k + \tau * v_{k+1}$$

- Pendulum now conserves energy *almost exactly*, forever



starts slow



keeps on ticking

Proof? The analysis isn't very easy…

Explicit Euler Methods

[Forward]

$$v_{k+1} = v_k + \tau * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_k$$

[Symplectic]

$$v_{k+1} = v_k + \tau * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$

[Verlet]

$$v_{k+1} = v_{k+0.5} + \frac{\tau}{2} * a(q_k)$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$
$$v_{k+1.5} = v_{k+1} + \frac{\tau}{2} * a(q_k)$$

[RK2]

$$v'_{k+1} = \tau * a(q_k)$$
$$v''_{k+1} = \tau * a(q_k + \frac{v'_{k+1}}{2})$$
$$v_{k+1} = v_k + v''_{k+1}$$
$$q_{k+1} = q_k + \tau * v_{k+1}$$

[RK4]

$$v'_{k+1} = \tau * a(q_k)$$
$$v''_{k+1} = \tau * a(q_k + \frac{v'_{k+1}}{2})$$
$$v'''_{k+1} = \tau * a(q_k + \frac{v''_{k+1}}{2})$$
$$v''''_{k+1} = \tau * a(q_k + v'''_{k+1})$$
$$q_{k+1} = q_k + \frac{1}{6}(v'_{k+1} + 2v''_{k+1} + 2v'''_{k+1} + v''''_{k+1})$$

- ~~Physically-Based Animation~~

- ~~ODE Solvers~~

- PDE Solvers

Partial Differential Equations

- **Partial Differential Equations (PDEs)** have a derivative with respect to multiple variables
  - ODE Simulations take derivatives w.r.t time

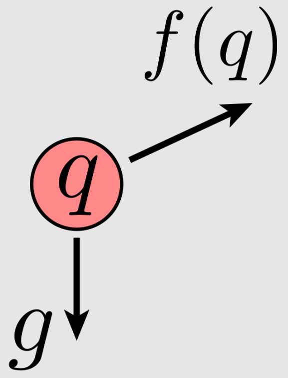
    $$q_{k+1} = q_k + \tau f(q)$$

  - PDE Simulations take derivatives w.r.t time + space

    $$q_{k+1} = q_k + \tau f(q)$$

    - Same function! But depends on how we parameterize the configuration q

- PDEs described with implicit definitions
  - Need to solve for the actual function



[ODE]

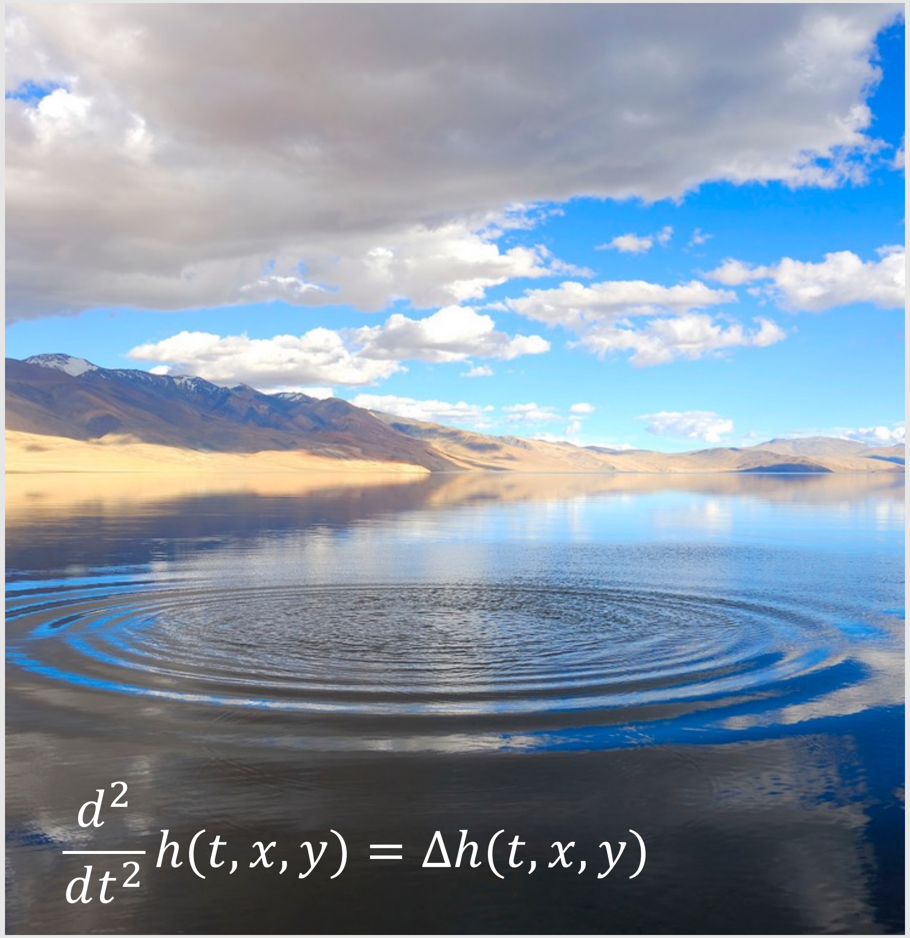
	1	
1	-4	1
	1	

$f(q)$

[PDE]

