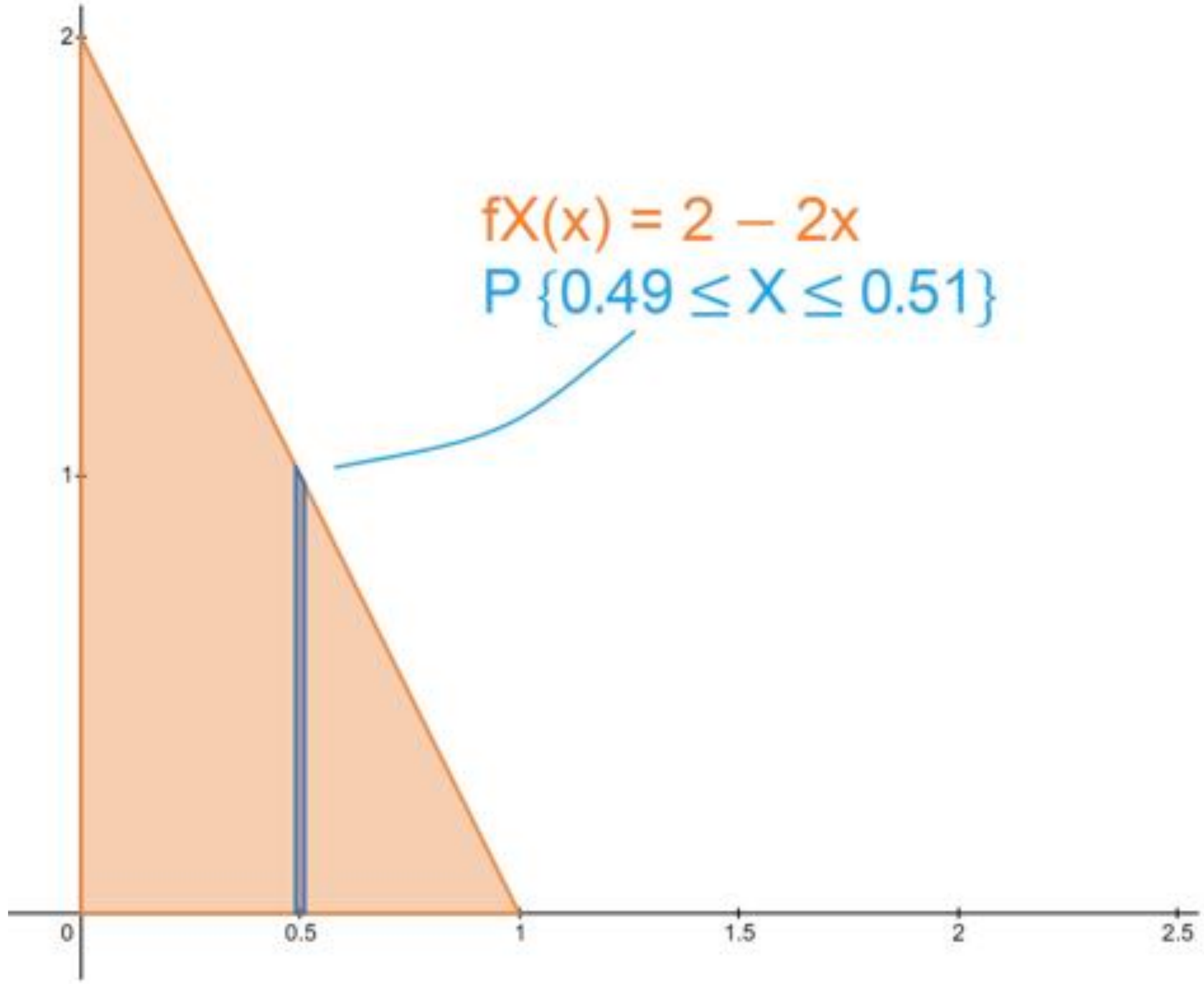
Variance Reduction

Computer Graphics
CMU 15-462/15-662

MiniHW 7: Continuous Probability

- **Due before class Monday**



Last time: Monte Carlo Ray Tracing

- Recursive description of incident illumination
- Difficult to integrate; tour de force of numerical integration
- Leads to lots of sophisticated integration strategies:
  - sampling strategies
  - variance reduction
  - Markov chain methods
  - ...
- Today: get a glimpse of these ideas
- Also valuable outside rendering!
  - Monte Carlo one of the "Top 10 Algorithms of the 20th Century"!

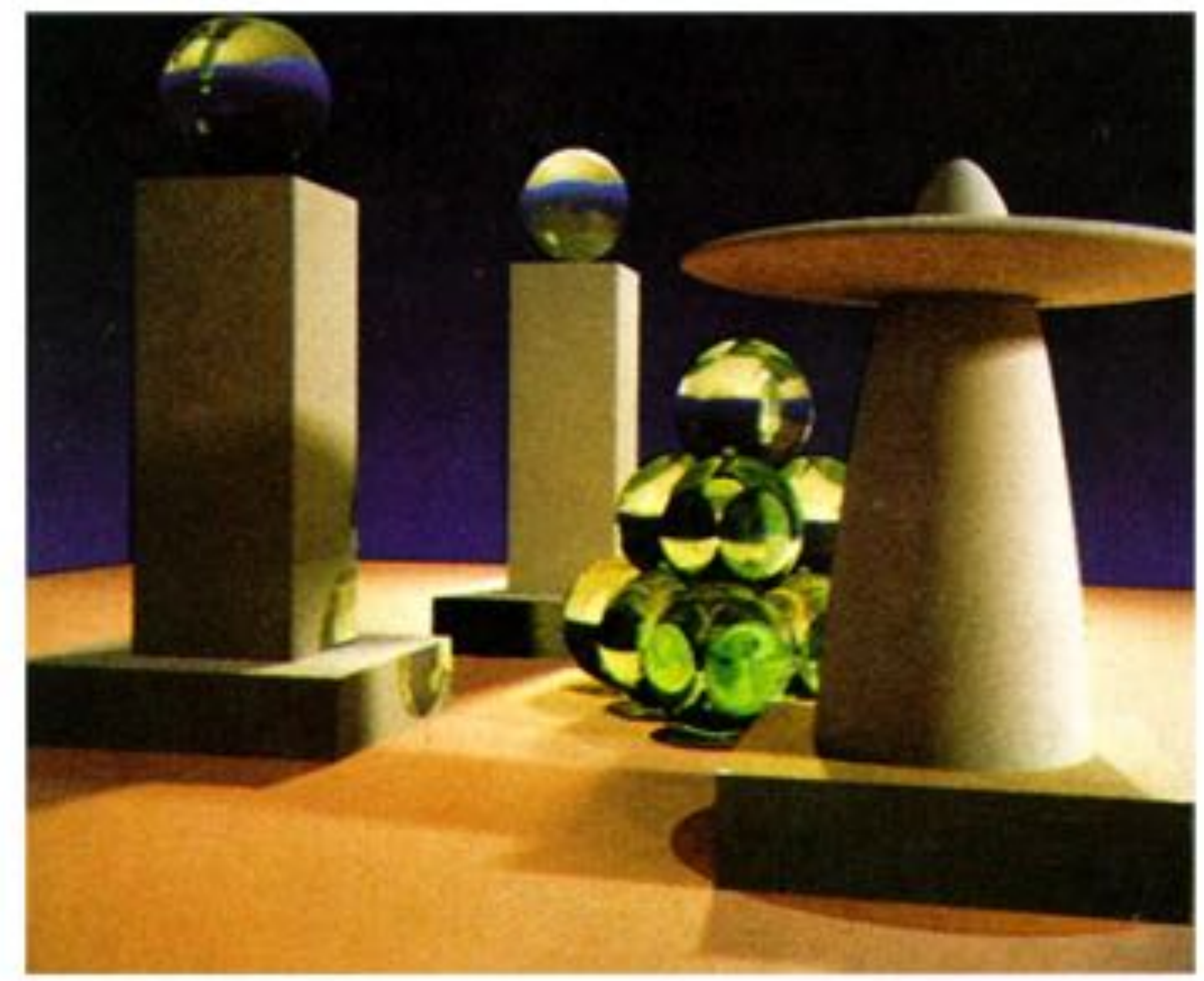


Figure 6. A sample image. All objects are neutral grey. Color on the objects is due to caustics from the green glass balls and color bleeding from the base polygon.

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_\Omega f_r(\mathbf{x}, \omega_i, \omega_o)\, L_i(\mathbf{x}, \omega_i)\, (\omega_i \cdot \mathbf{n})\, d\omega_i$$

Review: Monte Carlo Integration

Want to integrate:

$$I := \int_{\Omega} f(x)\; dx$$

any function*

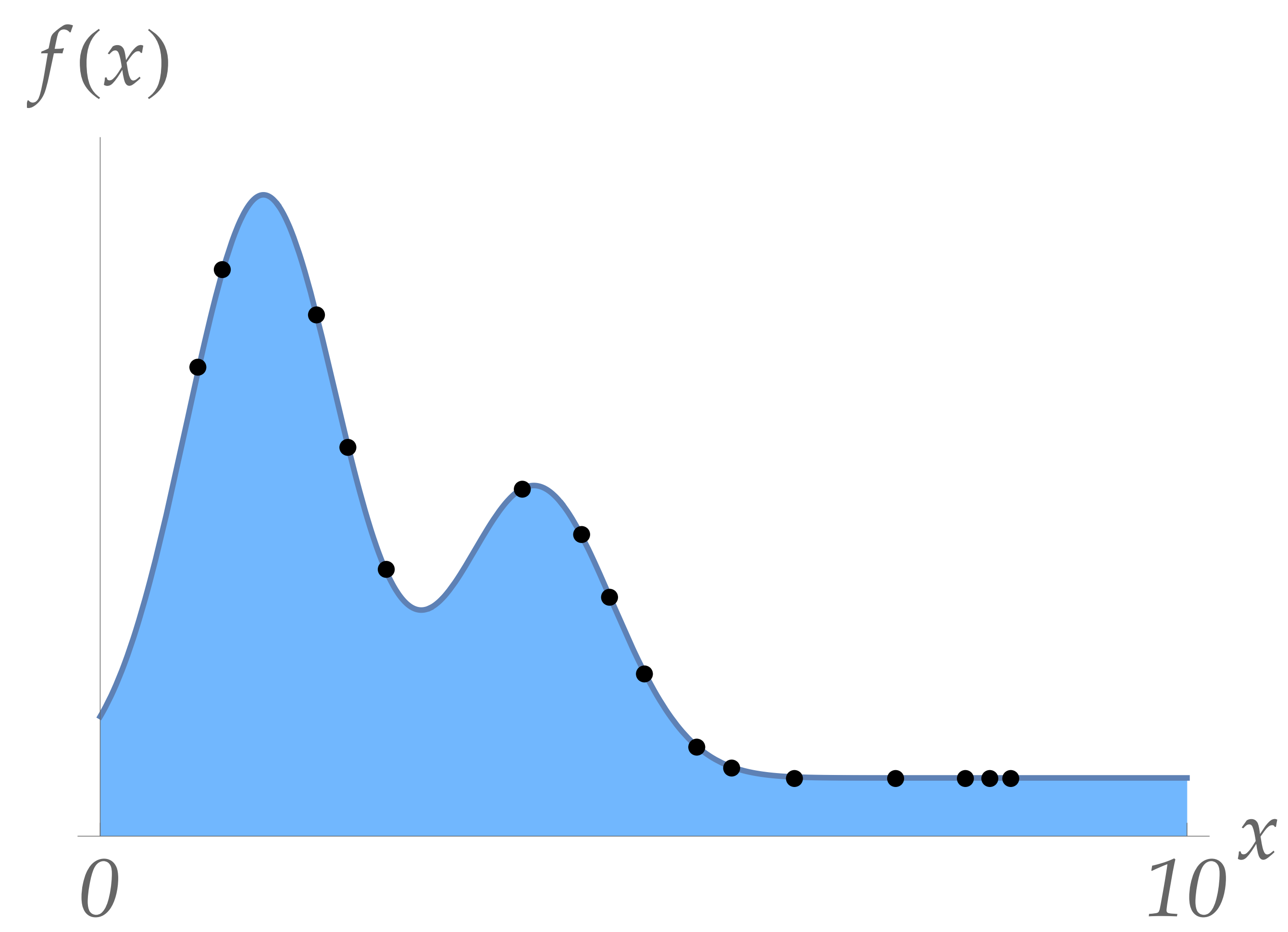
any domain

General-purpose hammer: Monte-Carlo integration

$$I = \lim_{n\to\infty} V(\Omega)\frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

volume of the domain

uniformly random samples of domain



*Must of course have a well-defined integral!

Review: Expected Value (DISCRETE)

A discrete random variable X has n possible outcomes x_i, occuring w/ probabilities $0 \le p_i \le 1$, $p_1 + \ldots + p_n = 1$

probability of event i

$$E(X) := \sum_{i=1}^{n} p_i x_i$$

value of event i

expected value

(just the "weighted average"!)

E.g., what's the expected value for a fair coin toss?

$p_1 = 1/2$
$x_1 = 1$

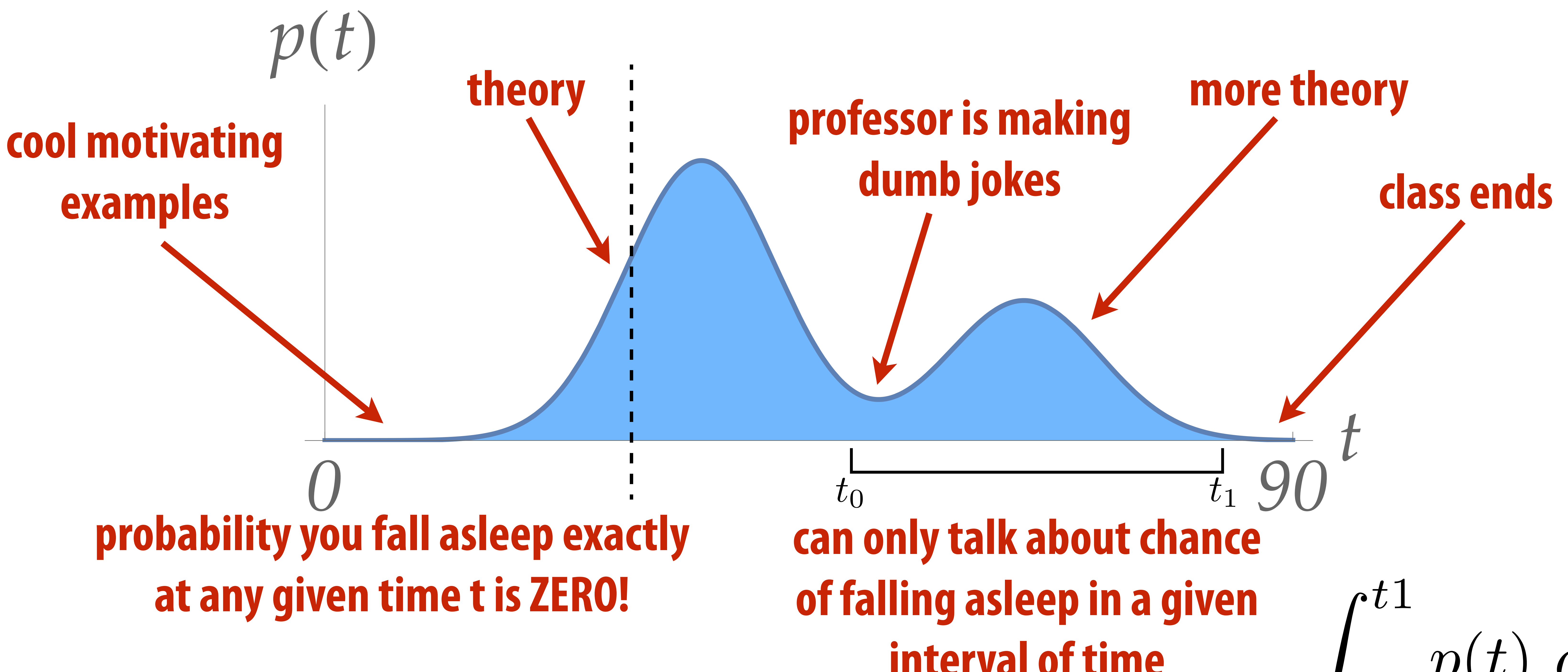
$p_2 = 1/2$
$x_2 = 0$

Continuous Random Variables

A continuous random variable X takes values x anywhere in a set Ω

Probability **density** p gives probability x appears in a given region.

E.g., probability you fall asleep at time t in a 15-462 lecture:



$$\int_{t_0}^{t1} p(t)\ dt$$

Review: Expected Value (CONTINUOUS)

Expected value of continuous random variable again just the "weighted average" with respect to probability p:

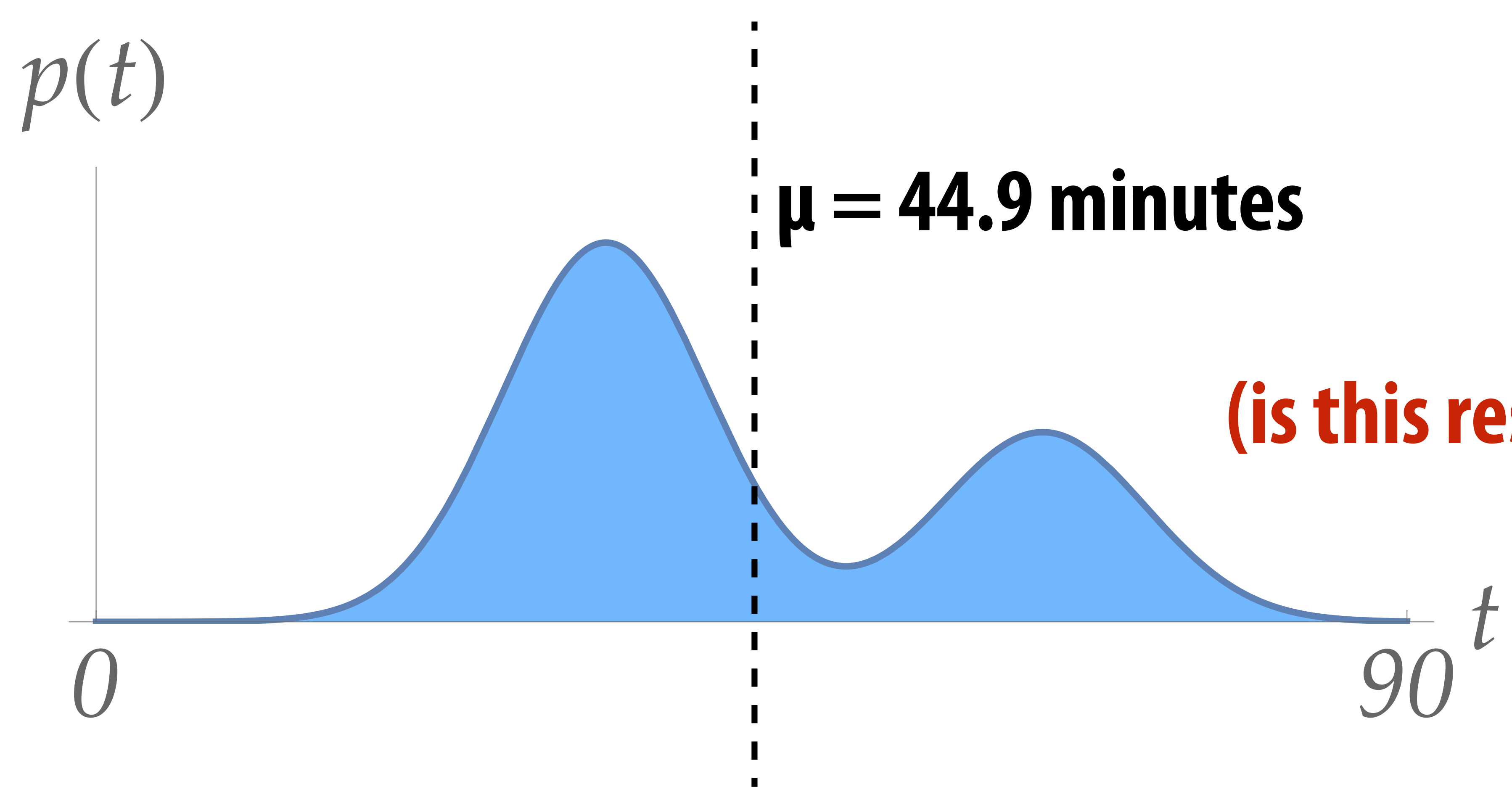
probability density at point x

$$E(X) := \int_\Omega x p(x)\, dx$$

expected value

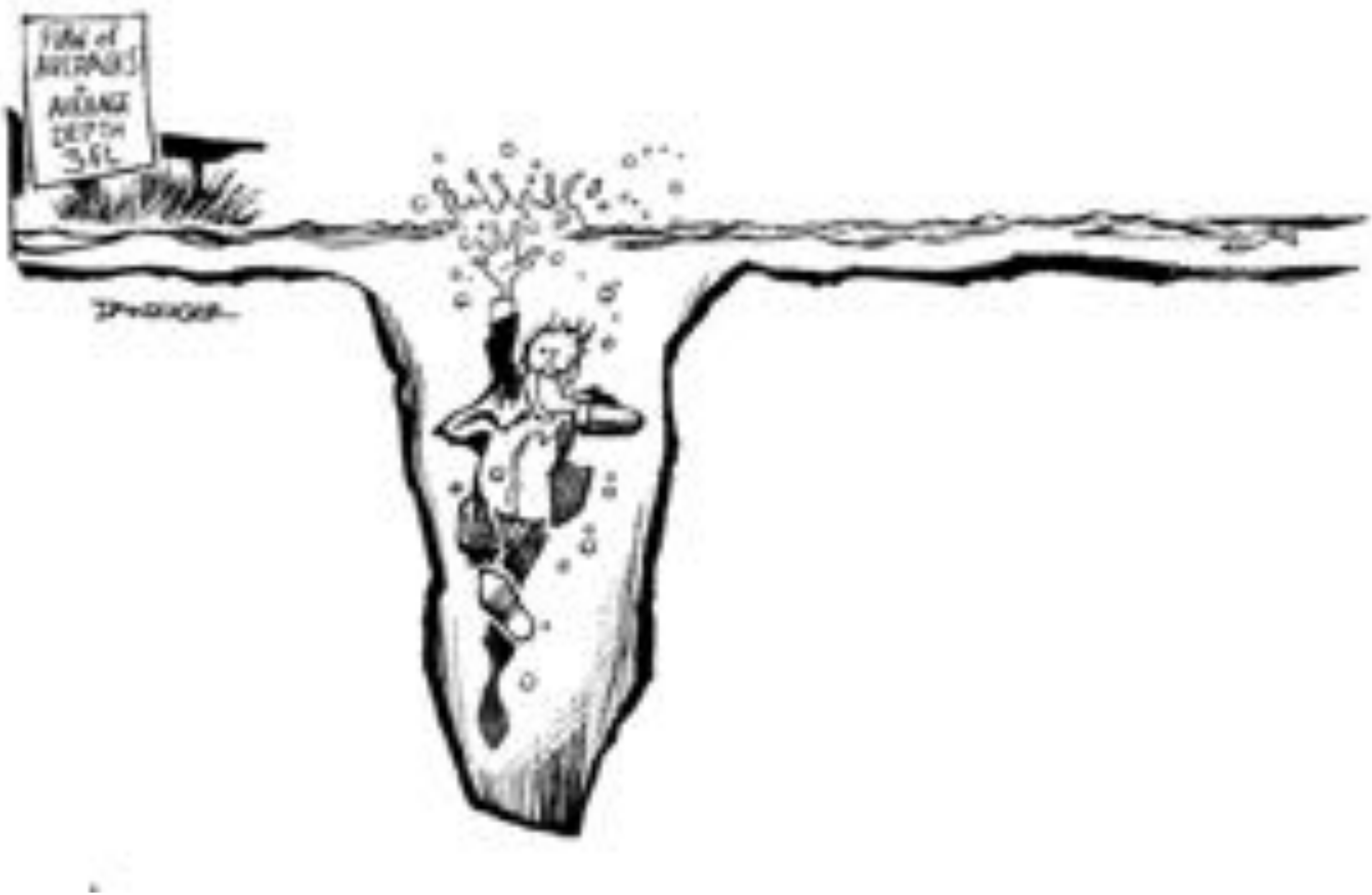
sometimes just use "μ" (for "mean")

E.g., expected time of falling asleep?



(is this result counter-intuitive?)

Flaw of Averages



Review: Variance

- Expected value is the "average value"
- Variance is how far we are from the average, on average!

$$\mathrm{Var}(X) := E[(X - E[X])^2]$$

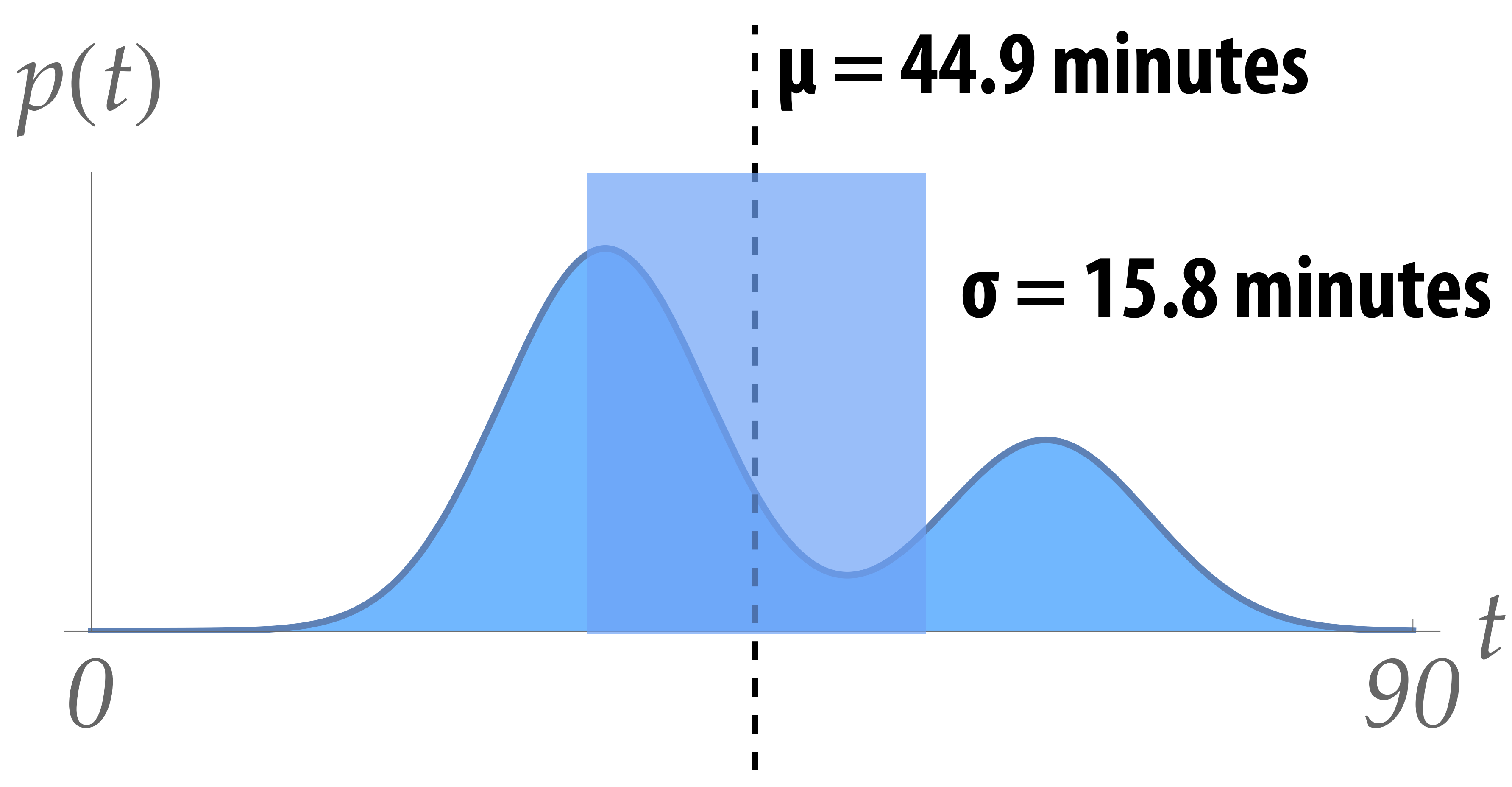
DISCRETE

$$\sum_{i=1}^{n} p_i (x_i - \sum_j p_j x_j)^2$$

CONTINUOUS

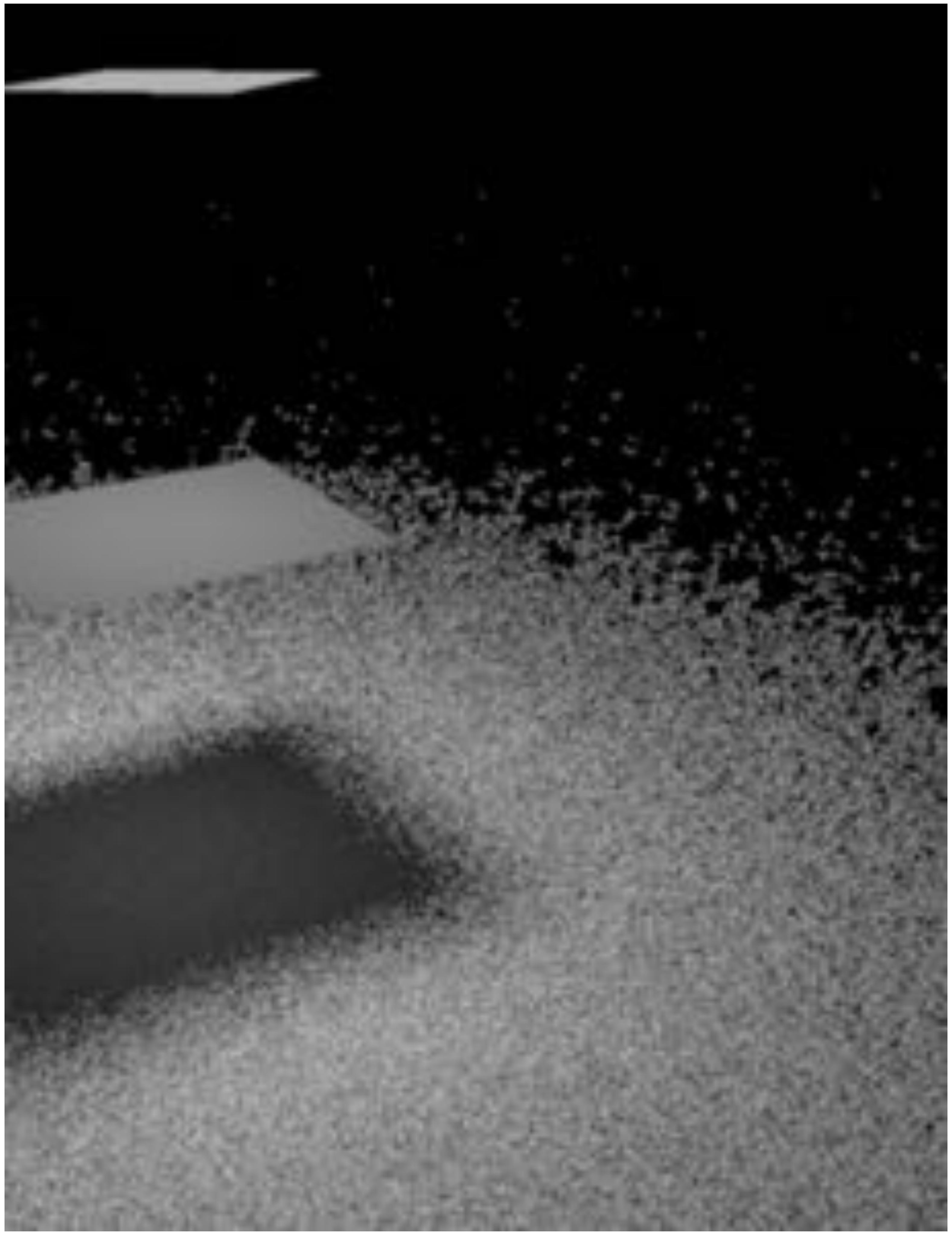
$$\int_\Omega p(x)(x - \int_\Omega y p(y)\ dy)^2\ dx$$

- Standard deviation σ is just the square root of variance

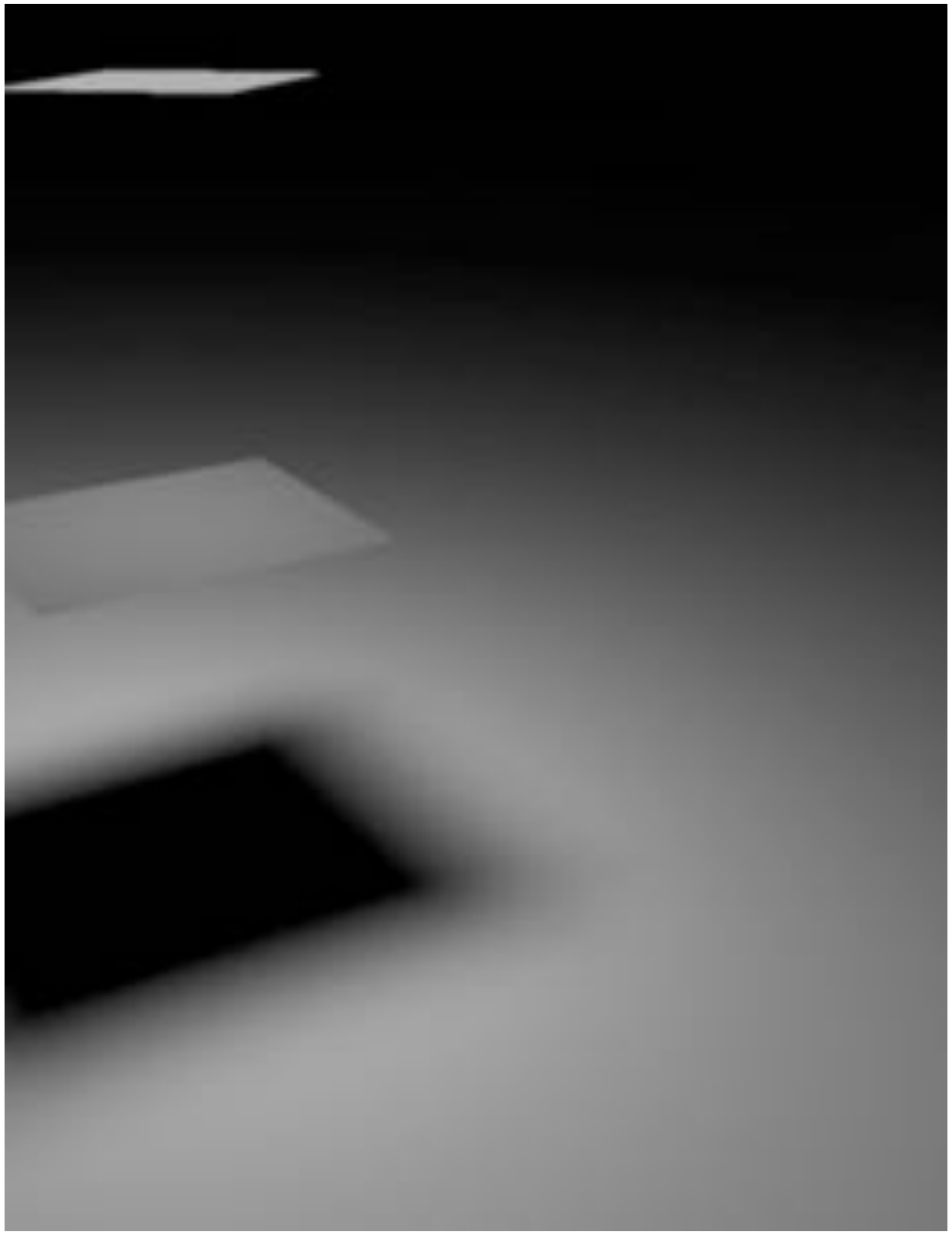


(More intuitive perhaps?)

