Variance Reduction

Computer Graphics
CMU 15-462/15-662

Last time: Monte Carlo Ray Tracing

- Recursive description of incident illumination
- Difficult to integrate; tour de force of numerical integration
- Leads to lots of sophisticated integration strategies:
  - sampling strategies
  - variance reduction
  - Markov chain methods
  - ...
- Today: get a glimpse of these ideas
- Also valuable outside rendering!
  - Monte Carlo one of the "Top 10 Algorithms of the 20th Century"!

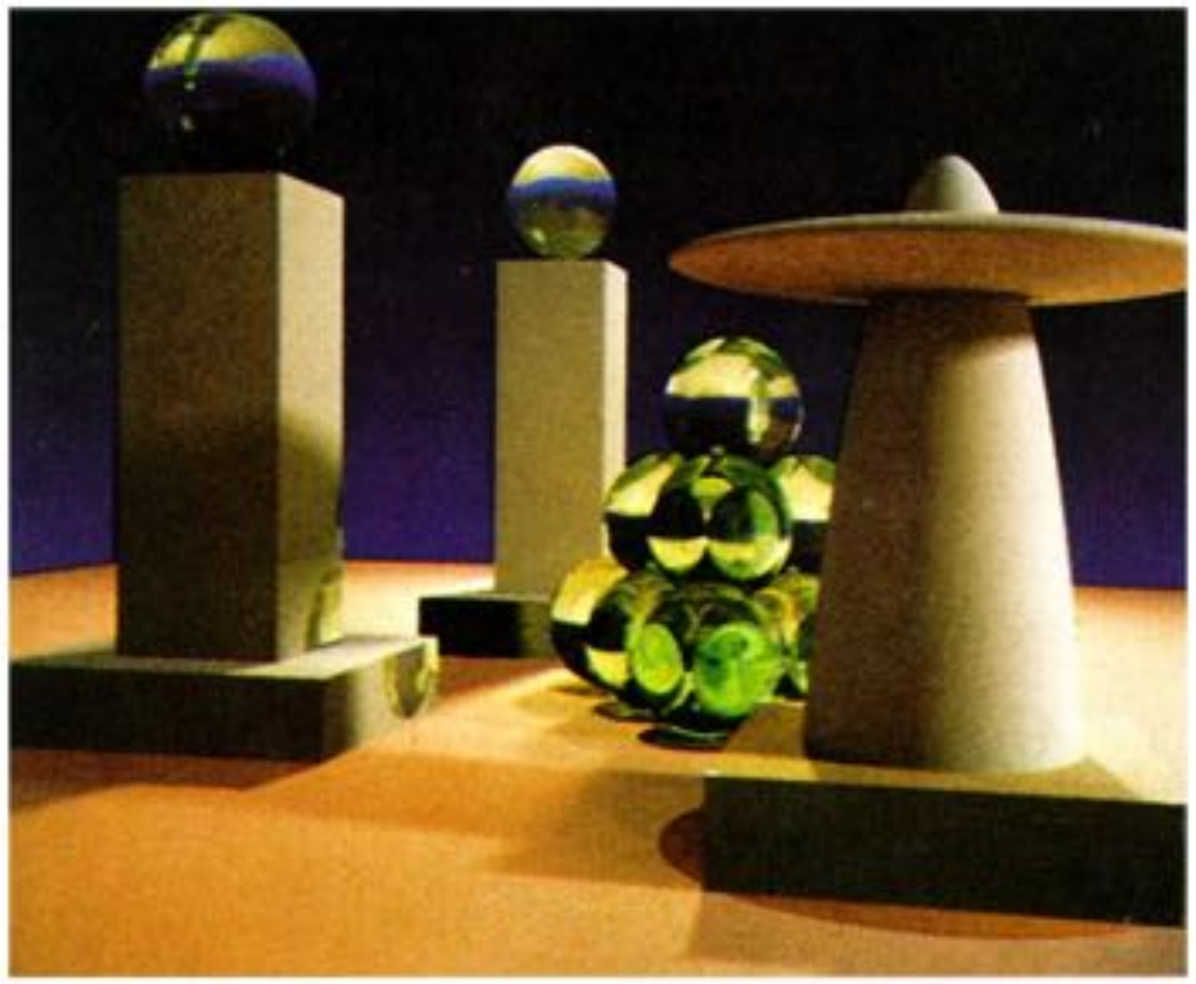


Figure 6. A sample image. All objects are neutral grey. Color on the objects is due to caustics from the green glass balls and color bleeding from the base polygon.

$$L_o(\mathbf{x}, \omega_o) = L_e(\mathbf{x}, \omega_o) + \int_\Omega f_r(\mathbf{x}, \omega_i, \omega_o)\, L_i(\mathbf{x}, \omega_i)\, (\omega_i \cdot \mathbf{n})\, \mathrm{d}\,\omega_i$$

Review: Monte Carlo Integration

Want to integrate:

any function*

$$I := \int_{\Omega} f(x)\, dx$$

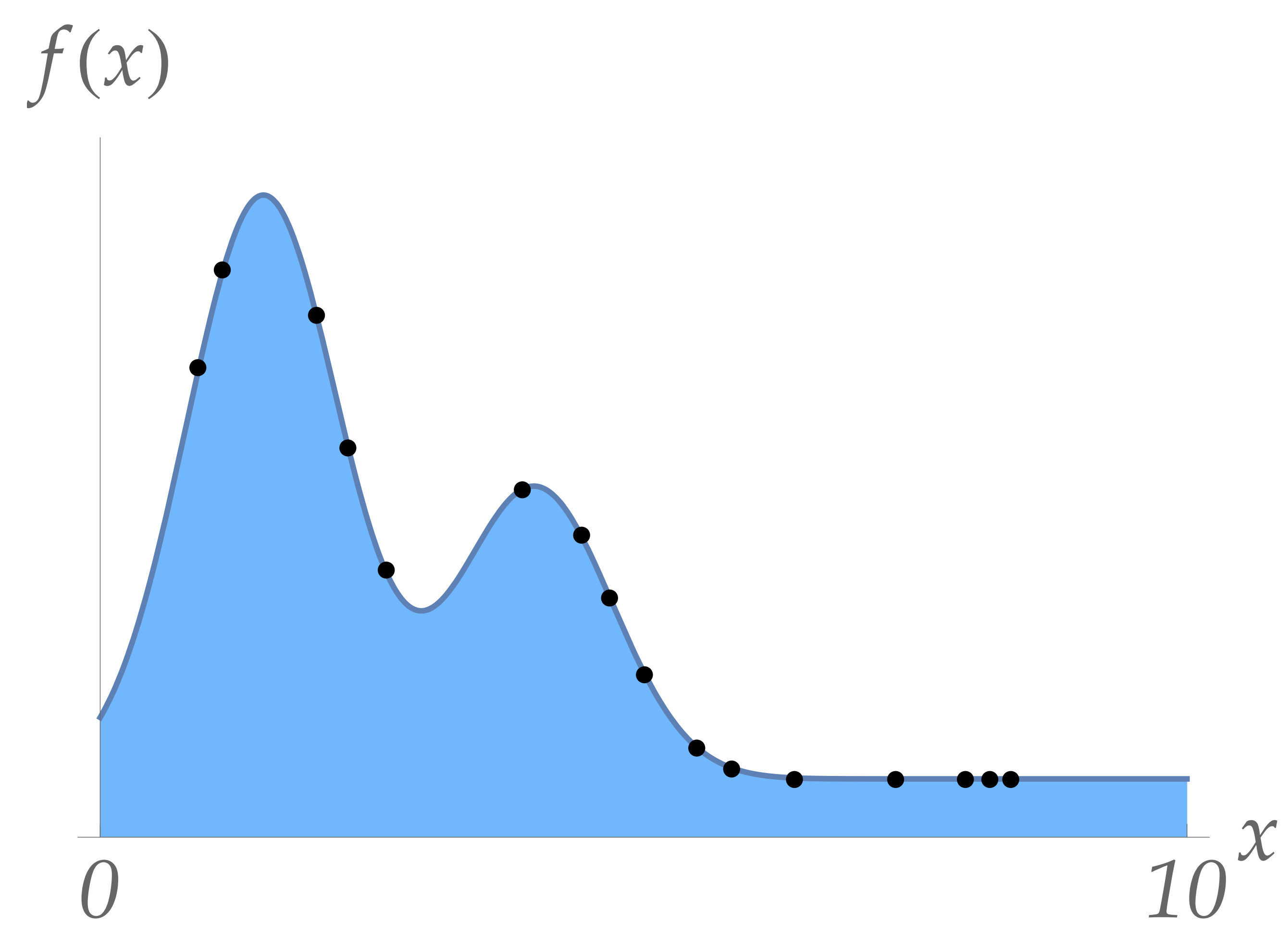
any domain

General-purpose hammer: Monte-Carlo integration

$$I = \lim_{n\to\infty} V(\Omega)\frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

volume of the domain

uniformly random samples of domain



*Must of course have a well-defined integral!