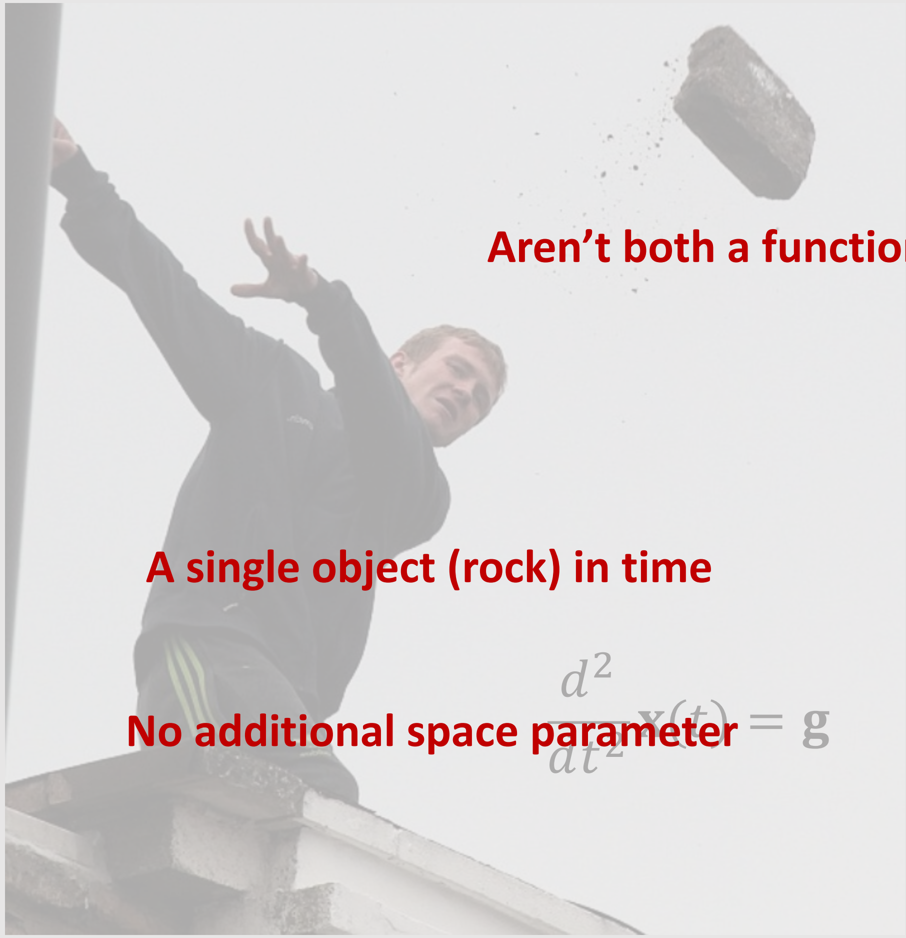
ODEs vs. PDEs

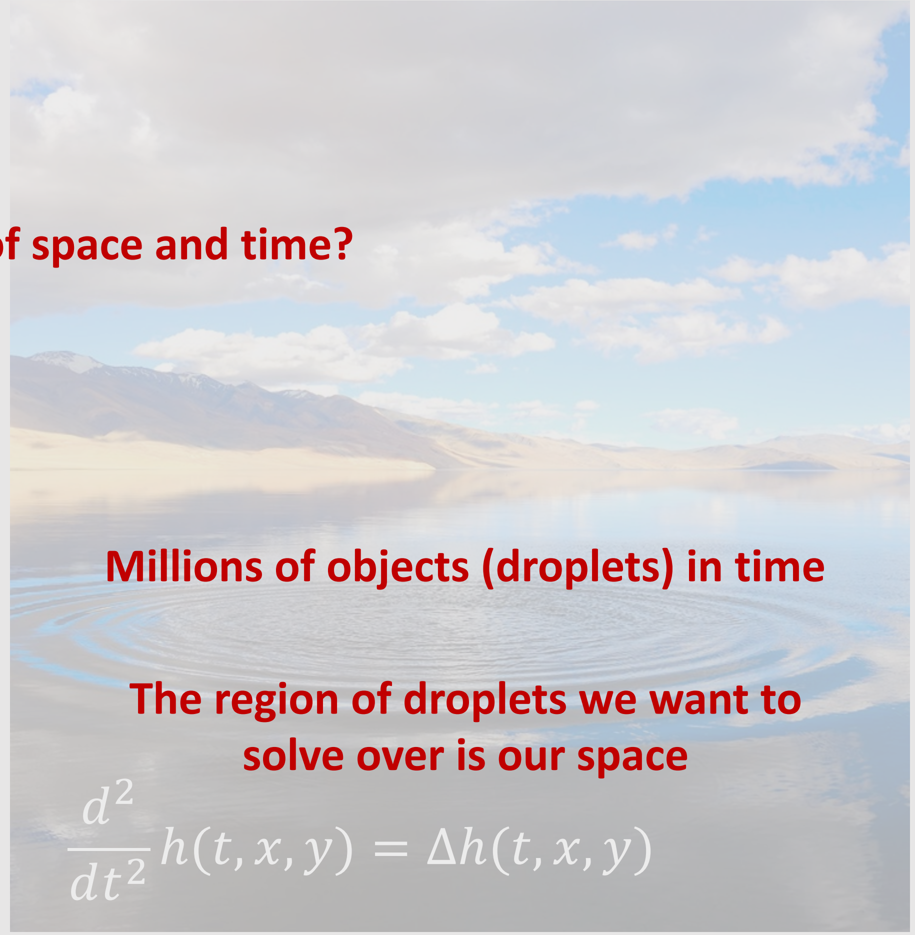
[ODE] throwing a rock



[PDE] throwing rock lands in pond

ODEs vs. PDEs





Aren’t both a function of space and time?

A single object (rock) in time

No additional space parameter

$$\frac{d^2}{dt^2}\mathbf{x}(t) = \mathbf{g}$$

Millions of objects (droplets) in time

The region of droplets we want to solve over is our space

$$\frac{d^2}{dt^2}h(t,x,y) = \Delta h(t,x,y)$$

[ODE] throwing a rock

[PDE] throwing rock lands in pond

Moving forward, we will denote:

$$u(t, x)$$

As the vales for which our PDE will solve for, and:

$$\dot{u}, \ddot{u}, \dddot{u} \ldots$$

As temporal derivatives, and:

$$u', u'', u''' \ldots$$

As spatial derivatives

The Laplacian Operator

- All of our model equations used the Laplace operator
  - Laplace Equation $\Delta u = 0$
  - Heat Equation $\dot{u} = \Delta u$
  - Wave Equation $\ddot{u} = \Delta u$

- Unbelievably important object showing up everywhere across physics, geometry, signal processing, and more

- What does the Laplacian mean?
  - **Differential operator:** eats a function, spits out its 2nd derivative
  - What does that mean for a function: $u: \mathbb{R}^n \to \mathbb{R}$?
    - Divergence of gradient

$$\Delta u = \nabla \cdot \nabla u$$

    - Sum of second derivatives

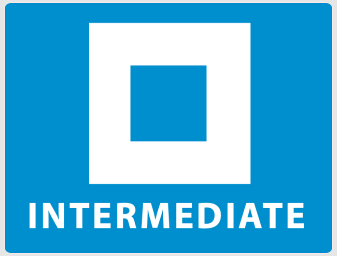
$$\Delta u = \frac{\partial u^2}{\partial x_1^2} + \cdots + \frac{\partial u^2}{\partial x_n^2}$$

    - Deviation from local average
    - …

Modeling PDE Equations

Laplace Equation [Elliptic]
"What's the smoothest function interpolating the given boundary data?"

$$\Delta u = 0$$



Heat Equation [Parabolic]
"How does an initial distribution of heat spread out over time?"

$$\dot{u} = \Delta u$$

Wave Equation [Hyperbolic]
"If you throw a rock into a pond, how does the wavefront evolve over time?"

$$\ddot{u} = \Delta u$$

Nonlinear + Hyperbolic + High-Order
"A lot of real life phenomenon"

? ? ?

Modeling PDE Equations

EASIER

Laplace Equation [Elliptic]
"What's the smoothest function interpolating the given boundary data?"

$$\Delta u = 0$$

INTERMEDIATE

Heat Equation [Parabolic]
"How does an initial distribution of heat spread out over time?"

$$\dot{u} = \Delta u$$

ADVANCED

Wave Equation [Hyperbolic]
"If you throw a rock into a pond, how does the wavefront evolve over time?"

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EXPERTS ONLY

Nonlinear + Hyperbolic + High-Order
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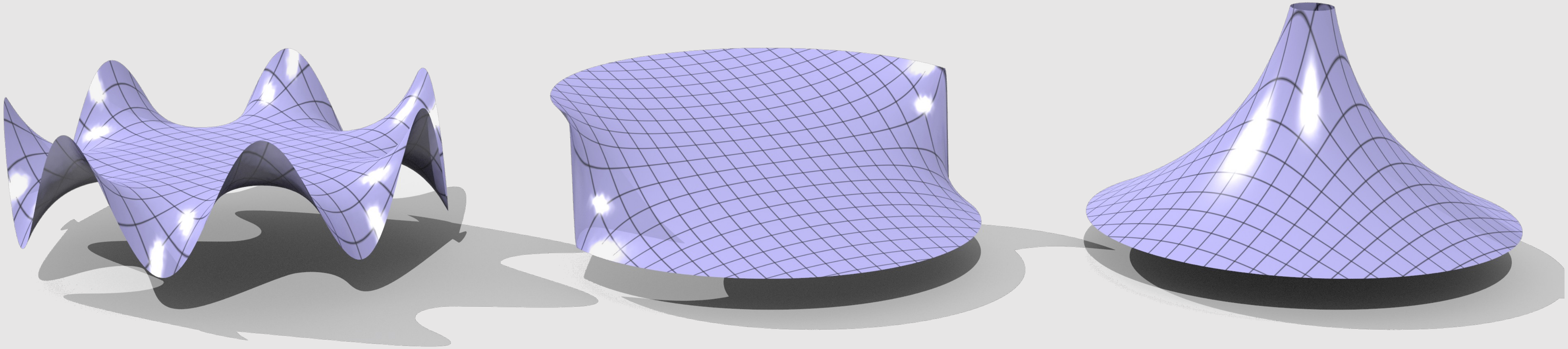
Laplace Equation

"What's the smoothest function interpolating the given boundary data?"



Laplace-Beltrami: The Swiss Army Knife of Geometry Processing (2014) Solomon, Crane, Vouga

- Conceptually each value is at the average of its "neighbors"
    - Very robust to errors: just keep averaging with neighbors
    - Errors will eventually get averaged out/diminish

Modeling PDE Equations

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Heat Equation

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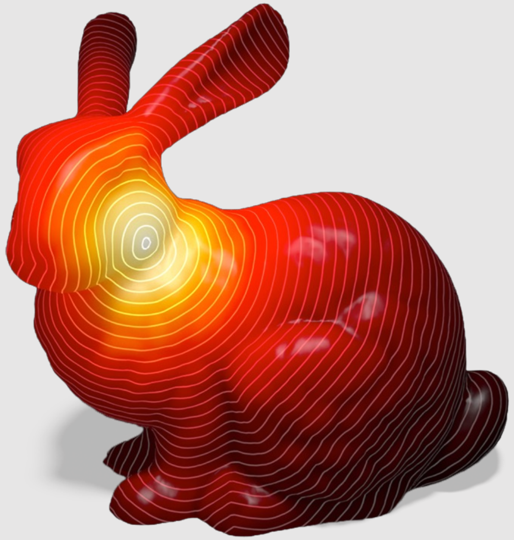
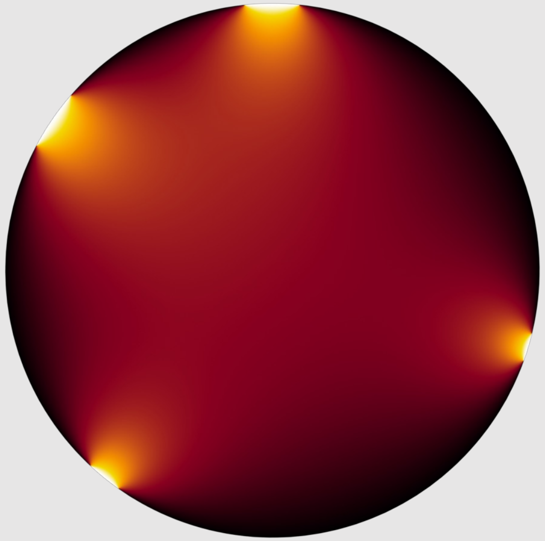




- After a long time, solution is same as Laplace equation
  - Treat 3D problem over a mesh as a 2D surface problem

- Models damping/viscosity in many physical system

Modeling PDE Equations

EASIER

Laplace Equation [Elliptic]
"What's the smoothest function interpolating the given boundary data?"

$$\Delta u = 0$$

INTERMEDIATE

Heat Equation [Parabolic]
"How does an initial distribution of heat spread out over time?"

$$\dot{u} = \Delta u$$

ADVANCED

Wave Equation [Hyperbolic]
"If you throw a rock into a pond, how does the wavefront evolve over time?"

$$\ddot{u} = \Delta u$$

EXPERTS ONLY

Nonlinear + Hyperbolic + High-Order
"A lot of real life phenomenon"

? ? ?

