Introduction to Optimization

Computer Graphics
CMU 15-462/15-662
Last time: physically-based animation

- Use dynamics to drive motion
- Complexity from simple models
- Technique: numerical integration
  - formulate equations of motion
  - take little steps forward in time
  - general, powerful tool
- Today: numerical optimization
  - another general, powerful tool
  - basic idea: “ski downhill” to get a better solution
  - used everywhere in graphics (not just animation)
  - image processing, geometry, rendering, ...
What is an optimization problem?

- Natural human desire: find the best solution among all possibilities (subject to certain constraints)
- E.g., cheapest flight, shortest route, tastiest restaurant ...
- Has been studied since antiquity, e.g., isoperimetric problem:

“The first optimization problem known in history was practically solved by Dido, a clever Phoenician princess, who left her Tyrian home and emigrated to North Africa, with all her property and a large retinue, because her brother Pygmalion murdered her rich uncle and husband Acerbas, and plotted to defraud her of the money which he left. On landing in a bay about the middle of the north coast of Africa she obtained a grant from Hiarbas, the native chief of the district, of as much land as she could enclose with an ox-hide. She cut the ox-hide into an exceedingly long strip, and succeeded in enclosing between it and the sea a very valuable territory on which she built Carthage.”

—Lord Kelvin, 1893

“Obvious” solution is a circle...
Optimization in Graphics

Sumit Jain, Yuting Ye, and C. Karen Liu, “Optimization-based Interactive Motion Synthesis”
Optimization in Graphics

Niloy J. Mitra, Leonidas Guibas, Mark Pauly, “Symmetrization”
Optimization in Graphics

Moritz Bächer, Emily Whiting, Bernd Bickel, Olga Sorkine-Hornung,
“Spin-It: Optimizing Moment of Inertia for Spinnable Objects”
Optimization in Graphics

Nobuyuki Umetani, Yuki Koyama, Ryan Schmidt & Takeo Igarashi,
“Pteromys: Interactive Design and Optimization of Free-formed Free-flight Model Airplanes”
Continuous vs. Discrete Optimization

**DISCRETE:**
- domain is a discrete set (e.g., finite or integers)
- Example: best vegetable to put in a stew
  - Basic strategy? Try them all! (exponential)
  - sometimes clever strategy (e.g., MST)
  - more often, NP-hard (e.g., TSP)

**CONTINUOUS:**
- domain is not discrete (e.g., real numbers)
- Example: best temperature to cook an egg
- still many (NP-)hard problems, but also large classes of “easy” problems (e.g., convex)
Optimization Problem in Standard Form

- Can formulate most continuous optimization problems this way:

  "objective": how much does solution $x$ cost?

  \[
  \min_{x \in \mathbb{R}^n} f_0(x)
  \]

  subject to \( f_i(x) \leq b_i, \ i = 1, \ldots, m \)

  "constraints": what must be true about $x$? ("$x$ is feasible")

- Optimal solution $x^*$ has smallest value of $f_0$ among all feasible $x$

- Q: What if we want to maximize something instead?
  - A: Just flip the sign of the objective!

- Q: What if we want equality constraints, rather than inequalities?
  - A: Include two constraints: $g(x) \leq c$ and $g(x) \leq -c$

\( (f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 0, \ldots, m) \)

often (but not always) continuous, differentiable, ...
Local vs. Global Minima

- Global minimum is absolute best among all possibilities
- Local minimum is best “among immediate neighbors”

Philosophical question: does a local minimum “solve” the problem? Depends on the problem! (E.g., real protein folding is local minimum) Other times, local minima can be really bad (e.g., path planning)
Optimization Problem, Visualized

Q: Is this an optimization problem in standard form?  
A: Yes.

Q: Where is the optimal solution?  
A: There are two, (0,1), (0,-1).

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad x_1^2 - x_2^2 \\
\text{s.t.} & \quad x_1^2 + x_2^2 - 1 \leq 0
\end{align*}
\]
Existence & Uniqueness of Minimizers

- Already saw that (global) minimizer is not unique.
- Does it always exist? Why?
- Just consider all possibilities and take the smallest one, right?

\[ f_0(x) \]

- WRONG! Not all objectives are bounded from below.
- It’s like that old adage: “no matter how good you are, there will always be someone better than you.”
Feasibility

- Ok, but suppose the objective is bounded from below.
- Then we can just take the best feasible solution, right?