Review: Expected Value (DISCRETE)

A discrete random variable X has n possible outcomes x_i, occuring w/ probabilities $0 \leq p_i \leq 1$, $p_1 + \ldots + p_n = 1$

probability of event i

$$E(X) := \sum_{i=1}^{n} p_i x_i$$

expected value

value of event i

(just the "weighted average"!)

E.g., what's the expected value for a fair coin toss?

$p_1 = 1/2$

$x_1 = 1$

$p_2 = 1/2$

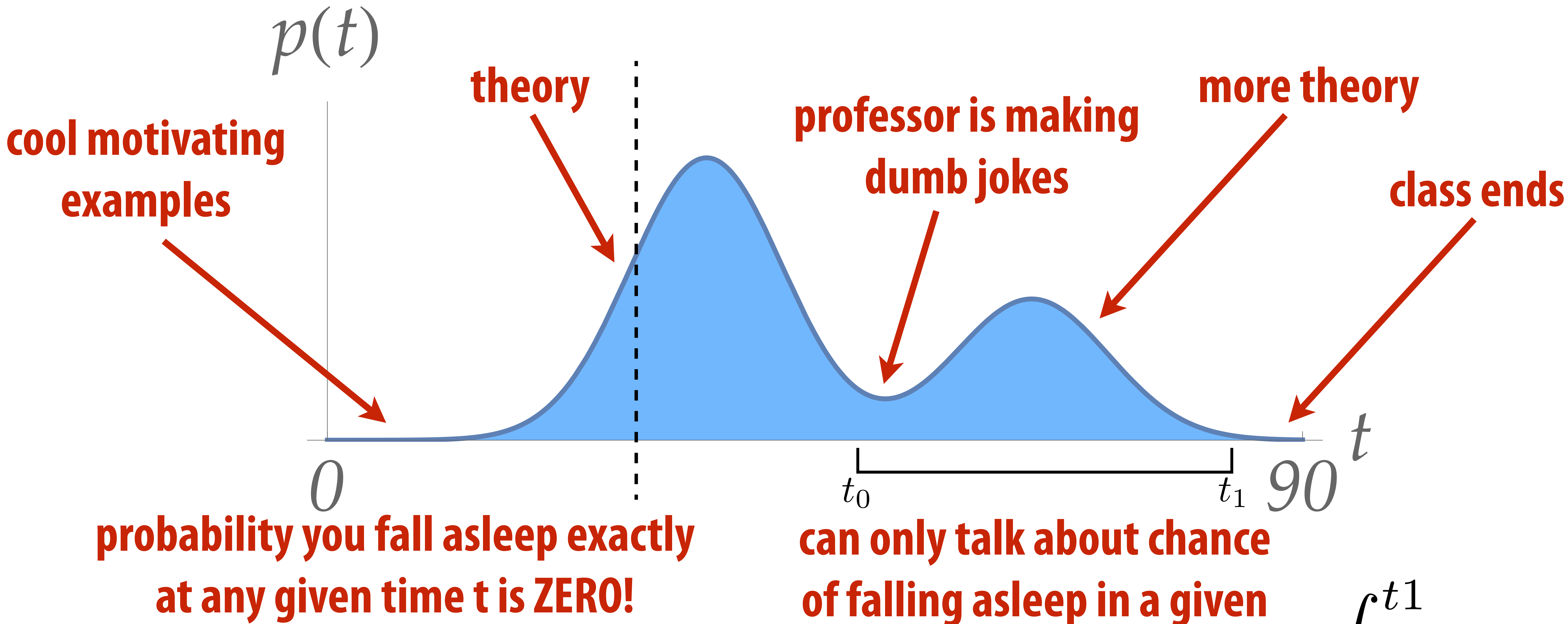
$x_2 = 0$

Continuous Random Variables

A continuous random variable X takes values x anywhere in a set Ω

Probability **density** p gives probability x appears in a given region.

E.g., probability you fall asleep at time t in a 15-462 lecture:



$$\int_{t_0}^{t1} p(t)\ dt$$

Review: Expected Value (CONTINUOUS)

Expected value of continuous random variable again just the "weighted average" with respect to probability p:

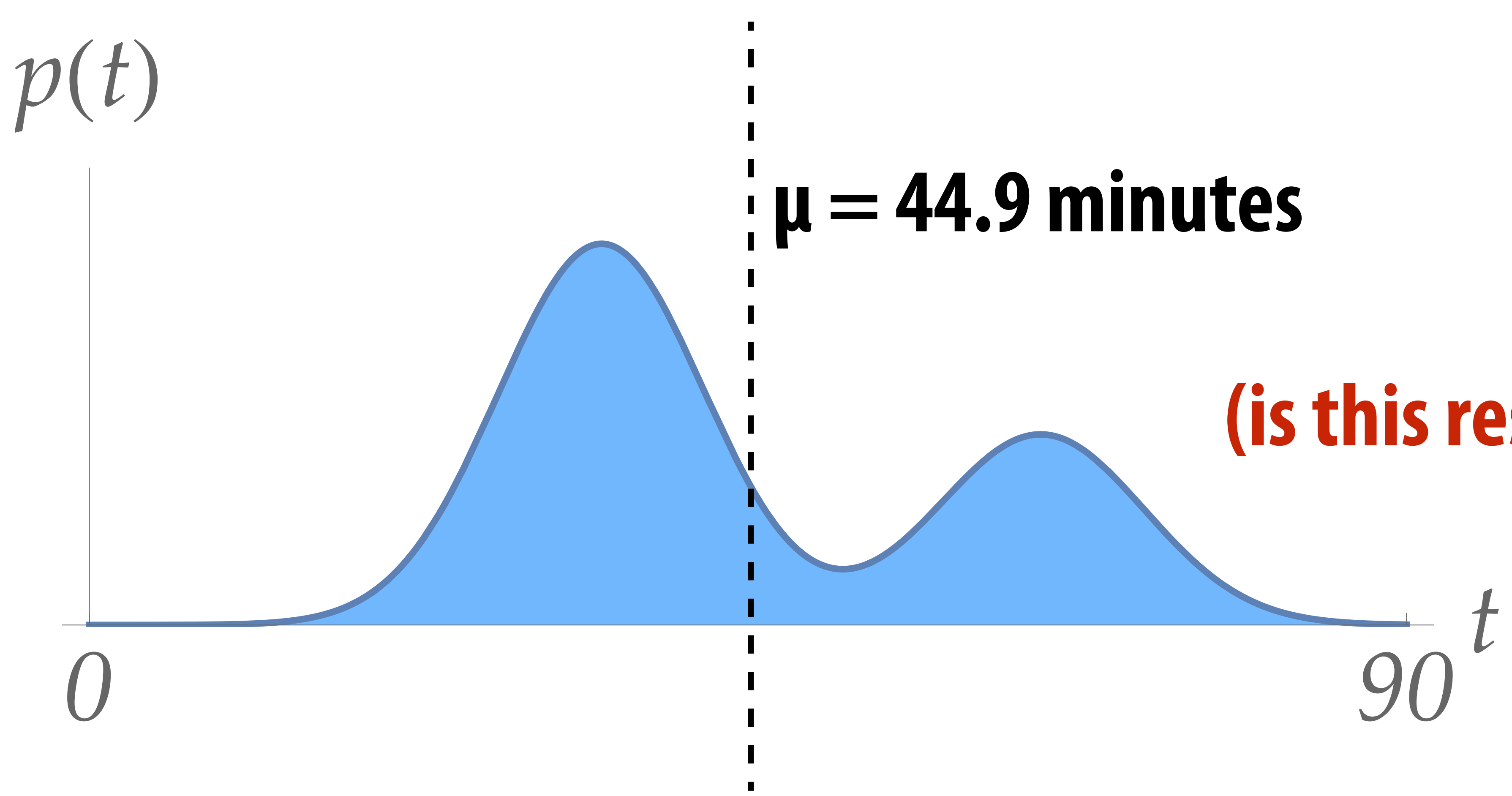
probability density at point x

$$E(X) := \int_{\Omega} x p(x) \; dx$$

expected value

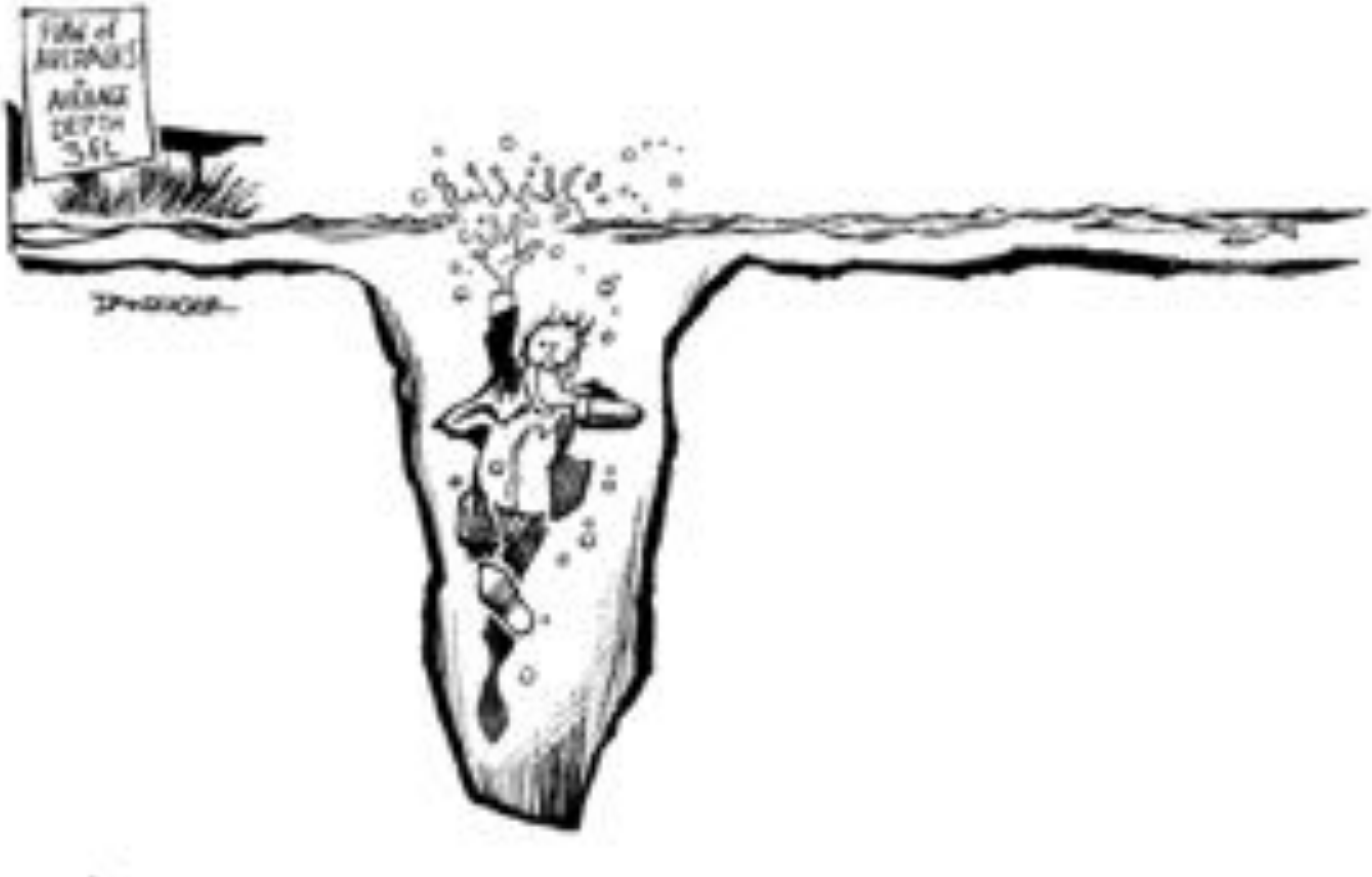
sometimes just use "μ" (for "mean")

E.g., expected time of falling asleep?



(is this result counter-intuitive?)

Flaw of Averages



Review: Variance

- Expected value is the "average value"
- Variance is how far we are from the average, on average!