Wave Equation

"If you throw a rock into a pond, how does the wavefront evolve over time?"

- Difficult! Errors made at the beginning will persist for a long time
  - Errors may even compound and explode/break simulation

Modeling PDE Equations

EASIER	**Laplace Equation [Elliptic]** *"What's the smoothest function interpolating the given boundary data?"*	$\Delta u = 0$
INTERMEDIATE	**Heat Equation [Parabolic]** *"How does an initial distribution of heat spread out over time?"*	$\dot{u} = \Delta u$
ADVANCED	**Wave Equation [Hyperbolic]** *"If you throw a rock into a pond, how does the wavefront evolve over time?"*	$\ddot{u} = \Delta u$
EXPERTS ONLY	**Nonlinear + Hyperbolic + High-Order** *"A lot of real life phenomenon"*	**? ? ?**

PDE Anatomy

- How are derivatives combined?
  - **Linear:** derivatives are not multiplied with each other
  - **Nonlinear:** derivatives multiplied with each other

nonlinear

$$\dot{u} + uu' = au''$$

[Burger's equation]

$$\dot{u} = au''$$

[diffusion equation]

- What is the highest order derivative in space and time?

1st order time

2nd order space

$$\dot{u} + uu' = au''$$

[Burger's equation]

$$\dot{u} = au''$$

[diffusion equation]

2nd order time

2nd order space

- The higher the order, the harder to solve!

Great, but how do we solve PDEs?

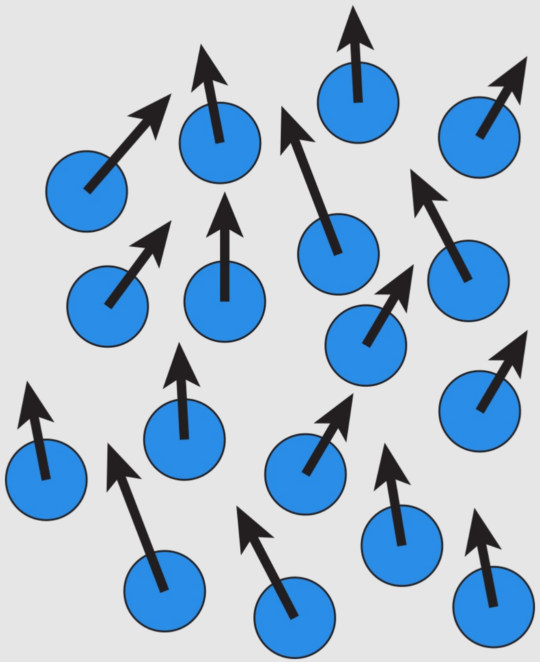
Numerically Solving a PDE

- PDEs are (near) impossible to solve analytically
  - Need to solve numerically
- **Algorithm:**
  - Pick a time discretization to compute temporal derivatives
    - Forward Euler, Symplectic, …
  - Pick a spatial discretization to compute spatial derivatives
    - Lagrangian, Eulerian, …
  - Perform time-stepping to advance solution
- Historically, very expensive
  - Only for "hero shots" in movies
- Computers are even faster nowadays
  - Can solve PDEs in real-time

What discretization formats do we have?

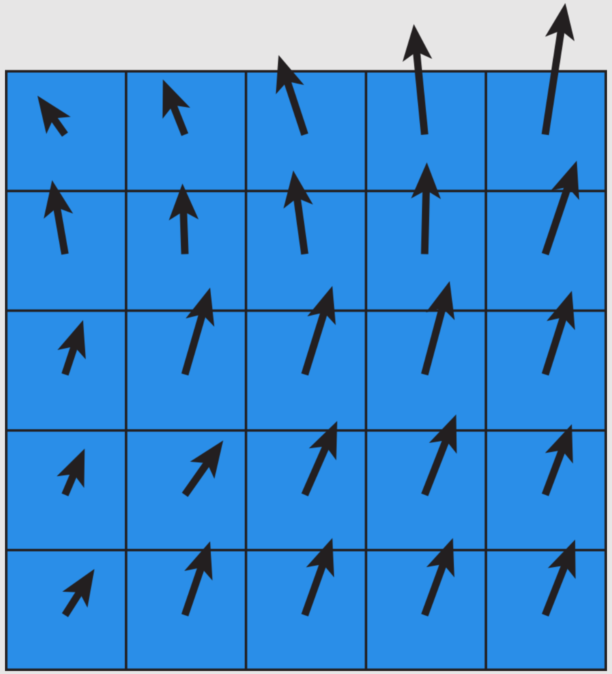
Titanic (1997) James Cameron

Lagrangian vs. Eulerian



[Lagrangian]

track position & velocity
of moving particles



[Eulerian]

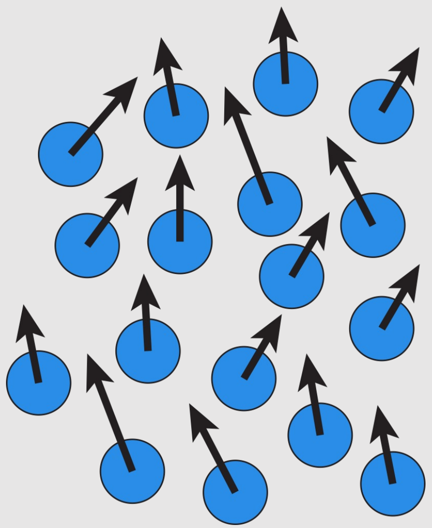
track velocity (or flux)
at fixed grid locations

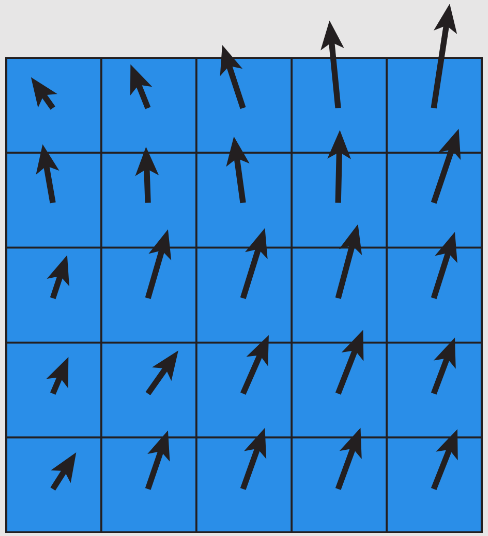
Lagrangian vs. Eulerian

- **Lagrangian:**
  - [+] Conceptually easy (like polygon soup!)
  - [+] Resolution/domain not limited by grid
  - [-] Good particle distribution can be tough
  - [-] Finding neighbors can be expensive

- **Eulerian:**
  - [+] Fast, regular computation
  - [+] Good cache coherence
  - [+] Easy to represent
  - [-] Simulation “trapped” in grid
  - [-] Grid causes “numerical diffusion” (blur/aliasing)
  - [-] Need to understand PDEs (but you will!)