Variance Reduction in Rendering



higher variance



lower variance

Q: How do we reduce variance?

Variance of an Estimator

- An "estimator" is a formula used to approximate an integral
- Most important example: our Monte Carlo estimate:

$$I = \int_\Omega f(x)\ dx$$

true integral

$$\hat{I} := V(\Omega)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

Monte Carlo estimate

- Get different estimates for different collections of samples
- Want to reduce variance of estimate across different samples
- Why? Integral itself only has one value!
- Many, many (many) techniques for reducing variance
- We will review some key examples for rendering

Bias & Consistency

- Two important things to ask about an estimator
  - Is it consistent?
  - Is it biased?
- Consistency: "converges to the correct answer"

estimate

$$\lim_{n \to \infty} P(|I - \hat{I}_n| > 0) = 0$$

true integral

\# of samples

- Unbiased: "estimate is correct on average"

$$E[I - \hat{I}_n] = 0$$

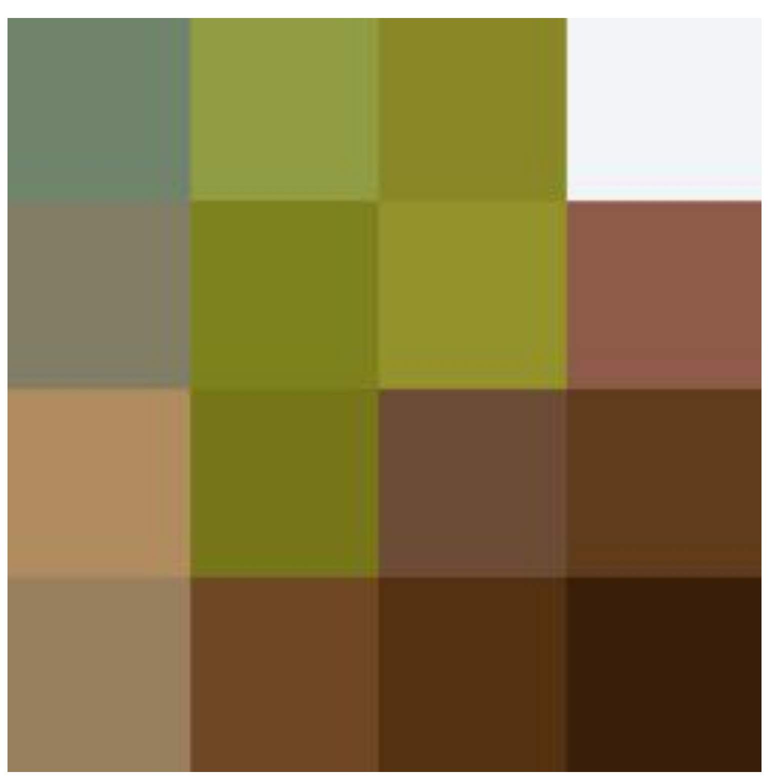
expected value

...even if n=1! (only one sample)

- Consistent does not imply unbiased!

Example 1: Consistent or Unbiased?

- My estimator for the integral over an image:
  - take n = m x m samples at fixed grid points
  - sum the contributions of each box
  - let m go to ∞



m = 4

m = 16

m = 64

m = ∞

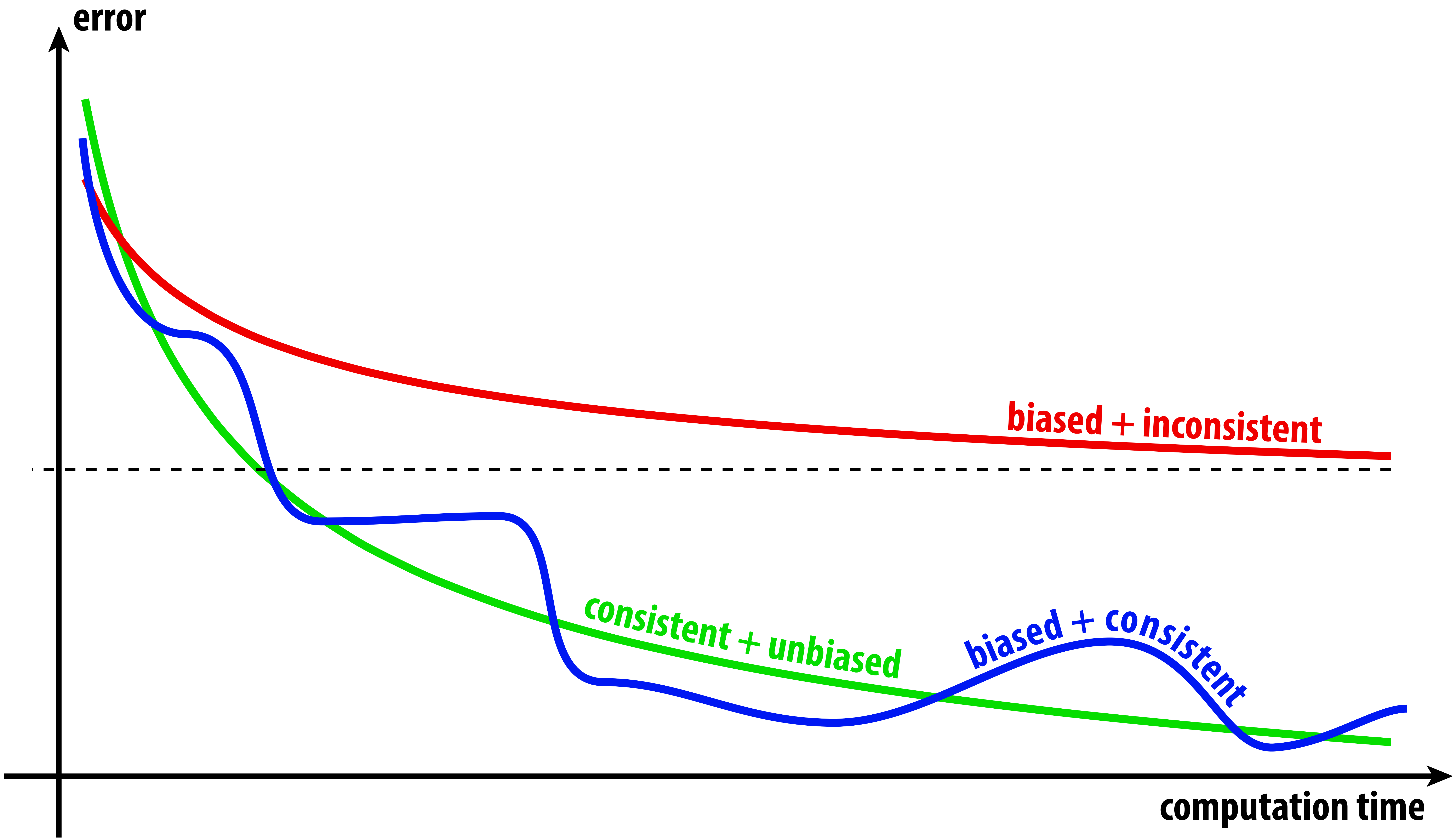
Is this estimator consistent? Unbiased?

Example 2: Consistent or Unbiased?

- My estimator for the integral over an image:
  - take only a **single** random sample of the image (n=1)
  - multiply it by the image area
  - use this value as my estimate

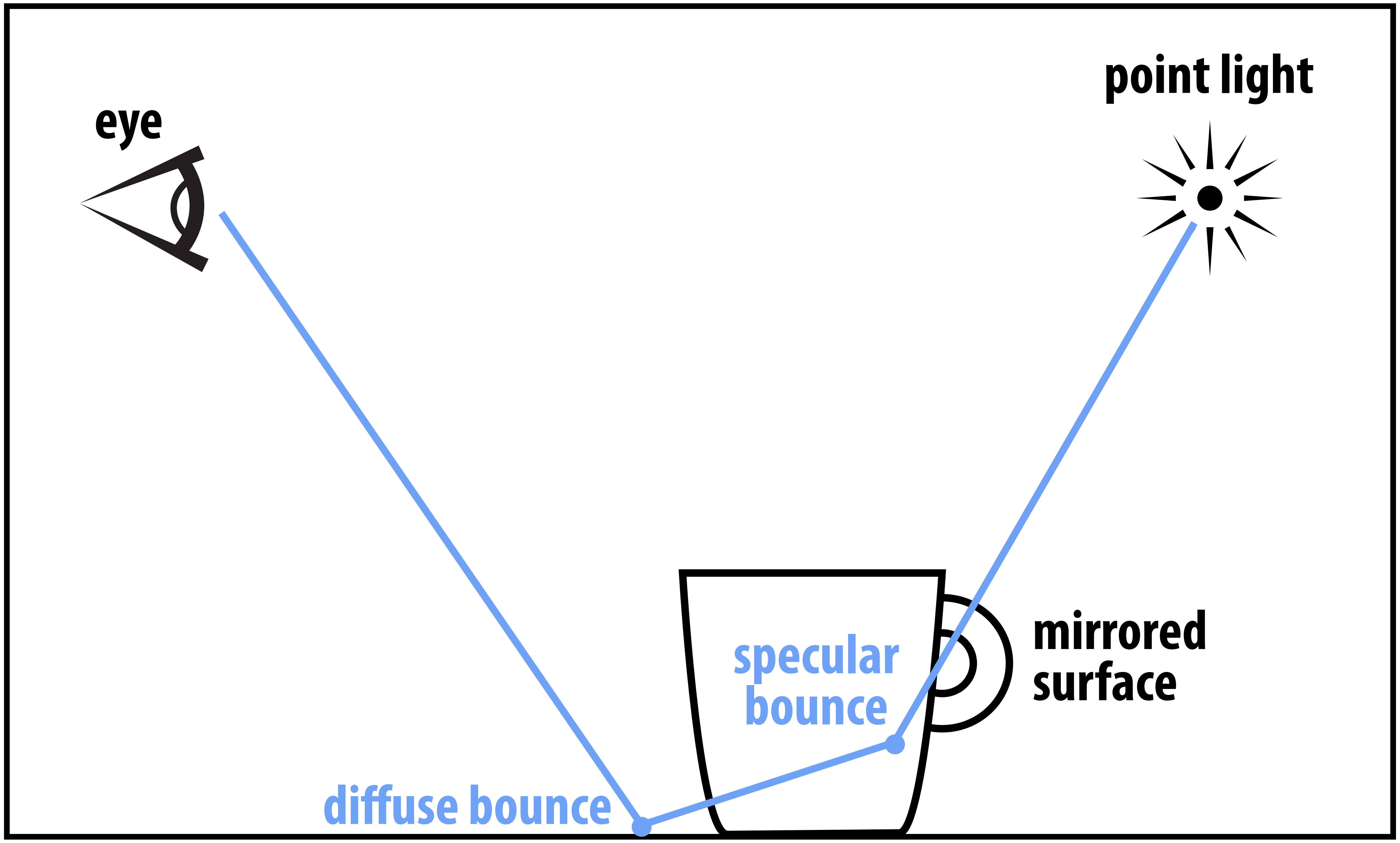
Is this estimator consistent? Unbiased?
(What if I then let n go to ∞?)

Why does it matter?



Rule of thumb: unbiased estimators have more predictable behavior / fewer parameters to tweak to get correct result (which says nothing about performance...)

Naïve Path Tracing: Which Paths Can We Trace?



"caustic" (focused light) from reflection

Q: What's the probability we sample the reflected direction?

A: ZERO.

Q: What's the probability we hit a point light source?

A: ZERO.

Naïve path tracing misses important phenomena!
(Formally: the result is biased.)

...But isn't this example pathological?
No such thing as point light source, perfect mirror.

Real lighting can be close to pathological

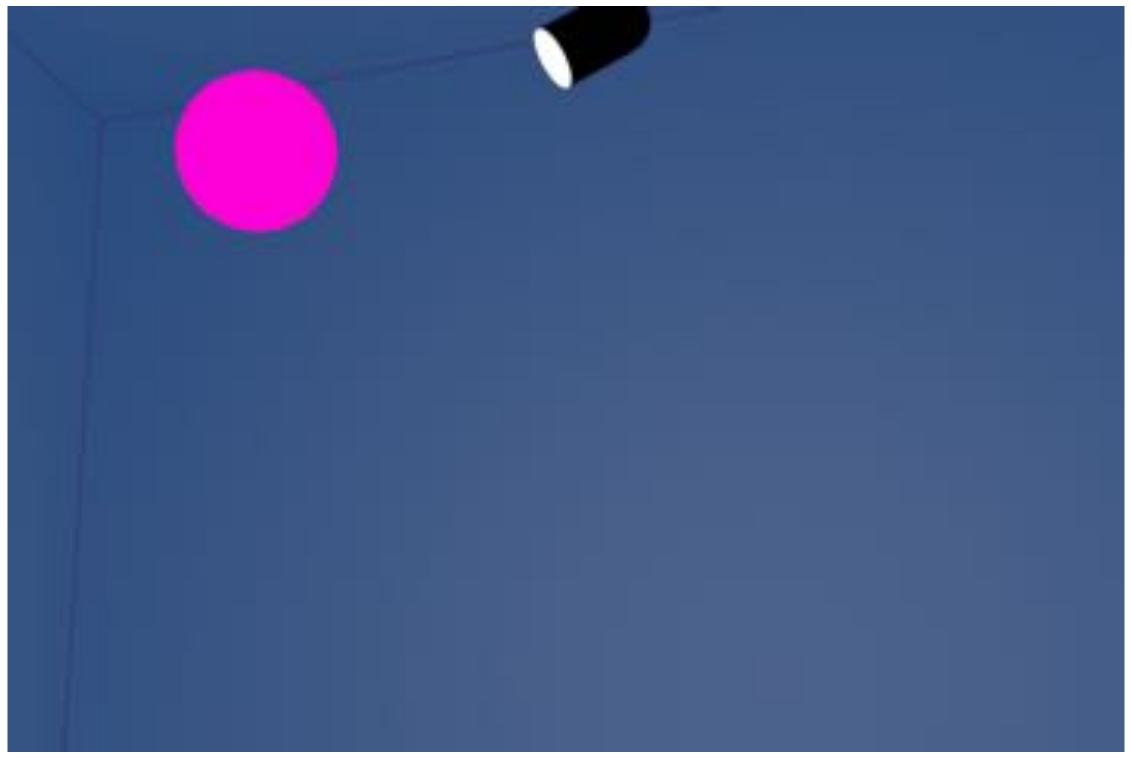
small directional
light source

near-perfect mirror

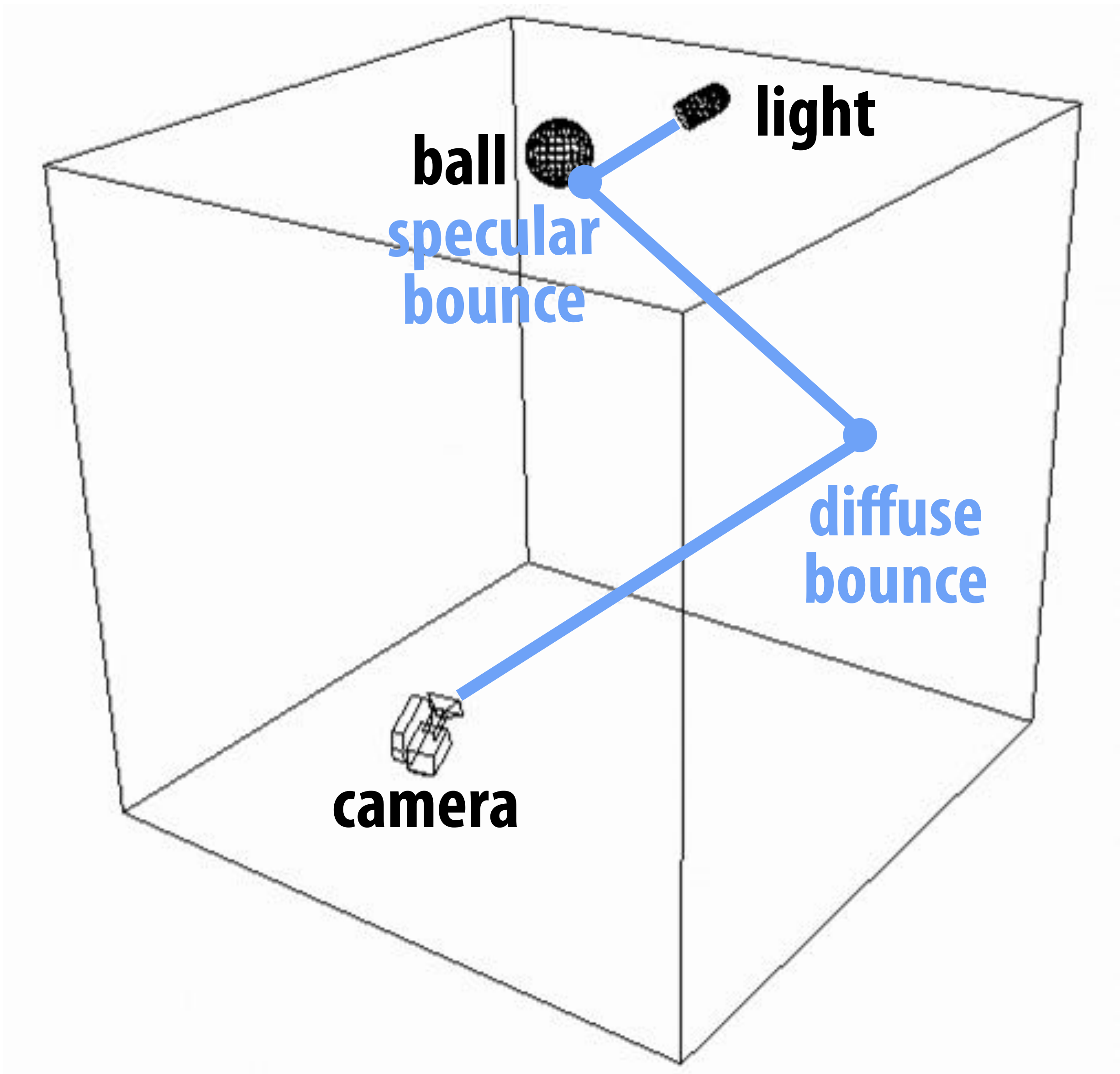
Still want to render this scene!

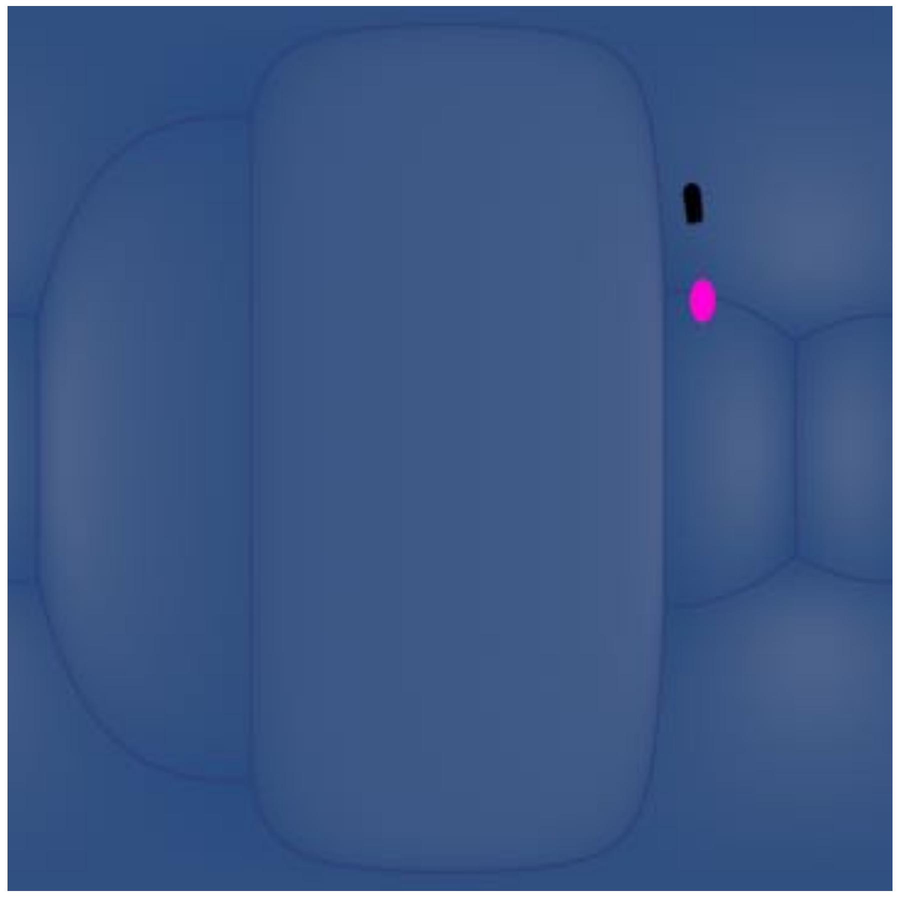
Light has a very "spiky" distribution

- Consider the view from each bounce in our disco scene:

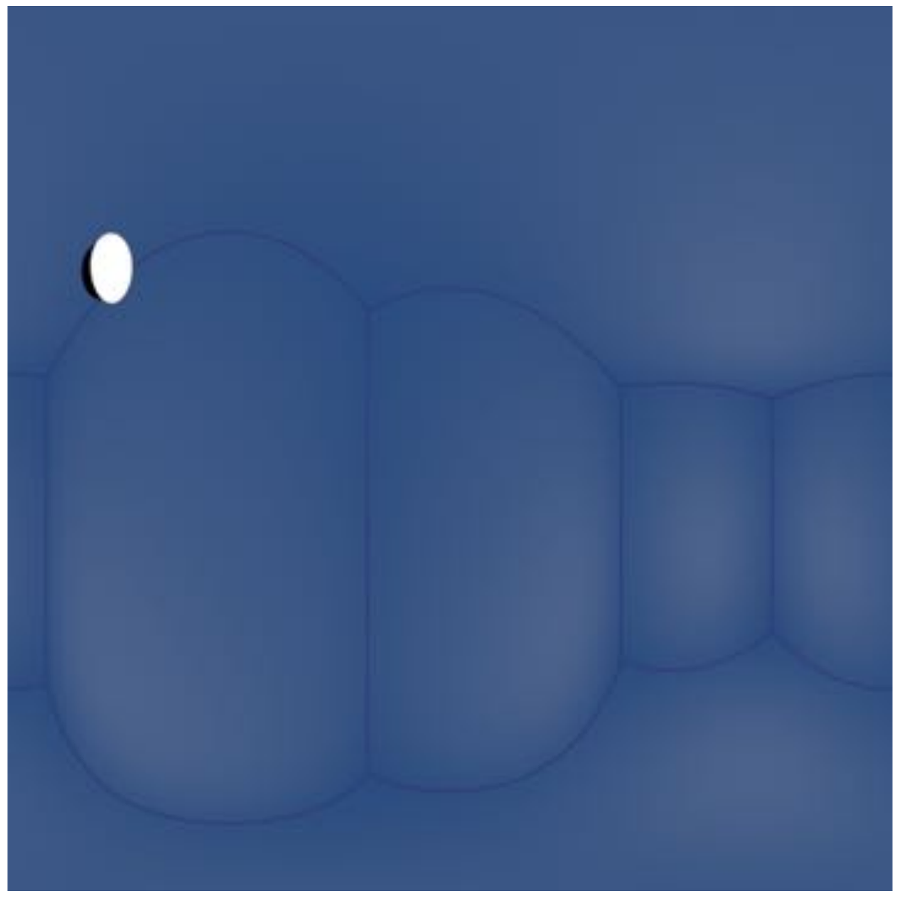


view from camera





view from diffuse bounce
mirrored ball (pink) covers small percentage of solid angle

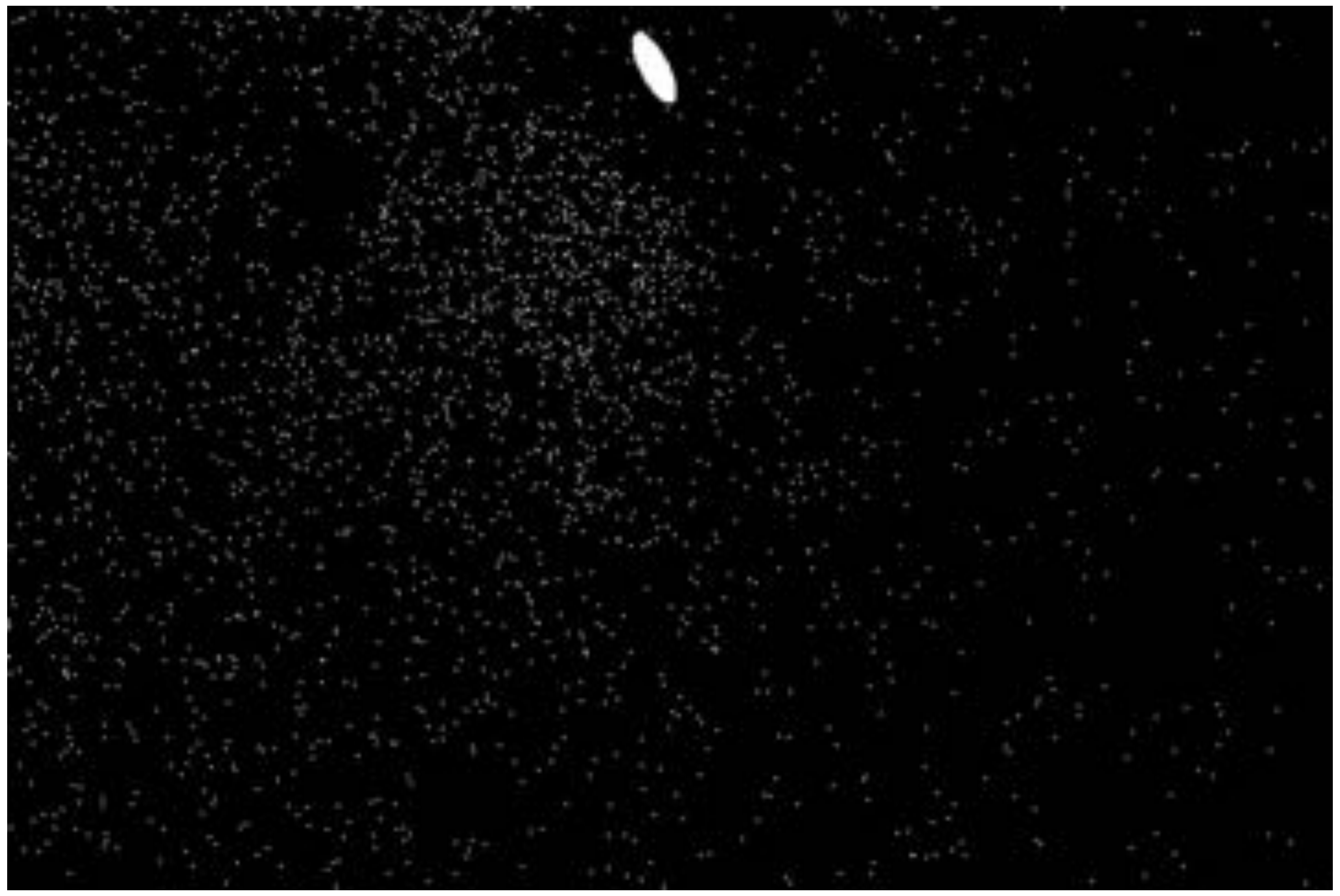


view from specular bounce
area light (white) covers small percentage of solid angle

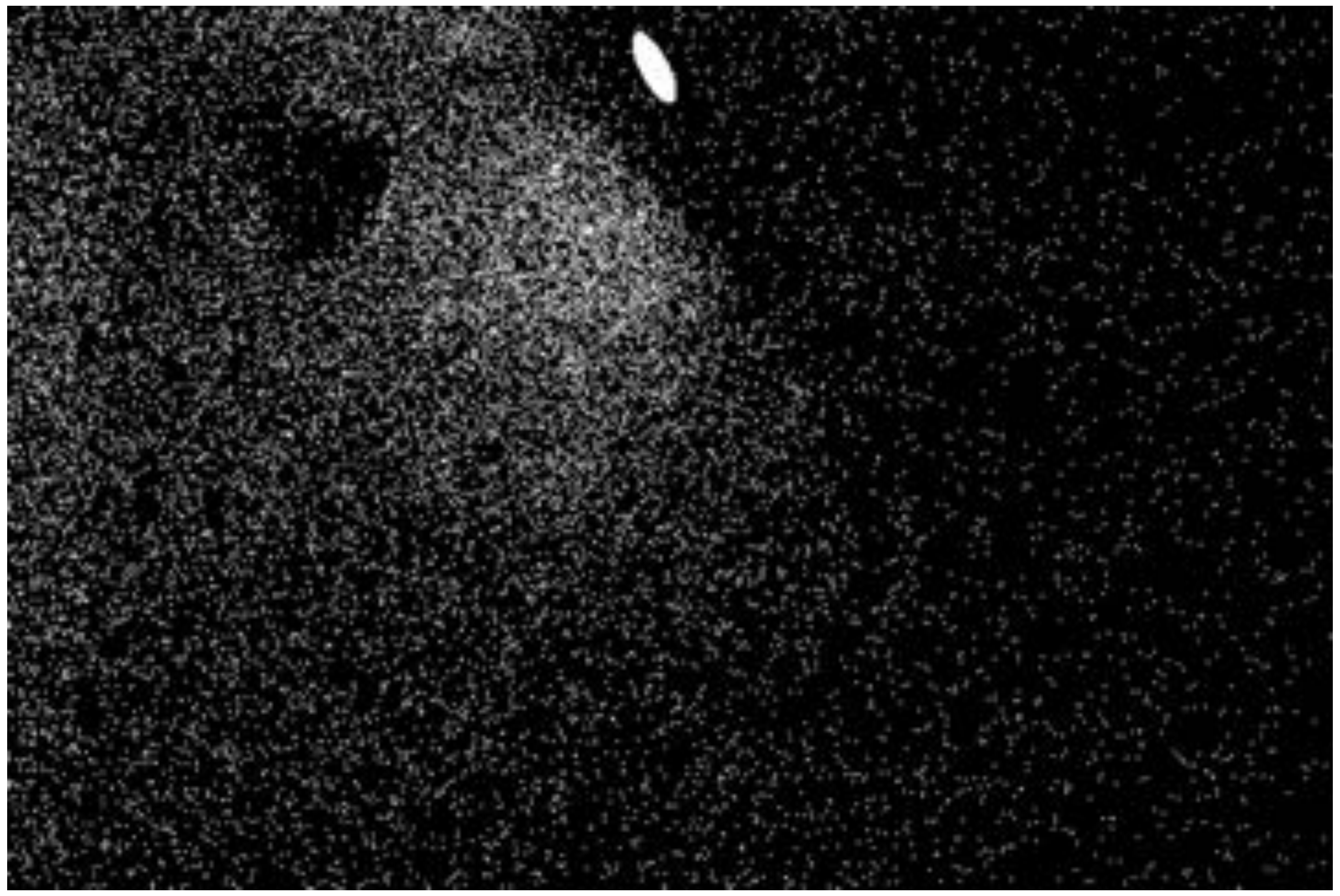
Probability that a uniformly-sampled path carries light is the product of the solid angle fractions. (Very small!)

Then consider even more bounces...

Just use more samples?



path tracing - 16 samples/pixel



path tracing - 128 samples/pixel

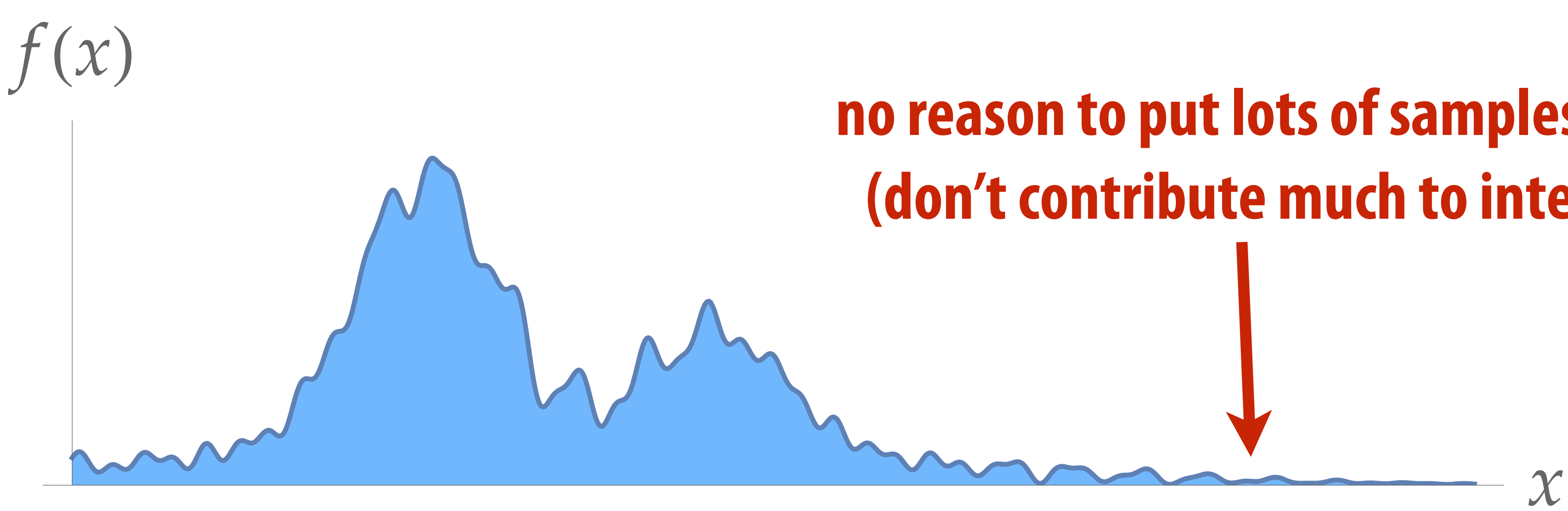
path tracing - 8192 samples/pixel

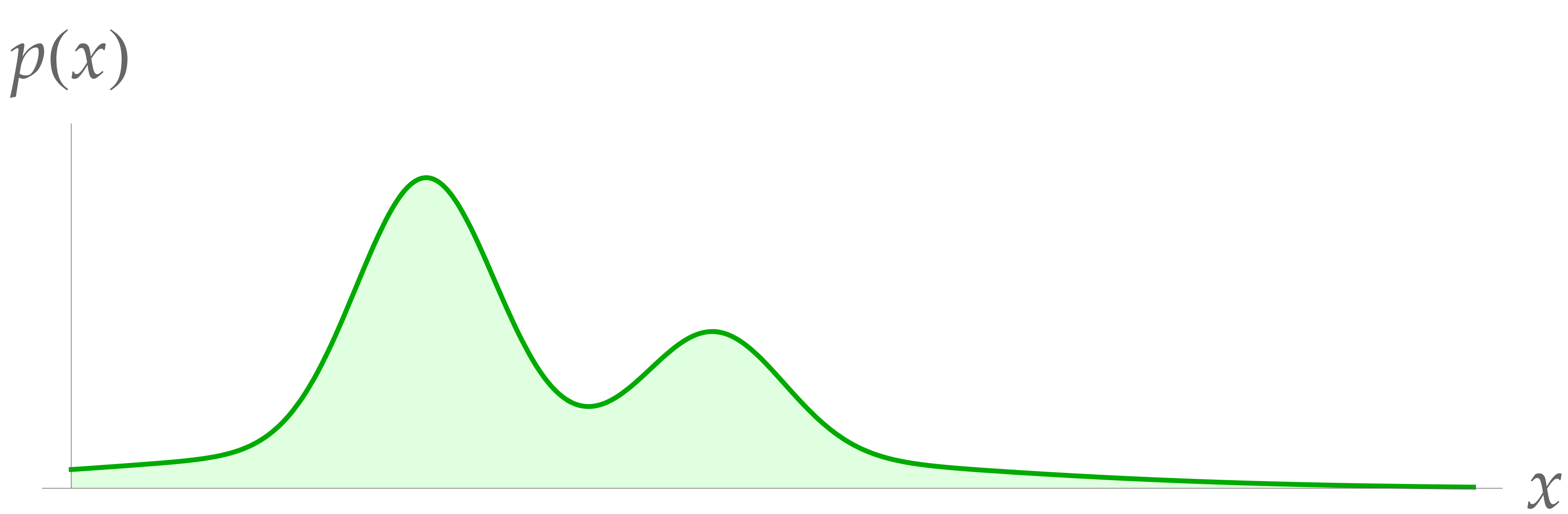
how do we get here? (photo)

We need better sampling strategies!