Not if there aren’t any!

Every system of equations is an optimization problem.

But not all problems have solutions!

(You’ll appreciate this more as you get older.)
Q: Is this problem feasible?

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad \sin(x_1) + x_2^2 \\
\text{s.t.} & \quad (x_1 - 2)^2 + x_2^2 \leq 1, \\
& \quad x_1 \leq -1
\end{align*}
\]

A: No—the two sublevel sets (points where \( f_i(x) \leq 0 \)) have no common points, i.e., they do not overlap.
Existence & Uniqueness of Minimizers, cont.

- Even being bounded from below is not enough:

\[
\min_{x \in \mathbb{R}} e^{-x}
\]

No matter how big \( x \) is, we never achieve the lower bound (0)

- So when does a solution exist? Two sufficient conditions:
  - Extreme value theorem: continuous objective & compact domain
  - Coercivity: objective goes to \(+\infty\) as we travel (far) in any direction
Characterization of Minimizers

- Ok, so we have some sense of when a minimizer might exist.
- But how do we know a given point $x$ is a minimizer?

- Checking if a point is a global minimizer is (generally) hard.
- But we can certainly test if a point is a local minimum (ideas?)
- (Note: a global minimum is also a local minimum!)
Characterization of Local Minima

- Consider an objective $f_0 : \mathbb{R} \rightarrow \mathbb{R}$. How do you find a minimum?
- (Hint: you may have memorized this formula in high school!)

Also need to check second derivative (how?)
Make sure it’s positive
Ok, but what does this all mean for more general functions $f_0$?

$$f_0(x)$$

$$f_0'(x^*) = 0$$

$$f_0''(x^*) \geq 0$$

...but what about this point?

Find points where $f_0'(x^*) = 0$.
Optimality Conditions (Unconstrained)

- In general, our objective is $f_0: \mathbb{R}^n \rightarrow \mathbb{R}$
- How do we test for a local minimum?
- 1st derivative becomes gradient; 2nd derivative becomes Hessian

\[ \nabla f := \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \nabla^2 f := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial f}{\partial x_n^2} \end{bmatrix} \]

**Gradient** (measures “slope”)

**Hessian** (measures “curvature”)

- Optimality conditions?

\[ \nabla f_0(x^*) = 0 \quad \text{(1st order)} \]

\[ \nabla^2 f_0(x^*) \succeq 0 \quad \text{(2nd order)} \]

positive semidefinite (PSD) \( (u^T A u \geq 0 \text{ for all } u) \)
Optimality Conditions (Constrained)

- What if we have constraints?
- Is gradient at minimizer still zero?
- Is Hessian at minimizer still PSD?
- Not necessarily! (See example above)
- In general, any (local or global) minimizer must at least satisfy the Karush–Kuhn–Tucker (KKT) conditions:

  \[
  \exists \lambda_i \text{ s.t.} \quad \nabla f_0(x^*) = - \sum_{i=1}^{n} \lambda_i \nabla f_i(x^*) \quad \text{stationarity}
  \]
  \[
  f_i(x^*) \leq 0, \quad i = 1, \ldots, n \quad \text{primal feasibility}
  \]
  \[
  \lambda_i \geq 0, \quad i = 1, \ldots, n \quad \text{dual feasibility}
  \]
  \[
  \lambda_i f_i(x^*) = 0, \quad i = 1, \ldots, n \quad \text{complementary slackness}
  \]

- ...we won’t work with these in this class! (But good to know where to look.)
Convex Optimization

- Special class of problems that are almost always “easy” to solve (polynomial-time!)