$$\mathrm{Var}(X) := E[(X - E[X])^2]$$

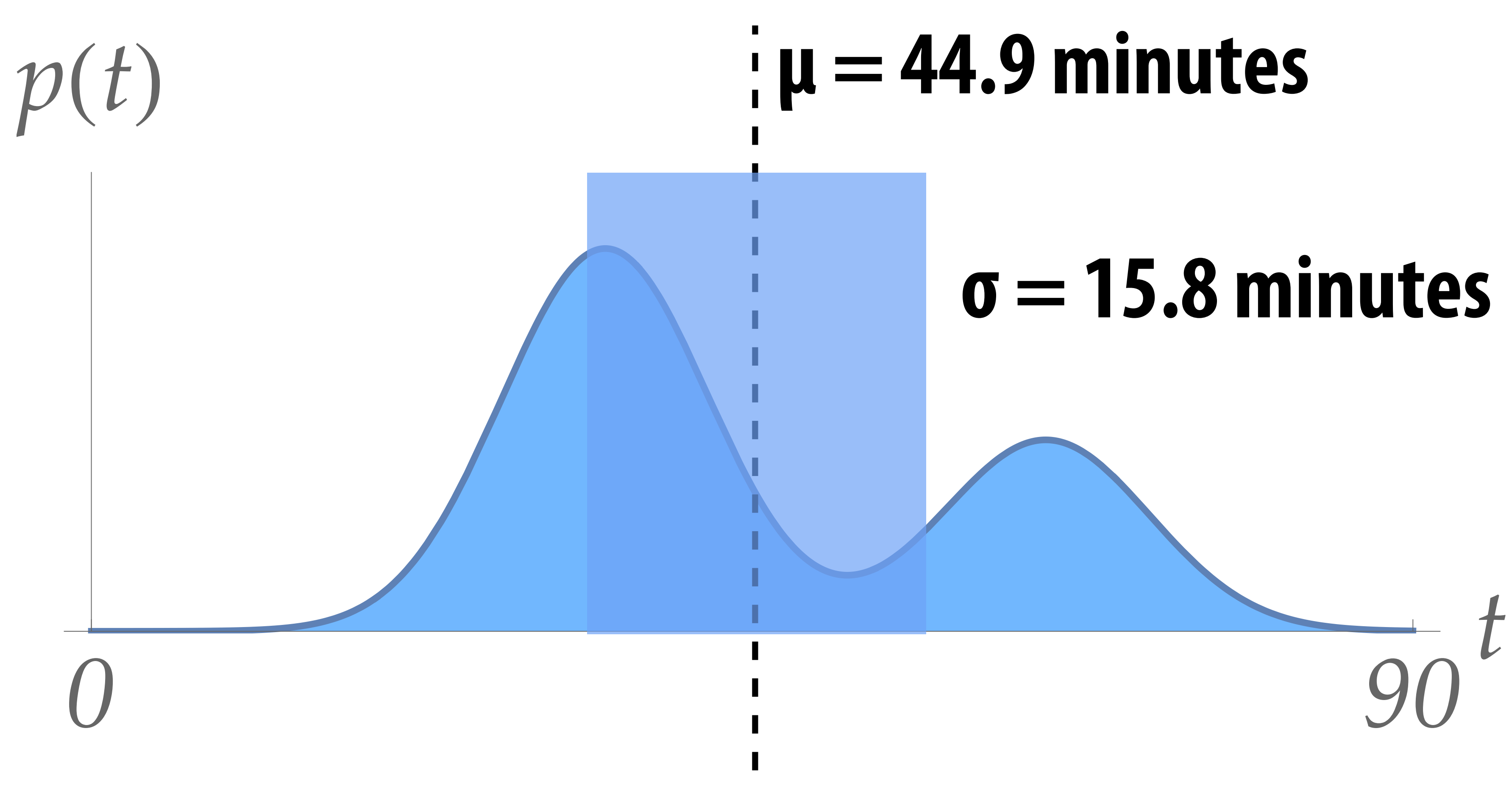
DISCRETE

$$\sum_{i=1}^{n} p_i (x_i - \sum_j p_j x_j)^2$$

CONTINUOUS

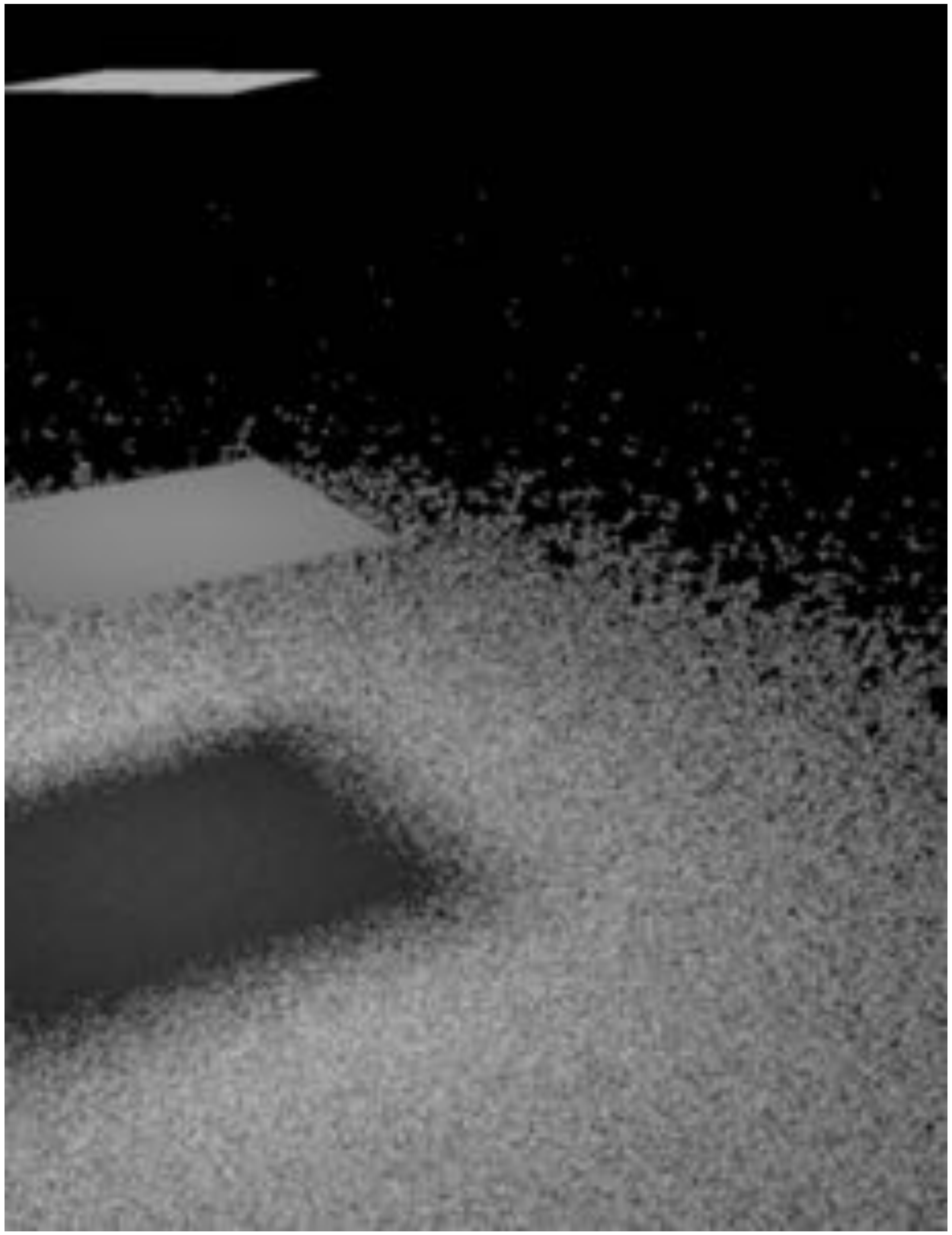
$$\int_\Omega p(x)(x - \int_\Omega y p(y)\ dy)^2\ dx$$

- Standard deviation σ is just the square root of variance

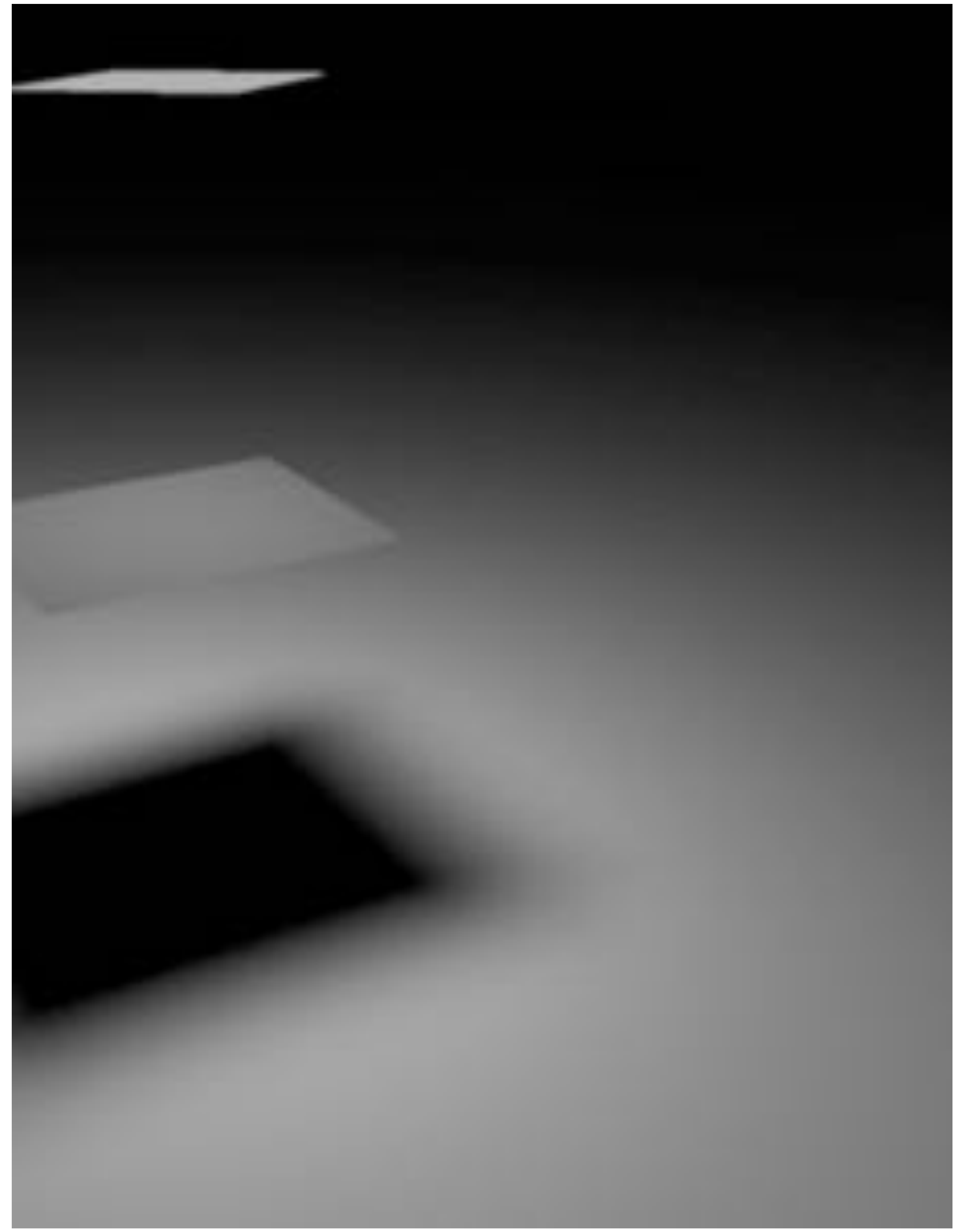


(More intuitive perhaps?)