Review: Importance Sampling

- **Simple idea: sample the integrand according to how much we expect it to contribute to the integral.**





naïve Monte Carlo:

$$V(\Omega)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

(x$_i$ are sampled uniformly)

importance sampled Monte Carlo:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{p(x_i)}$$

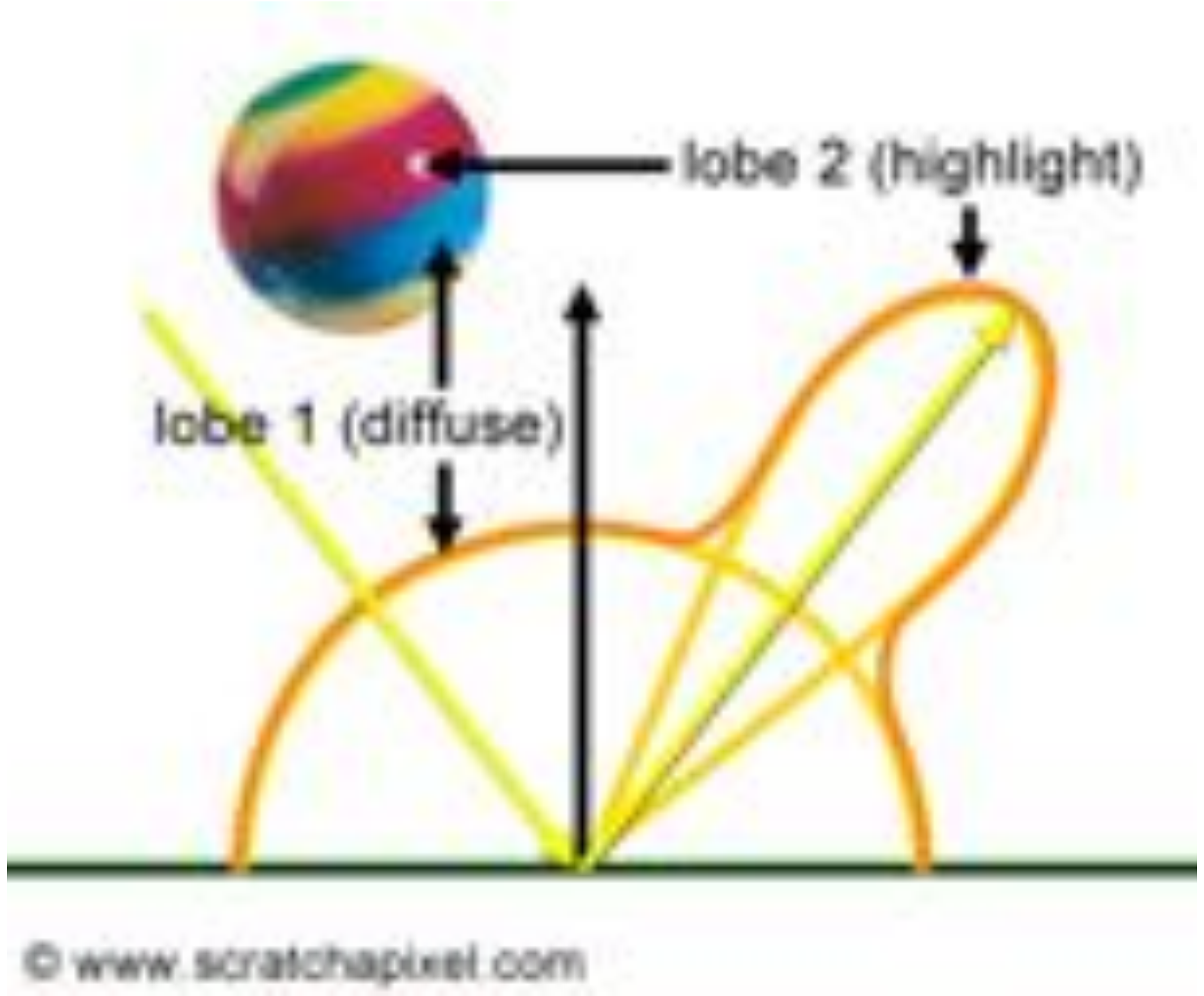
(x$_i$ are sampled proportional to p)

"If I sample x more frequently, each sample should count for less; if I sample x less frequently, each sample should count for more."

Q: What happens when p is proportional to f (p = cf)?

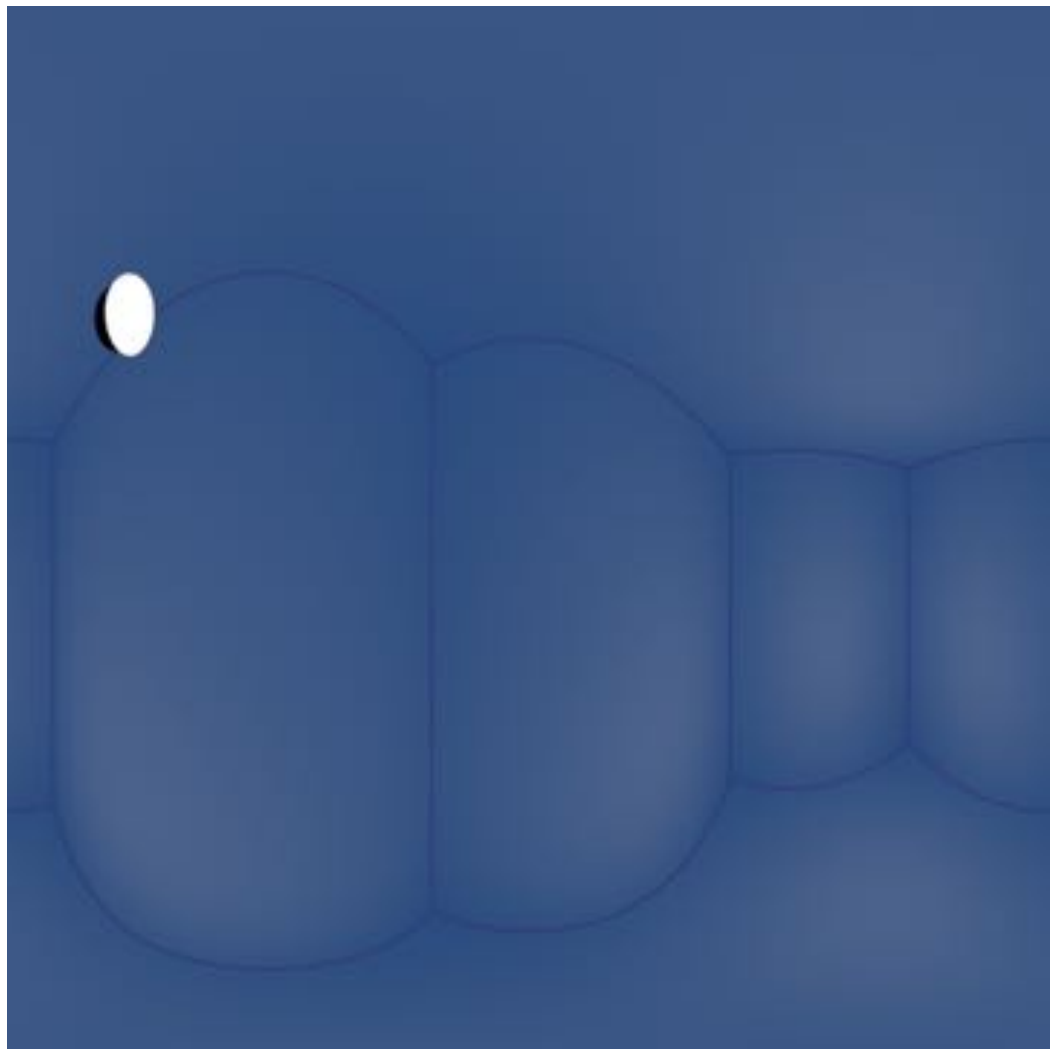
Importance Sampling in Rendering

materials: sample important "lobes"



(important special case: perfect mirror!)

illumination: sample bright lights



(important special case: point light!)

Q: How else can we re-weight our choice of samples?

Path Space Formulation of Light Transport

- **So far have been using recursive rendering equation:**

$$L_{\mathrm{o}}(\mathbf{x},\,\omega_{\mathrm{o}}) \,=\, L_e(\mathbf{x},\,\omega_{\mathrm{o}}) \,+\, \int_{\Omega} f_r(\mathbf{x},\,\omega_{\mathrm{i}},\,\omega_{\mathrm{o}})\, L_{\mathrm{i}}(\mathbf{x},\,\omega_{\mathrm{i}})\,(\omega_{\mathrm{i}}\,\cdot\,\mathbf{n})\,\mathrm{d}\,\omega_{\mathrm{i}}$$

- **Make intelligent "local" choices at each step (material/ lights)**
- **Alternatively, we can use a "path integral" formulation:**

how much "light" is carried by this path?

$$I = \int_{\Omega} f(\bar{x}) d\mu(\bar{x})$$

how much of path space does this path "cover"

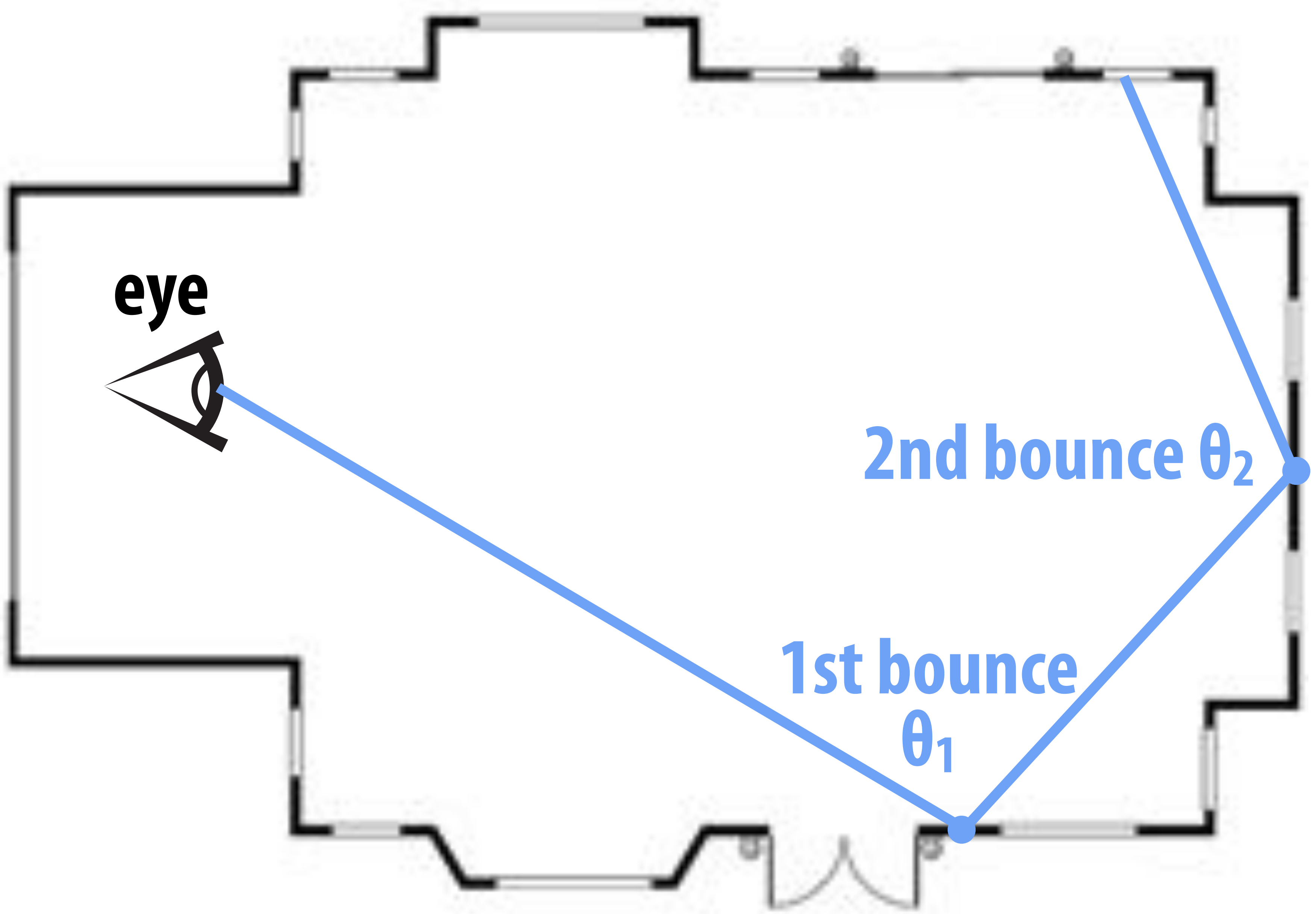
one particular path

all possible paths

- **Opens the door to intelligent "global" importance sampling. (But still hard!)**

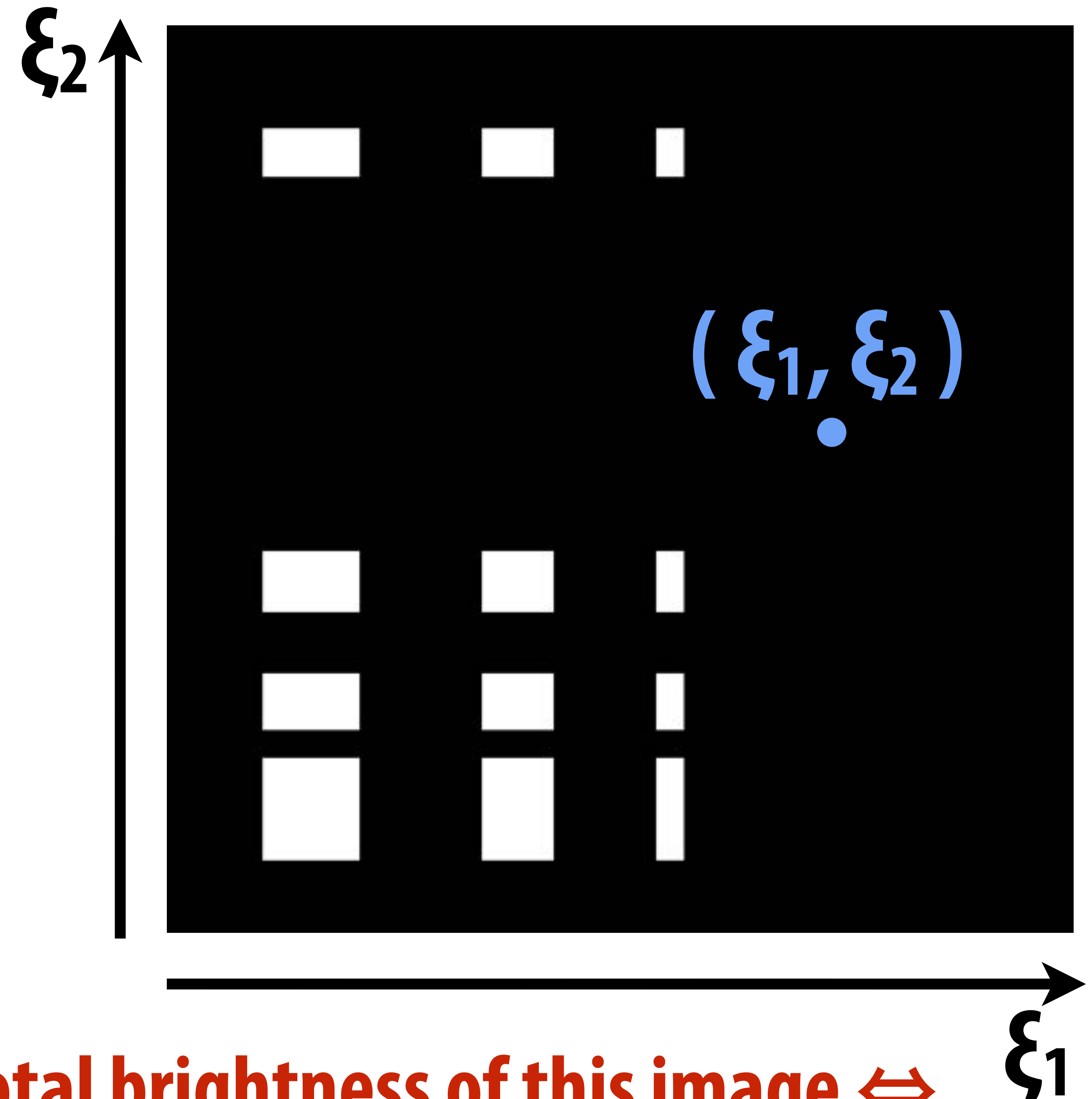
Unit Hypercube View of Path Space

- Paths determined by a sequence of random values ξ in [0,1]
- Hence, path of length k is a point in hypercube $[0,1]^k$
- "Just" integrate over cubes of each dimension k
- E.g., two bounces in a 2D scene:



each bounce: $\xi \in [0,1] \mapsto \theta \in [0,\pi]$

Each point is a path of length 2:

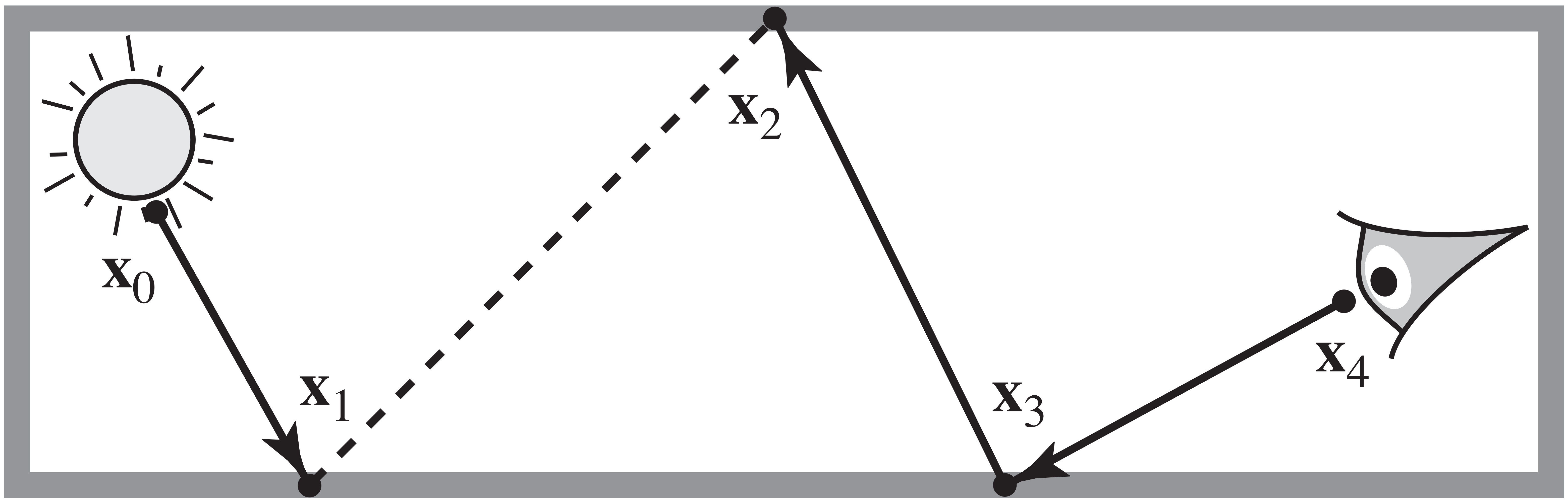


Total brightness of this image ⇔ total contribution of length-2 paths.

How do we choose paths—and path lengths?

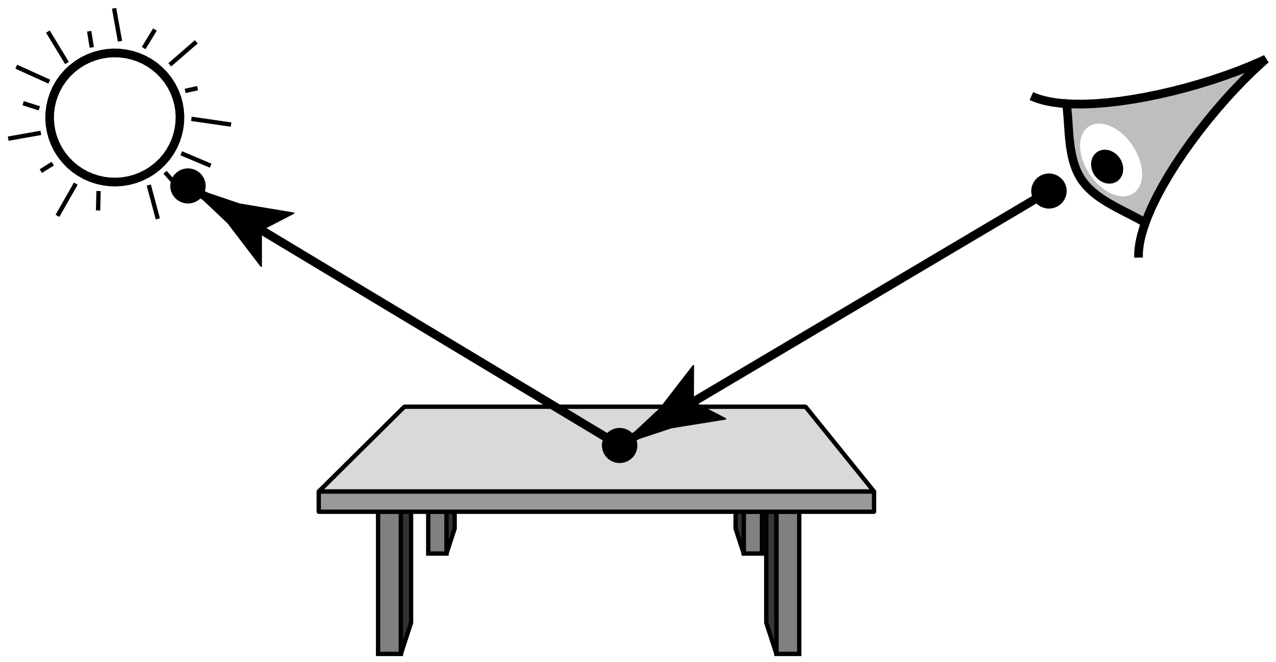
Bidirectional Path Tracing

- Forward path tracing: no control over path length (hits light after n bounces, or gets terminated by Russian Roulette)
- Idea: connect paths from light, eye (“bidirectional”)

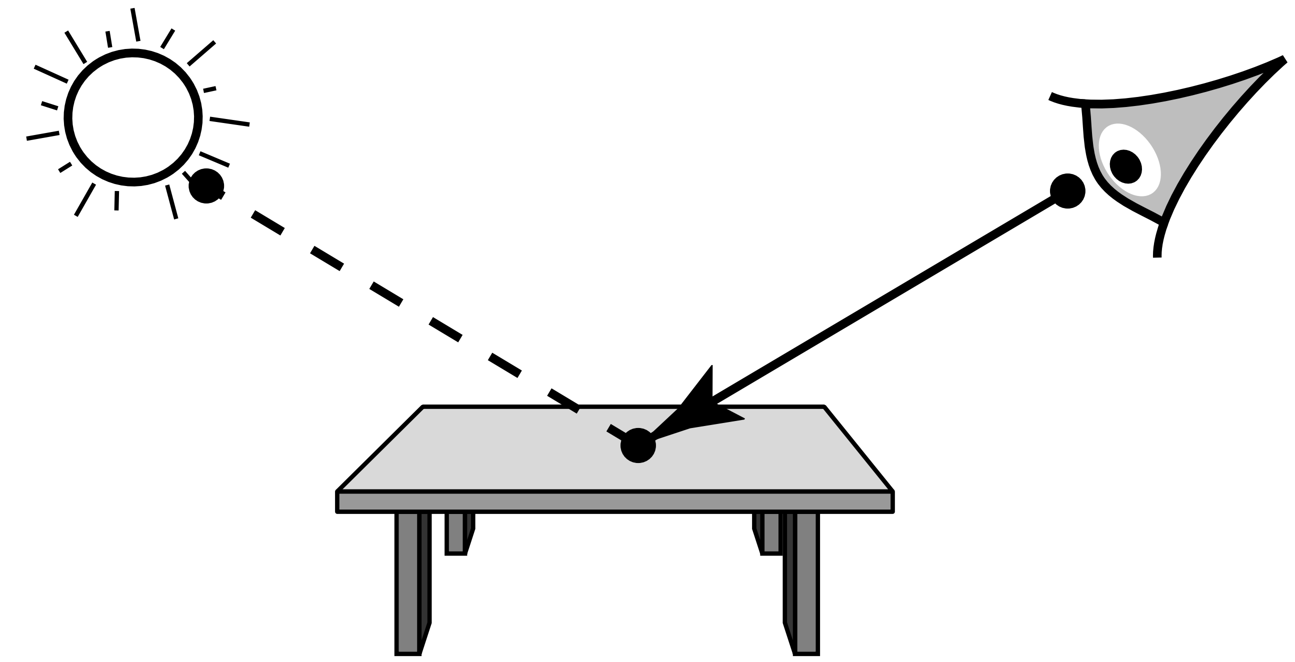


- Importance sampling? Need to carefully weight contributions of path according to sampling strategy.
- (Details in Veach & Guibas, “Bidirectional Estimators for Light Transport”)

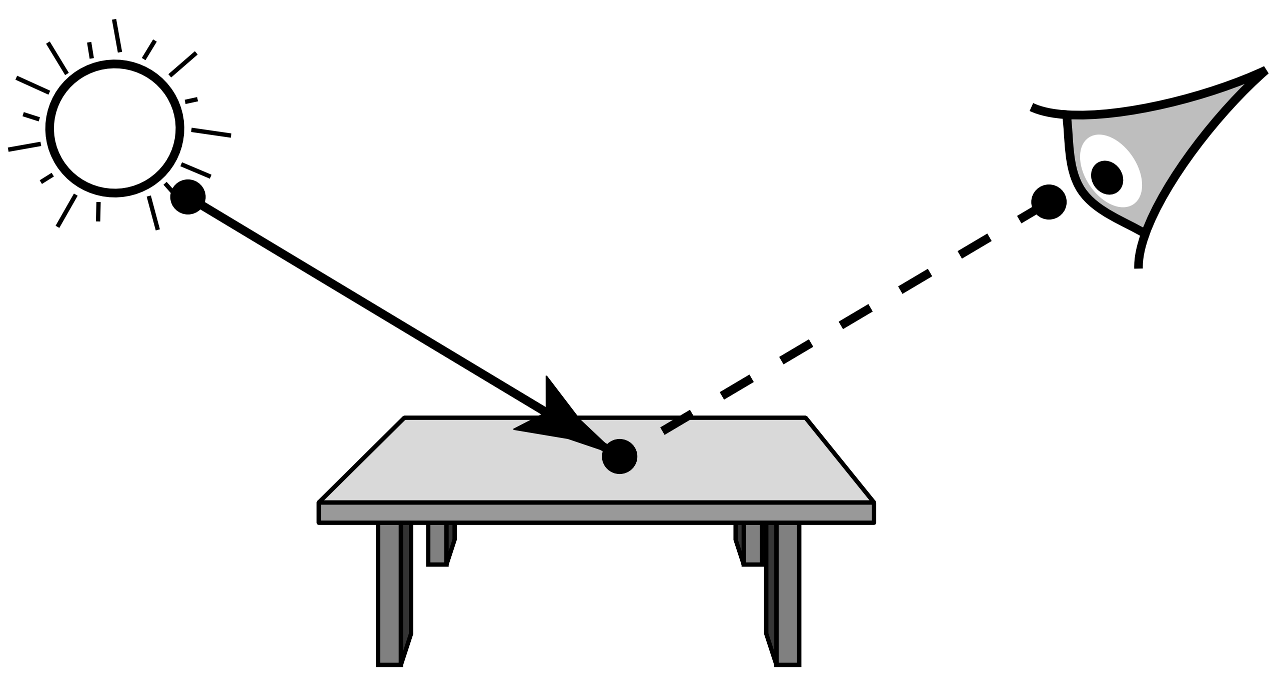
Bidirectional Path Tracing (Path Length=2)



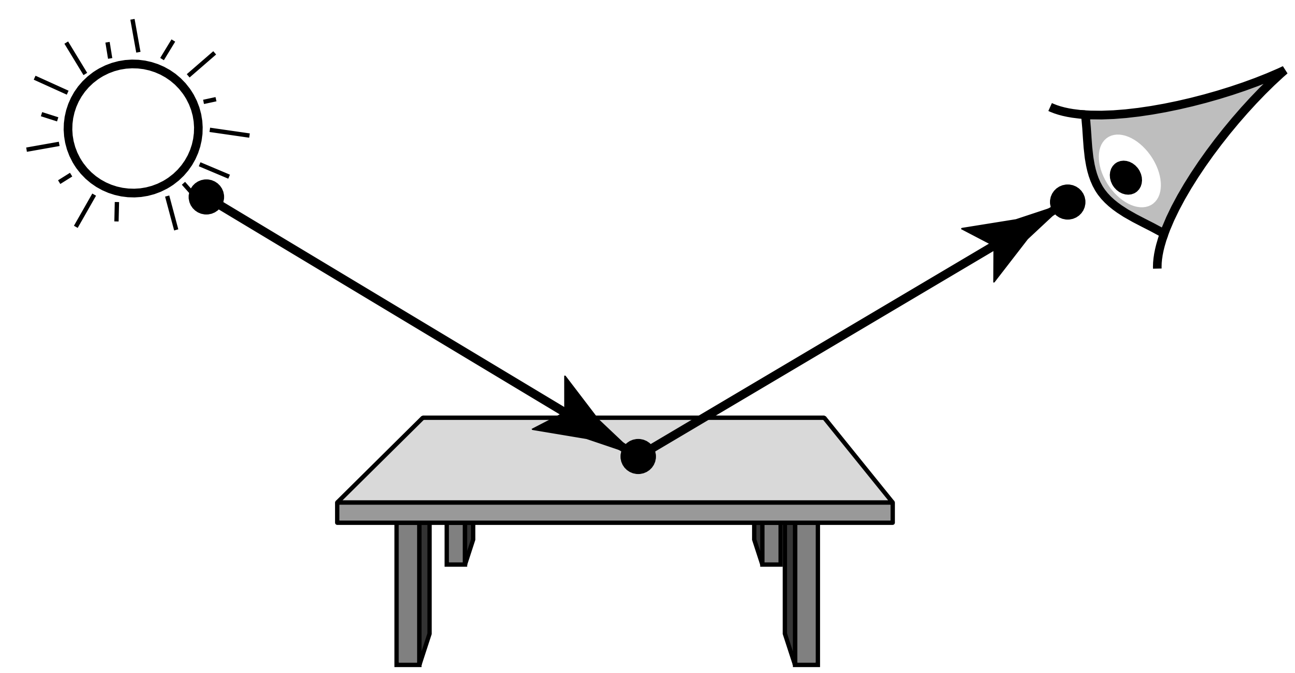
standard (forward) path tracing
fails for point light sources



direct lighting

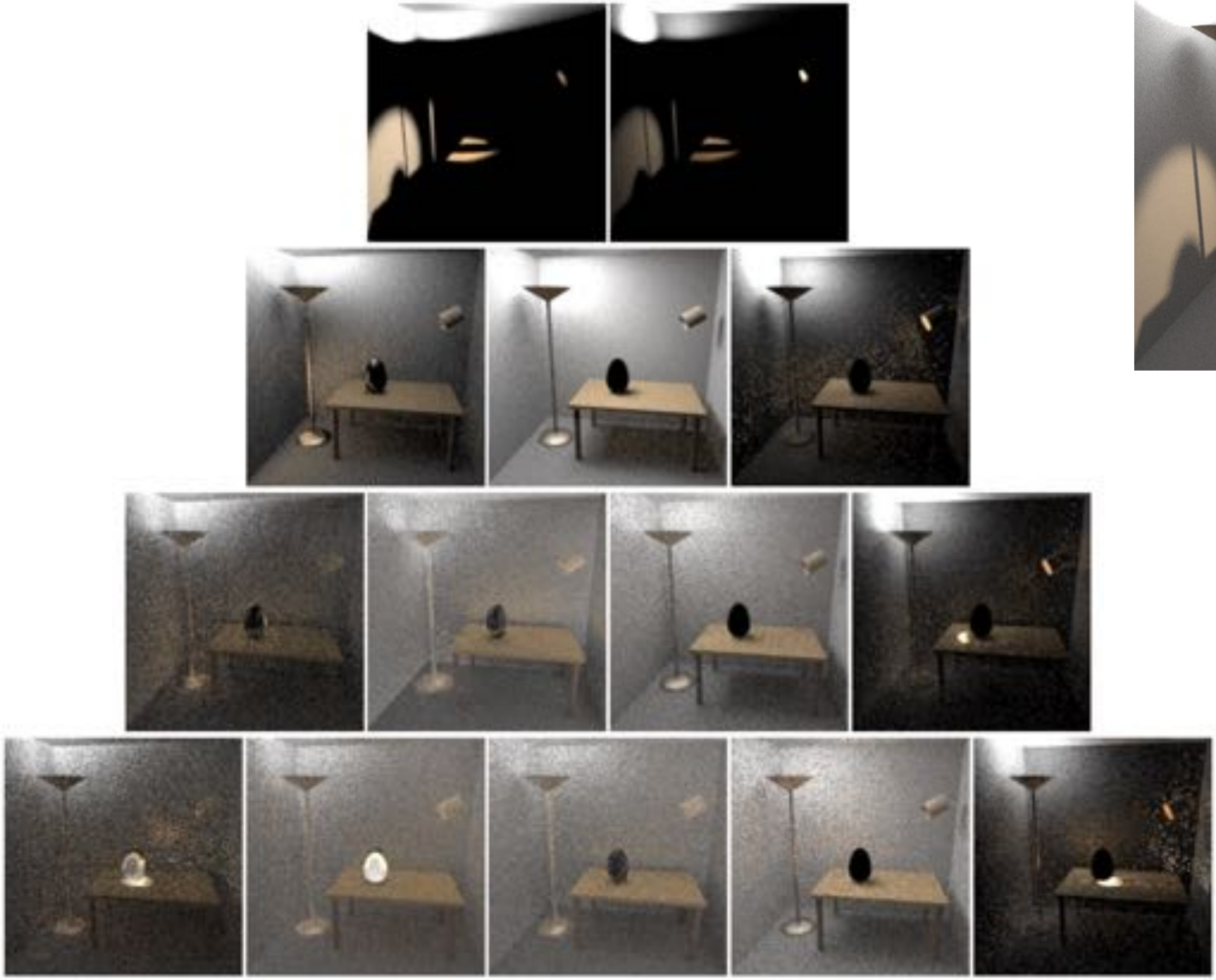


visualize particles from light



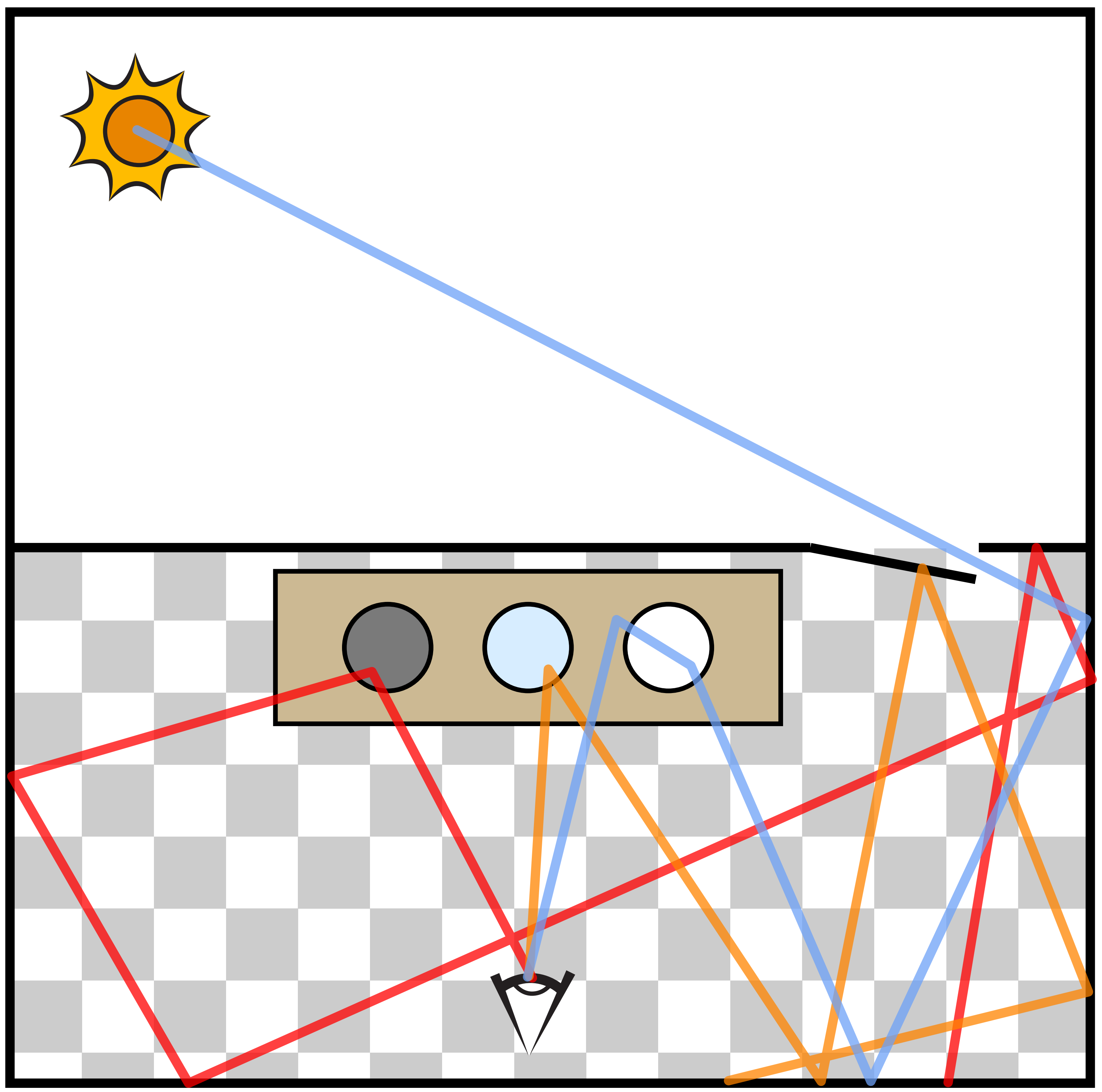
backward path tracing
fails for a pinhole camera

Contributions of Different Path Lengths



final image

Good paths can be hard to find!