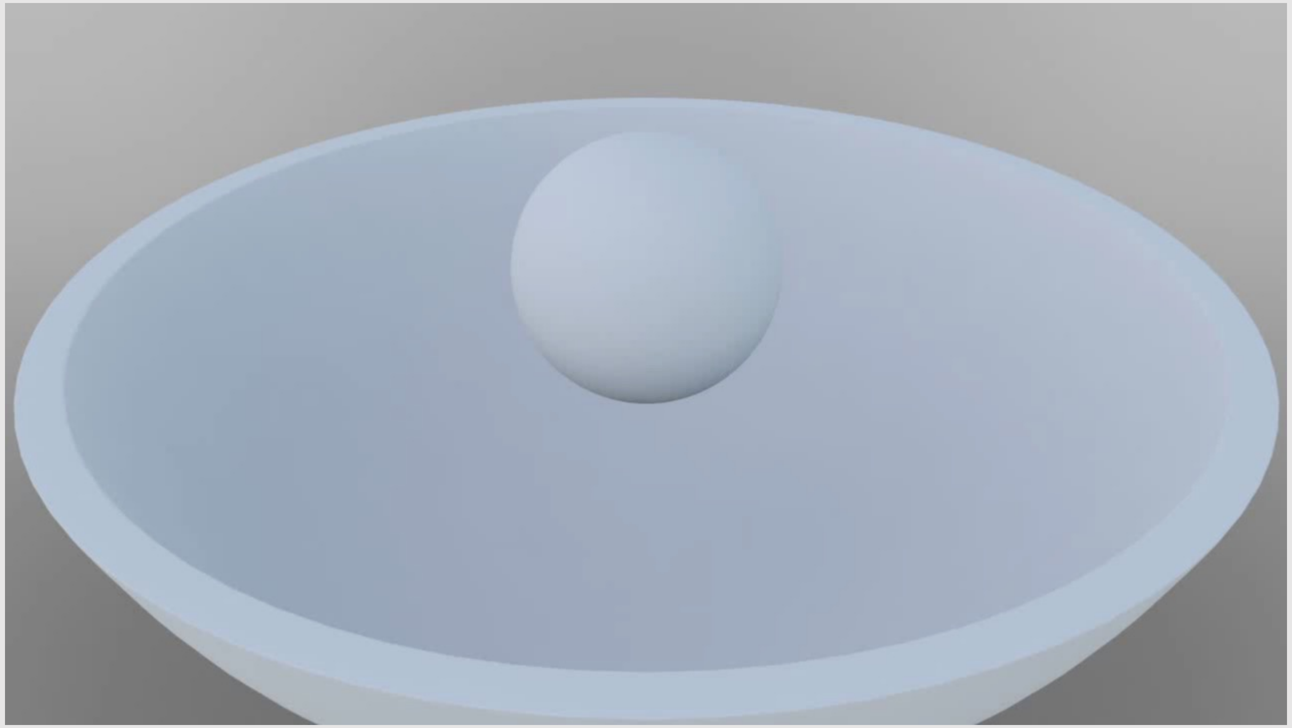
- Where have we seen these formats before?
  - Rasterization!
    - **Lagrangian** is the primitives in our scene
    - **Eulerian** is the pixel representations on our displays





Mixing Lagrangian & Eulerian

- Many modern methods mix Lagrangian & Eulerian:
  - PIC/FLIP, particle level sets, mesh-based surface tracking, Voronoi-based, arbitrary Lagrangian-Eulerian (ALE), ...
- Pick the right tool for the job!
  - If you can’t pick one, pick them all!



The Laplacian Operator

- All of our model equations used the Laplace operator
  - Laplace Equation $\Delta u = 0$
  - Heat Equation $\dot{u} = \Delta u$
  - Wave Equation $\ddot{u} = \Delta u$

- Unbelievably important object showing up everywhere across physics, geometry, signal processing, and more

- What does the Laplacian mean?
  - **Differential operator:** eats a function, spits out its 2nd derivative
  - What does that mean for a function: $u: \mathbb{R}^n \to \mathbb{R}$?
    - Divergence of gradient

$$\Delta u = \nabla \cdot \nabla u$$

    - Sum of second derivatives

$$\Delta u = \frac{\partial u^2}{\partial x_1^2} + \cdots + \frac{\partial u^2}{\partial x_n^2}$$

    - Deviation from local average
    - …

Discretizing The Laplacian

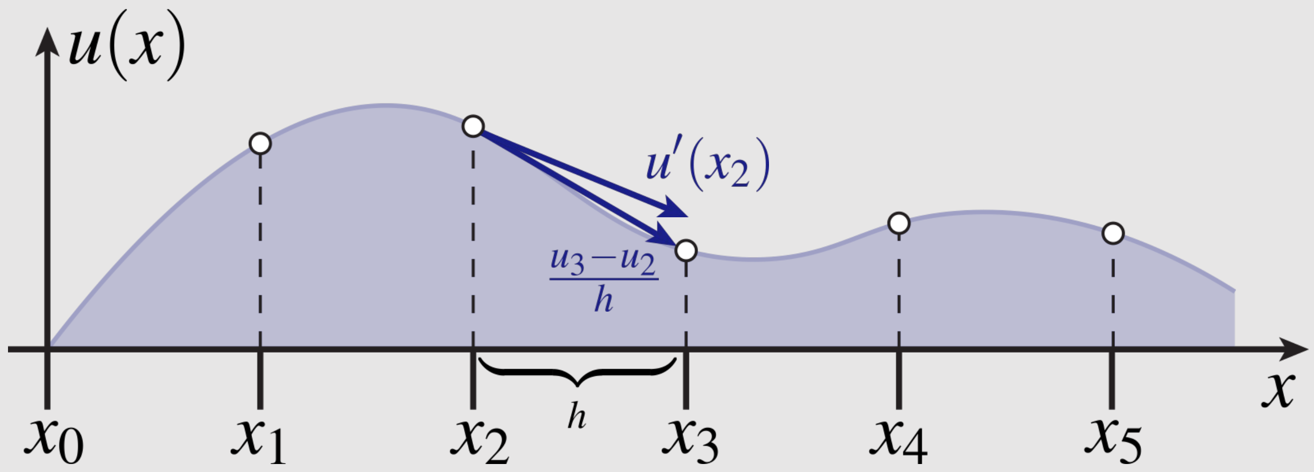
- Consider the Laplacian as a sum of second derivatives:

$$\Delta u = \frac{\partial u^2}{\partial x_1^2} + \cdots + \frac{\partial u^2}{\partial x_n^2}$$

- How do we compute this numerically?

- Consider a non-differentiable function with evaluated samples $x_0, x_1, \ldots$
  - The firs derivative approximated is:

$$u'(x_i) \approx \frac{u_{i+1} - u_i}{h}$$



  - The second derivative approximated is:

$$u''(x_i) \approx \frac{u_i' - u_{i-1}'}{h} \approx \frac{\left(\frac{u_{i+1} - u_i}{h}\right) - \left(\frac{u_i - u_{i-1}}{h}\right)}{h} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

- Known as the **finite difference** approach to PDEs

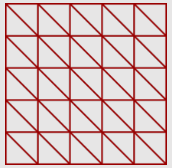
Discretizing The Laplacian

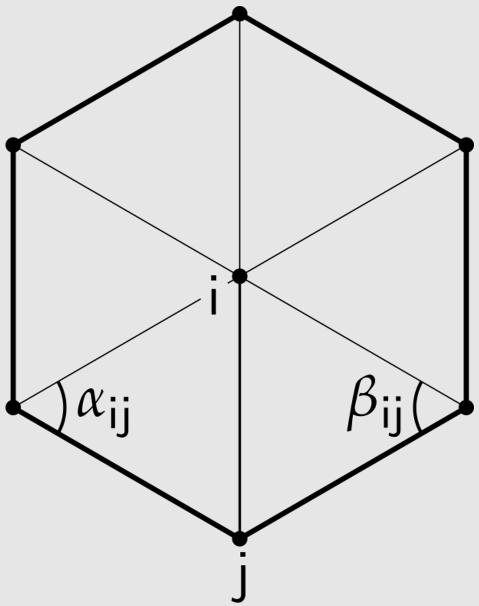
What if u is not a 1D function…

h

	1	
1	-4	1
	1	

If the mesh is a grid, equations become the same





$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2}$$

[Grid]

$$\frac{1}{2}\sum_j (\cot\alpha_{ij} + \cot\beta_{ij})(u_j - u_i)$$

[Triangle Mesh]

Numerically Solving The Laplacian

	$u_{i,j+1}$	
$u_{i-1,j}$	$u_{i,j}$	$u_{i+1,j}$
	$u_{i,j-1}$	

Want to solve $\Delta u = 0$:

$$\frac{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2} = 0$$

Can isolate for $u_{i,j}$:

$$\Leftrightarrow u_{i,j} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right)$$

- If u is a solution, then each value must be the average of the neighboring values
- How do we solve this?
  - **Idea:** keep averaging with neighbors! ("Jacobi method")
  - Correct, but slow
    - Much better to use modern linear solver

Linearly Solving The Laplacian

- We have a bunch of equations of the form:

$$4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

- Index 2D grid using 1D indices
  - Create a matrix with all equations (these are our constraints)

Make sure to use a sparse solver!

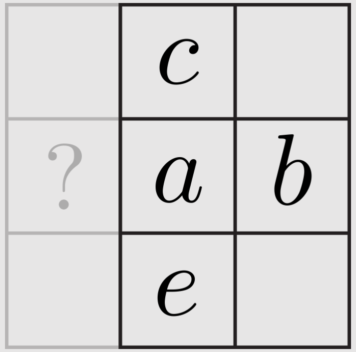
$$\begin{bmatrix}
-4 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -4 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -4
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

What is the issue with this?

Boundary Conditions

- We need boundary conditions that make our solution non-zero
  - Essentially, what is the data we want to interpolate?