Variance Reduction in Rendering



higher variance



lower variance

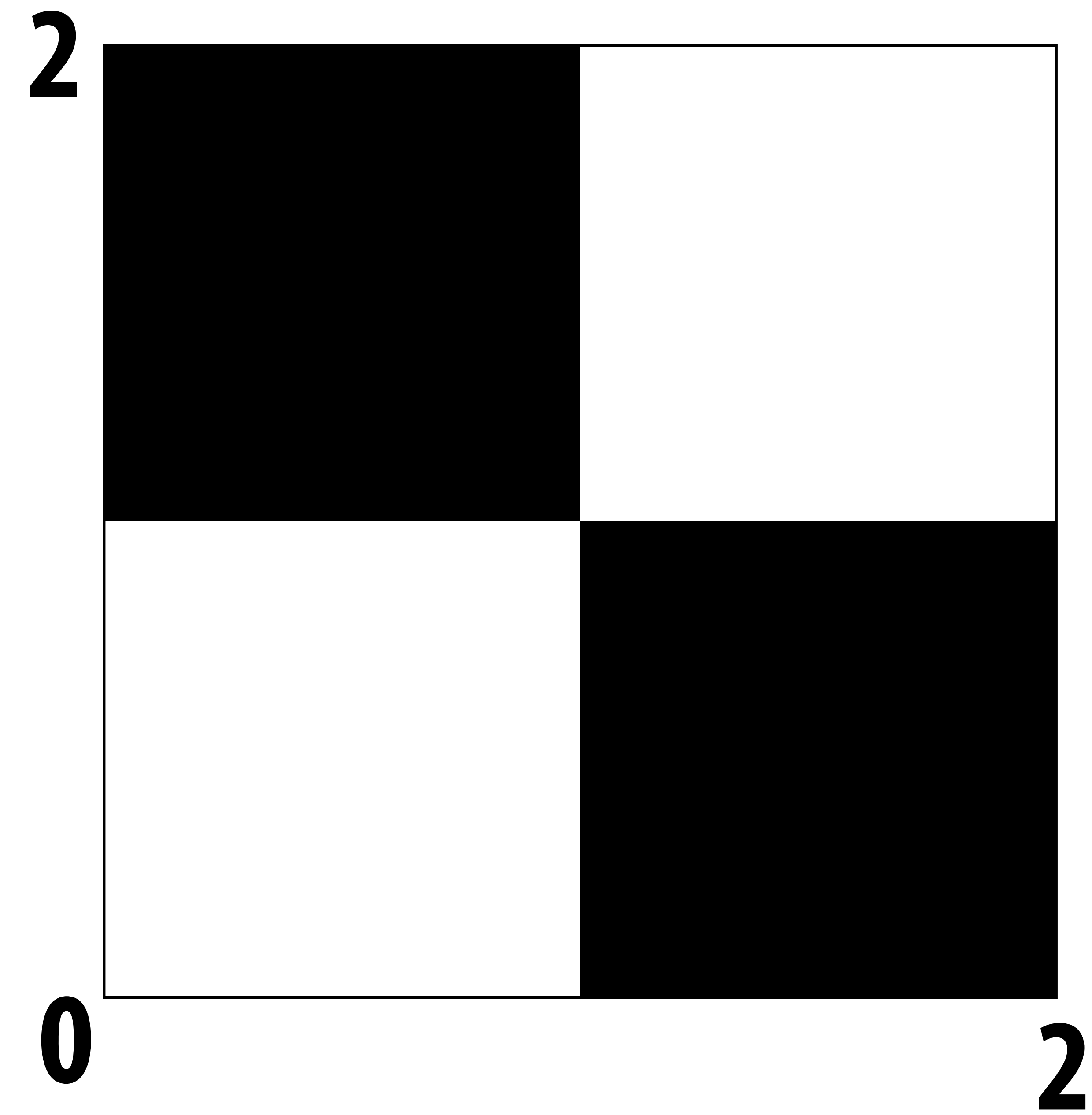
Q: How do we reduce variance?

Variance Reduction Example

$$\Omega := [0,2] \times [0,2]$$

$$f(x,y) := \begin{cases} 1 & \lfloor x \rfloor + \lfloor y \rfloor \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$

$$I := \int_\Omega f(x,y)\ dxdy$$



Q: What's the expected value of the integrand f?

A: Just by inspection, it's 1/2 (half white, half black!).

Q: What's its variance?

A: $(1/2)(0-1/2)^2 + (1/2)(1-1/2)^2 = (1/2)(1/4) + (1/2)(1/4) = 1/4$

Q: How do we reduce the variance?

That was a trick question.

You can't reduce variance of the integrand!
Can only reduce variance of an estimator.

Variance of an Estimator

- An "estimator" is a formula used to approximate an integral
- Most important example: our Monte Carlo estimate:

$$I = \int_\Omega f(x)\ dx \qquad\qquad \hat{I} := V(\Omega)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

true integral — Monte Carlo estimate

- Get different estimates for different collections of samples
- Want to reduce variance of estimate across different samples
- Why? Integral itself only has one value!
- Many, many (many) techniques for reducing variance
- We will review some key examples for rendering

Bias & Consistency

- Two important things to ask about an estimator
  - Is it consistent?
  - Is it biased?
- Consistency: "converges to the correct answer"

estimate

$$\lim_{n\to\infty} P(|I - \hat{I}_n| > 0) = 0$$

true integral

\# of samples

- Unbiased: "estimate is correct on average"

$$E[I - \hat{I}_n] = 0$$

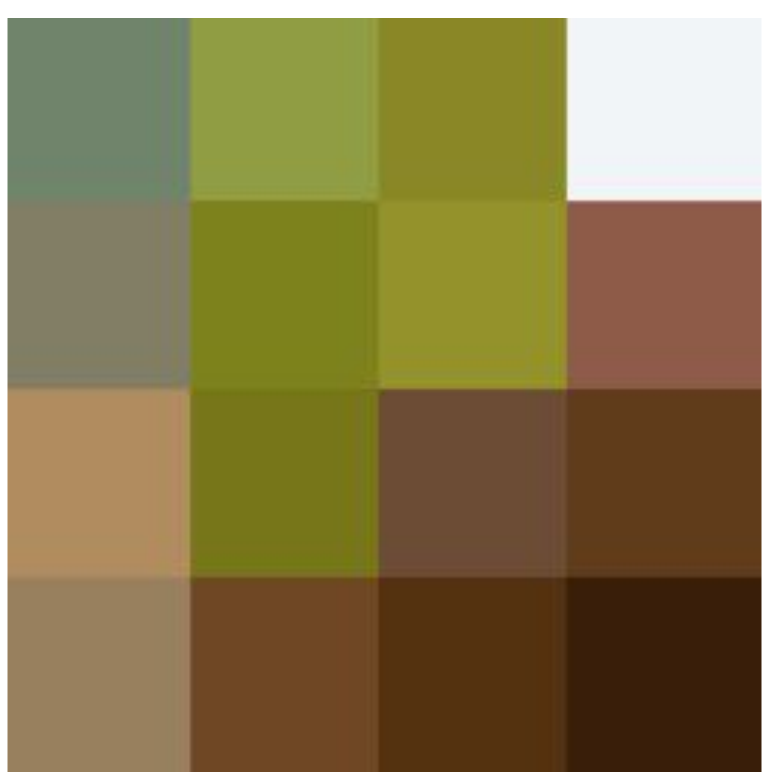
expected value

...even if n=1! (only one sample)

- Consistent does not imply unbiased!

Example 1: Consistent or Unbiased?

- My estimator for the integral over an image:
  - take $n = m \times m$ samples at fixed grid points
  - sum the contributions of each box
  - let m go to ∞



$m = 4$

$m = 16$

$m = 64$

$m = \infty$

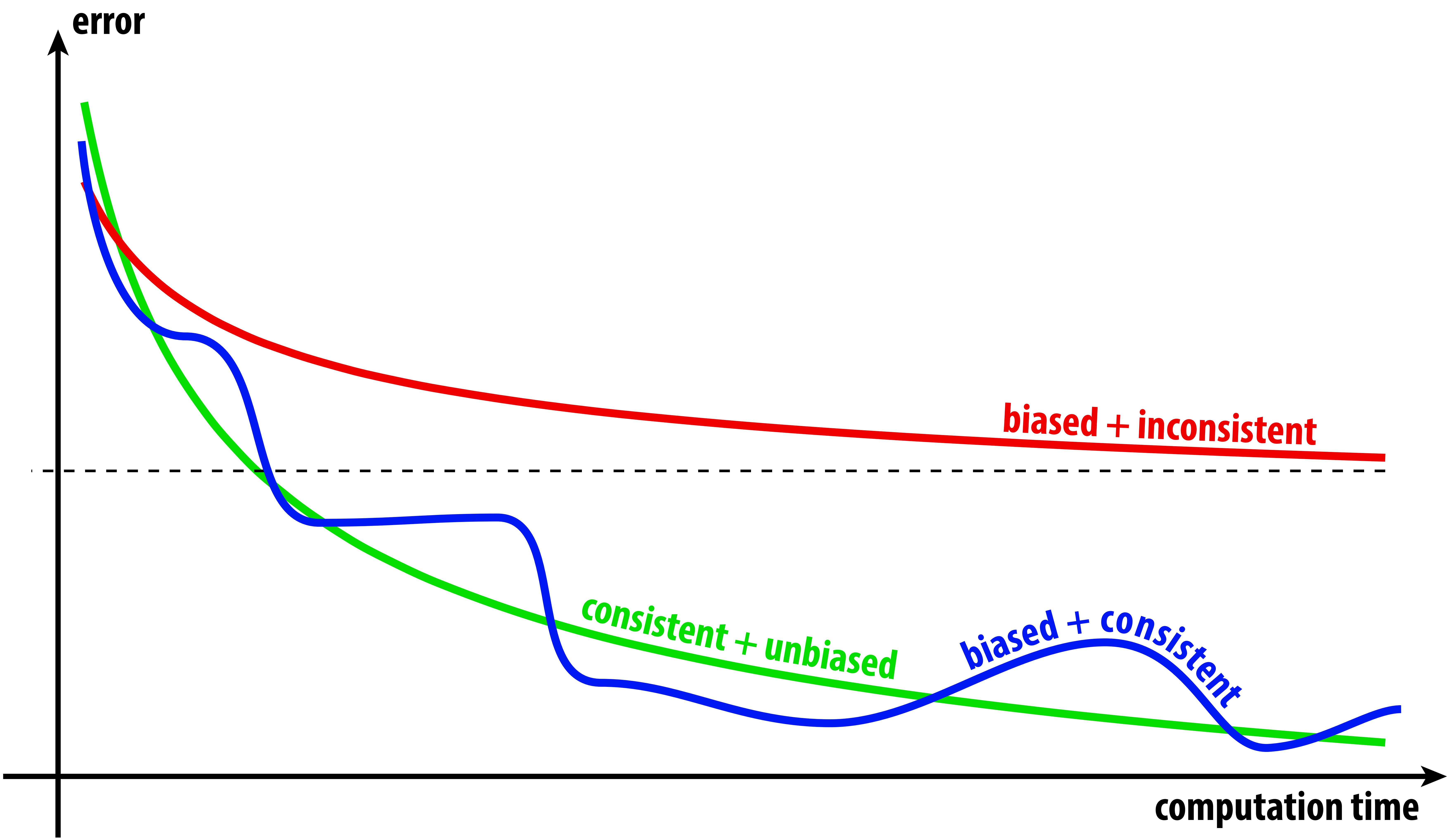
Is this estimator consistent? Unbiased?