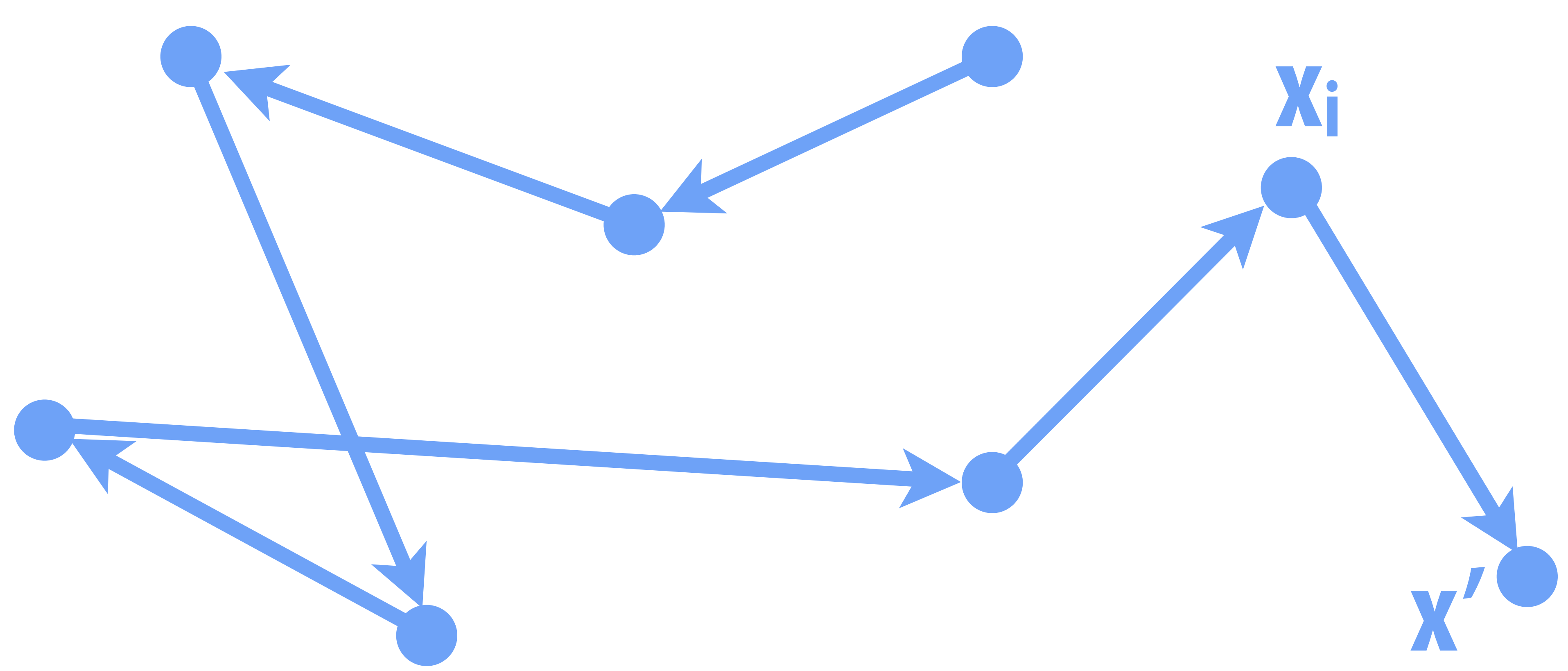
bidirectional path tracing

Metropolis light transport (MLT)

Idea:
Once we find a good path, perturb it to find nearby "good" paths.

Metropolis-Hastings Algorithm (MH)

- Standard Monte Carlo: sum up independent samples
- MH: take random walk of dependent samples ("mutations")
- Basic idea: prefer to take steps that increase sample value



$\alpha := f(x') / f(x_i)$ "transition probability"

if random # in [0,1] < α:

  $x_{i+1} = x'$

else:

  $x_{i+1} = x_i$

- If careful, sample distribution will be proportional to integrand
  - make sure mutations are "ergodic" (reach whole space)
  - need to take a long walk, so initial point doesn't matter ("mixing")

Metropolis-Hastings: Sampling an Image

- Want to take samples proportional to image density f
- Start at random point; take steps in (normal) random direction
- Occasionally jump to random point (ergodicity)
- Transition probability is "relative darkness" $f(x')/f(x_i)$

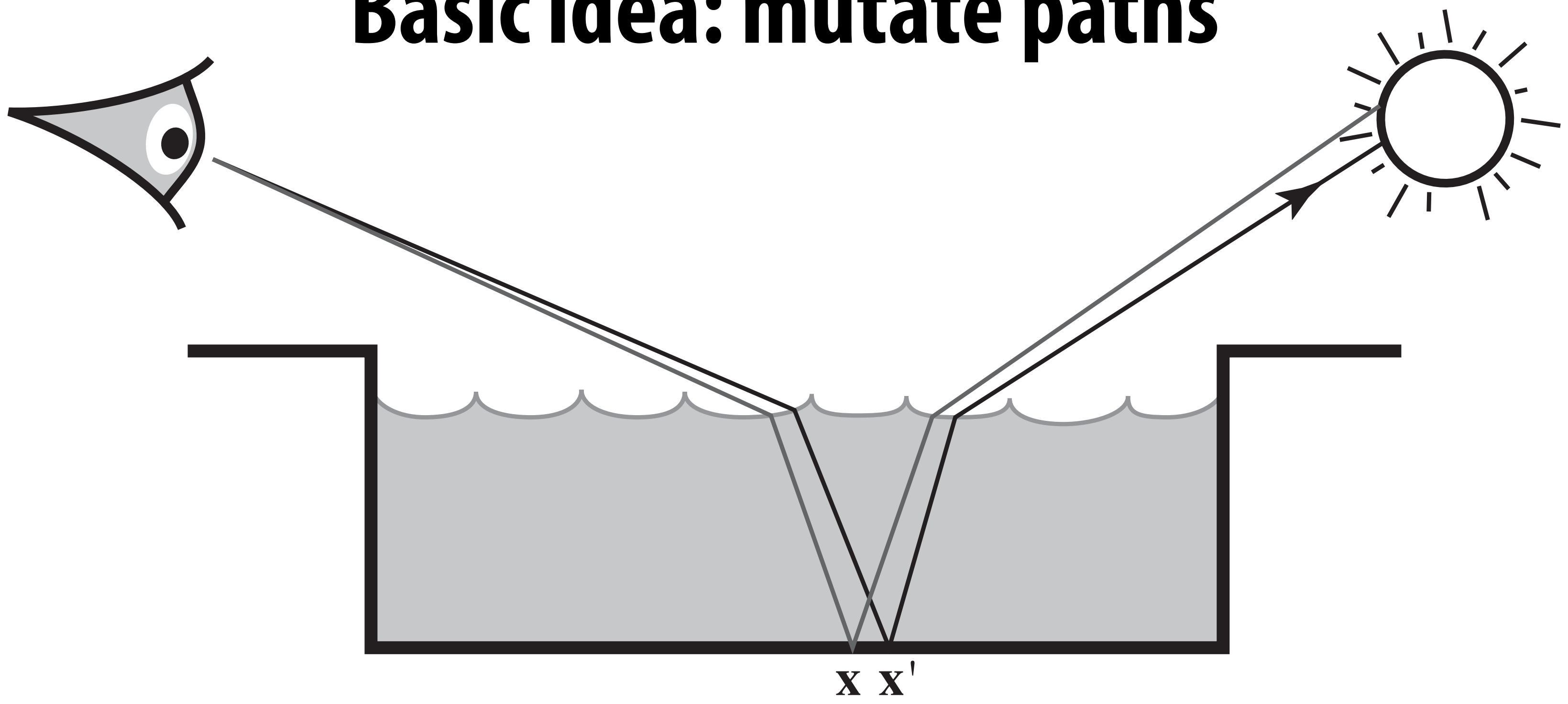
short walk

long walk

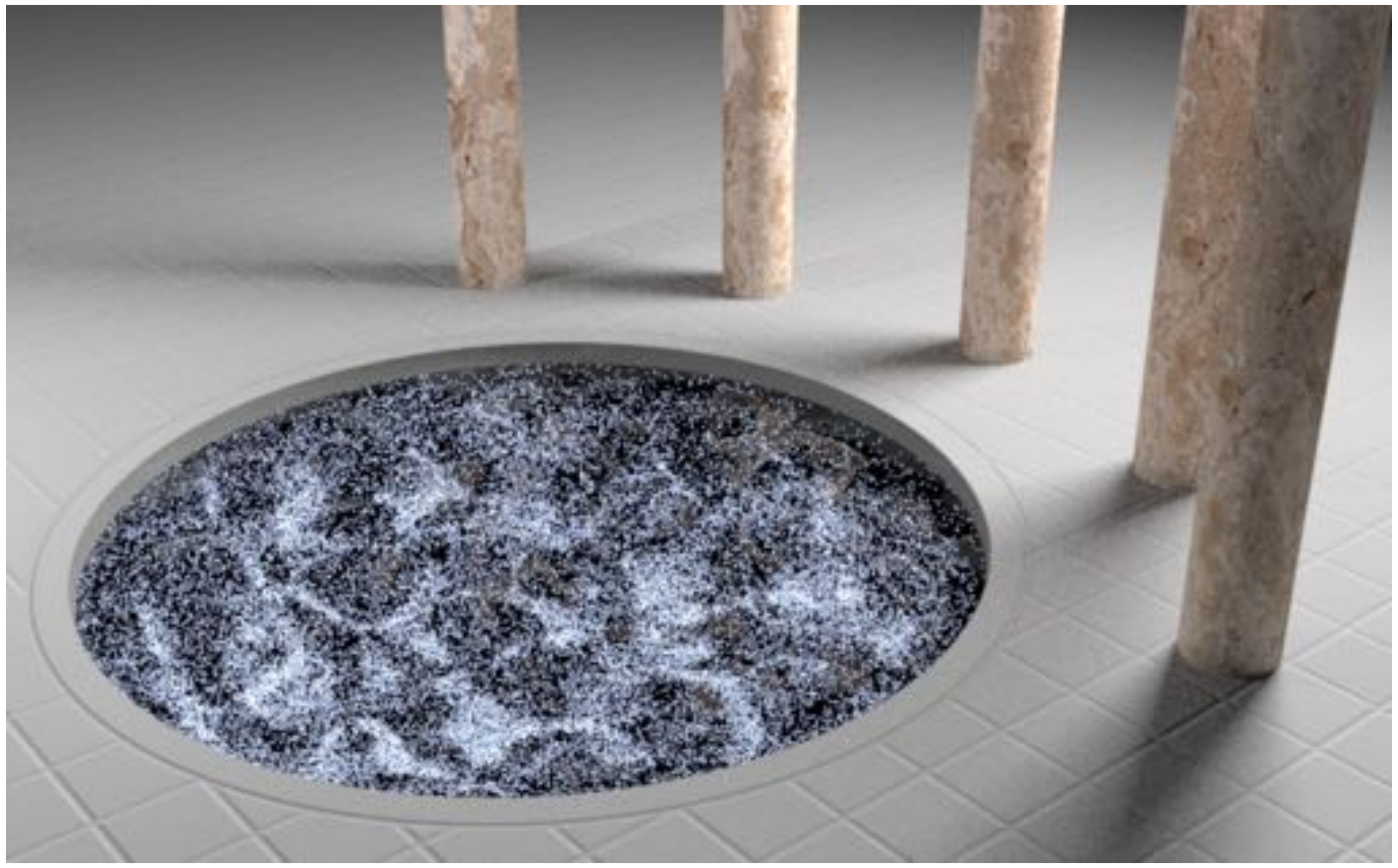
(original image)

Metropolis Light Transport

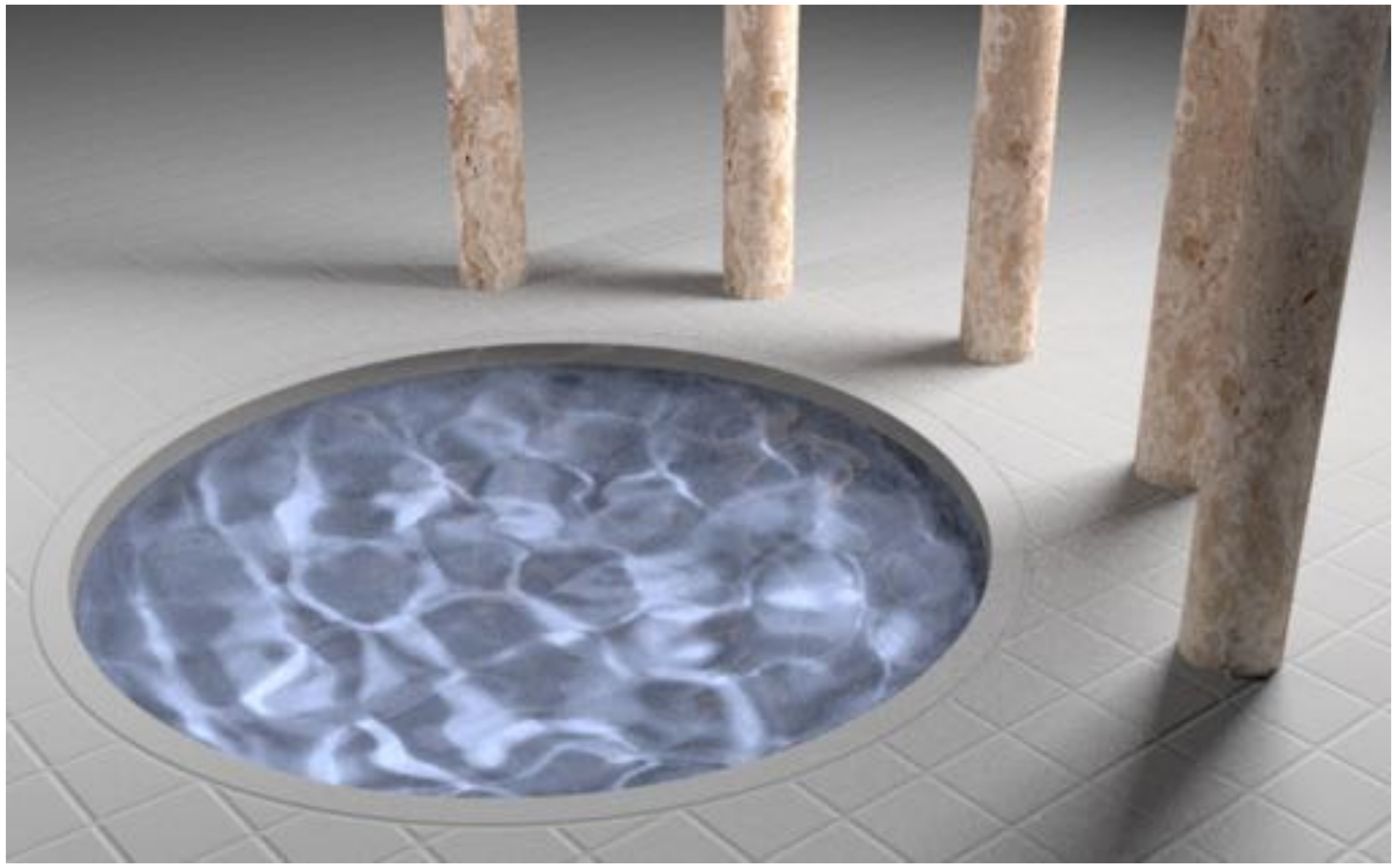
Basic idea: mutate paths



(For details see Veach, "Robust Monte Carlo Methods for Light Transport Simulation")



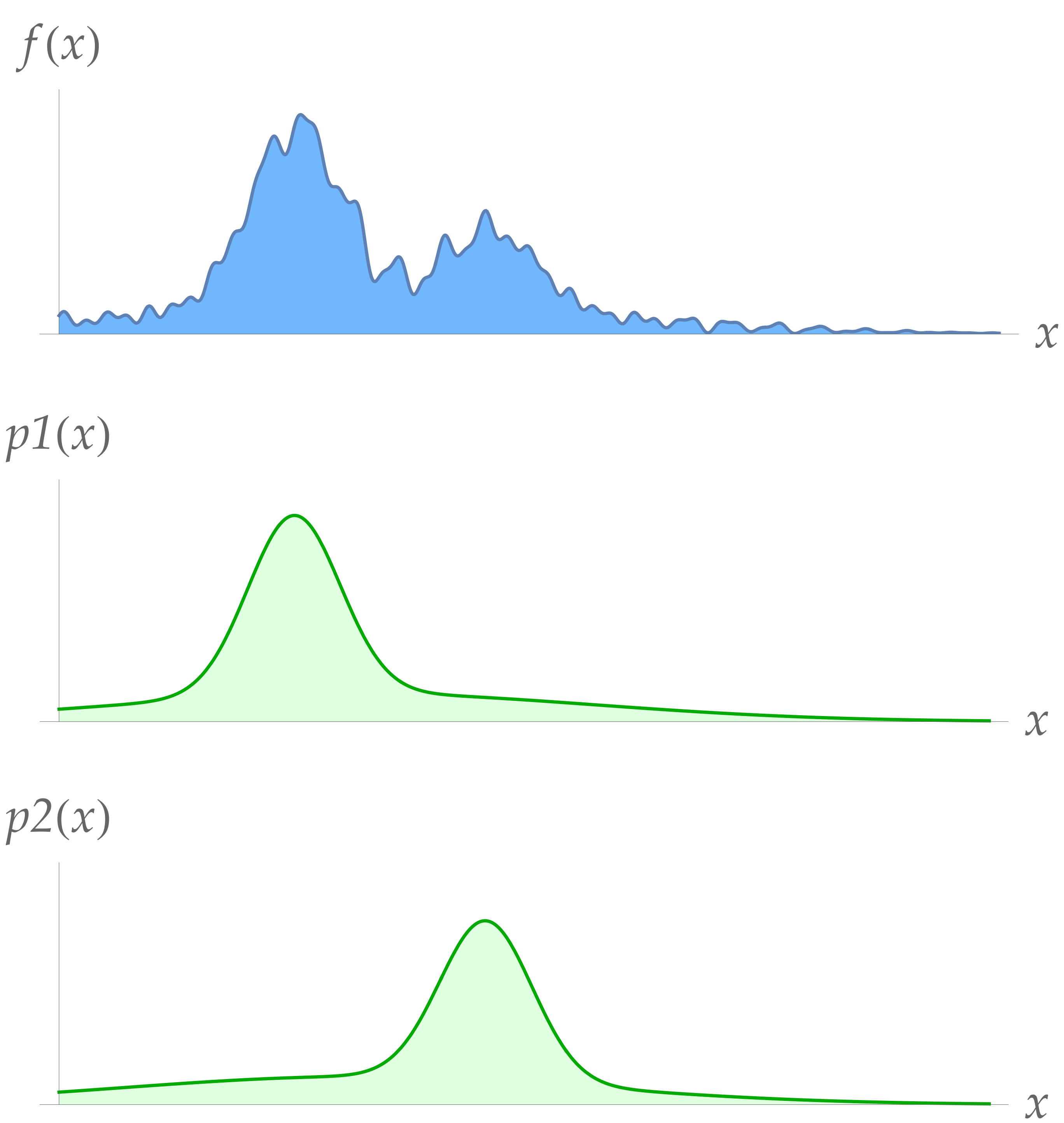
path tracing



Metropolis light transport (same time)

Multiple Importance Sampling (MIS)

- Many possible importance sampling strategies
- Which one should we use for a given integrand?
- MIS: combine strategies to preserve strengths of all of them



sum over strategies

sum over samples

jth sample taken with ith strategy

$$\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{f(x_{ij})}{\sum_k c_k p_k(x_{ij})}$$

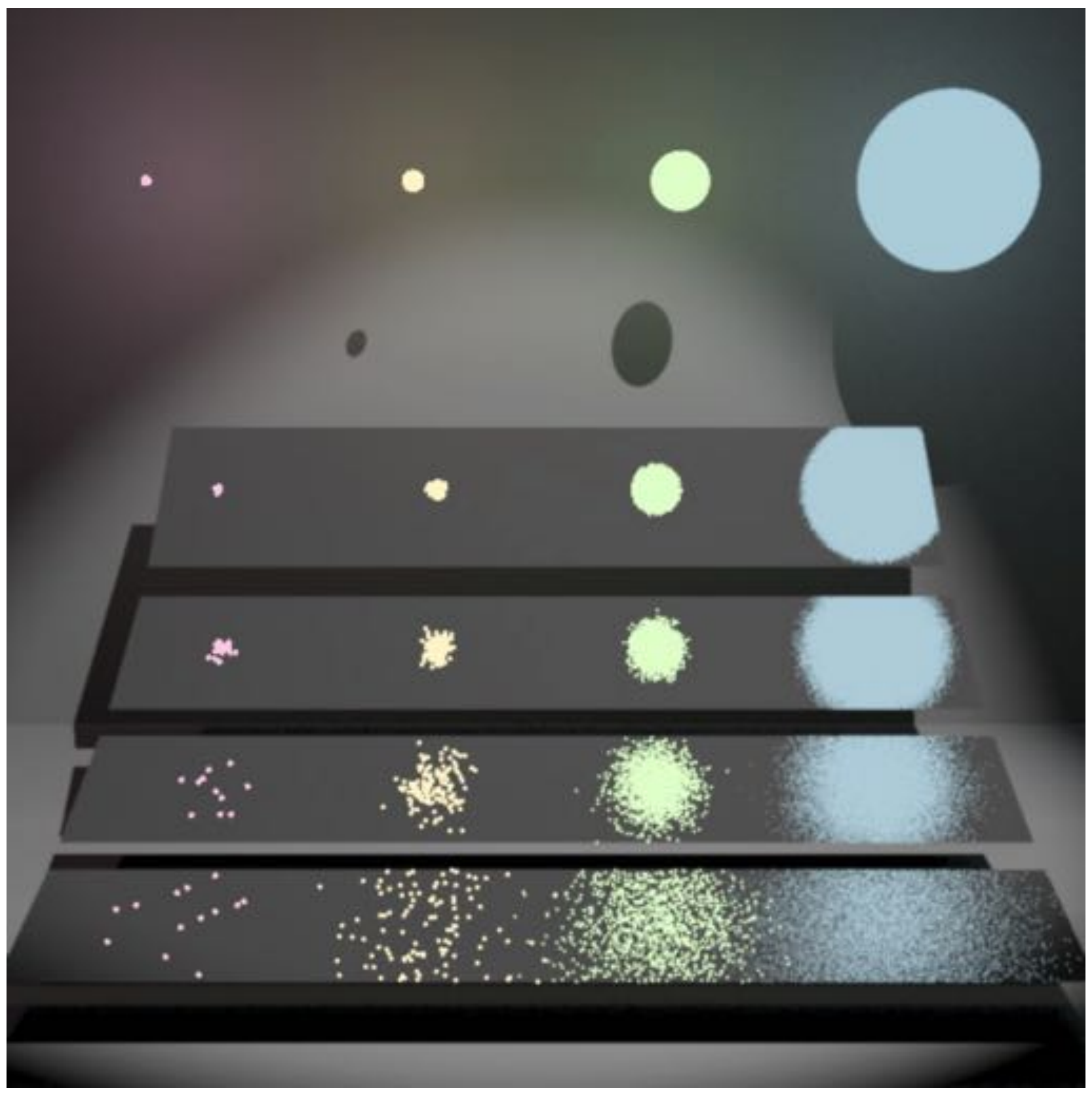
total # of samples

fraction of samples taken w/ kth strategy

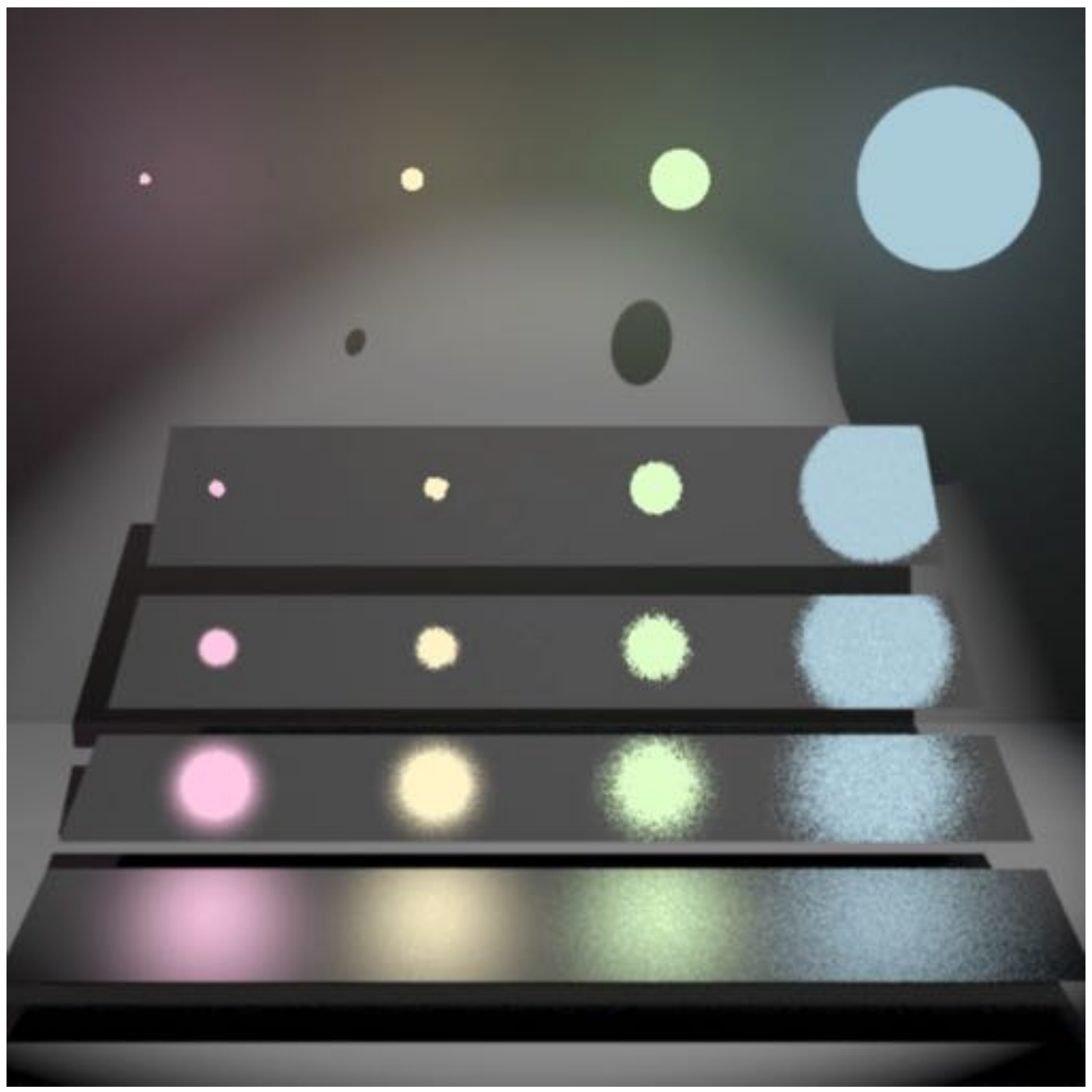
kth importance density

Still, several improvements possible (cutoff, power, max)—see Veach & Guibas.