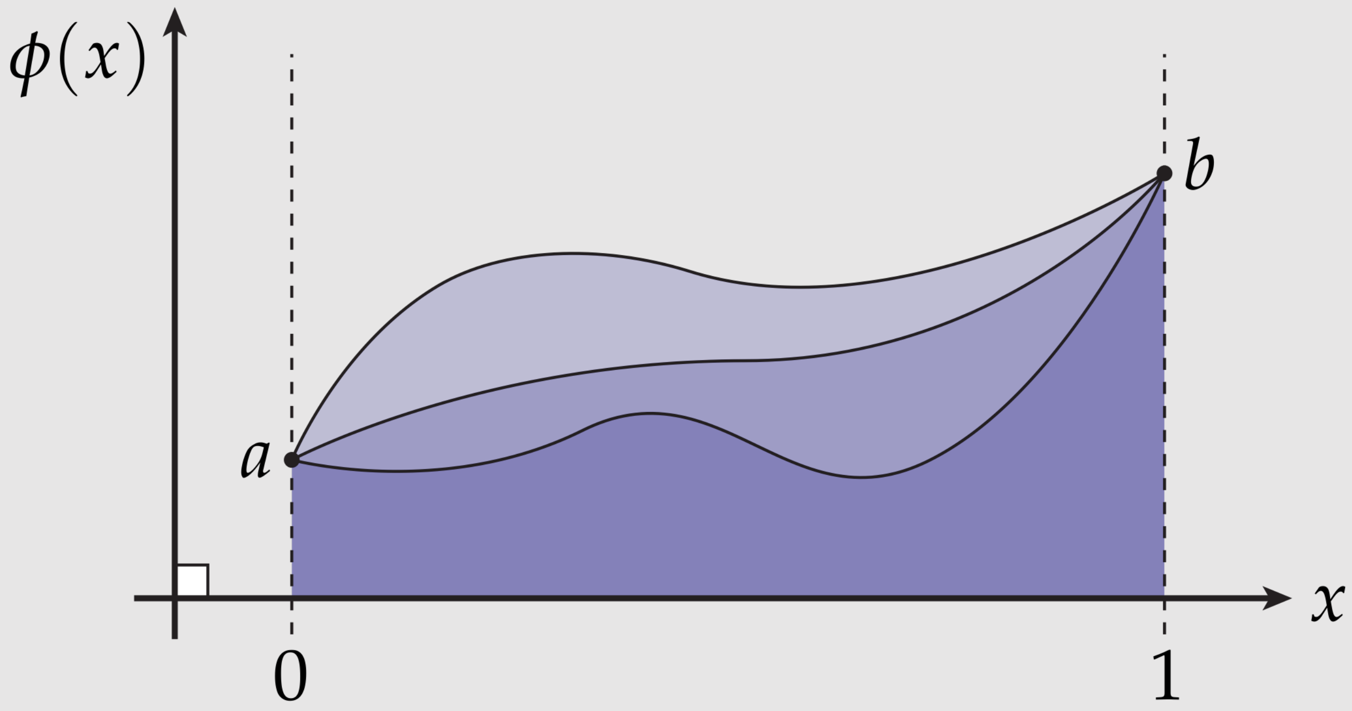


$$a = \frac{1}{4}(b + c + ? + e)$$

- Three types of boundary conditions:
  - **Dirichlet:** boundary data always set to fixed values
  - **Neumann:** specify derivatives across boundary
  - **Robin:** mix of fixed and derivative values
    - Many more in general, but this is all we will cover

Dirichlet Boundary Conditions

Dirichlet: boundary data always set to fixed values

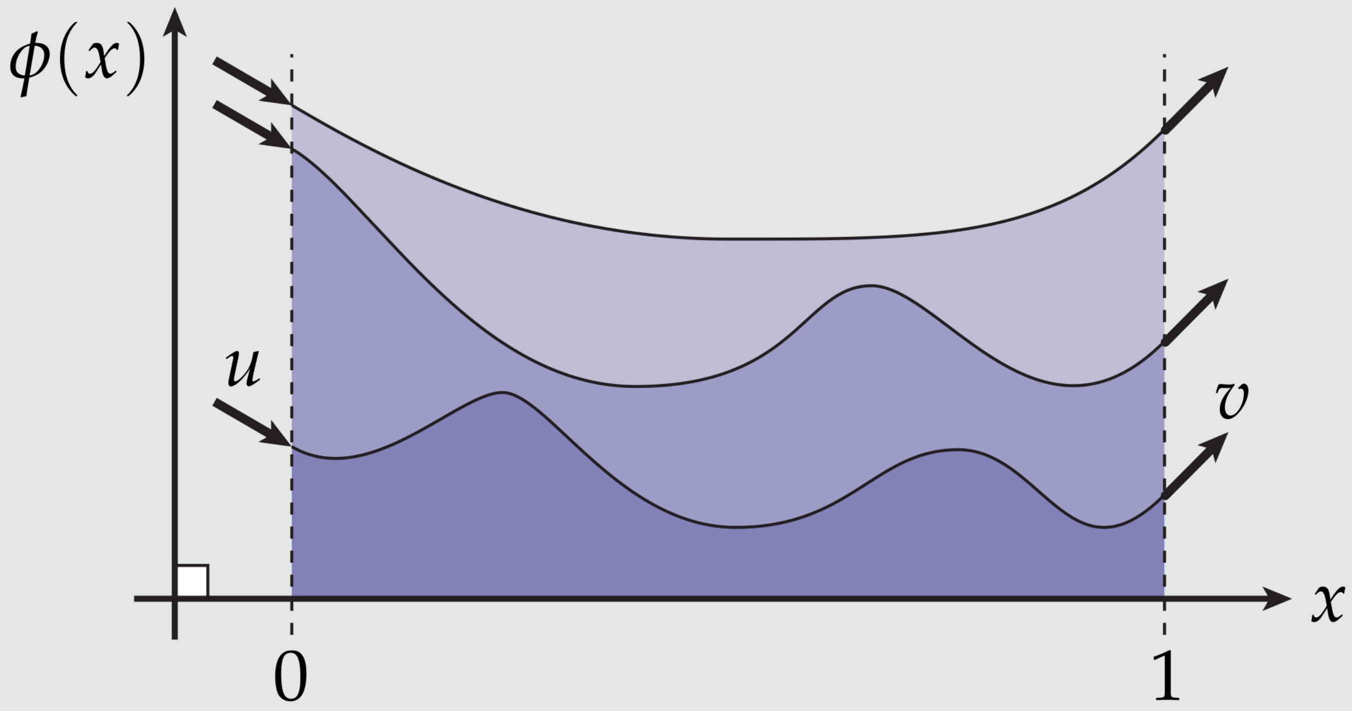


Example: $\phi(0) = a, \phi(1) = b$

Many possible functions interpolate values in between

Neumann Boundary Conditions

Neumann: specify derivatives across boundary



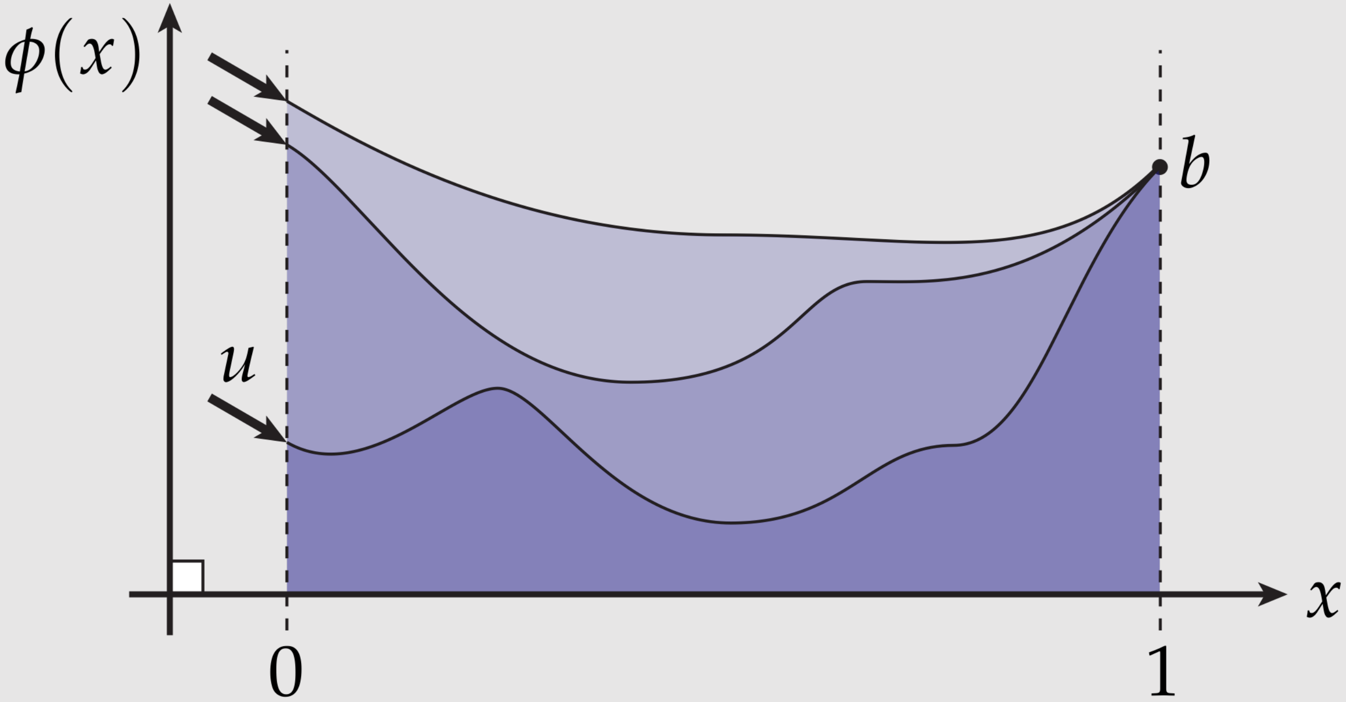
Example: $\phi'(0) = u,\ \ \phi'(1) = v$