Example 2: Consistent or Unbiased?

- My estimator for the integral over an image:
  - take only a single random sample of the image (n=1)
  - multiply it by the image area
  - use this value as my estimate

Is this estimator consistent? Unbiased?
(What if I then let n go to ∞?)

Why does it matter?



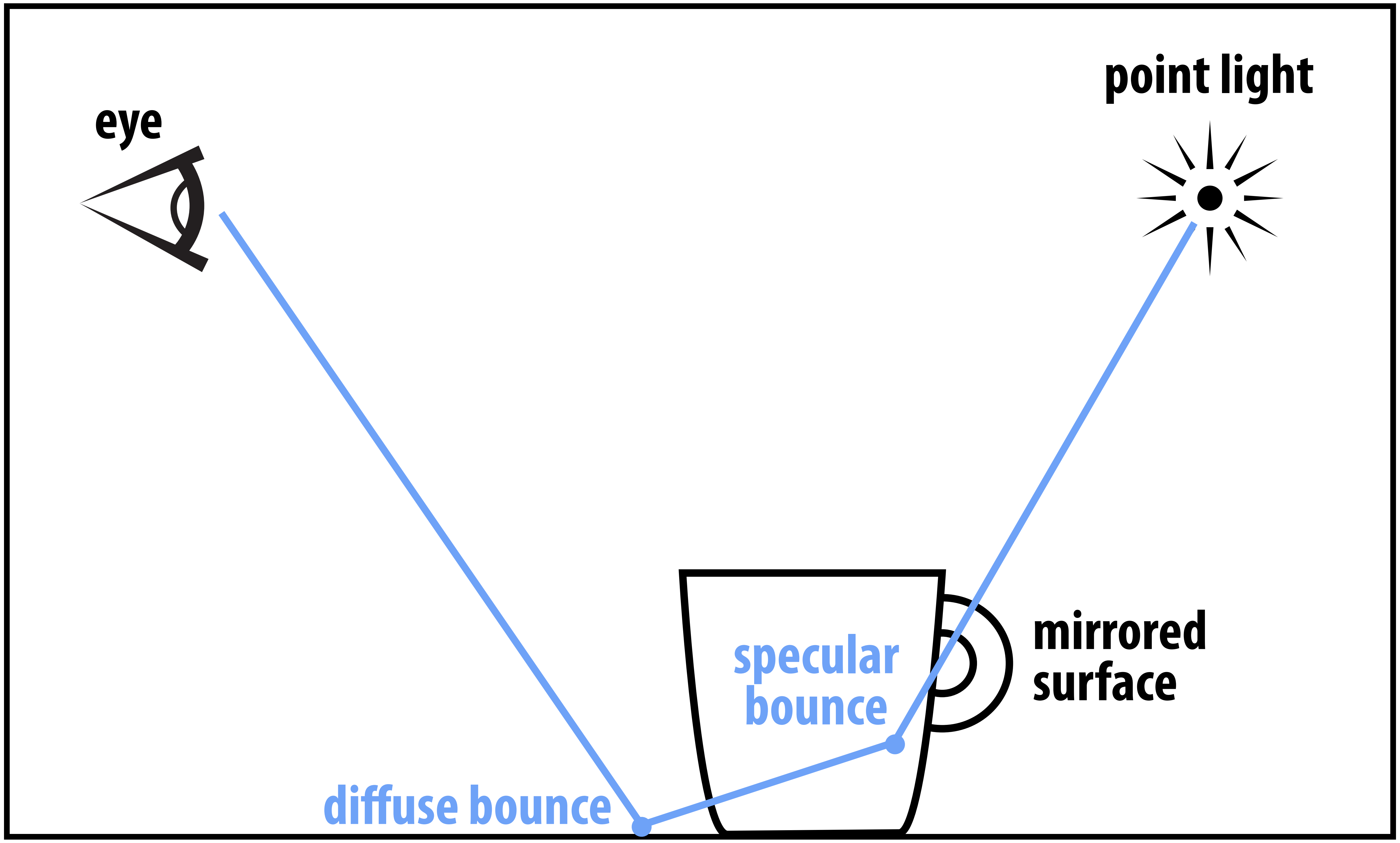
Rule of thumb: unbiased estimators have more predictable behavior / fewer parameters to tweak to get correct result (which says nothing about performance...)

Consistency & Bias in Rendering Algorithms

method	consistent?	unbiased?
rasterization*	NO	NO
path tracing	ALMOST	ALMOST
bidirectional path tracing	???	???
Metropolis light transport	???	???
photon mapping	???	???
radiosity	???	???

*But very high performance!

Naïve Path Tracing: Which Paths Can We Trace?



"caustic" (focused light) from reflection

Q: What's the probability we sample the reflected direction?

A: ZERO.

Q: What's the probability we hit a point light source?

A: ZERO.

Naïve path tracing misses important phenomena!

(Formally: the result is biased.)

...But isn't this example pathological?
No such thing as point light source, perfect mirror.