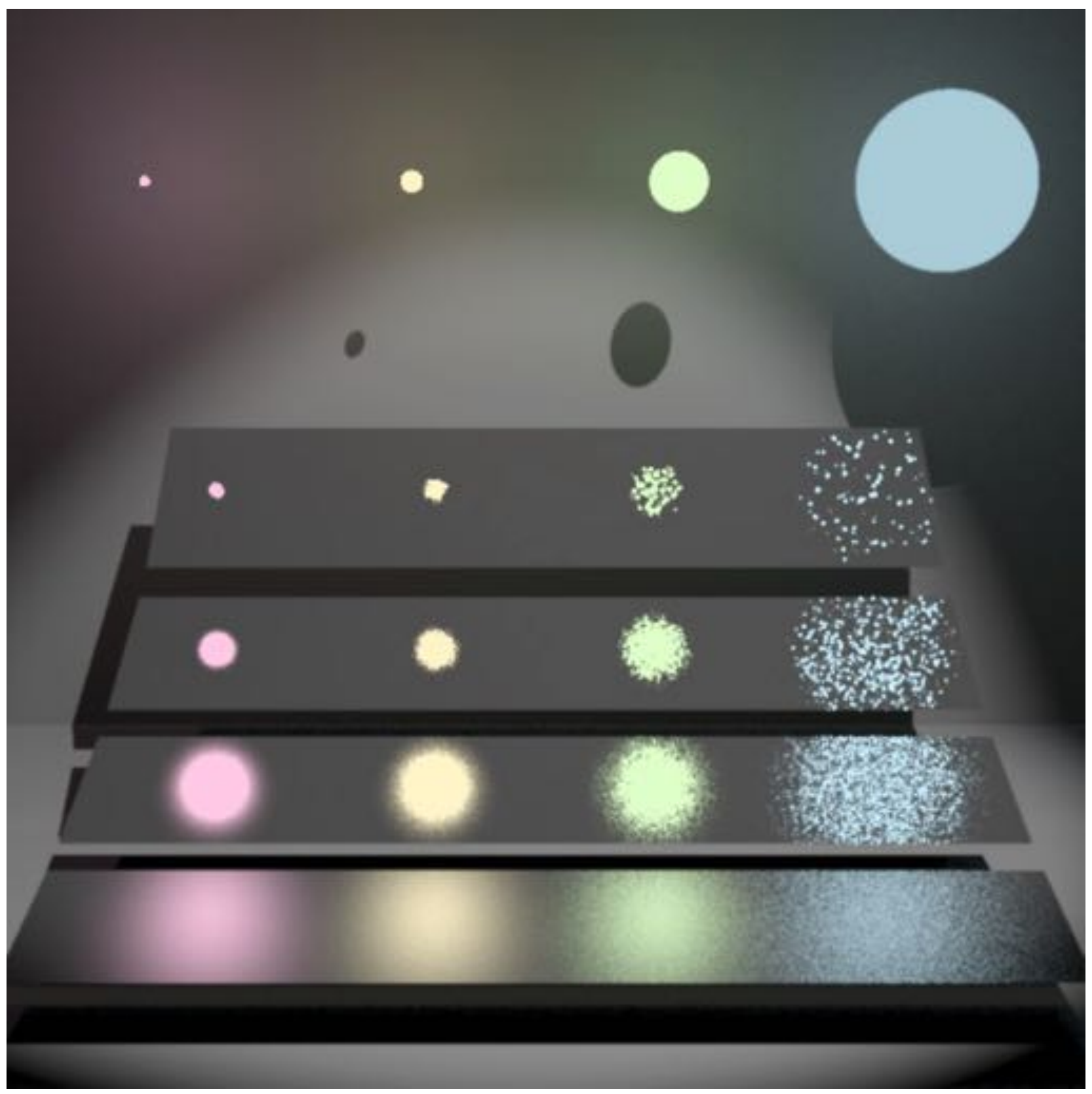
Multiple Importance Sampling: Example



sample materials



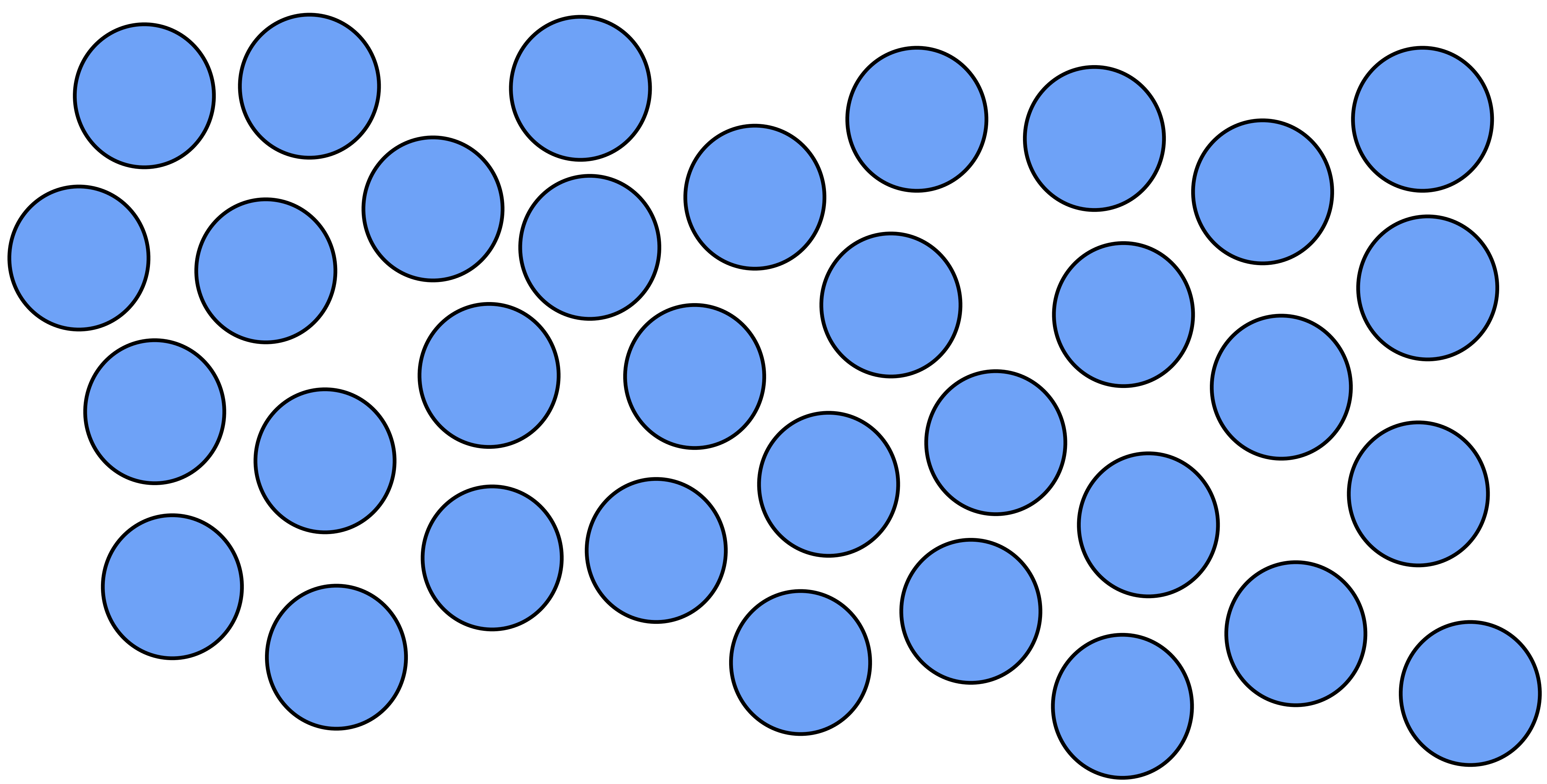
multiple importance sampling
(power heuristic)



sample lights

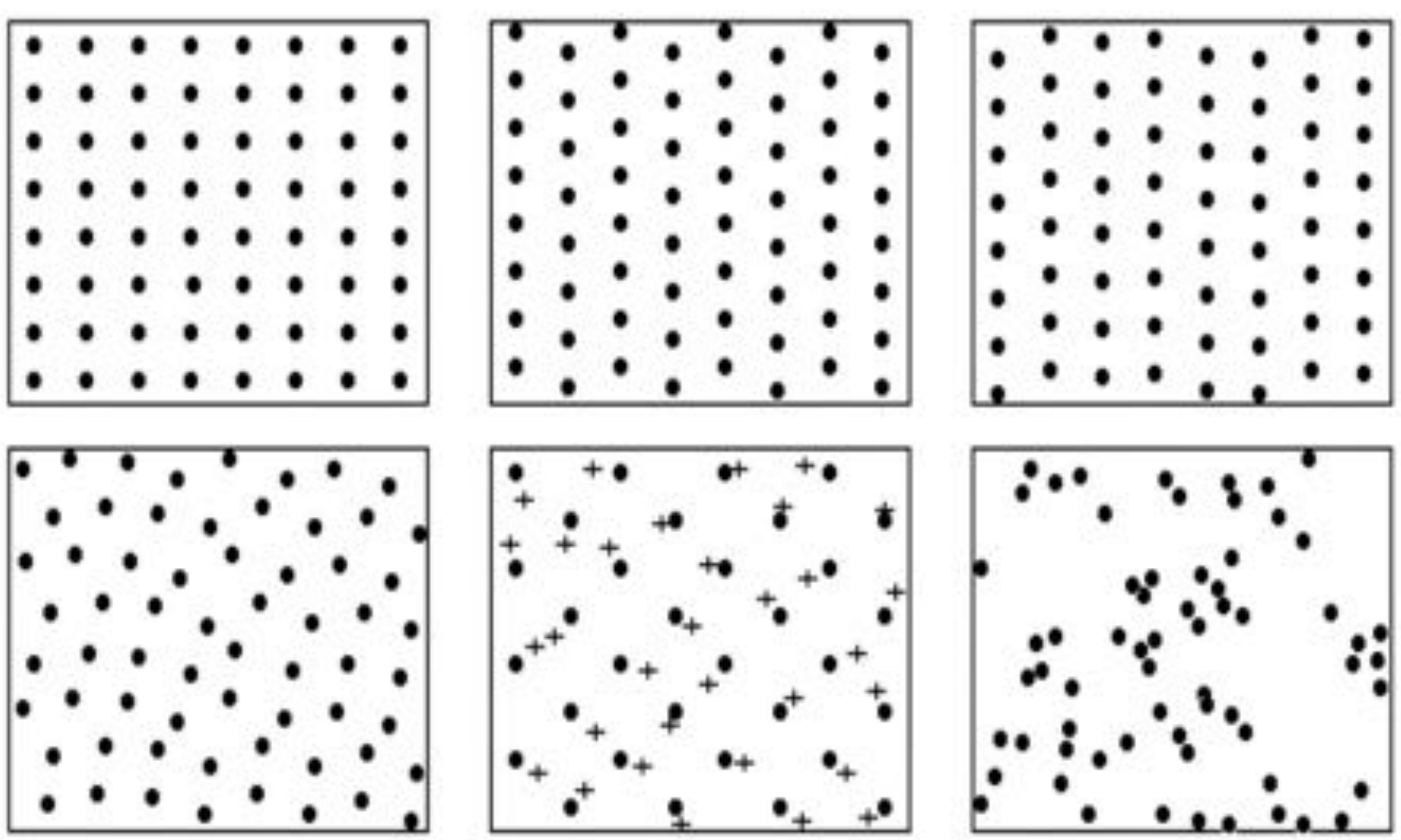
Ok, so importance is important.

But how do we sample our function in the first place?

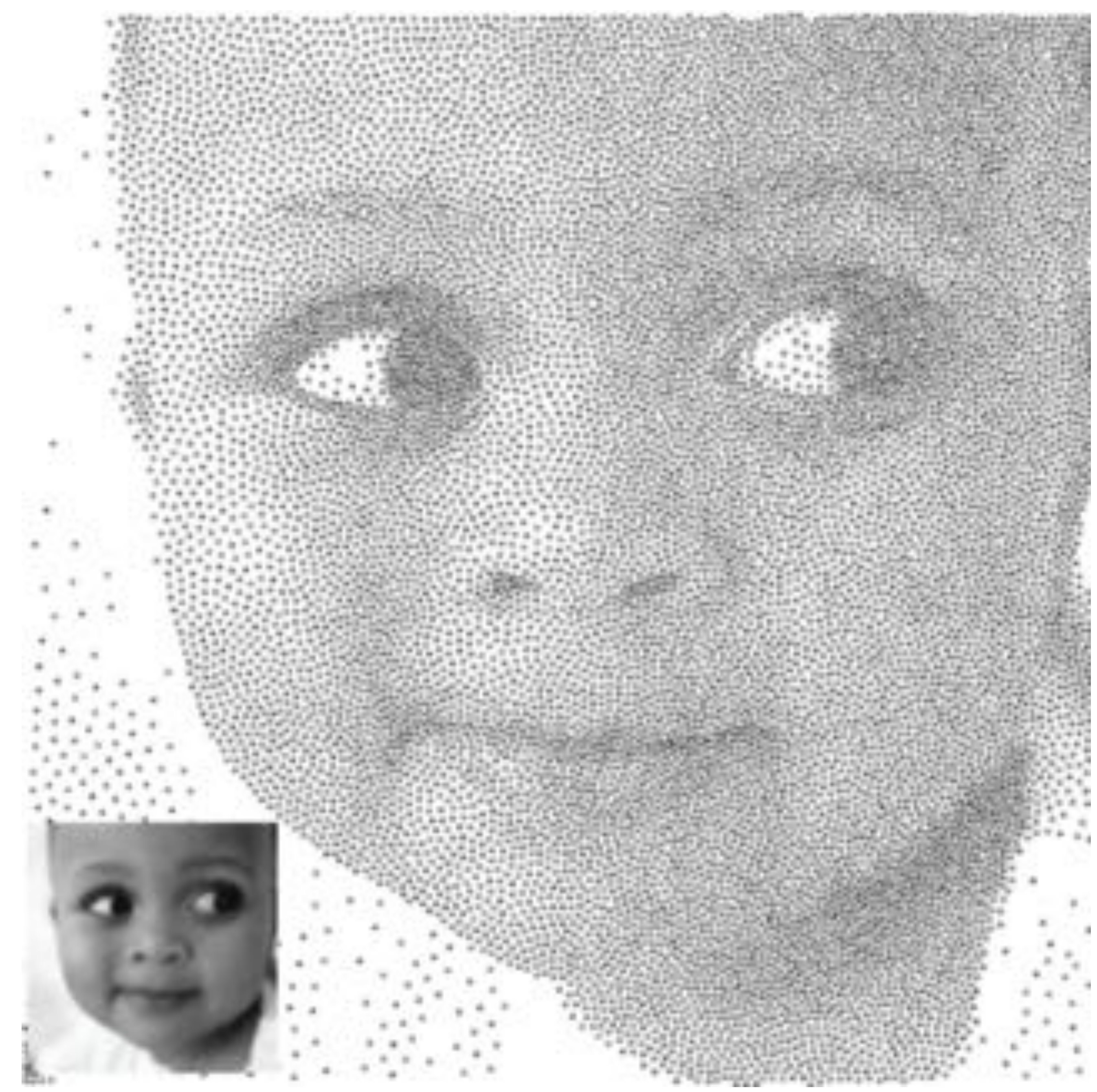


Sampling Patterns & Variance Reduction

- Want to pick samples according to a given density
- But even for uniform density, lots of possible sampling patterns
- Sampling pattern will affect variance (of estimator!)



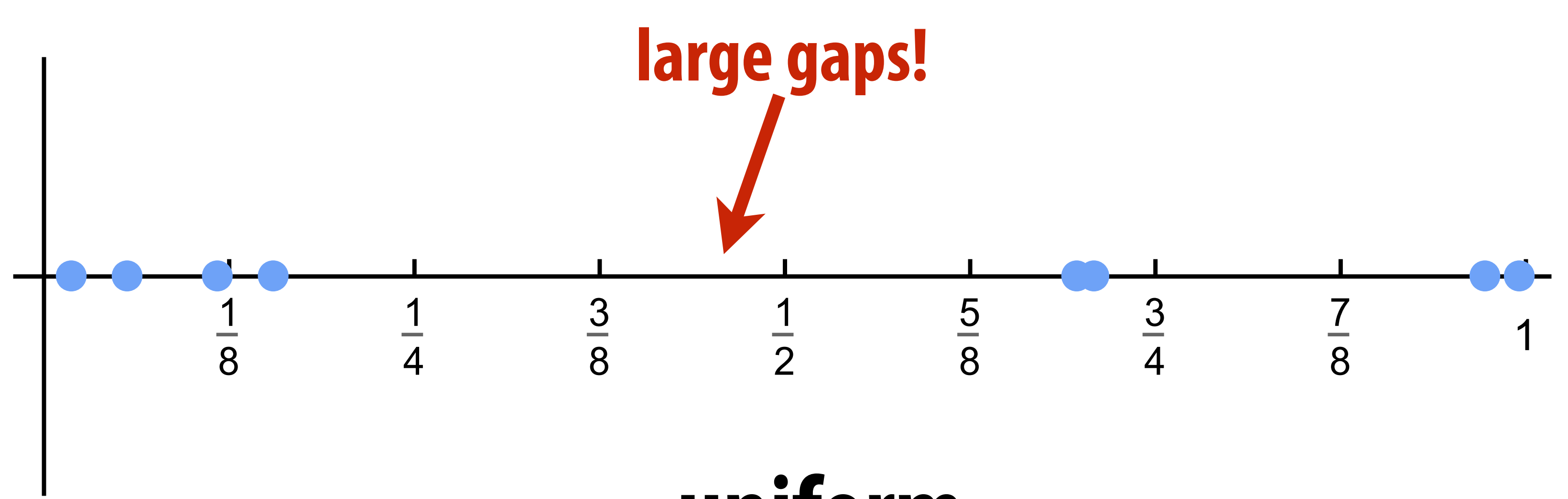
uniform sampling density



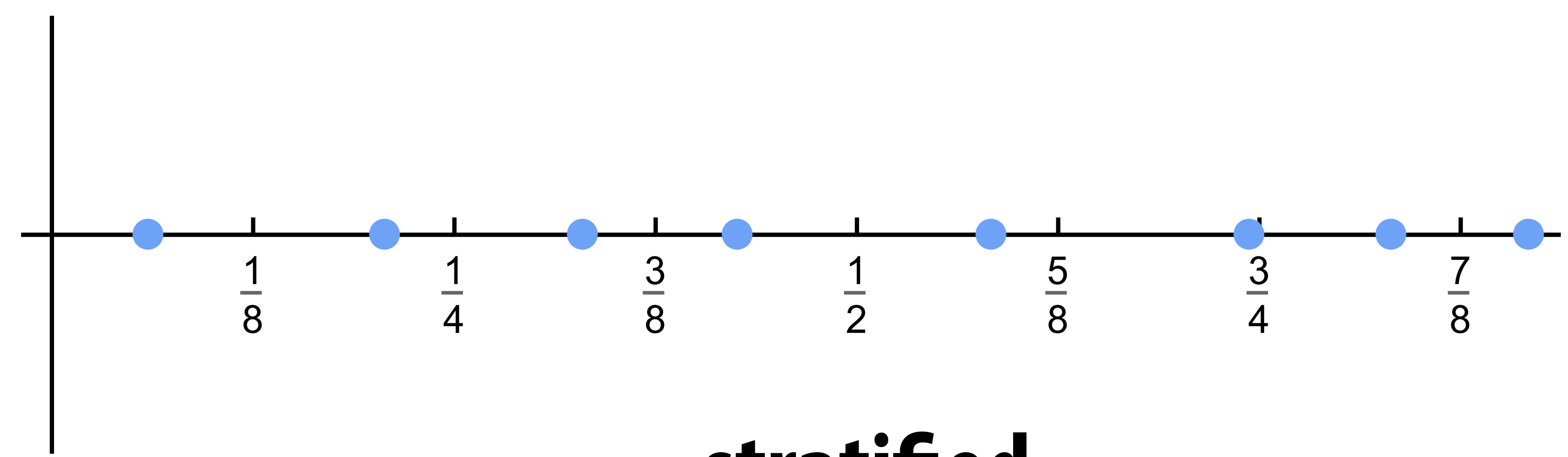
nonuniform sampling density

Stratified Sampling

- How do we pick n values from [0,1]?
- Could just pick n samples uniformly at random
- Alternatively: split into n bins, pick uniformly in each bin

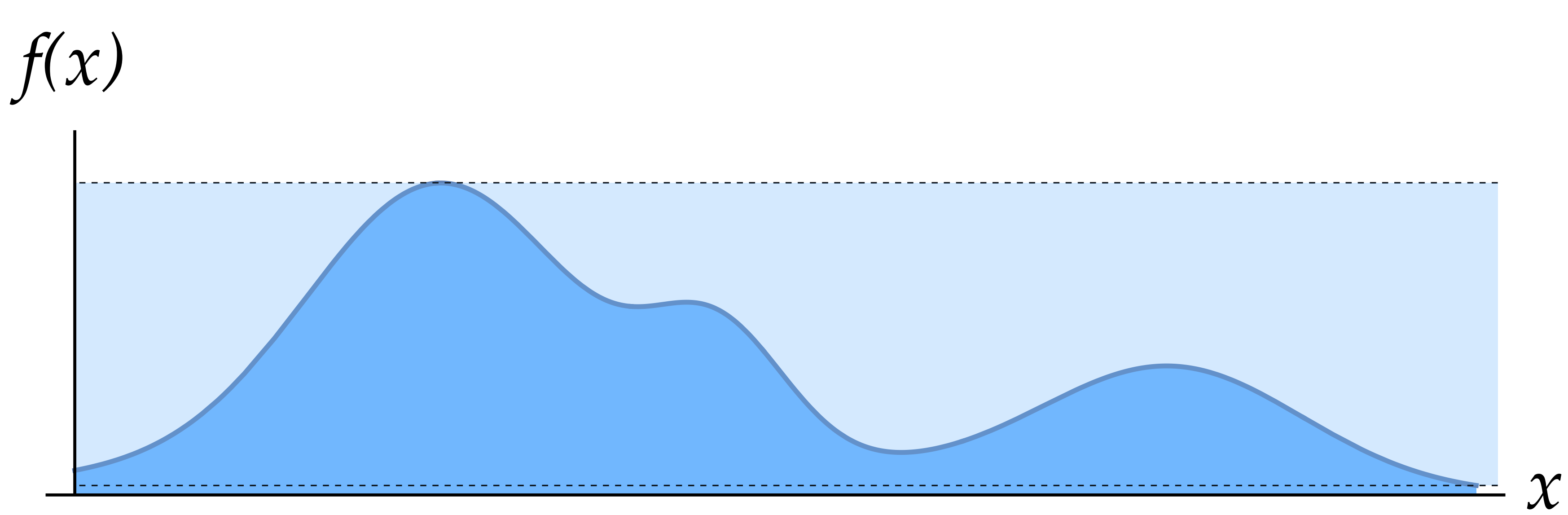


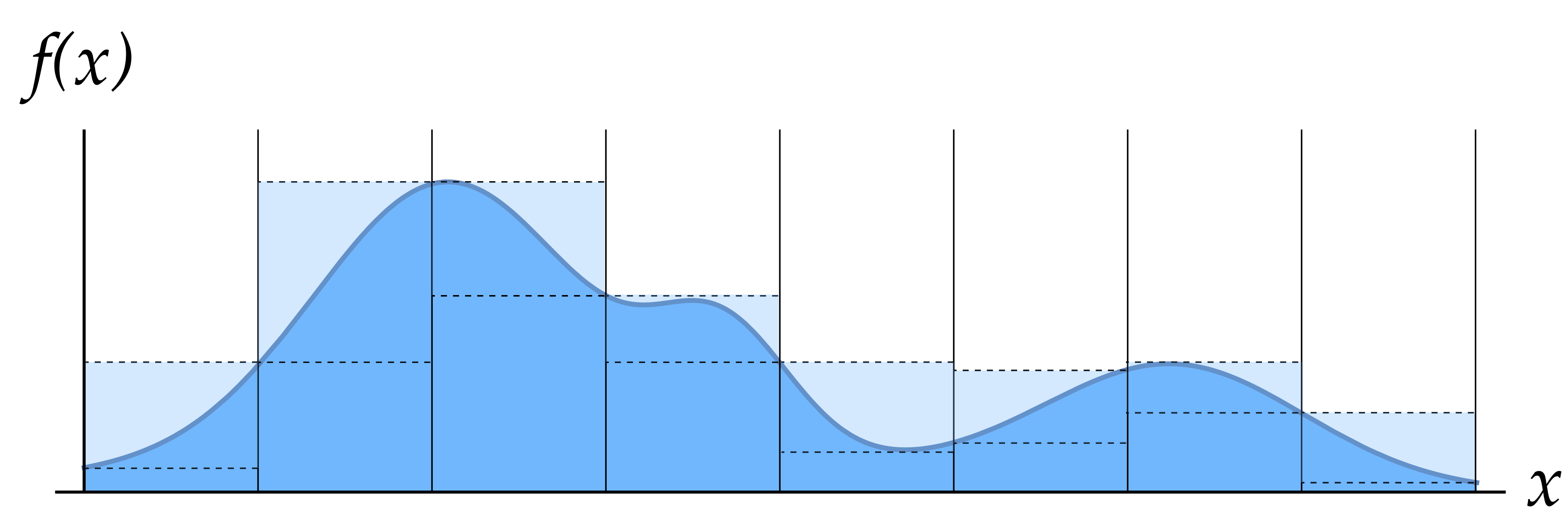
uniform



stratified

- FACT: stratified estimate never has larger variance (often lower)

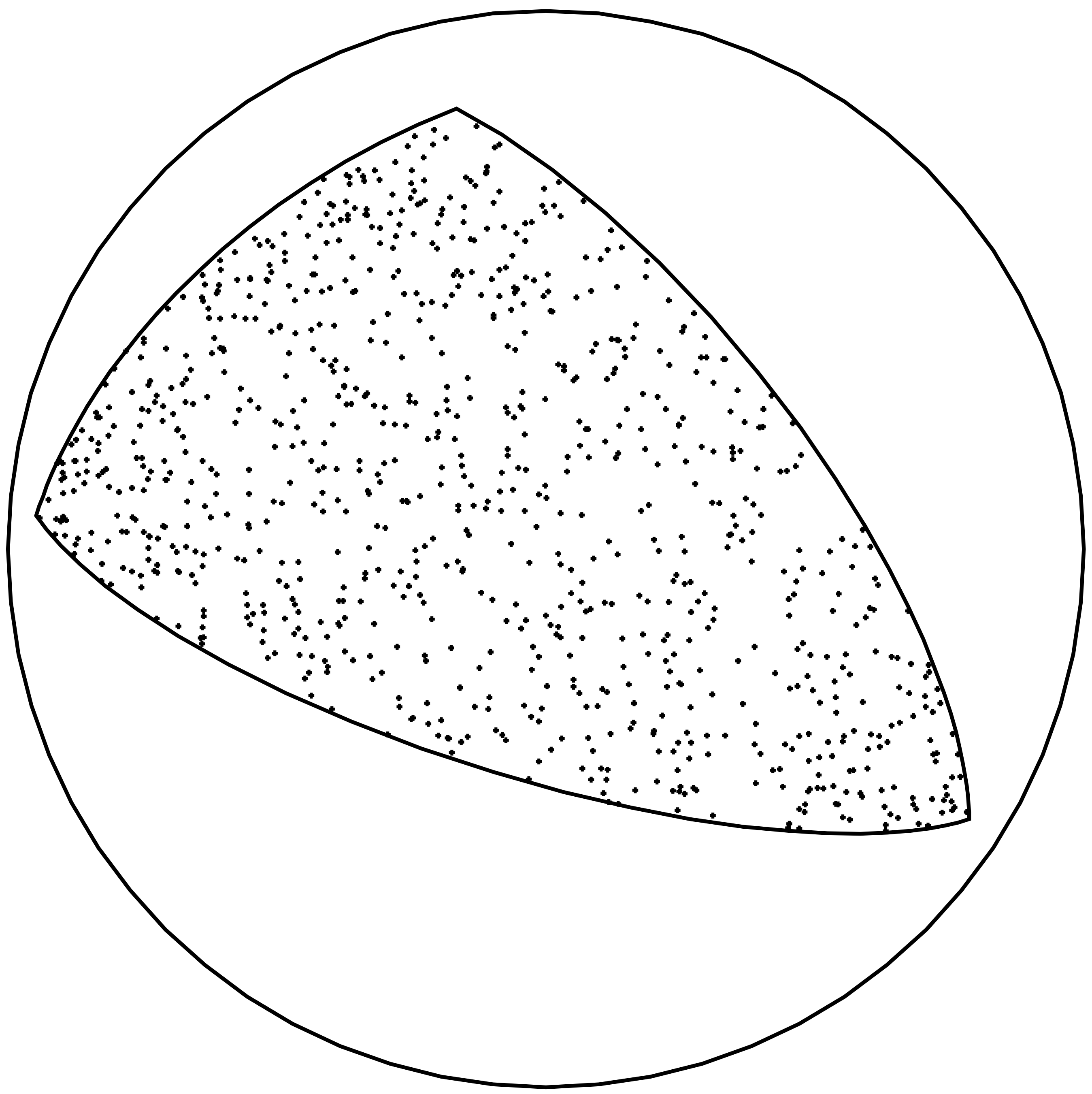


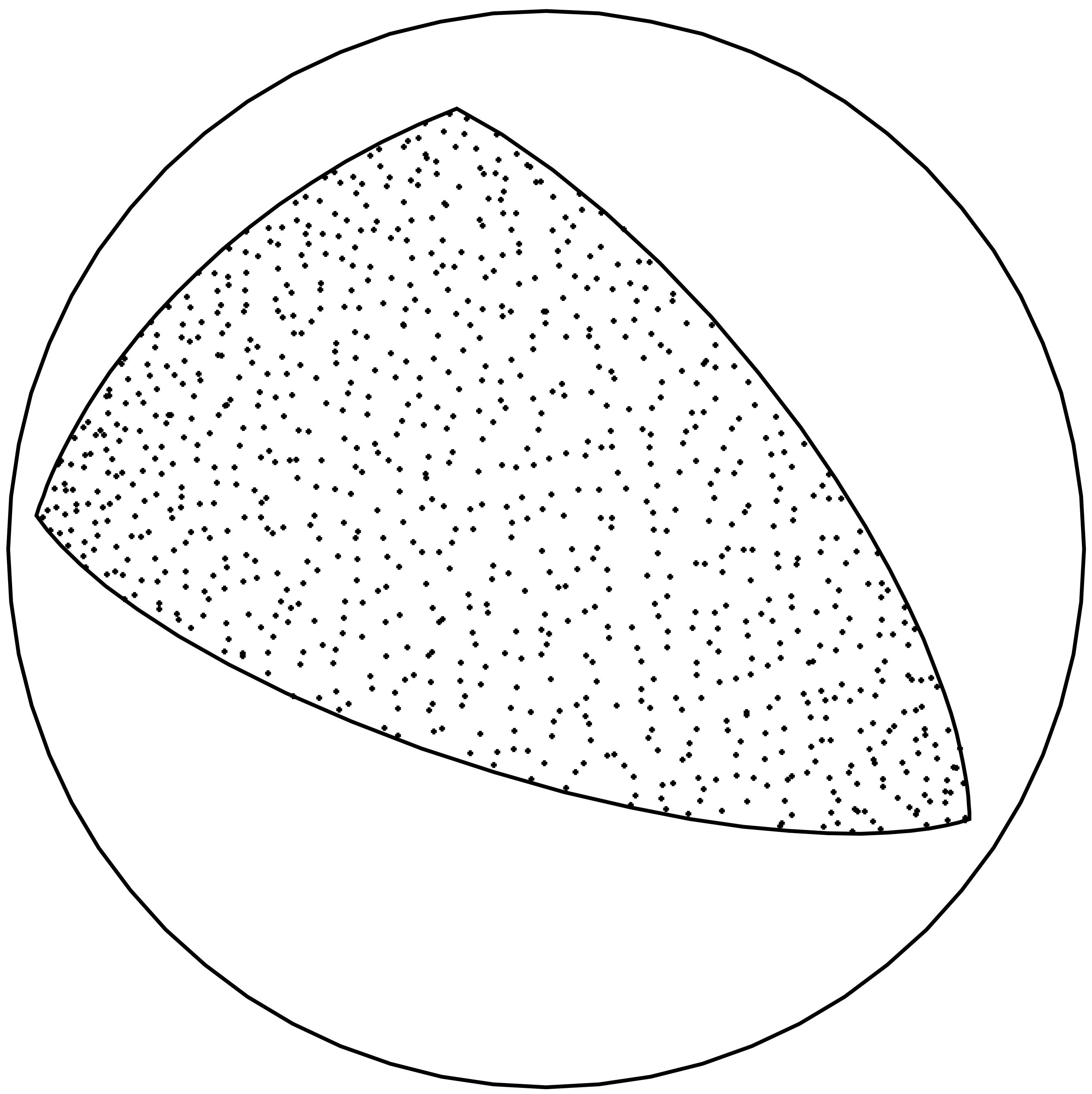


Intuition: each stratum has smaller variance. (Proof by linearity of expectation!)

Stratified Sampling in Rendering/Graphics

- Simply replacing uniform samples with stratified ones already improves quality of sampling for rendering (…and other graphics/visualization tasks!)





See especially: Jim Arvo, "Stratified Sampling of Spherical Triangles" (SIGGRAPH 1995)

Low-Discrepancy Sampling

- "No clumps" hints at one possible criterion for a good sample:
- Number of samples should be proportional to area
- Discrepancy measures deviation from this ideal

discrepancy of sample points X over a region S

number of samples in X covered by S

$$d_S(X) := \left| A(S) - \frac{n(S)}{|X|} \right|$$

area of S

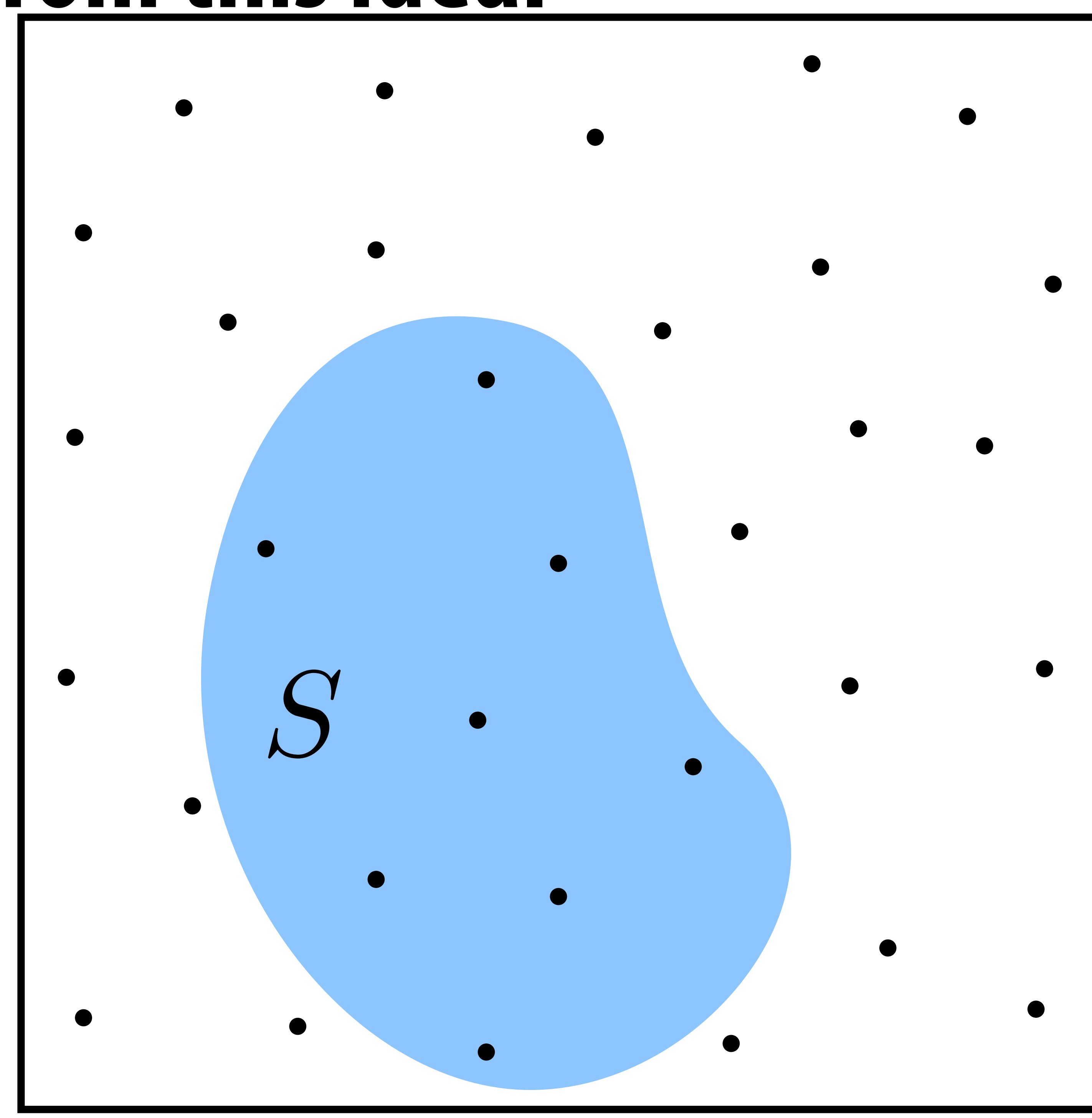
total # of samples in X

overall discrepancy of X

$$D(X) := \max_{S \in F} d_S(X)$$

(ideally equal to zero!)

some family of regions S (e.g., boxes, disks, ...)