Again, many possible functions

Dirichlet + Neumann Boundary Conditions

Dirichlet: boundary data always set to fixed values

Neumann: specify derivatives across boundary



Example: $\phi'(0) = u,\ \phi(1) = b$

Still, many possible functions

What about (Robin): $\phi'(0) + \phi(0) = p,\ \phi'(1) + \phi(1) = q$

We can generate a continuous function for any of the boundary conditions,
But does there exist a Laplacian solution for any set of boundary conditions?

Solution To The Laplacian

- Consider a 1D function
  - What is the solution to:

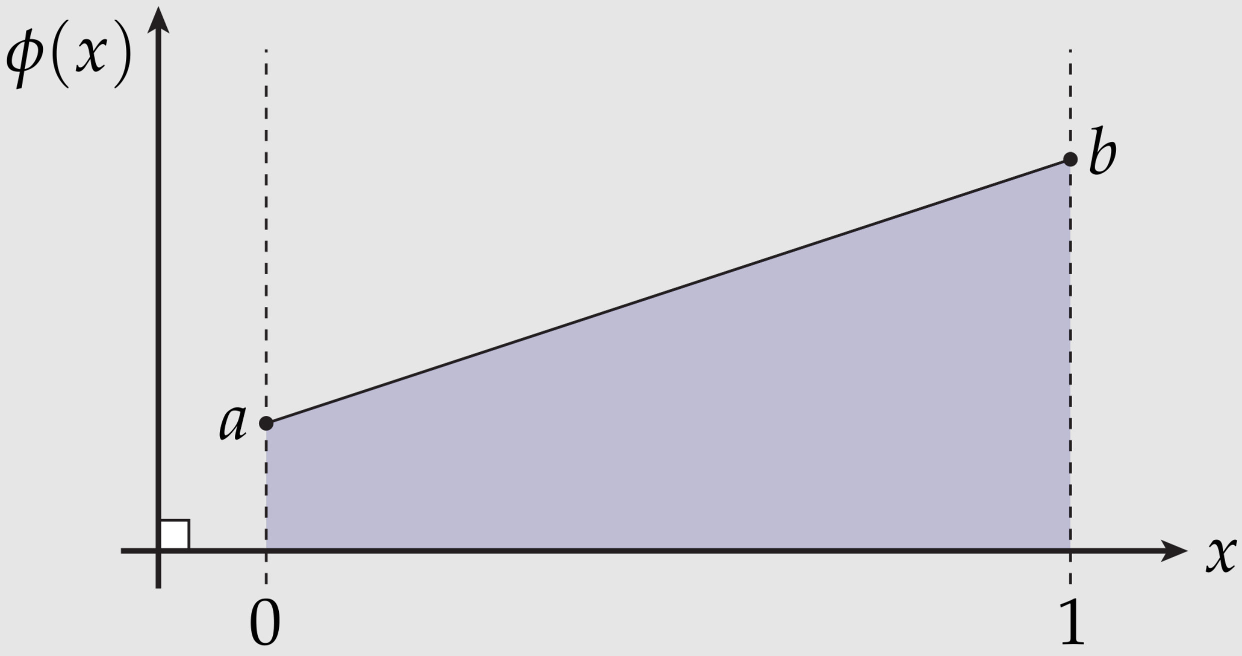
$$\Delta u = 0$$

  - Any function who's second derivative is 0

$$\partial^2 \phi / \partial x^2 = 0$$

  - Any function that is linear

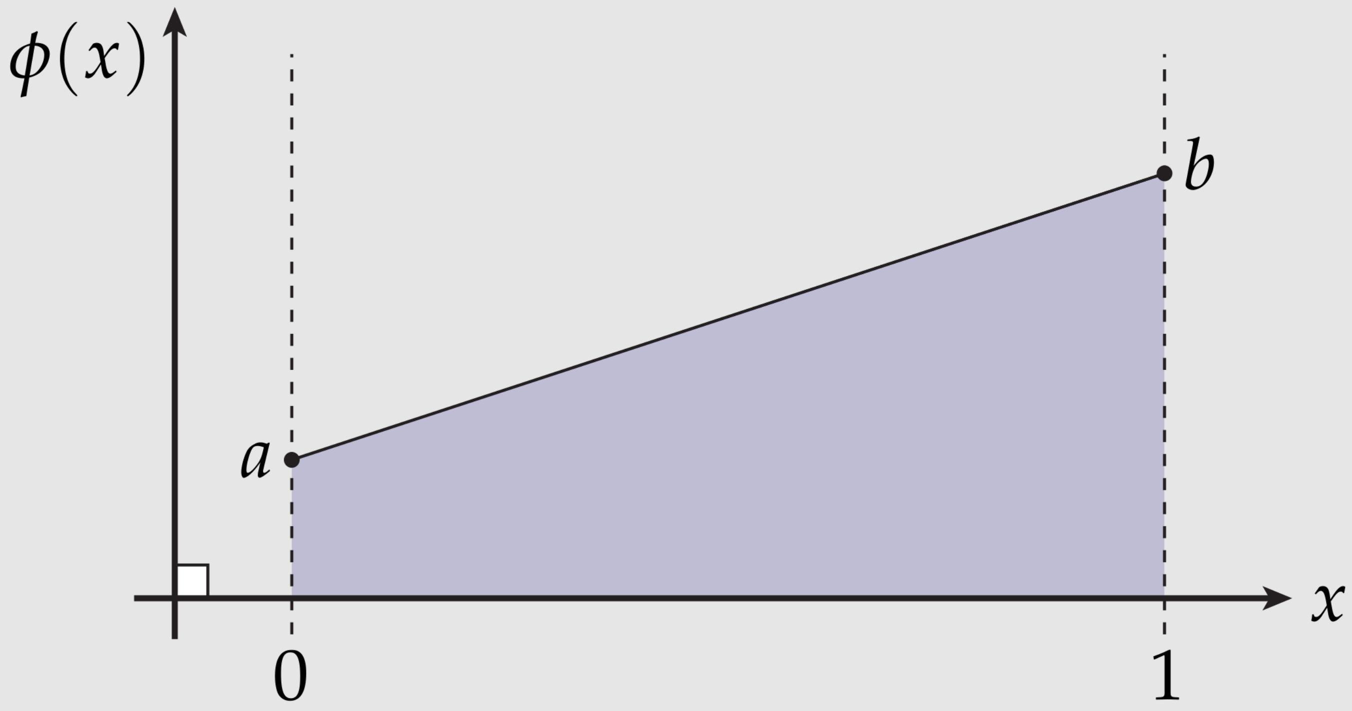
$$\phi(x) = cx + d$$



- Makes sense conceptually
  - The Laplacian gives us the resting state of diffusion
  - The resting state is a linear function between boundary conditions