Real lighting can be close to pathological

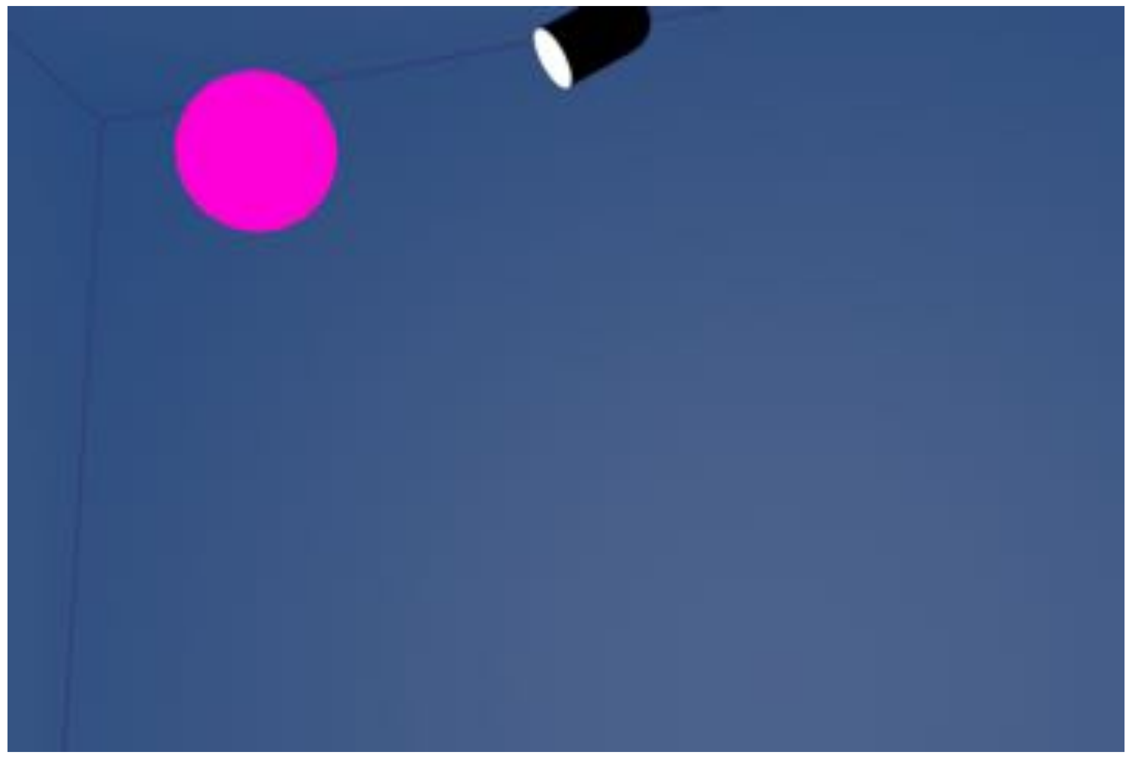
small directional
light source

near-perfect mirror

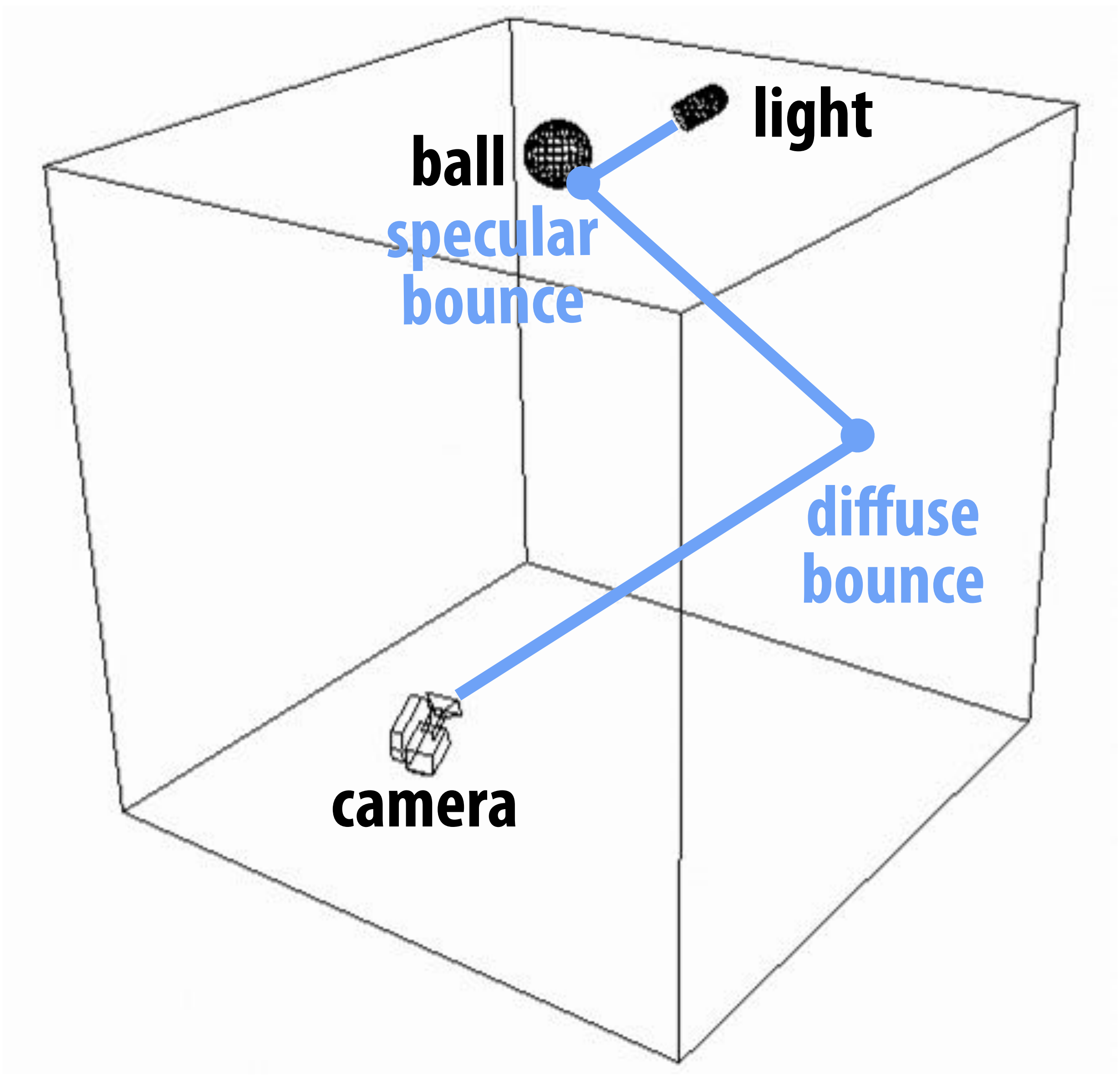
Still want to render this scene!

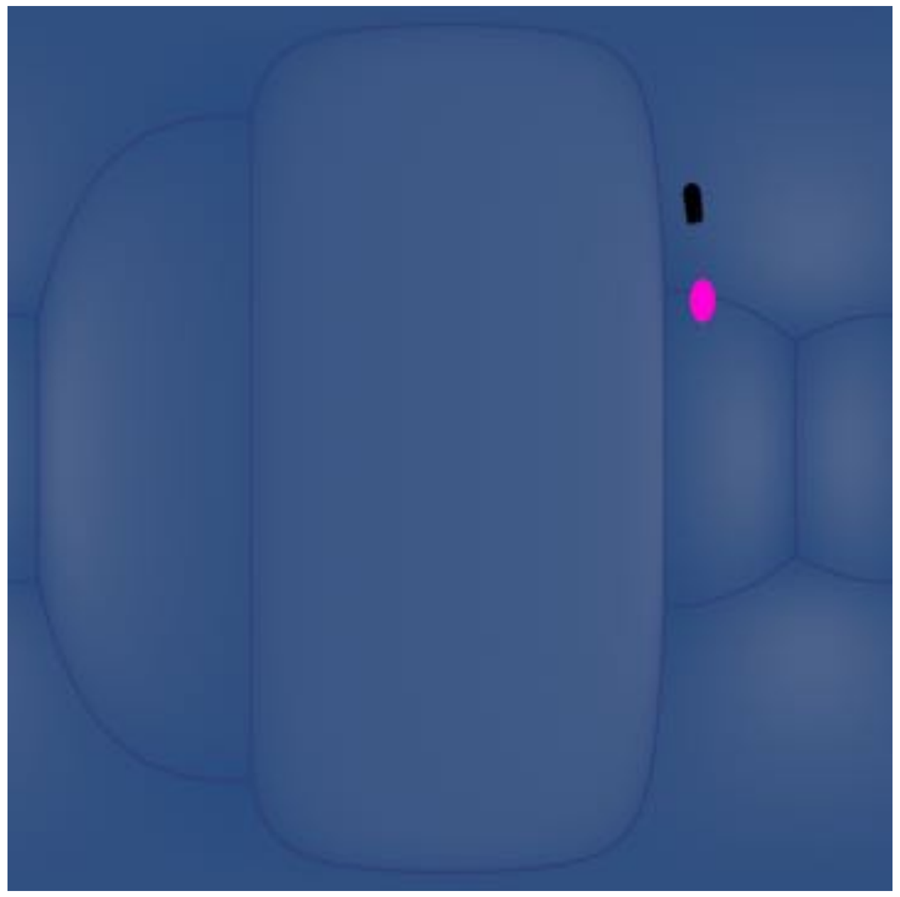
Light has a very "spiky" distribution

- Consider the view from each bounce in our disco scene:

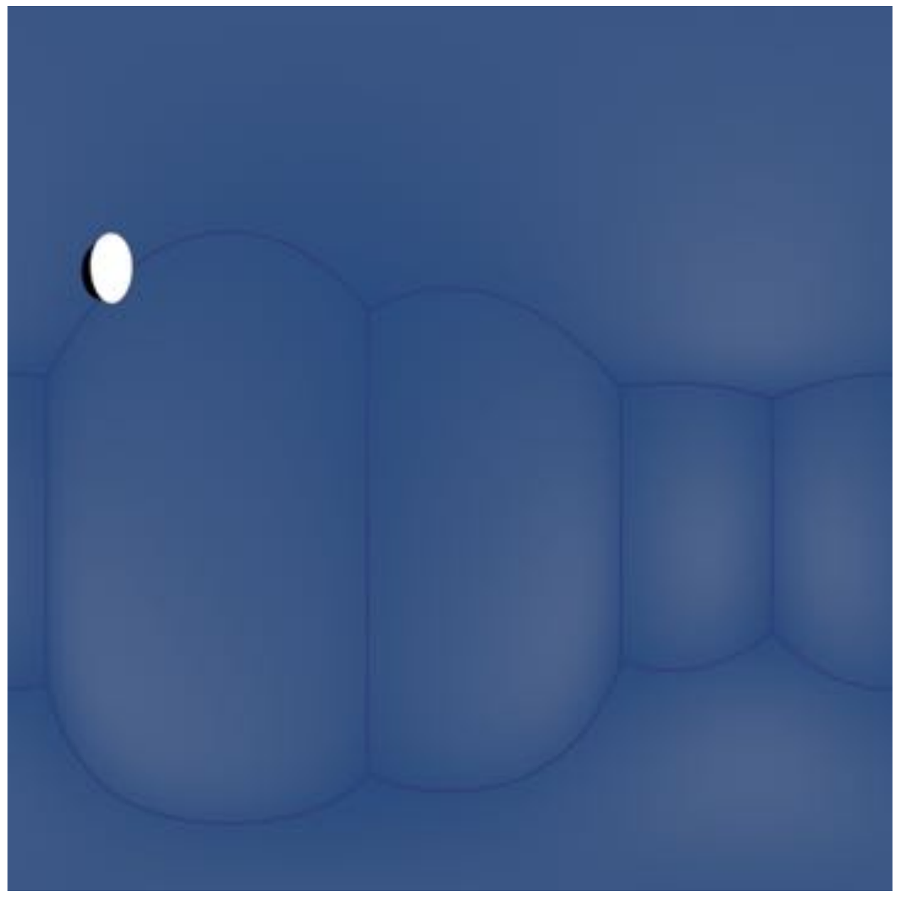


view from camera





view from diffuse bounce
mirrored ball (pink) covers small
percentage of solid angle