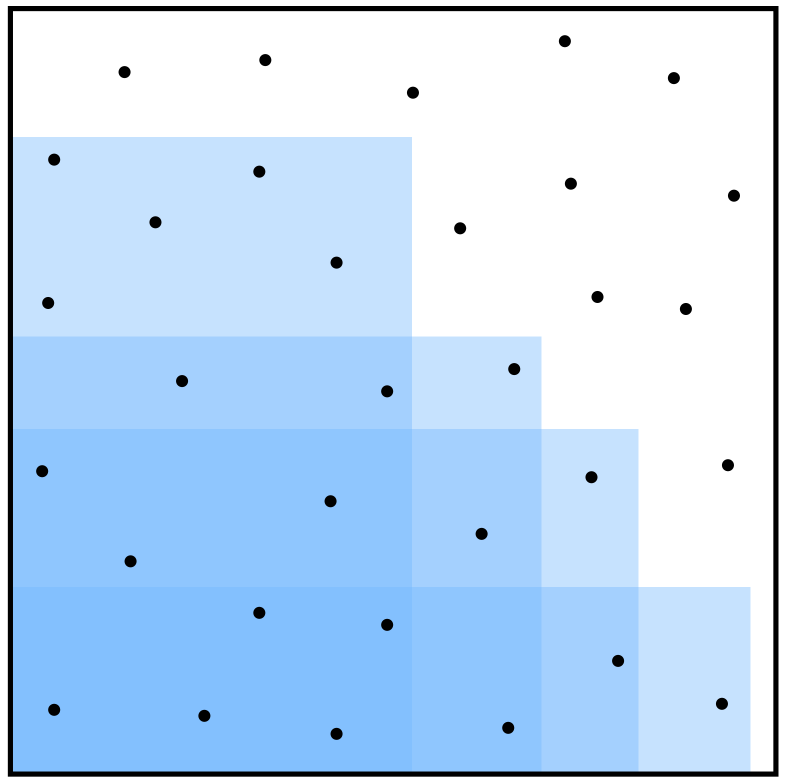
See especially: Dobkin et al, "Computing Discrepancy w/ Applications to Supersampling" (1996)

Quasi-Monte Carlo methods (QMC)

- **Replace truly random samples with low-discrepancy samples**
- **Why? Koksma's theorem:**

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(\overset{\text{sample points in X}}{x_i}) - \int_0^1 f(x)\ dx \right| \leq \overset{\text{total variation of f (integral of |f'|)}}{\mathcal{V}(f)}\ \overset{\text{discrepancy of sample X}}{D(X)}$$

- **I.e., for low-discrepancy X, estimate approaches integral**
- **Similar bounds can be shown in higher dimensions**
- **WARNING: total variation not always bounded!**
- **WARNING: only for family F of axis-aligned boxes S!**
- **Discrepancy still a very reasonable criterion in practice**



F

Hammersley & Halton Points

- Can easy generate samples with near-optimal discrepancy
- First define radical inverse $\varphi_r(i)$
- Express integer i in base r, then reflect digits around decimal
- E.g., $\varphi_{10}(1234) = 0.4321$
- Can get n Halton points $x_1, \ldots, x_n$ in k-dimensions via

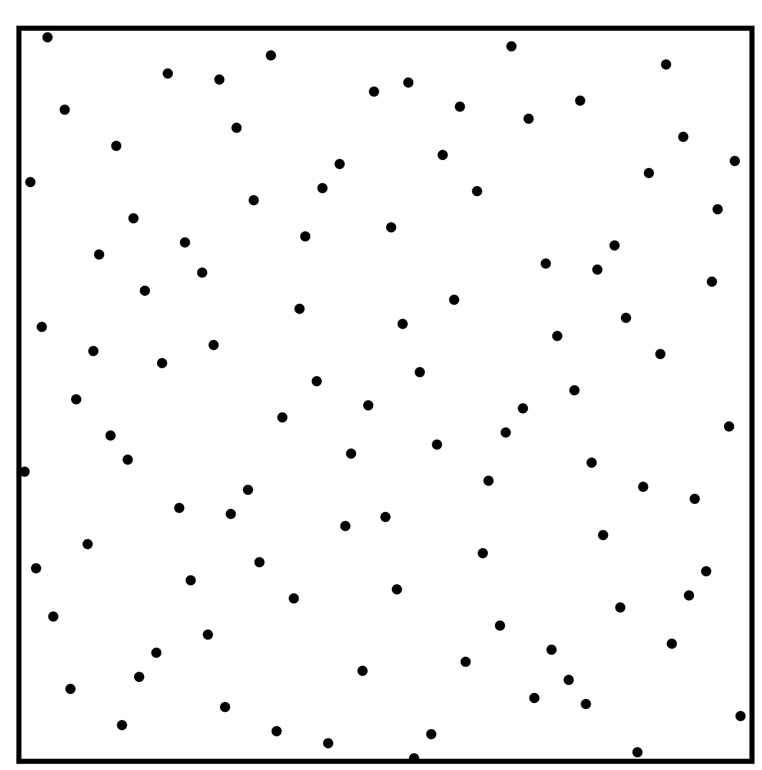
$$x_i = (\phi_{P_1}(i), \phi_{P_2}(i), \ldots, \phi_{P_k}(i))$$

kth prime number

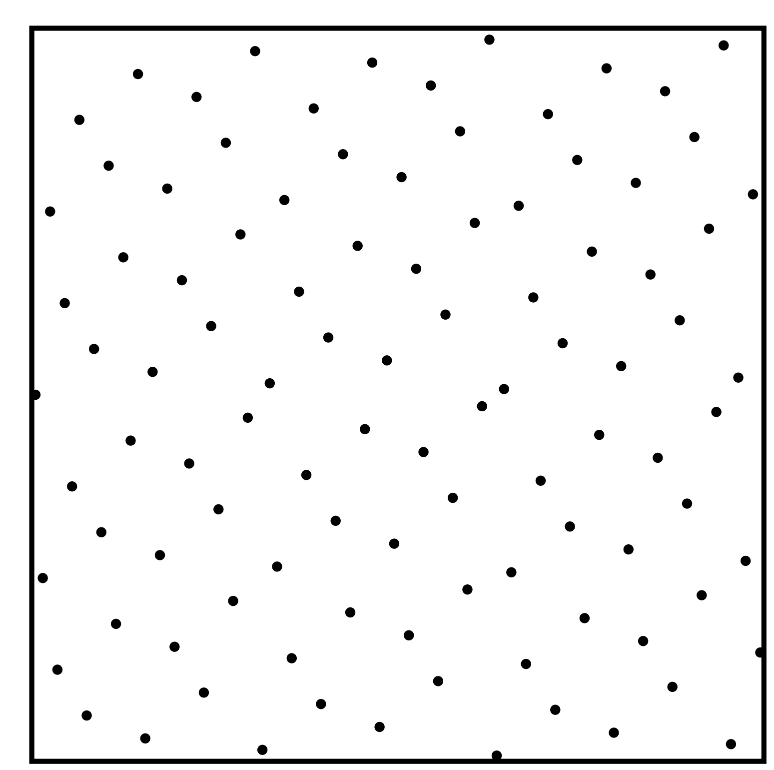
- Similarly, Hammersley sequence is

$$x_i = (i/n, \phi_{P_1}(i), \phi_{P_2}(i), \ldots, \phi_{P_{k-1}}(i))$$

n must be known ahead of time!

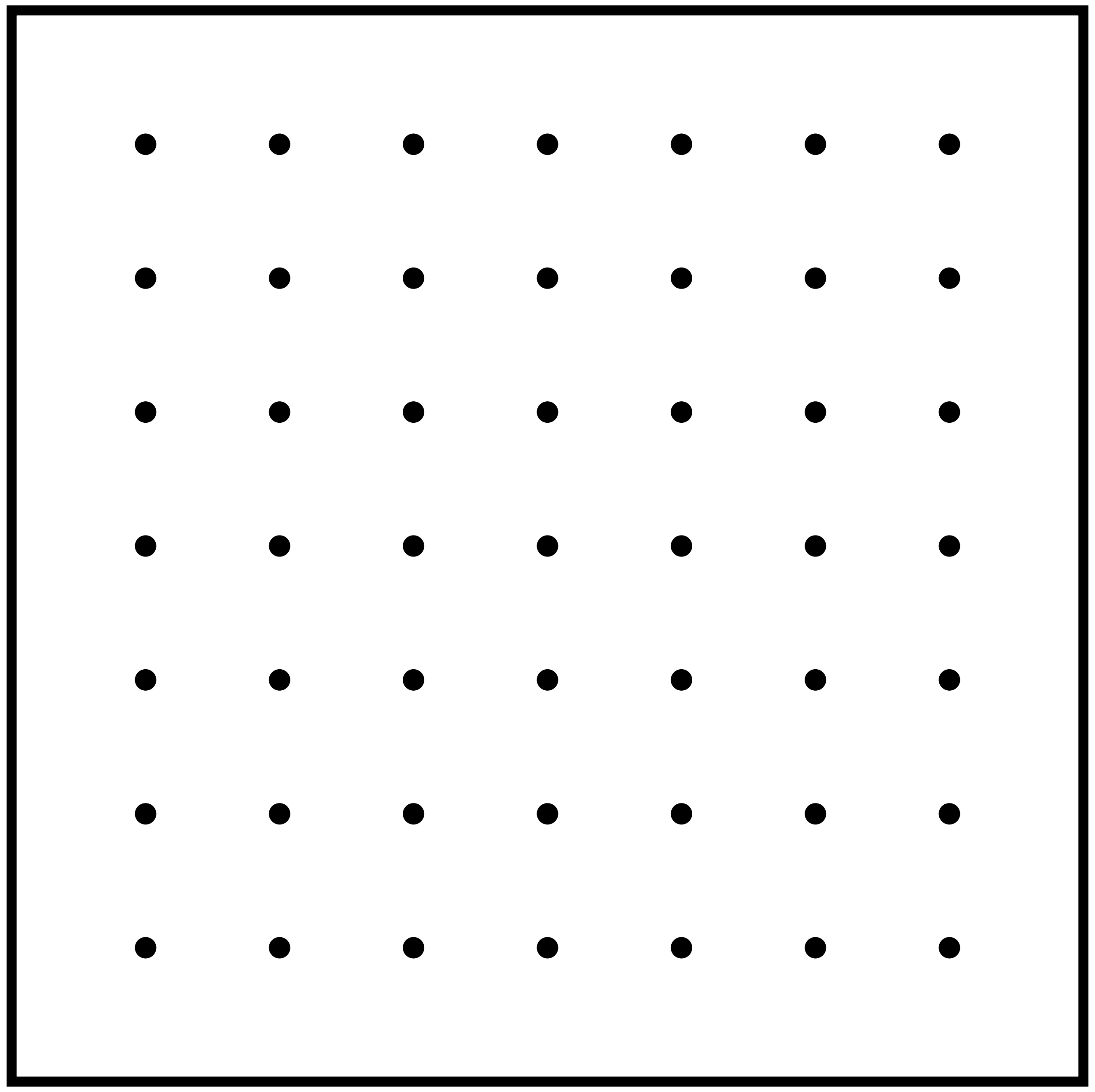


Halton



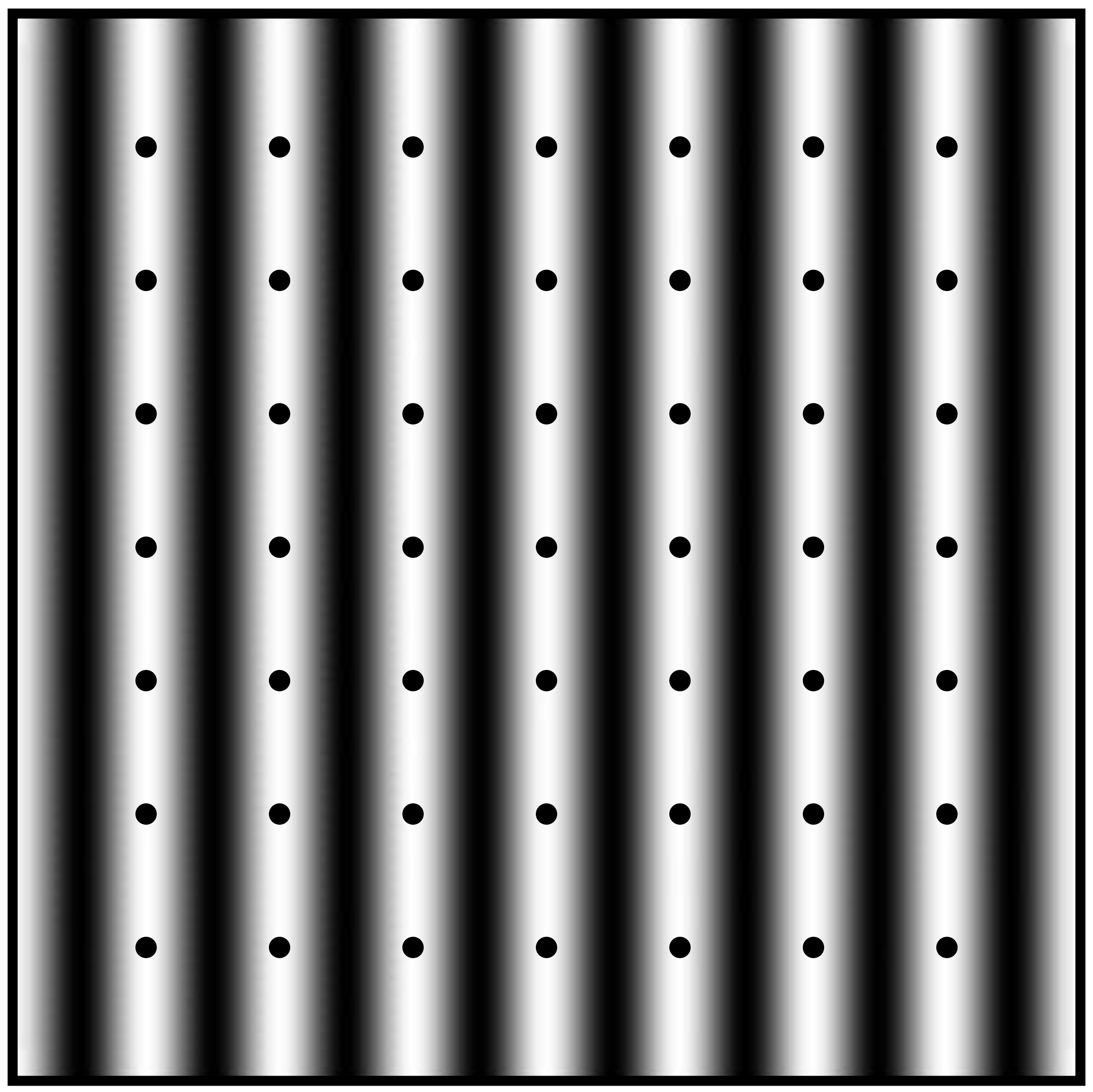
Hammersley

Wait, but doesn't a regular grid have really low discrepancy...?

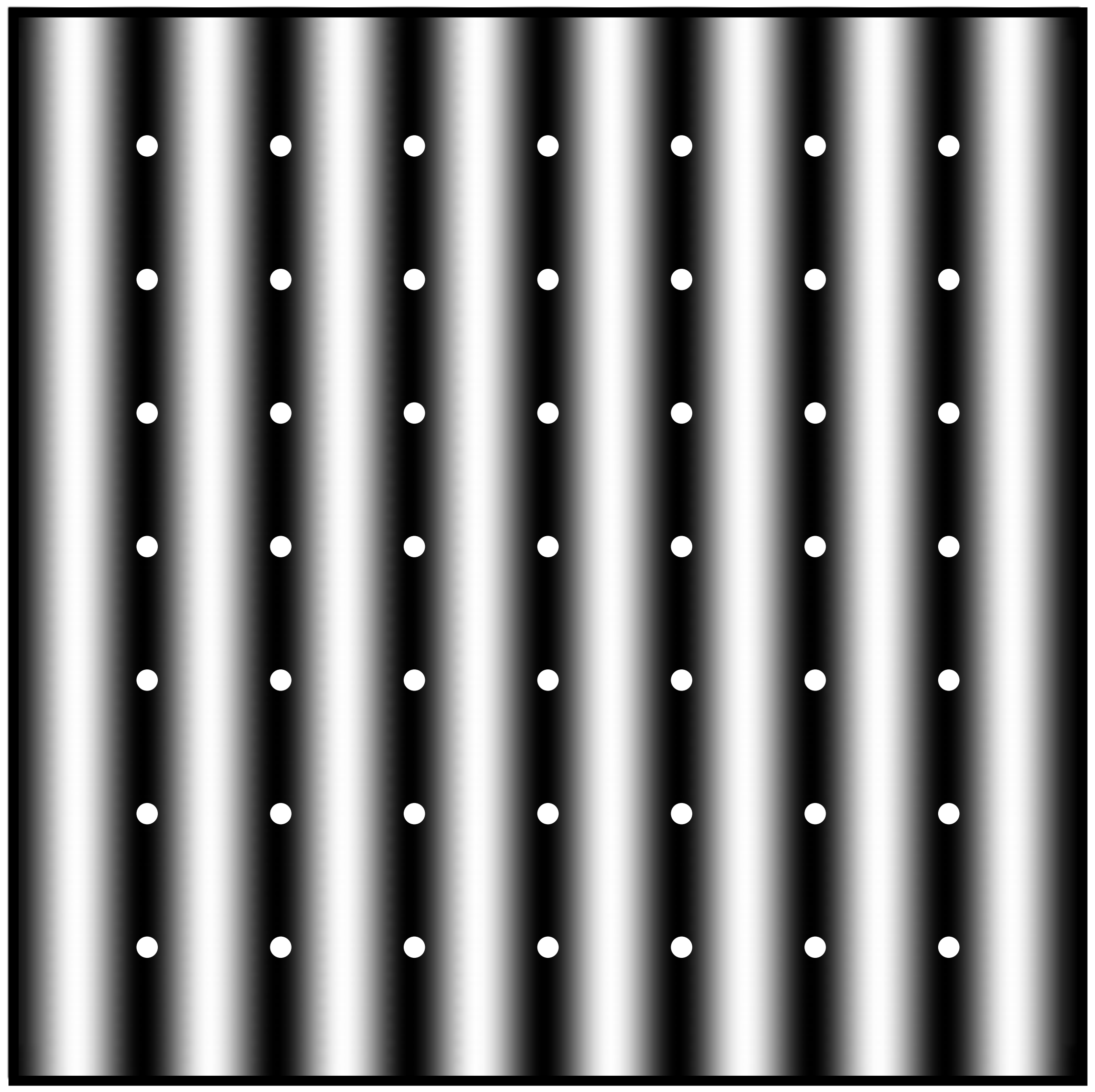


There's more to life than discrepancy

- **Even low-discrepancy patterns can exhibit poor behavior:**



$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = 1$$



$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) = 0$$

- **Want pattern to be anisotropic (no preferred direction)**
- **Also want to avoid any preferred frequency (see above!)**

Blue Noise - Motivation

- Can observe that monkey retina exhibits blue noise pattern [Yellott 1983]

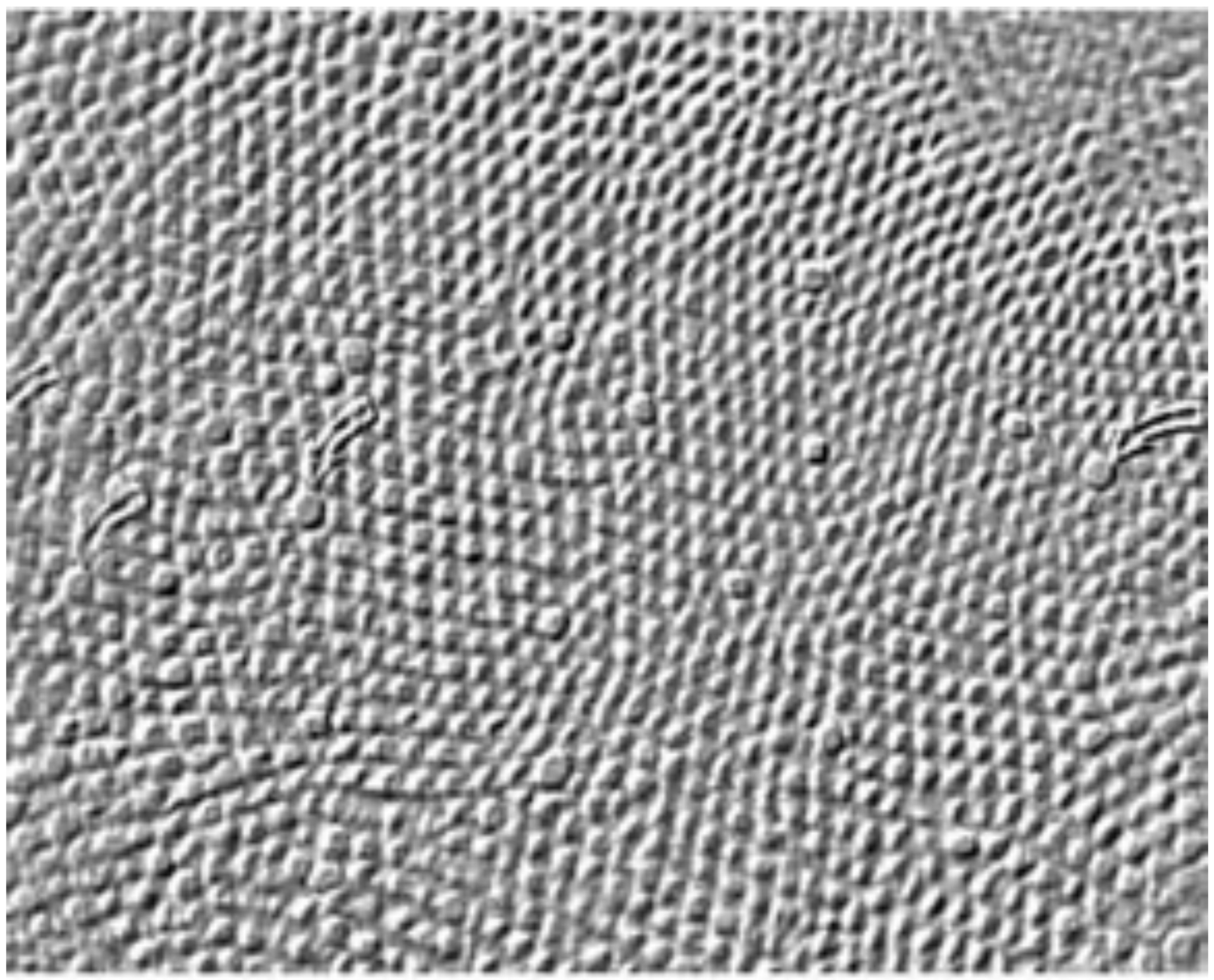
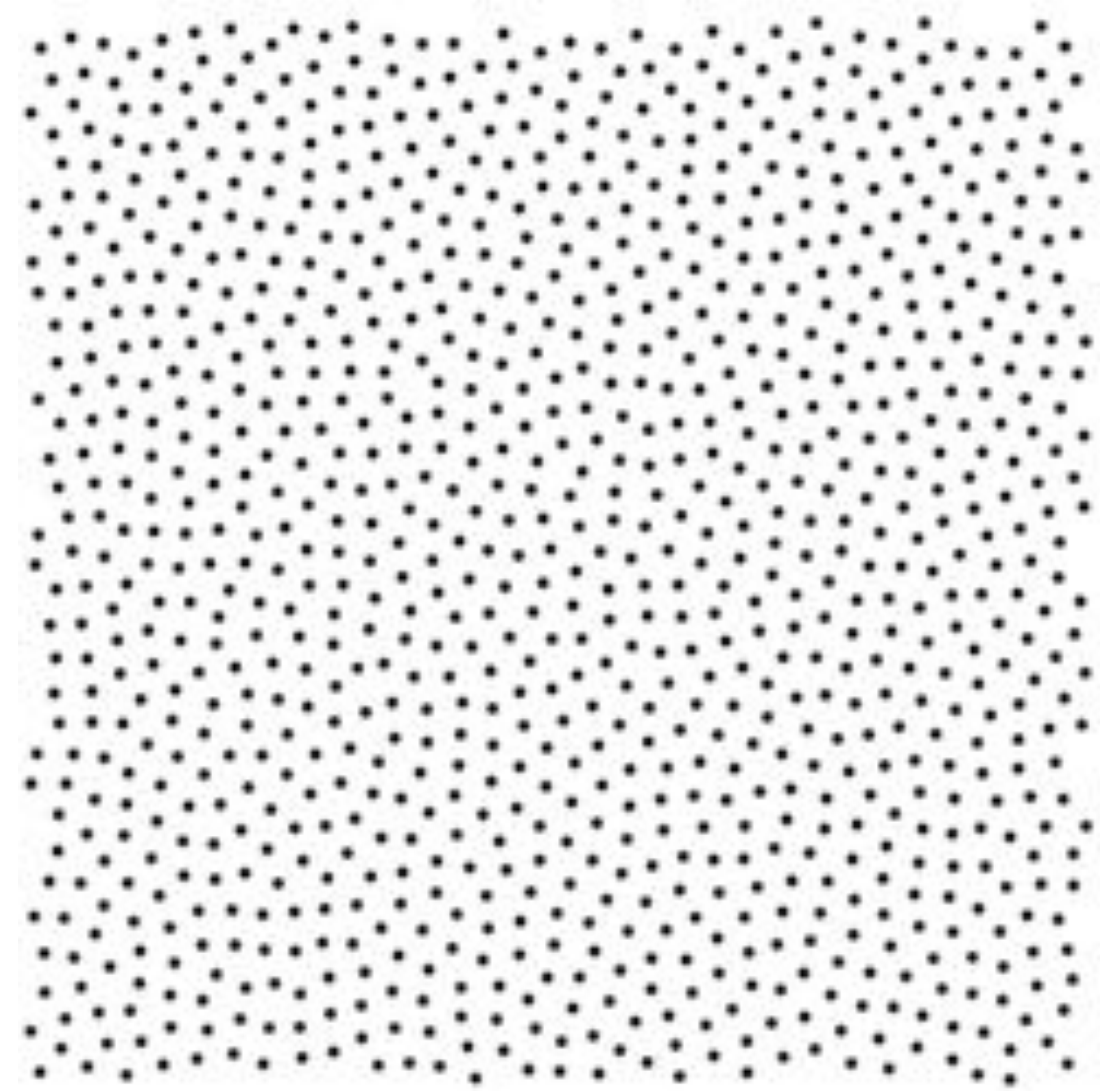


Fig. 13. Tangential section through the human fovea. Larger cones (arrows) are blue cones. From Ahnelt et al. 1987.

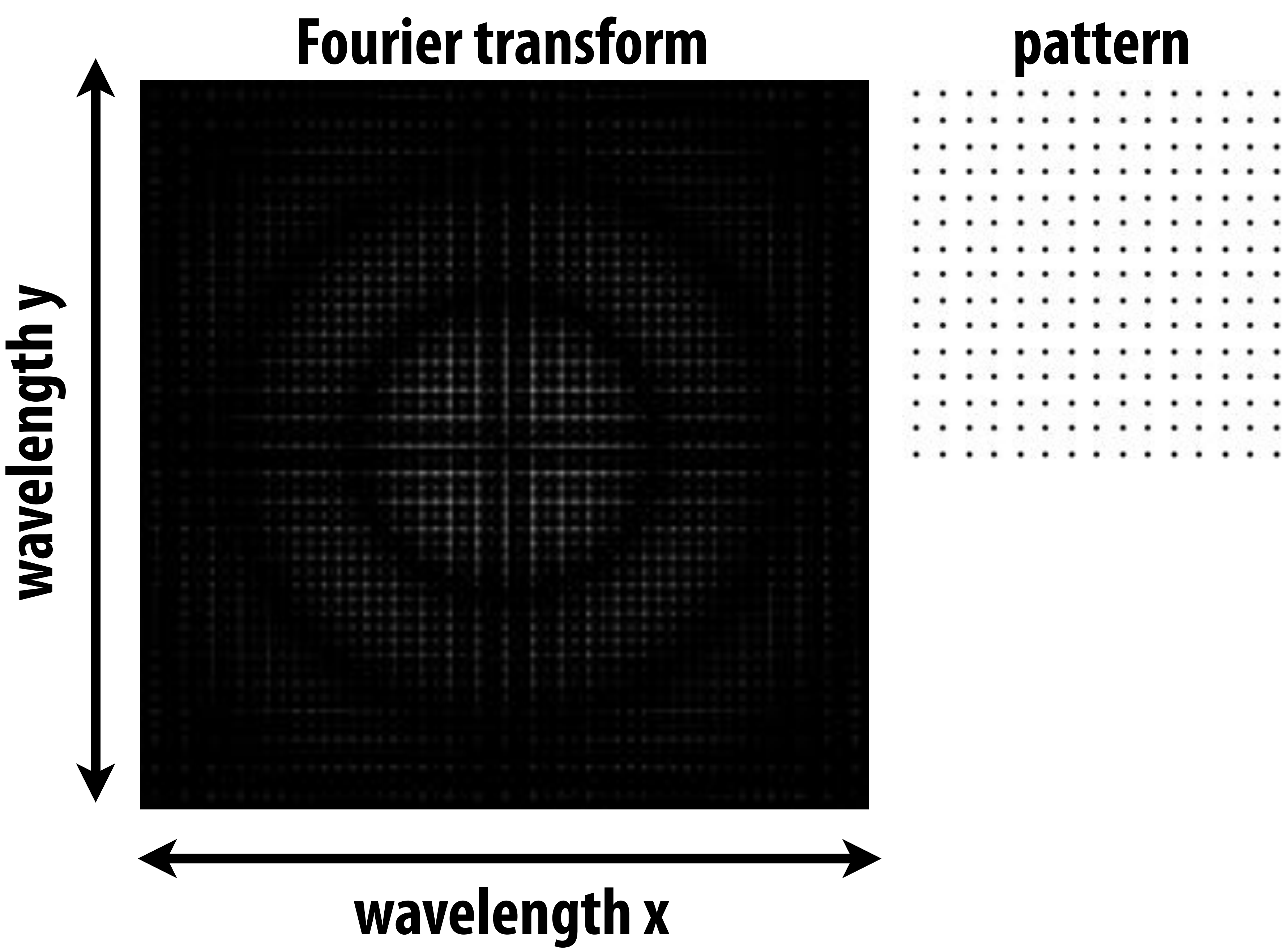


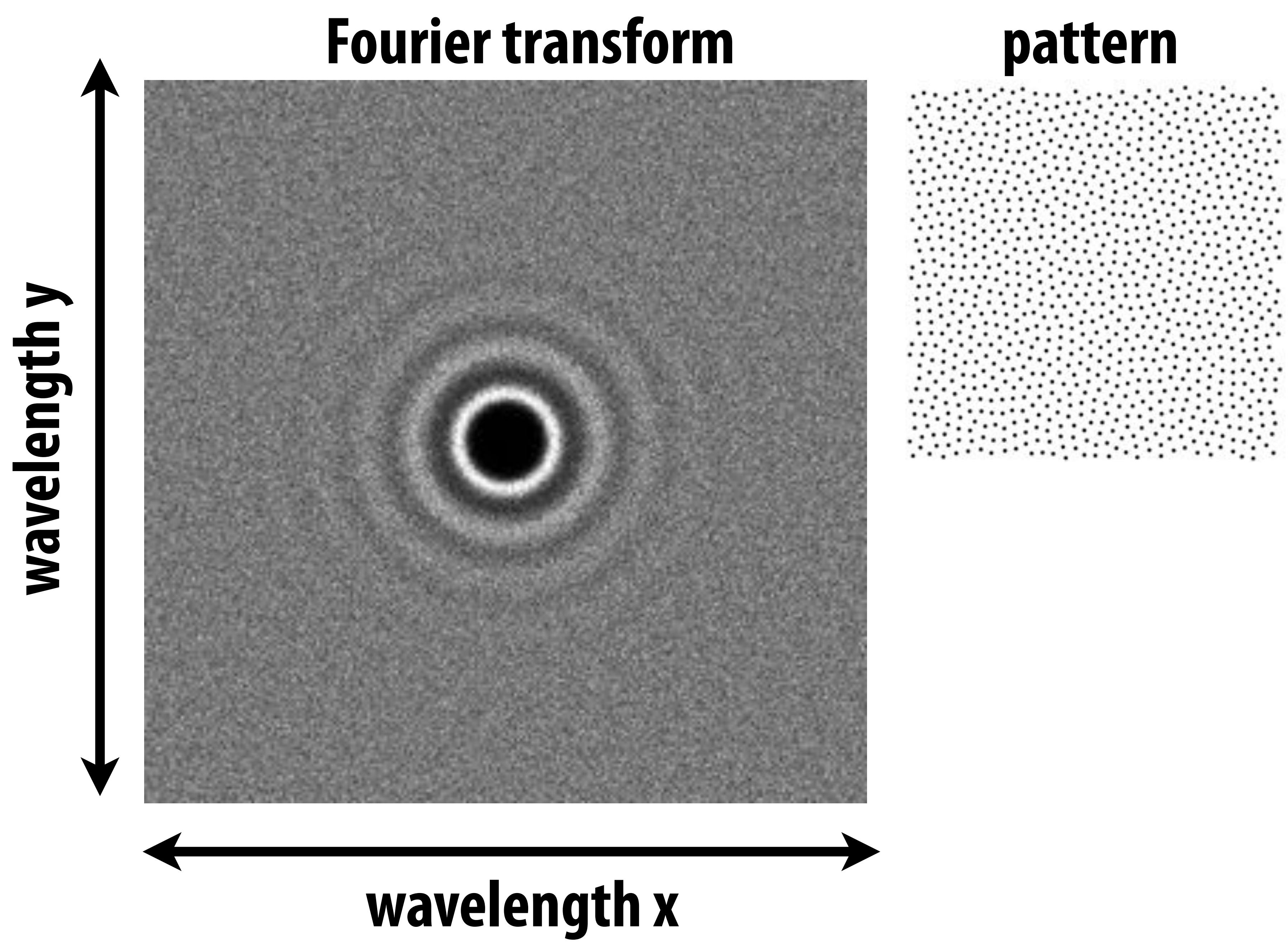
"blue noise"

- No obvious preferred directions (anisotropic)
- What about frequencies?