1D Laplacian With Dirichlet

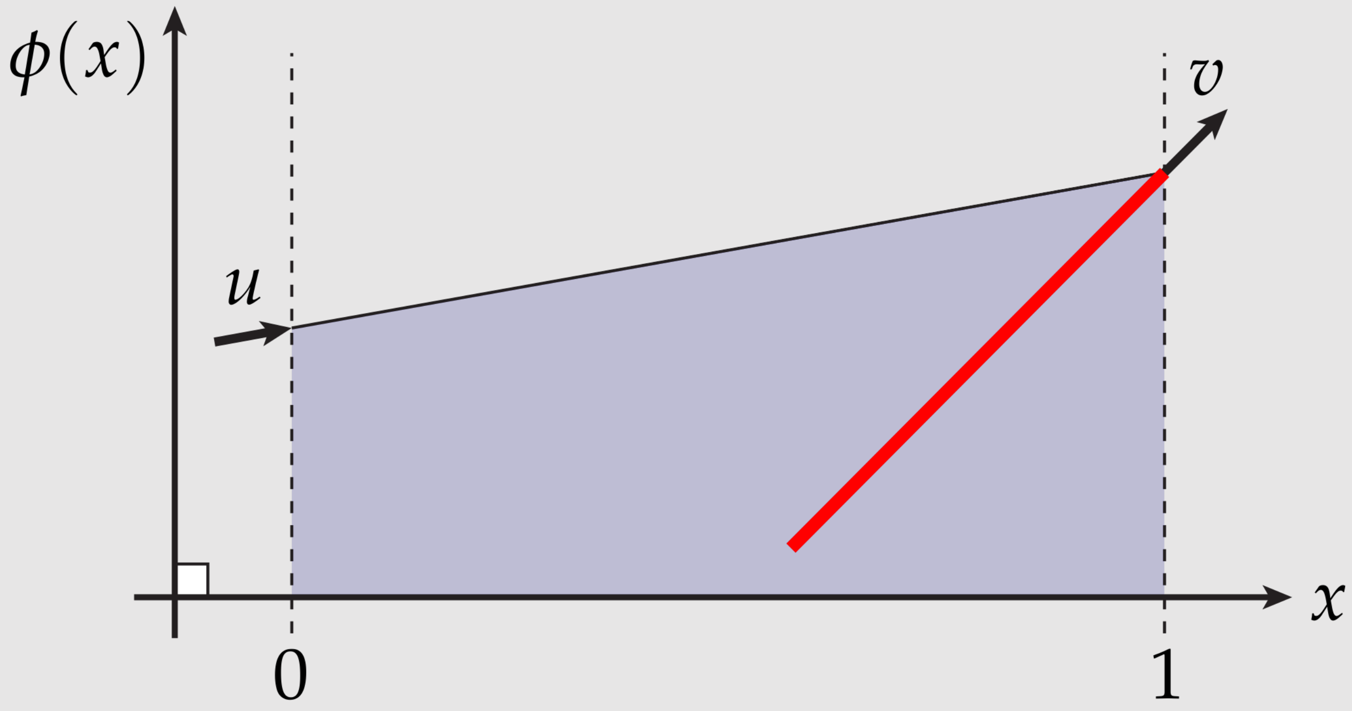
Can we always satisfy Dirichlet boundary conditions in 1D?



Yes! A line can always interpolate two points

1D Laplacian With Neumann

Can we always satisfy Neumann boundary conditions in 1D?

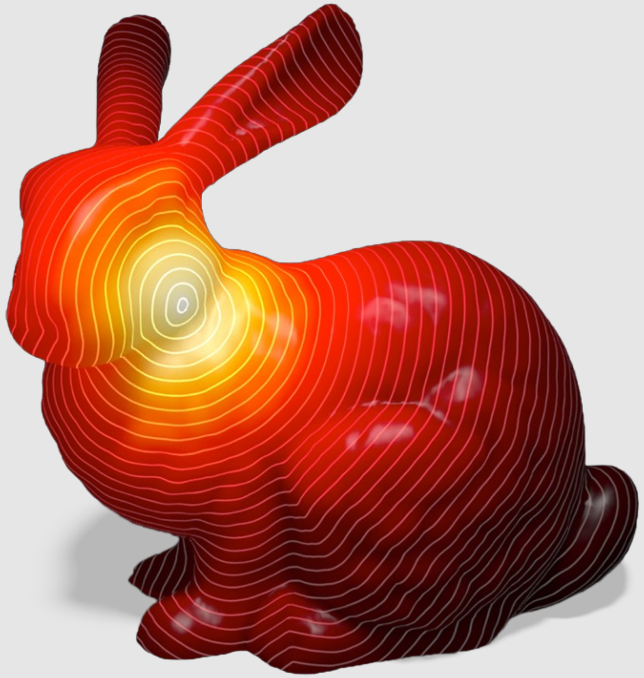


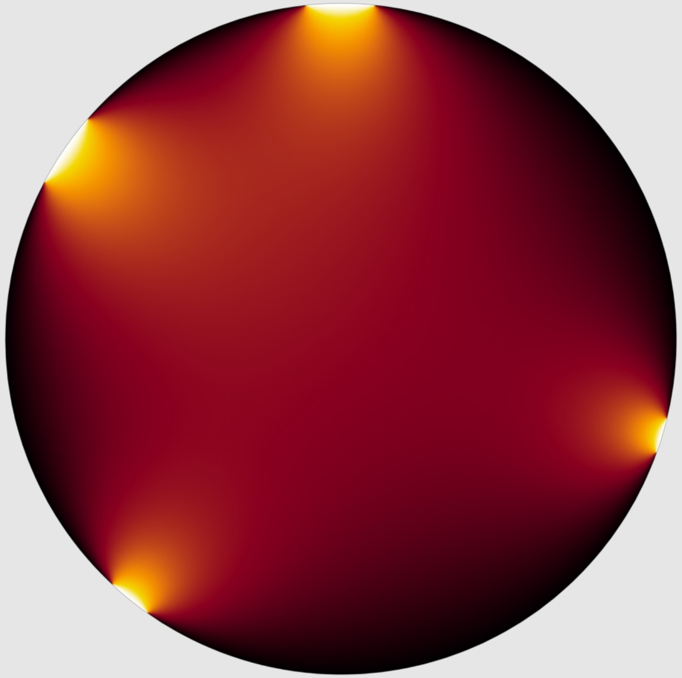
No! A line can only have one slope

Not always guaranteed that a PDE has a solution for given boundary conditions…

2D Laplacian With Dirichlet

Can we always satisfy Dirichlet boundary conditions in 2D?





Yes! Laplacian is a long-time solution to heat flow
Data is “heat” at boundary. Will eventually diffuse to equilibrium

2D Laplacian With Neumann

Can we always satisfy Neumann boundary conditions in 2D?

Neumann BCs prescribe derivative in normal direction: $n \cdot \nabla\phi$

Want to solve for $\Delta\phi = 0$

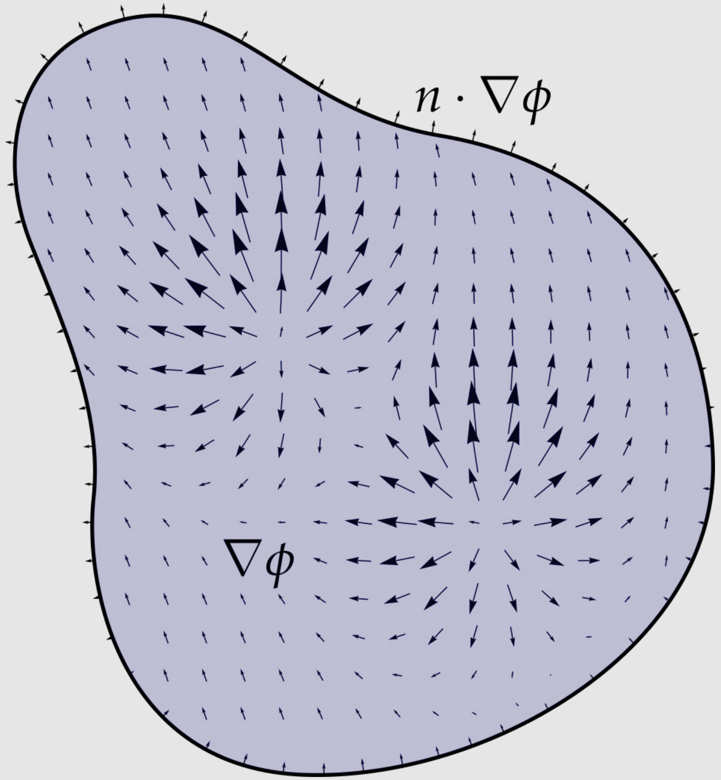
In 2D, we have the **divergence theorem:**

$$\int_{\partial\Omega} n \cdot \nabla\phi = \int_{\Omega} \nabla \cdot \nabla\phi = \int_{\Omega} \Delta\phi \overset{!}{=} 0$$

integrating ϕ over boundary

integrating divergence of ϕ over interior

what goes in, must come out!



Can't have a solution unless the net flux through the boundary is zero!

Numerical libraries will not always tell you that there is a problem with your boundary conditions

Need to verify yourself. If solving $Ax = b$, verify $||b - Ax||$

Modeling PDE Equations

EASIER

Laplace Equation [Elliptic]
"What's the smoothest function interpolating the given boundary data?"

$$\Delta u = 0$$

INTERMEDIATE

Heat Equation [Parabolic]
"How does an initial distribution of heat spread out over time?"

$$\dot{u} = \Delta u$$

ADVANCED

Wave Equation [Hyperbolic]
"If you throw a rock into a pond, how does the wavefront evolve over time?"

$$\ddot{u} = \Delta u$$

EXPERTS ONLY

Nonlinear + Hyperbolic + High-Order
"A lot of real life phenomenon"

? ? ?

How to solve these?