- Problem convex if it has a convex domain and convex objective

- Why care about convex problems in graphics?
  - can make guarantees about solution (always the best)
  - doesn’t depend on initialization (strong convexity)
  - often quite efficient, but not always
Convex Quadratic Objectives & Linear Systems

- Very important example: convex quadratic objective
- Already saw this with, e.g., quadric error simplification
- Valuable “variational” way of looking at many common equations
- Can be expressed via positive-semidefinite (PSD) matrix:

\[ f_0(x) = \frac{1}{2} x^T A x - x^T b, \quad A \succeq 0 \]

- Q: 1st-order optimality condition?
  \[ A x = b \]
  satisfied by definition

- Q: 2nd-order optimality condition?
  \[ A \succeq 0 \]
  just solve a linear system!

\[ f_0(x) = \frac{1}{2} x^T A x - x^T b, \quad A \succeq 0 \]
Sadly, life is not usually that easy. How do we solve optimization problems in general?
Descent Methods

An idea as old as the hills:
Gradient Descent (1D)

- Basic idea: follow the gradient “downhill” until it’s zero
- (Zero gradient was our 1st-order optimality condition)

\[
\frac{d}{dt} x(t) = - f_0'(x(t))
\]

- Do we always end up at a (global) minimum?
- How do we compute gradient descent in practice?
Gradient Descent Algorithm (1D)

- Did you notice that gradient descent equation is an ODE?
- Q: How do we solve it numerically? \( \frac{d}{dt} x(t) = -f_0'(x(t)) \)
- One way: forward Euler:
  \[ x_{k+1} = x_k - \tau f_0'(x_k) \]
- Q: How do we pick the time step?
- If we’re not careful, we’ll go zipping all over the place; won’t make any progress.

- Basic idea: use “step control” to determine step size based on value of objective & derivatives.

- A careful strategy (e.g., Armijo-Wolfe) can guarantee convergence at least to a local minimum.

- For now we will do something simpler: make \( \tau \) really small!
Gradient Descent Algorithm (nD)

- Q: How do we write gradient descent equation in general?
  \[
  \frac{d}{dt} x(t) = -\nabla f_0(x(t))
  \]

- Q: What’s the corresponding discrete update?
  \[
  x_{k+1} = x_k - \tau \nabla f_0(x_k)
  \]

- Basic challenge in nD:
  - solution can “oscillate”
  - takes many, many small steps
  - very slow to converge
Higher Order Descent

- General idea: apply a coordinate transformation so that the local energy landscape looks more like a “round bowl”
- Gradient now points directly toward nearby minimizer
- Most basic strategy: Newton’s method:

$$x_{k+1} = x_k - \tau (\nabla^2 f_0(x_k))^{-1} \nabla f_0(x_k)$$

- Great for convex problems (even proofs about # of steps!)
- For nonconvex problems, need to be more careful
- In general, nonconvex optimization is a BLACK ART
- Meta-strategy: try lots of solvers, see what works!
  - quasi-Newton, trust region, L-BFGS, ...
Example: Inverse Kinematics

Example 12: IK-driven robot claw
Forward Kinematics

- Many systems well-described by a kinematic chain
  - collection of rigid bodies, connected by joints
  - joints have various behaviors (ball, piston, ...)
  - also have constraints (e.g., range of angles)
  - hierarchical structure (body → leg → foot)
- In animation, often called a rig
- How do we specify the configuration of a “rig”?
  - One way: artist sets each joint individually
  - Another way: ...optimization!
Simple Kinematic Chain

- Consider a simple path-like chain in 2D
- Q: How do we write $p_1$ in terms of the root position $p_0$, angles, & vectors $u := c_{i+1} - c_i$?

\[
p_1 = p_0 + \begin{bmatrix} \cos \theta_0 & \sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{bmatrix} u_0
\]

- (For brevity, can use complex numbers:)

\[
p_1 = p_0 + e^{i\theta_0} u_0
\]

- Q: How about $p_2$?

\[
p_2 = p_0 + e^{i\theta_0} u_0 + e^{i\theta_0} e^{i\theta_1} u_1
\]
Simple IK Algorithm

- Basic idea behind our IK algorithm:
  - write down distance between final point and “target”
  - compute gradient with respect to angles
  - apply gradient descent

- Objective?

\[ f_0(\theta) = \frac{1}{2} |\tilde{p}_n - p_n|^2 \]

- Constraints?
  - None! The joint angle can take any value.
  - Though we could limit joint angles (for instance)
Coming up next: PDEs in Computer Graphics

Frank Losasso, Jerry O. Talton, Nipun Kwatra, and Ron Fedkiw, “Two-Way Coupled SPH and Particle Level Set Fluid Simulation”