Blue Noise - Fourier Transform

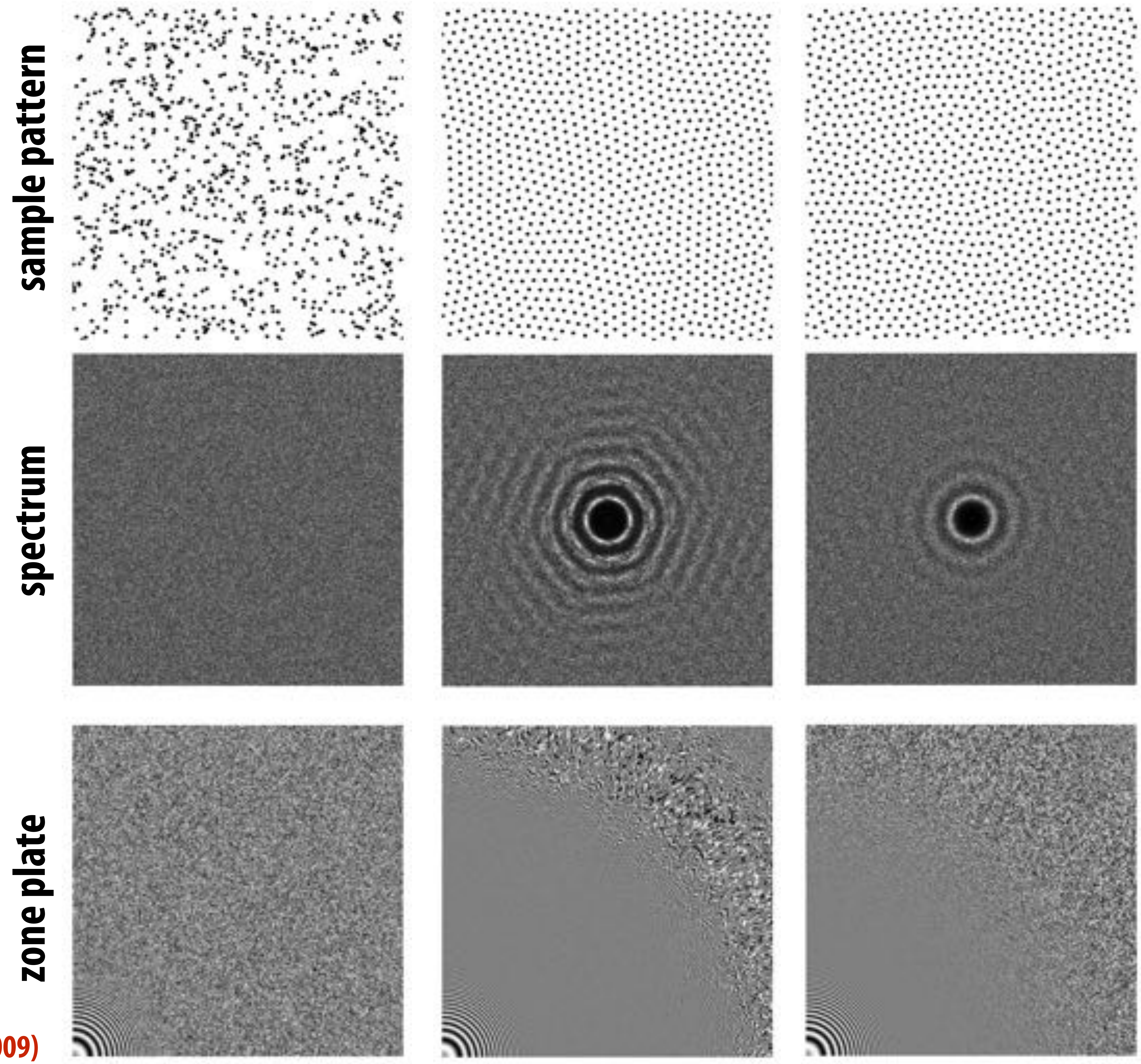
- Can analyze quality of a sample pattern in Fourier domain





- Regular pattern has "spikes" at regular intervals
- Blue noise is spread evenly over all frequencies in all directions
- bright center "ring" corresponds to sample spacing

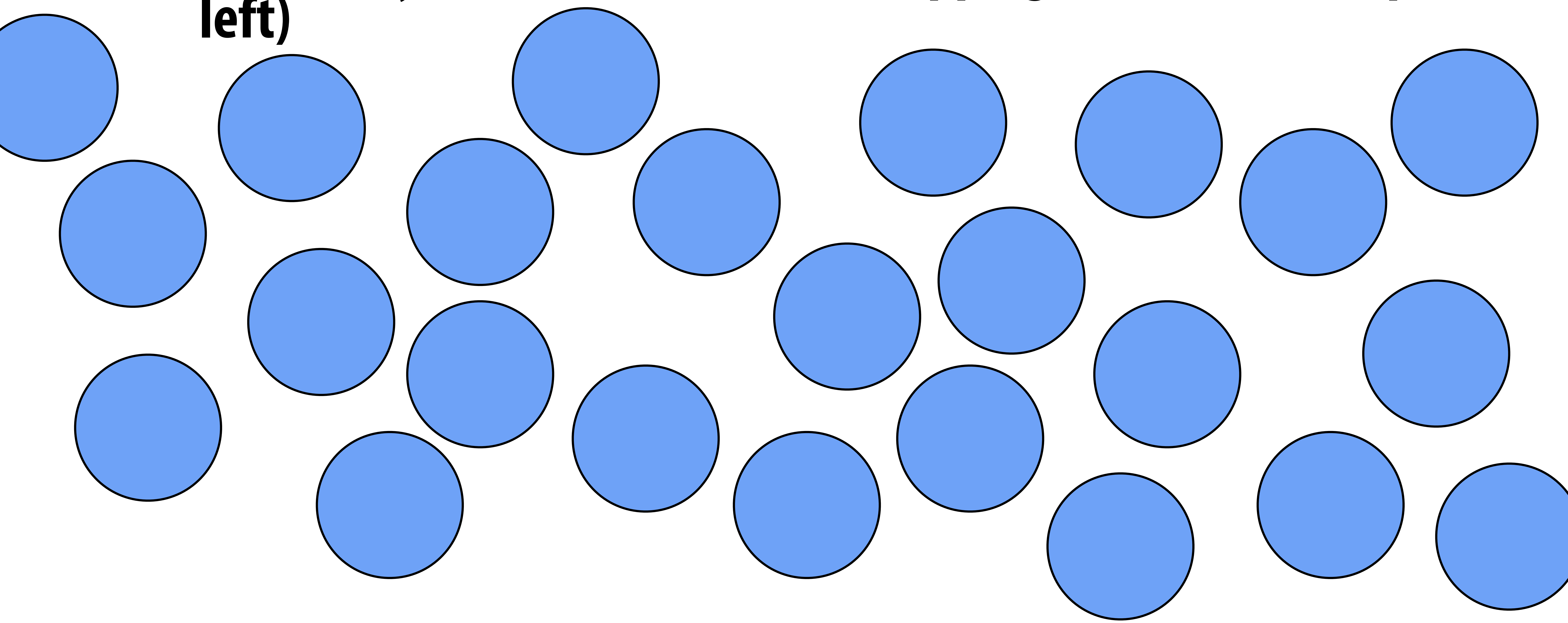
Spectrum affects reconstruction quality



(from Balzer et al 2009)

Poisson Disk Sampling

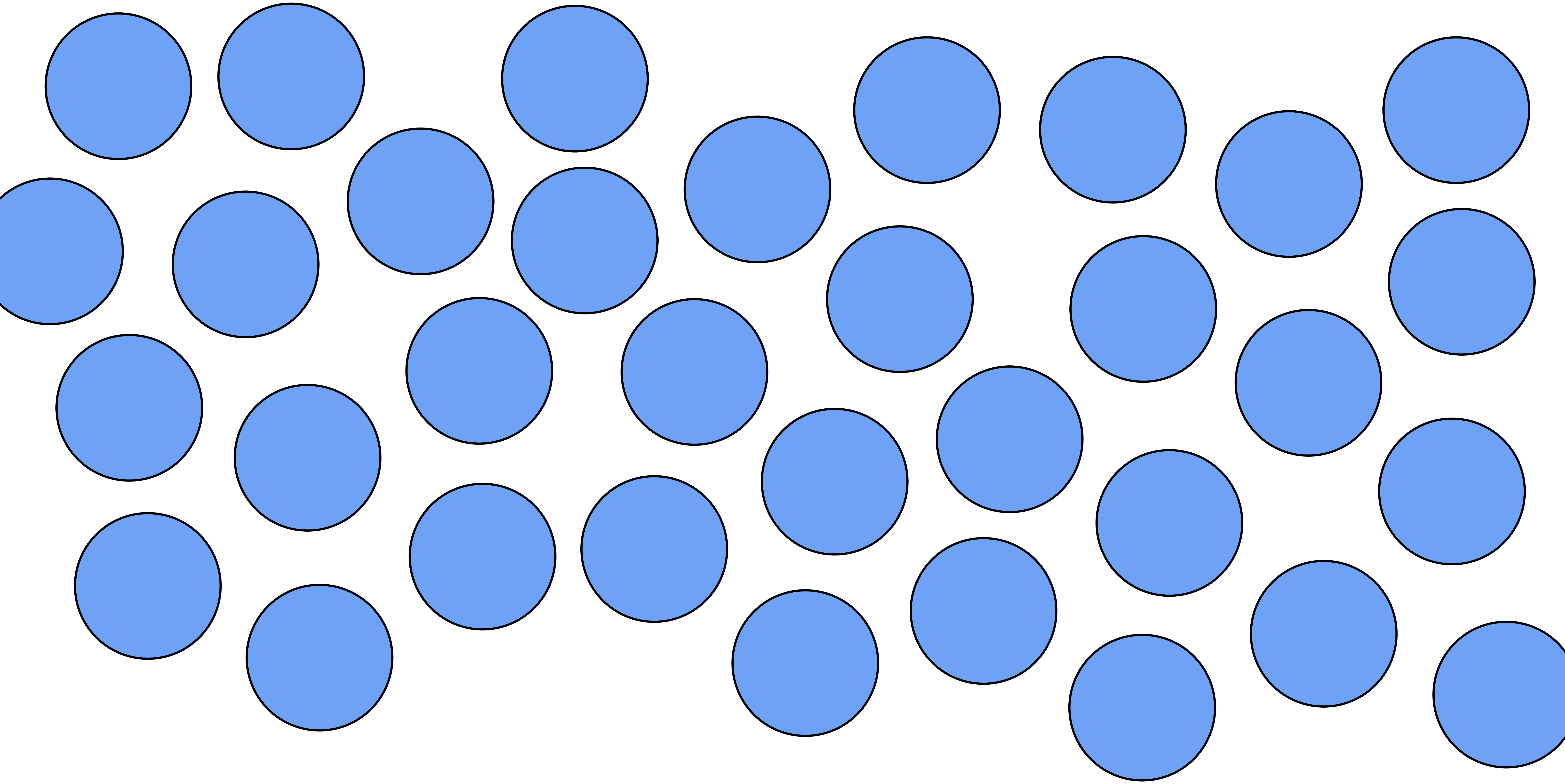
- How do you generate a “nice” sample?
- One of the earliest algorithms: Poisson disk sampling
- Iteratively add random non-overlapping disks (until no space left)



Decent spectral quality, but we can do better.

Lloyd Relaxation

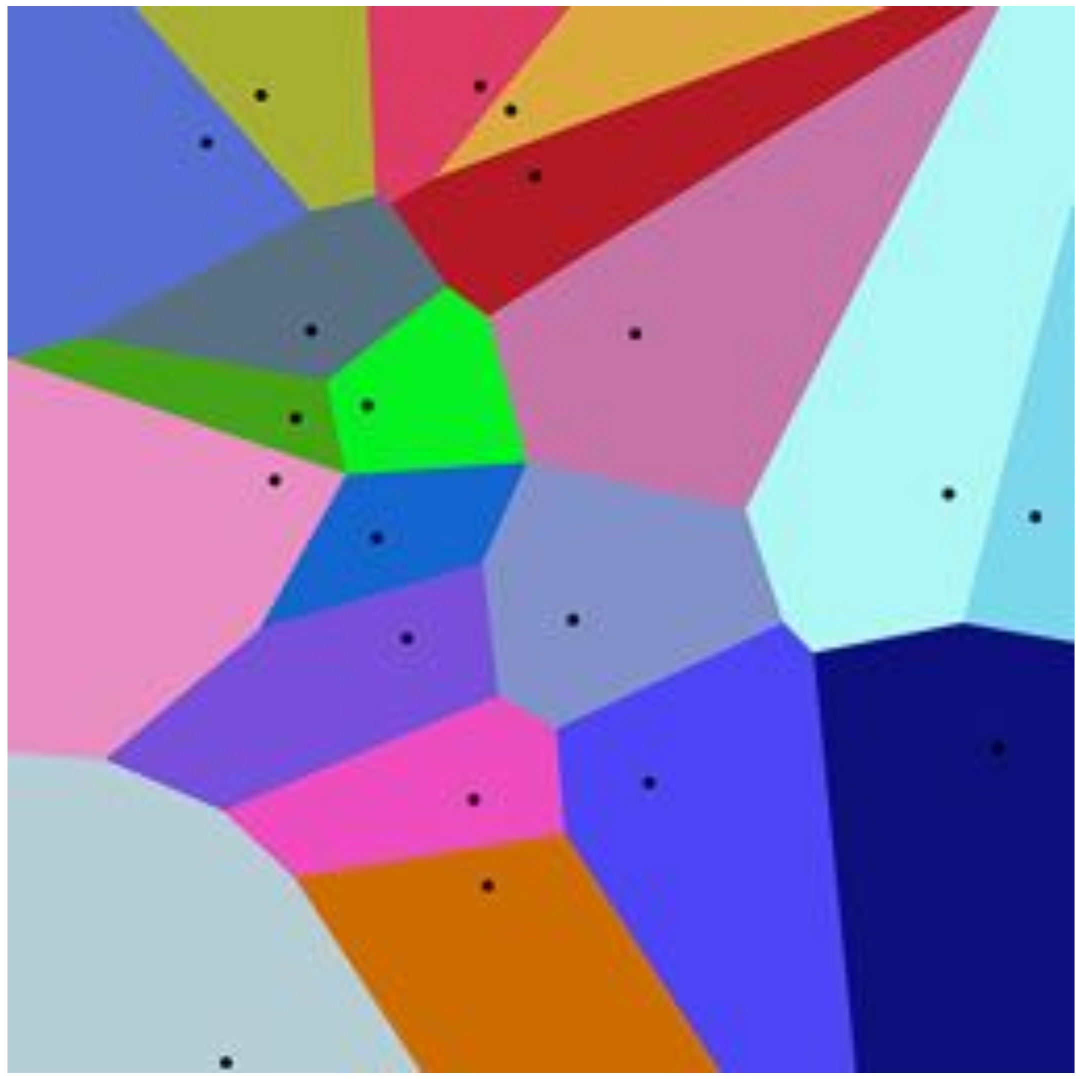
- Iteratively move each disk to the center of its neighbors



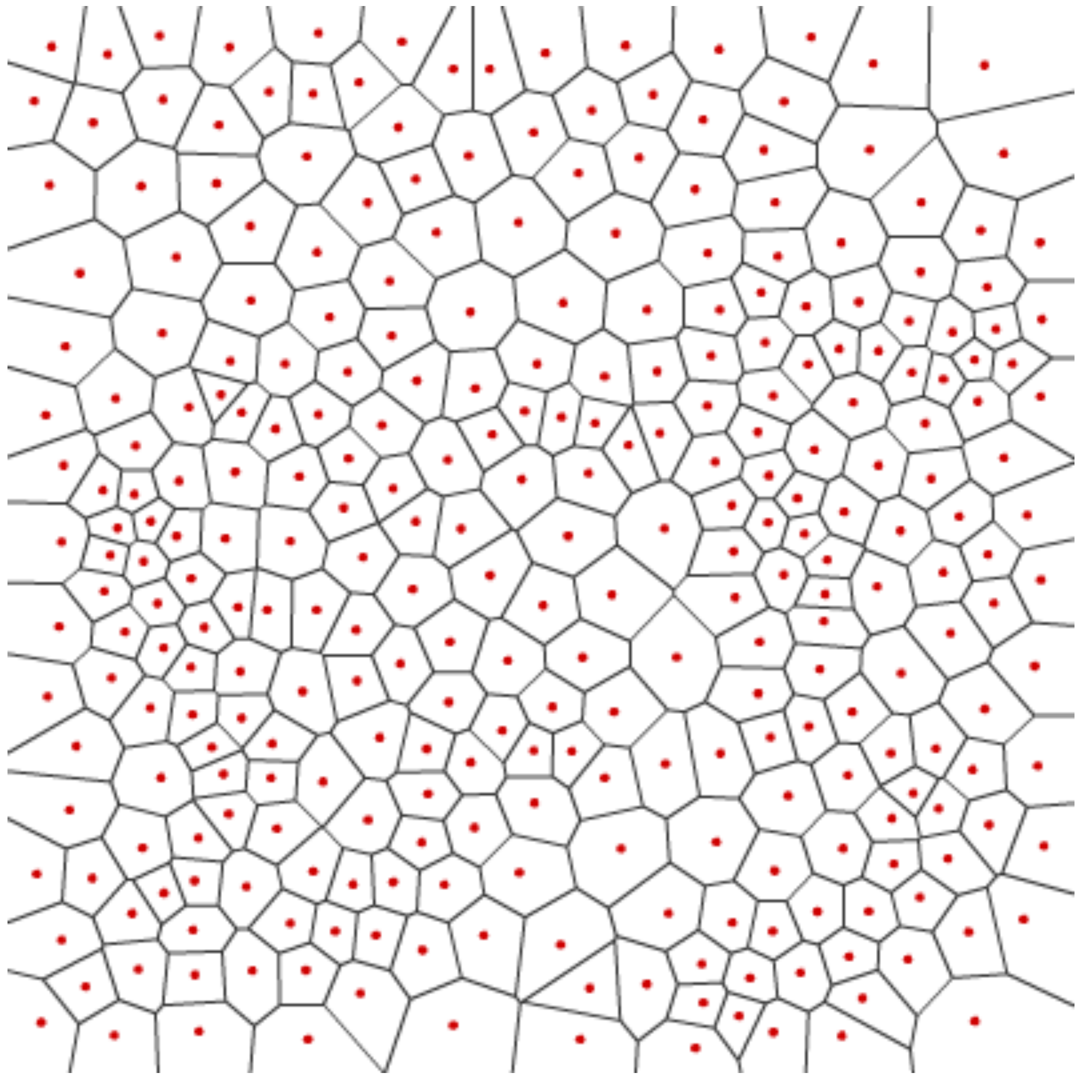
Better spectral quality, slow to converge. Can do better yet...

Voronoi-Based Methods

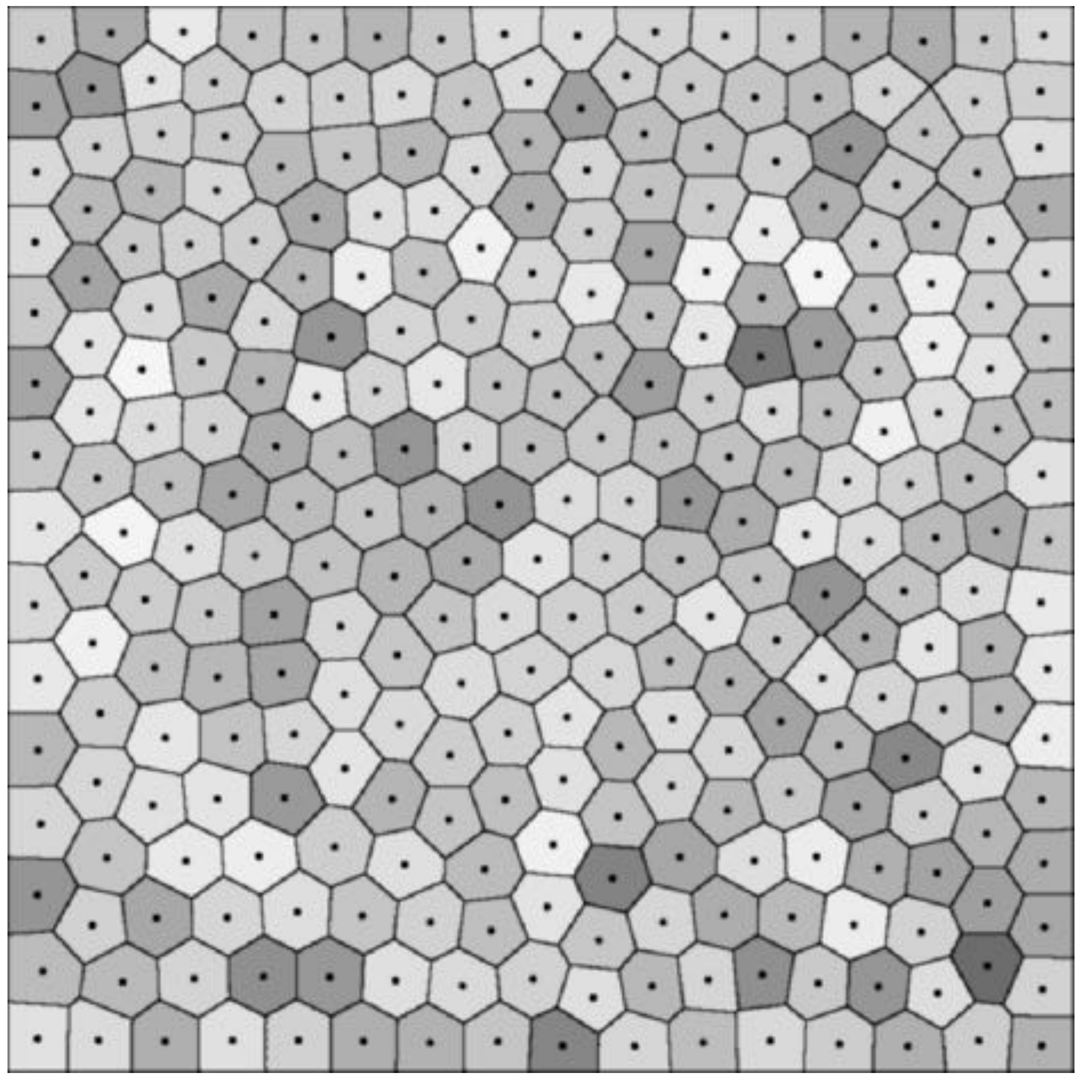
- Natural evolution of Lloyd
- Associate each sample with set of closest points (Voronoi cell)
- Optimize qualities of this Voronoi diagram
- E.g., sample is at cell's center of mass, cells have same area, etc.



Voronoi



centroidal



equal area

Adaptive Blue Noise

- Can adjust cell size to sample a given density (e.g., importance)

Computational tradeoff: expensive* precomputation / efficient sampling.

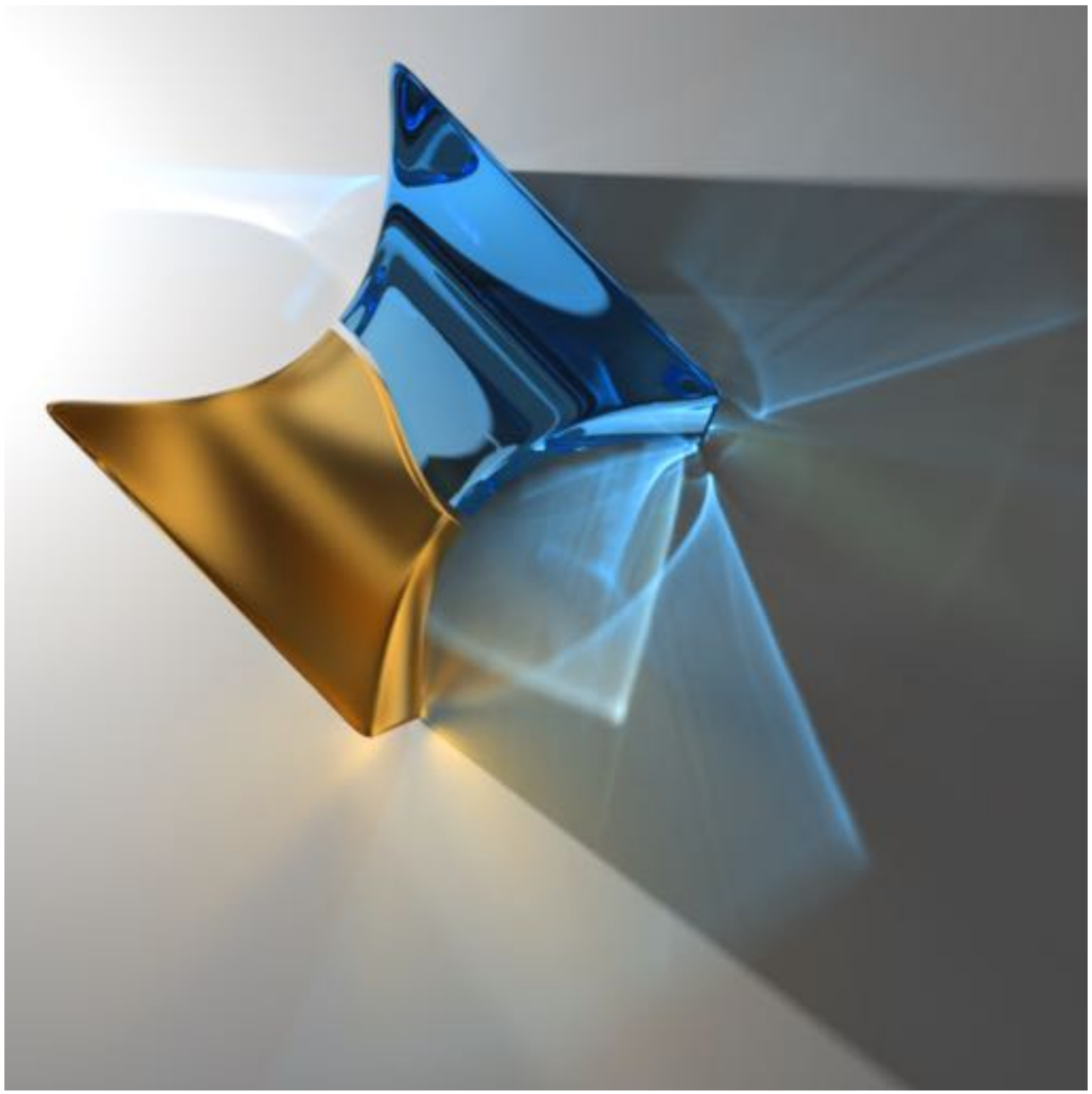
*But these days, not that expensive...

Ok, great!
Now we’ve wrapped up sampling
and we’ve reached the end of our discussion
of Monte Carlo path tracing

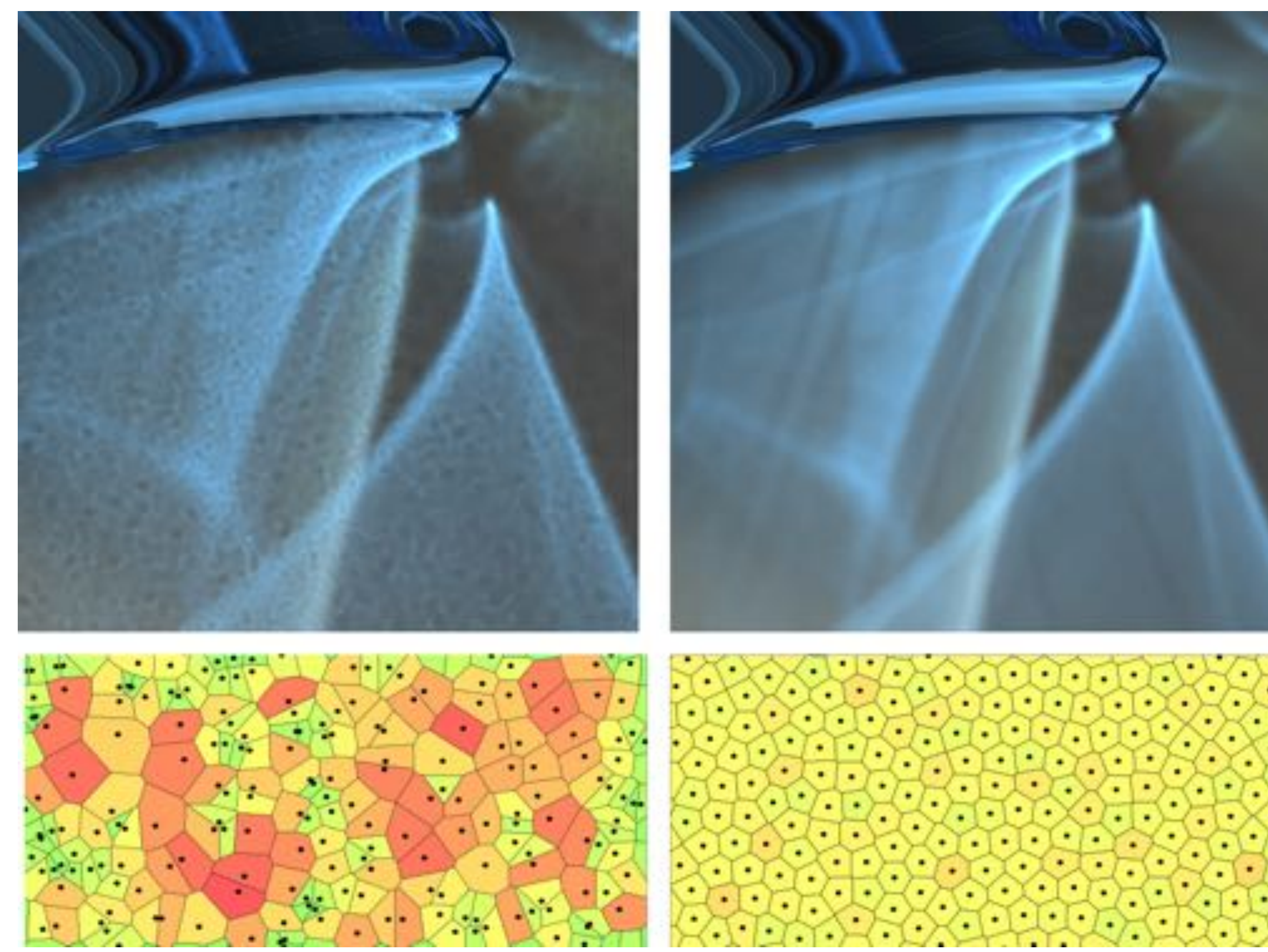
What other techniques are there?

Photon Mapping

- Trace particles from light, deposit "photons" in kd-tree
- Especially useful for, e.g., caustics, participating media (fog)

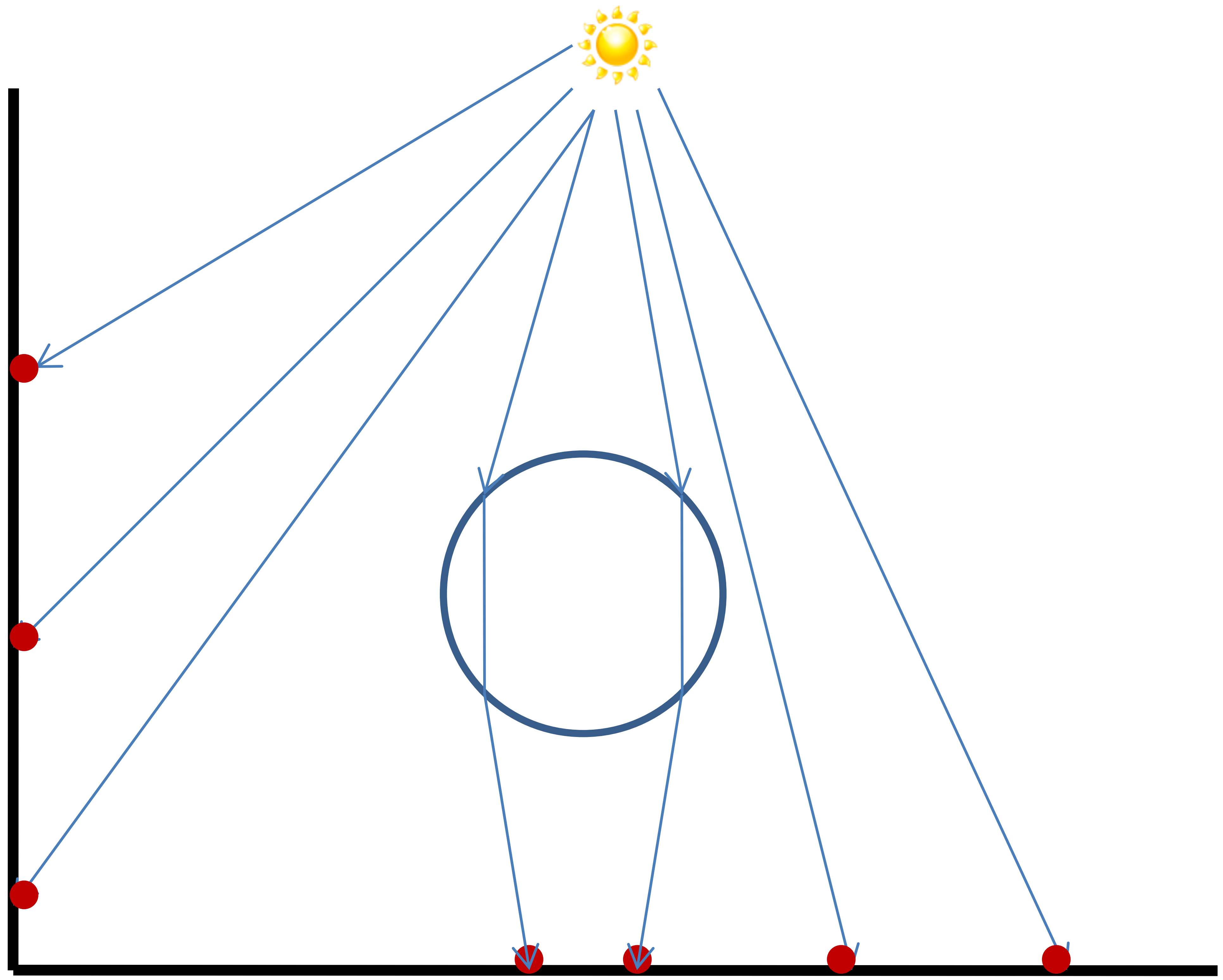


Voronoi diagrams can be used to improve photon distribution

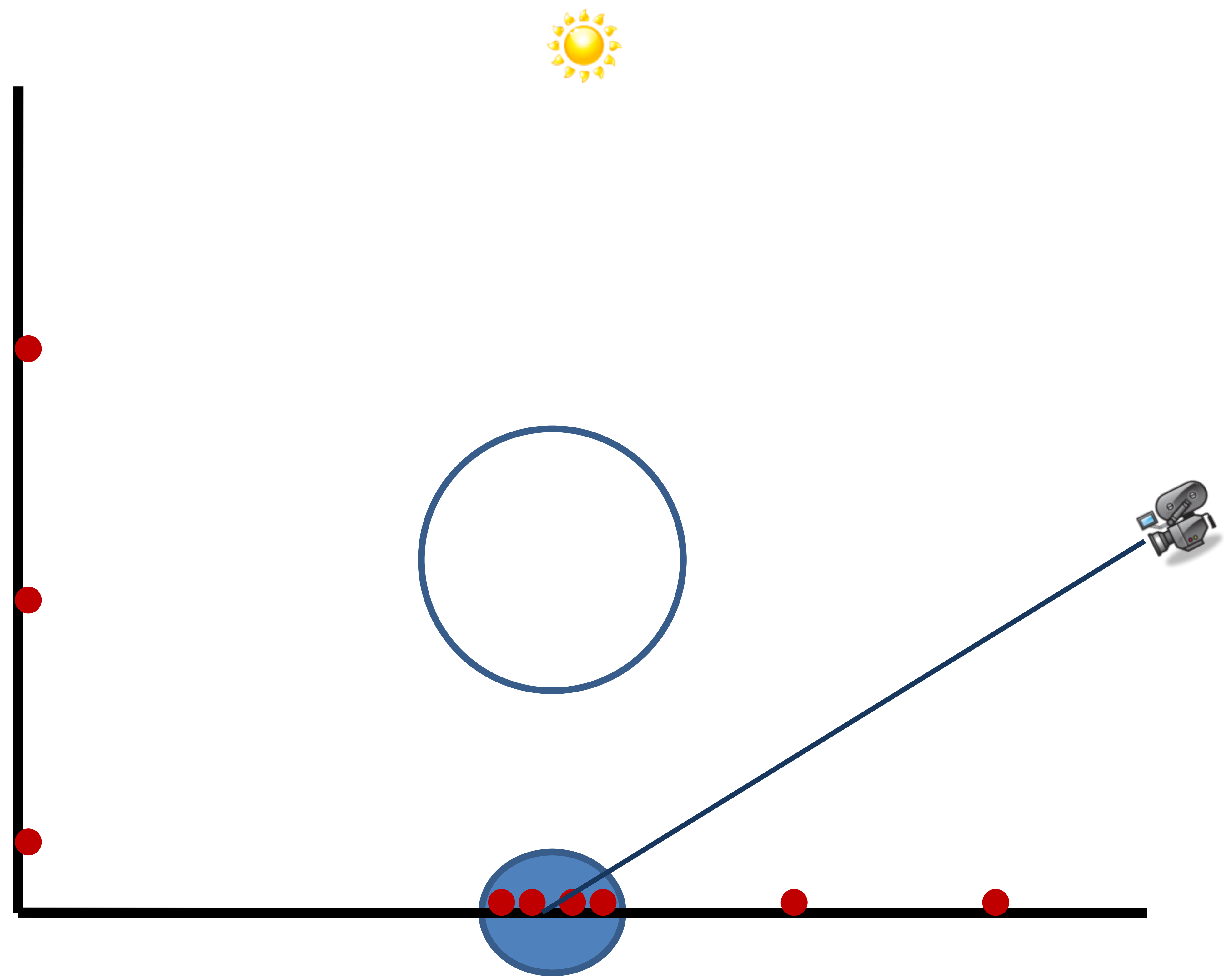


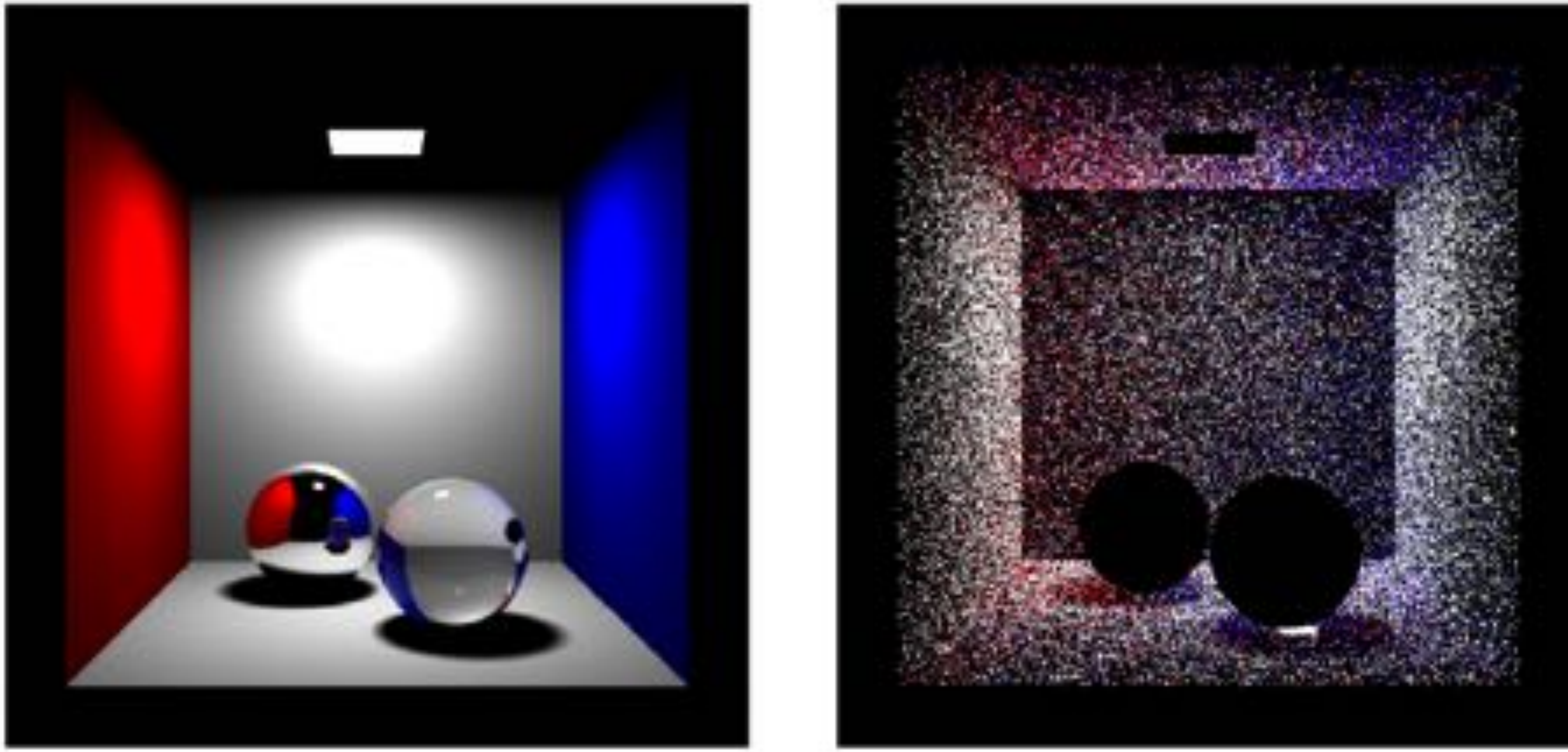
(from Spencer & Jones 2013)