Solving The Heat Equation

Heat equation tells us the Laplacian is the first temporal derivative:

$$\dot{u} = \Delta u$$

Compute the Laplacian as normal (Ex: on a grid):

$$u_{i,j}^{k+1} = u^k + \tfrac{\tau}{h^2}\left(4u_{i,j}^k - u_{i+1,j}^k - u_{i-1,j}^k - u_{i,j+1}^k - u_{i,j-1}^k\right)$$

Propagate using the first temporal derivative Δu (Ex: forward Euler):

$$u^{k+1} = u^k + \Delta u^k$$

Solving The Wave Equation

Wave equation tells us the Laplacian is the second temporal derivative:

$$\ddot{u} = \Delta u$$

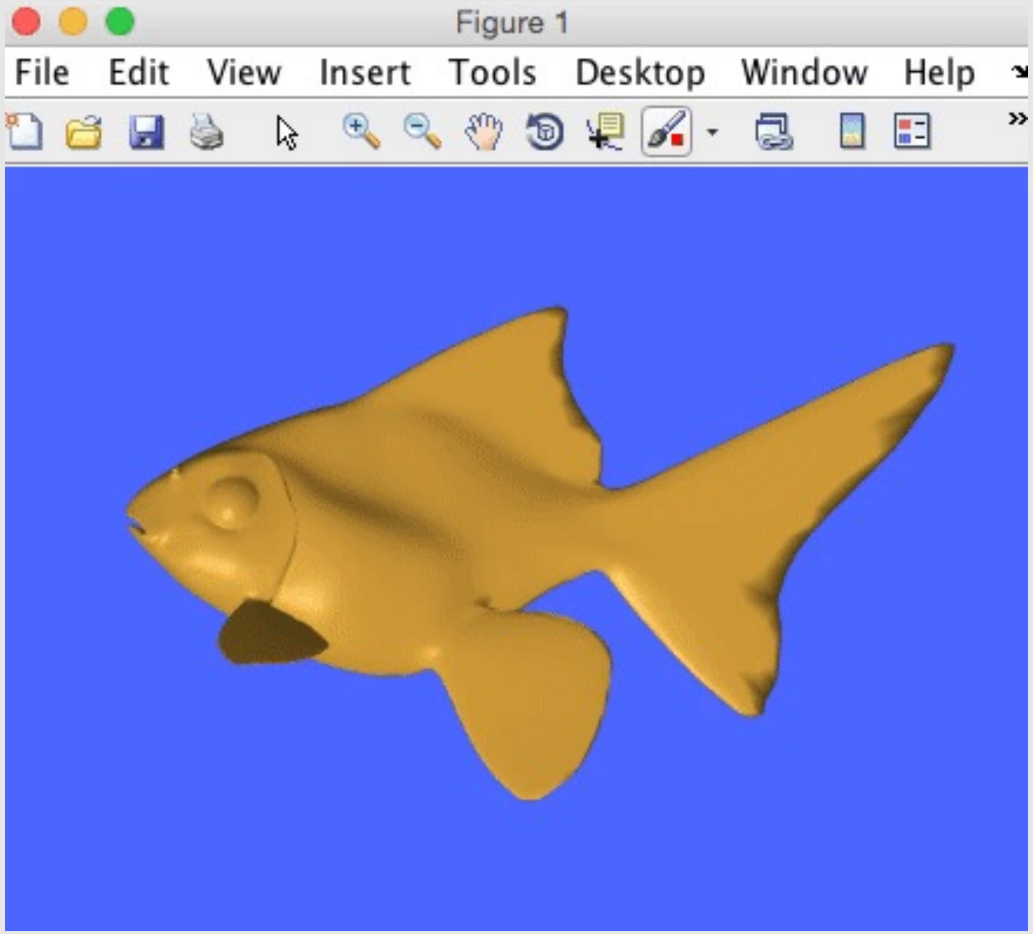
Compute the Laplacian as normal (Ex: on a grid):

$$u_{i,j}^{k+1} = u^k + \frac{\tau}{h^2}\left(4u_{i,j}^k - u_{i+1,j}^k - u_{i-1,j}^k - u_{i,j+1}^k - u_{i,j-1}^k\right)$$

Propagate using the second temporal derivative Δu (Ex: forward Euler):

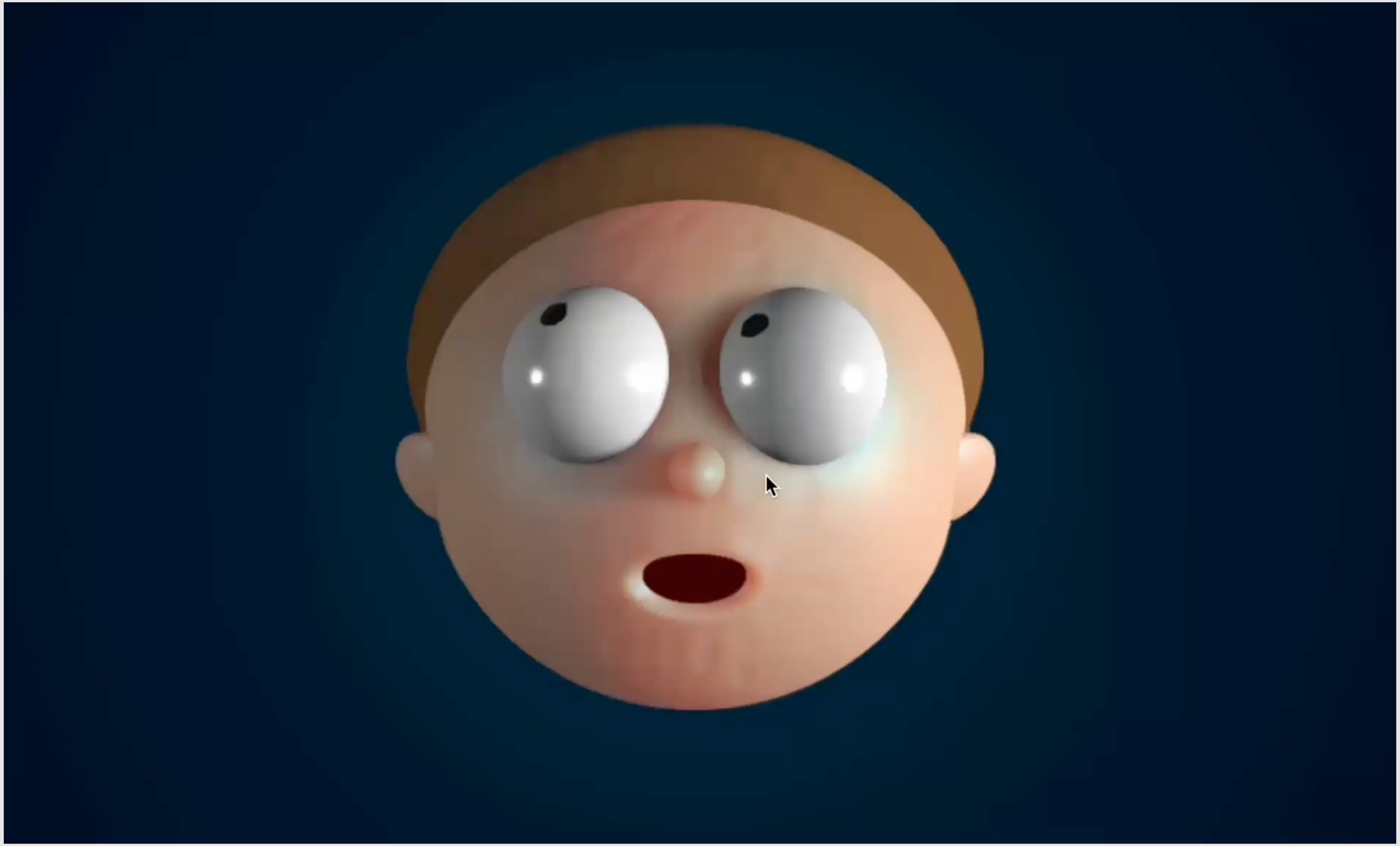
$$\dot{u} = v, \quad \dot{v} = \Delta u$$

Wave Equation On A Triangle Mesh



Wave Equation On Surfaces (2016) Alec Jacobson

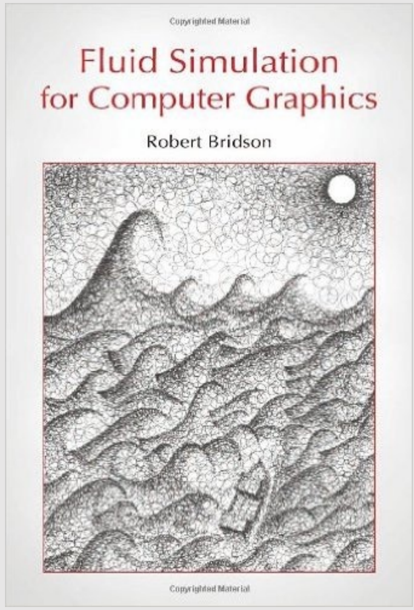
Wave Equation On A Triangle Mesh



https://www.adultswim.com/etcetera/elastic-man/

Want To Know More?

Plenty of books and papers on Simulation



What did the folks who wrote these papers/books read?