Building the photon map: Photon tracing



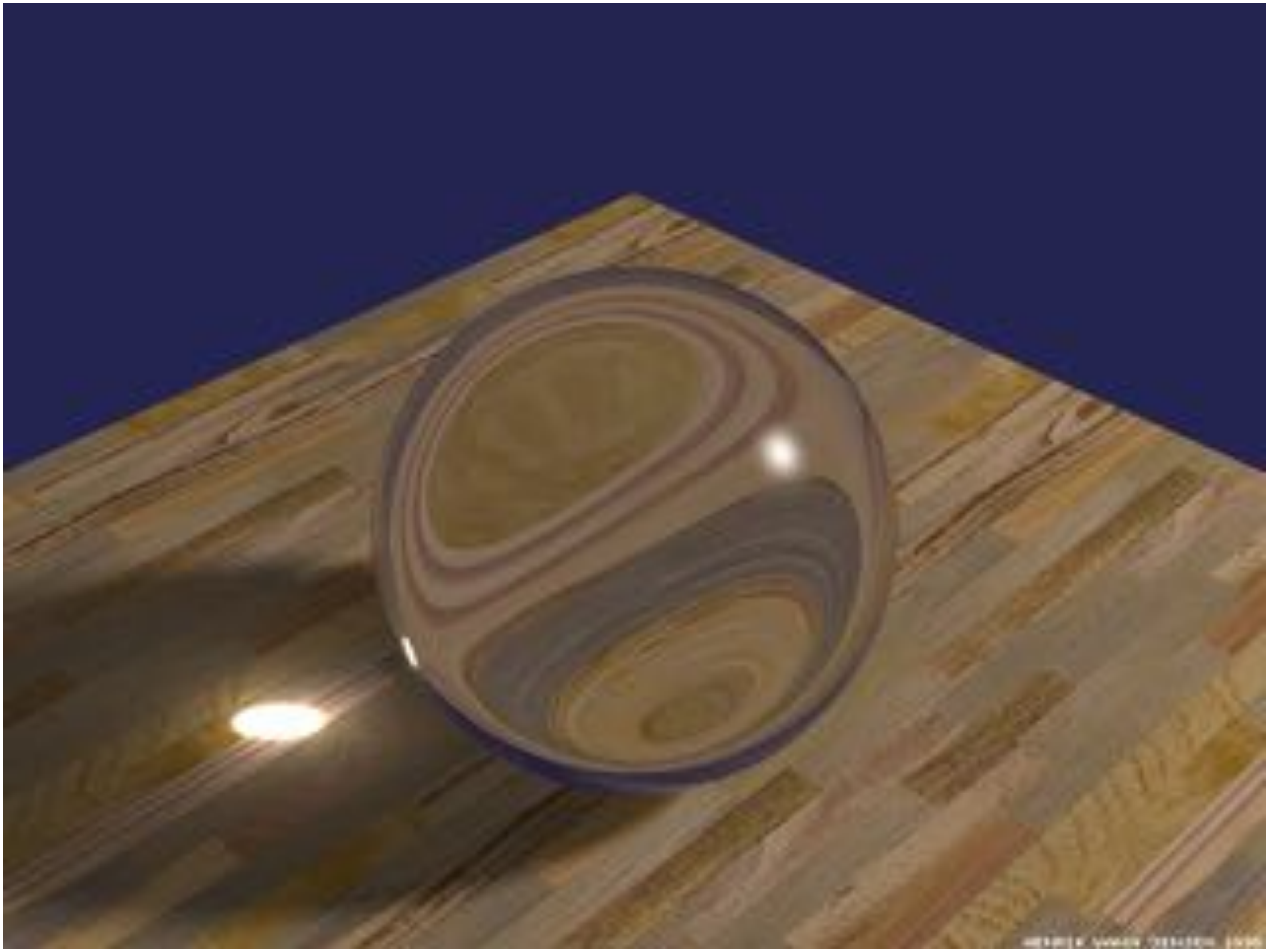
Rendering using the photon map



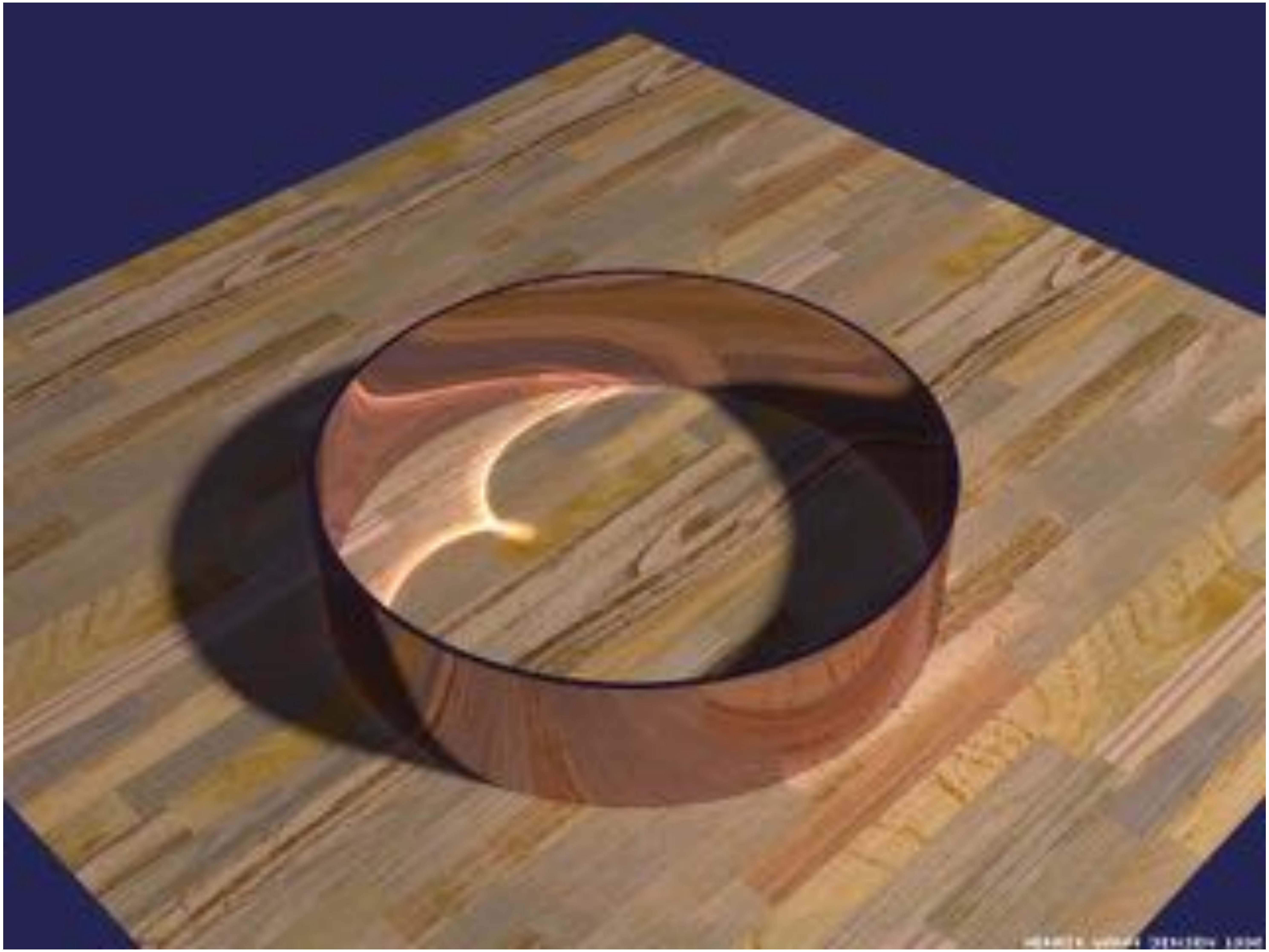


(a) (b)

Figure 4.4: "Cornell box" with glass and chrome spheres: (a) ray traced image (direct illumination and specular reflection and transmission), (b) the photons in the corresponding photon map.



30000 photons / 50 photons in radiance estimate



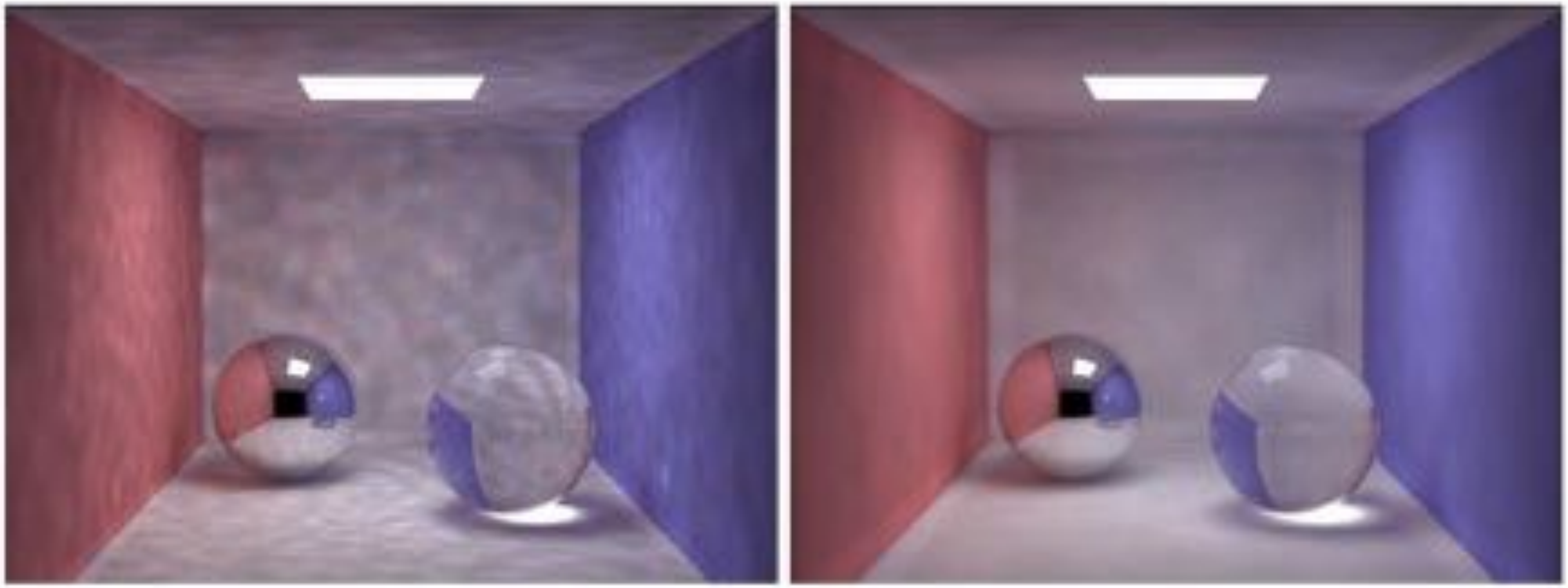
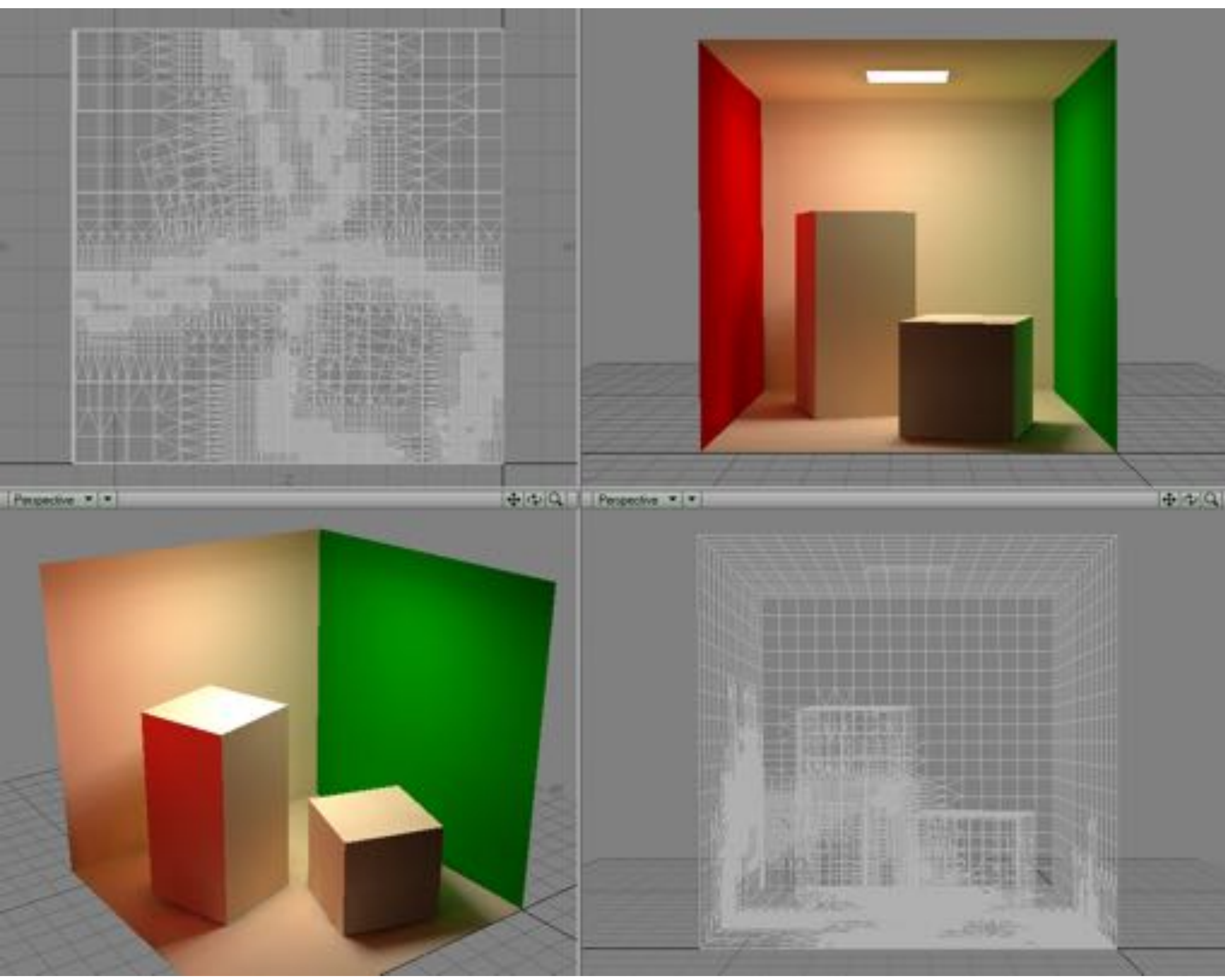


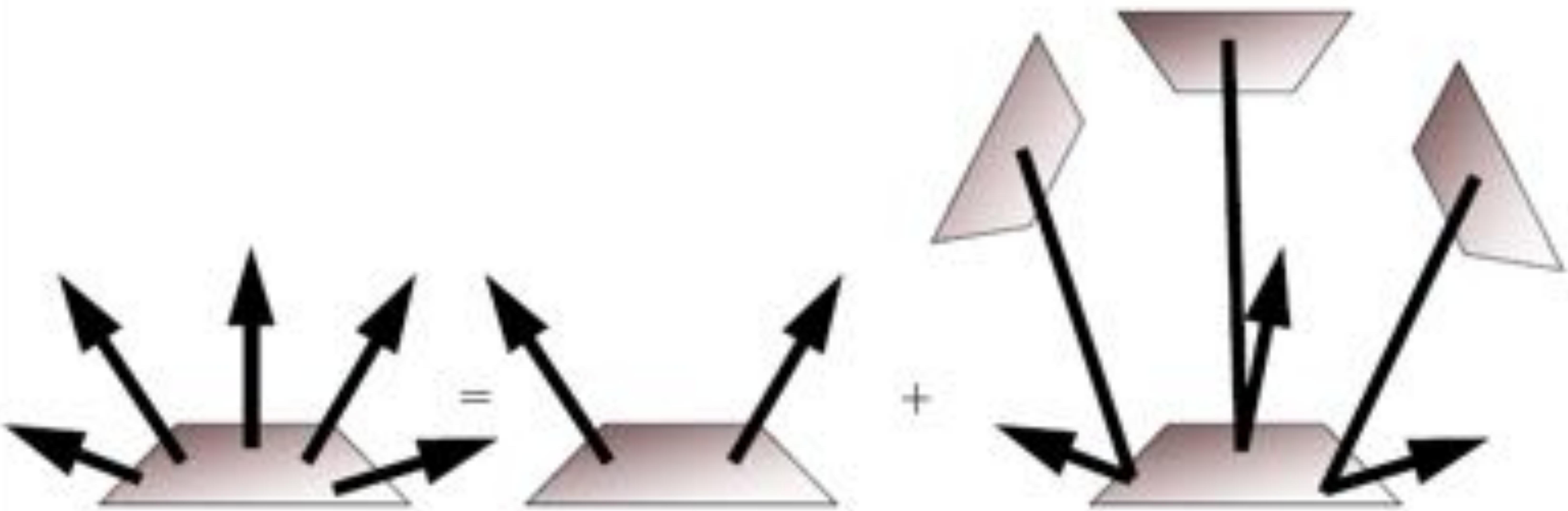
Figure 4.20: Global photon map radiance estimates visualized directly using 100 photons (left) and 500 photons (right) in the radiance estimate.

Finite Element Radiosity

- Very different approach: transport between patches in scene
- Solve large linear system for equilibrium distribution
- Good for diffuse lighting; hard to capture other light paths



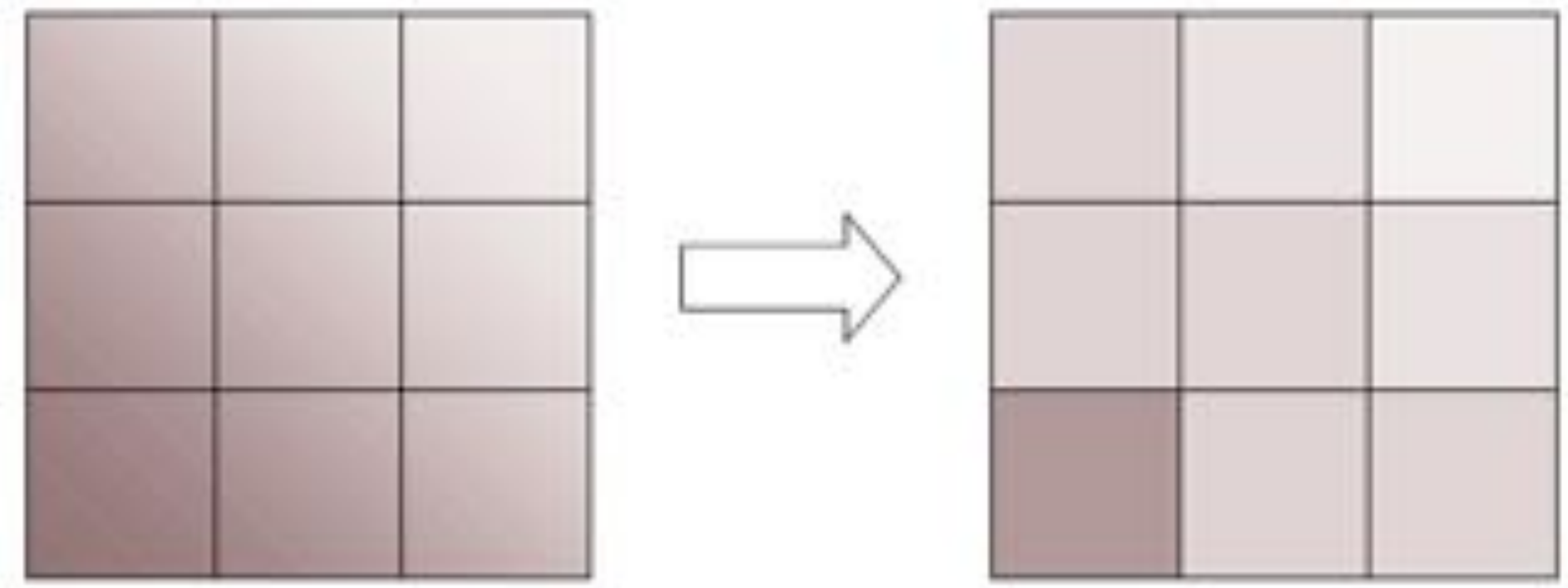
Conservation of Energy



Emitted power = self-emitted power + received & reflected power

Planar piecewise constancy assumption

- Subdivide scene into small "uniform" polygons



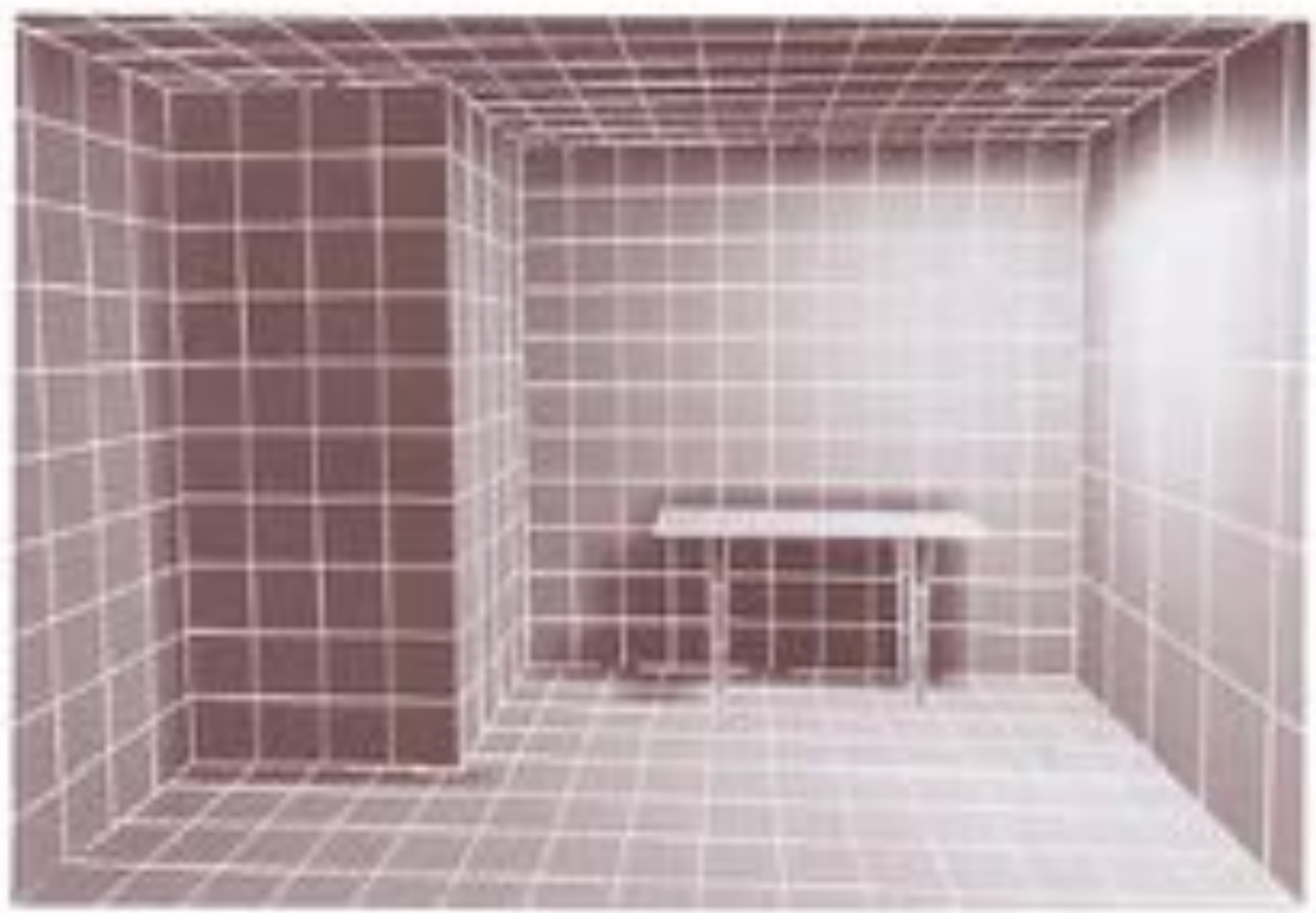


Table in room sequence from Cohen and Wallace

Power Equation

- Power from each polygon:

$$\forall i : \Phi_i = \Phi_{ei} + \rho_i \sum_{j=1}^{N} \Phi_j \, F(j \rightarrow i)$$

- Linear System of Equations:

  - Φ_i : power of patch i (unknown)
  - $\Phi_{e,i}$: emission of patch i (known)
  - ρ_i : reflectivity of patch i (known)
  - $F(j \rightarrow i)$: form-factor (coefficients of matrix)

Linear System of Radiosity Equations

$\forall$patches i: $$B_i = B_{ei} + \rho_i \sum_j F_{i \to j} B_j$$

$$\begin{bmatrix} 1-\rho_1 F_{1\to 1} & -\rho_1 F_{1\to 2} & \cdots & -\rho_1 F_{1\to n} \\ -\rho_2 F_{2\to 1} & 1-\rho_2 F_{2\to 2} & \cdots & -\rho_2 F_{2\to n} \\ \cdots & \cdots & \cdots & \cdots \\ -\rho_n F_{n\to 1} & -\rho_n F_{n\to 2} & \cdots & 1-\rho_n F_{n\to n} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \cdots \\ B_n \end{bmatrix} = \begin{bmatrix} B_{e1} \\ B_{e2} \\ \cdots \\ B_{en} \end{bmatrix}$$

Known Unknown Known

- Matrix Inversion to Solve for Radiosities.

Iterative approaches

- Jacobi iteration
- Start with initial guess for energy distribution (light sources)
- Update radiosity/power of all patches based on the previous guess

$$B_i = B_{e,i} + \rho_i \sum_{j=1}^{N} B_j F(i \rightarrow j)$$

new value

old values

- Repeat until converged

Sample Scenes

Sample Scenes

From Cohen, Chen, Wallace and Greenberg 1988

Sample Scenes

Sample Scenes

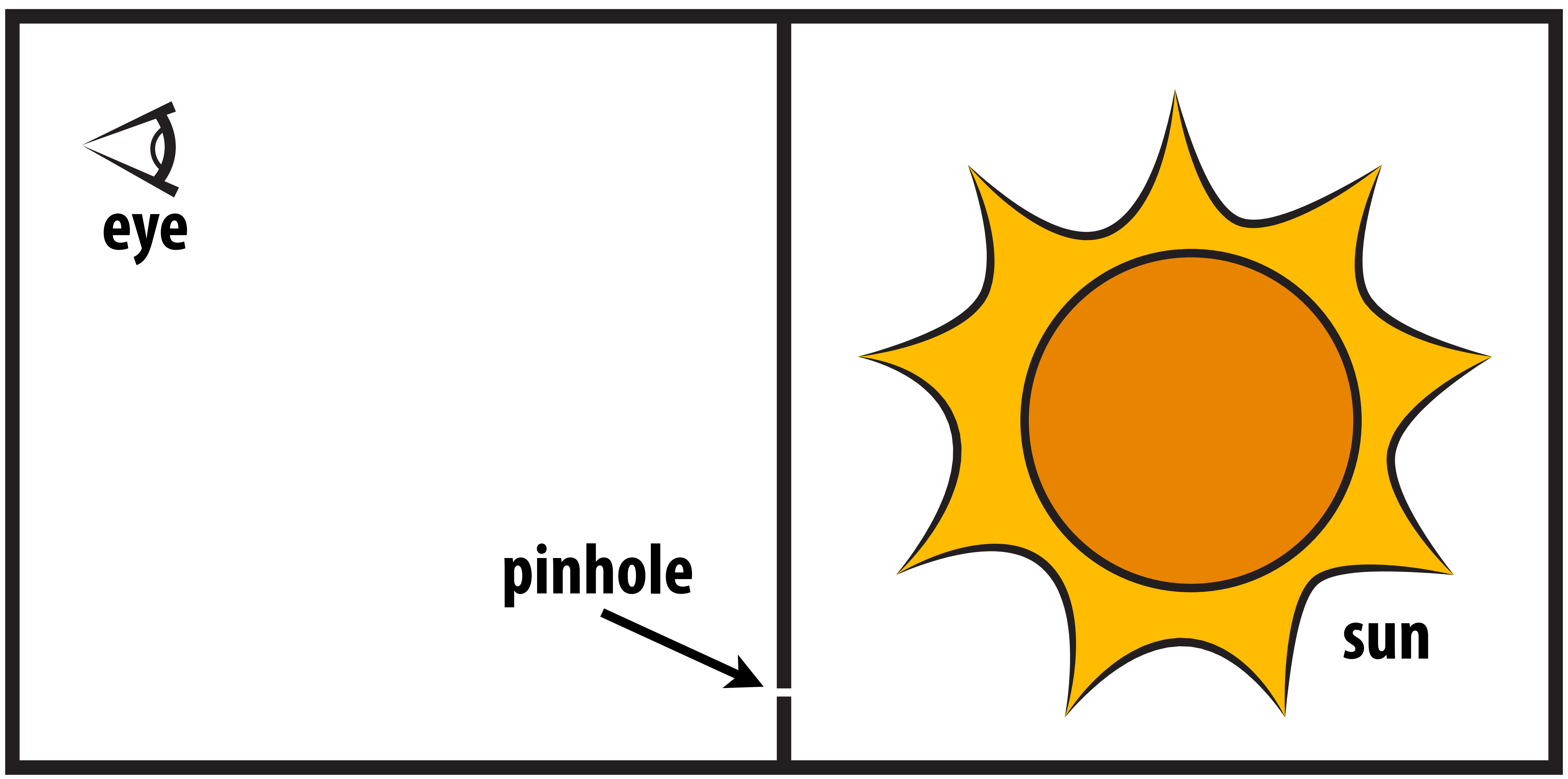
Sample Scenes

Consistency & Bias in Rendering Algorithms

method	consistent?	unbiased?
rasterization	NO	NO
path tracing	ALMOST	ALMOST
bidirectional path tracing	YES	YES
Metropolis light transport	YES	YES
photon mapping	YES	NO
radiosity	NO	NO

Can you certify a renderer?

- Grand challenge: write a renderer that comes with a certificate (i.e., provable, formally-verified guarantee) that the image produced represents the illumination in a scene.
- Harder than you might think!
- Inherent limitation of sampling: you can never be 100% certain that you didn't miss something important.



Can always make sun brighter, hole smaller...!