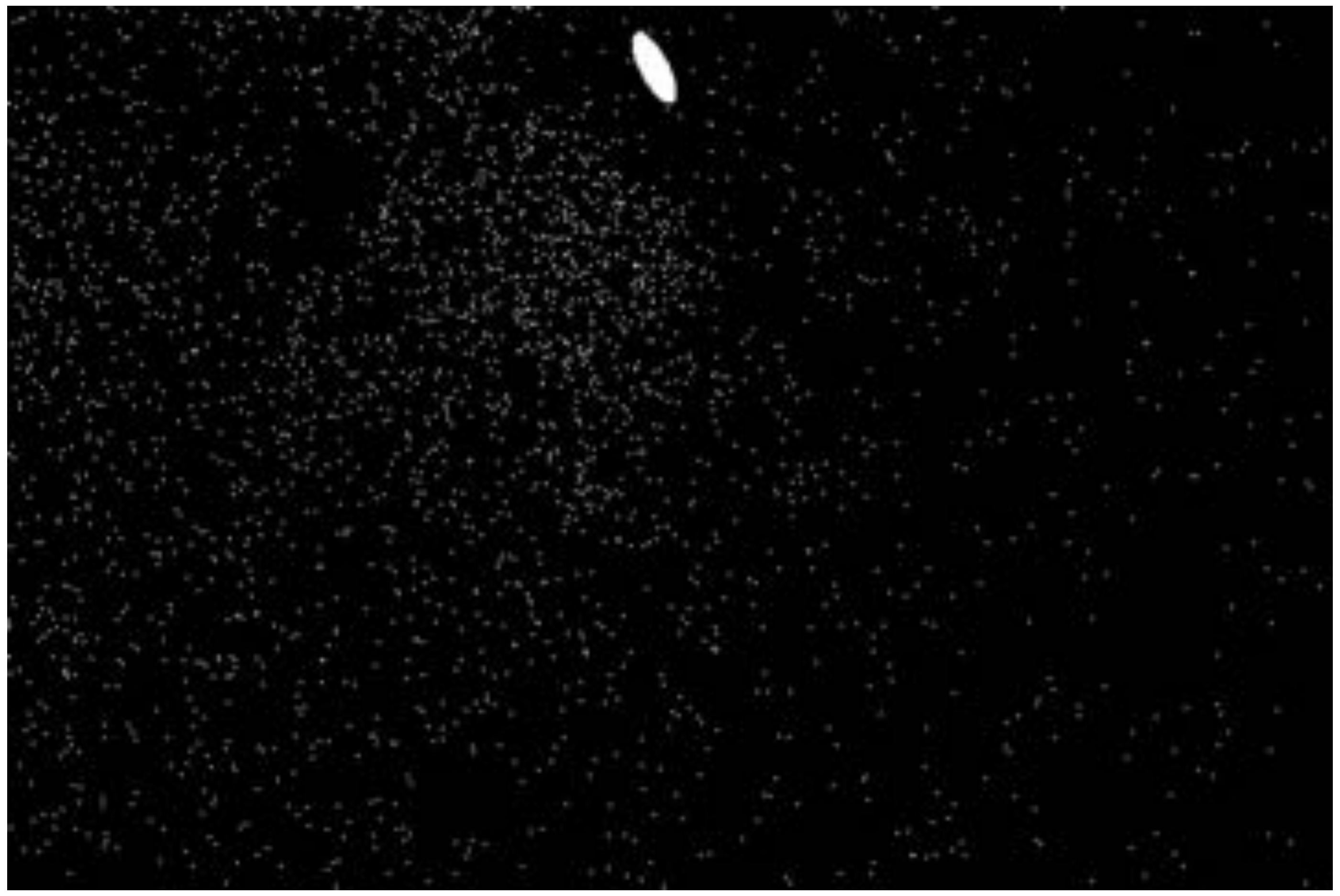


view from specular bounce
area light (white) covers small
percentage of solid angle

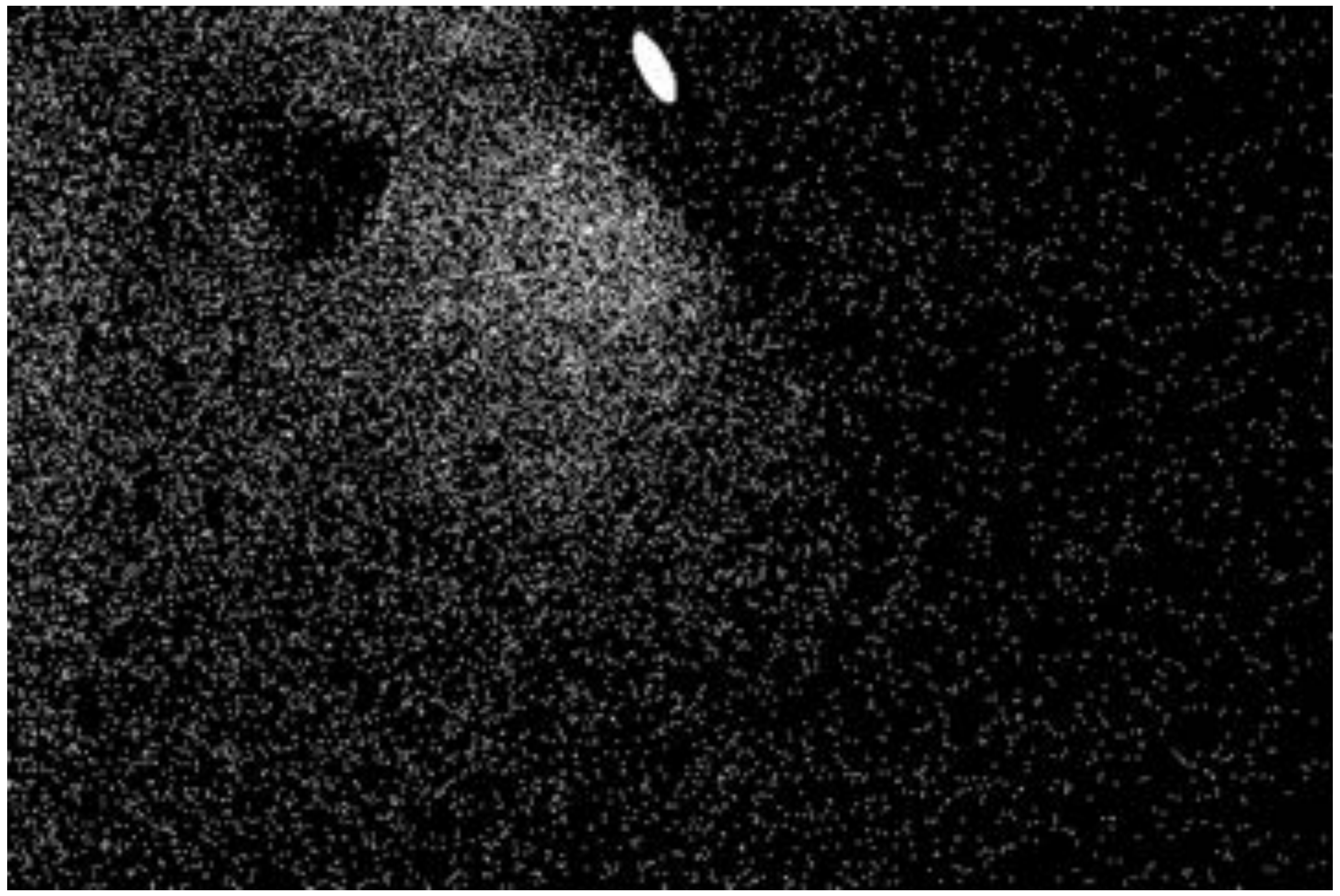
Probability that a uniformly-sampled path carries light is the product of the solid angle fractions. (Very small!)

Then consider even more bounces...

Just use more samples?



path tracing - 16 samples/pixel



path tracing - 128 samples/pixel

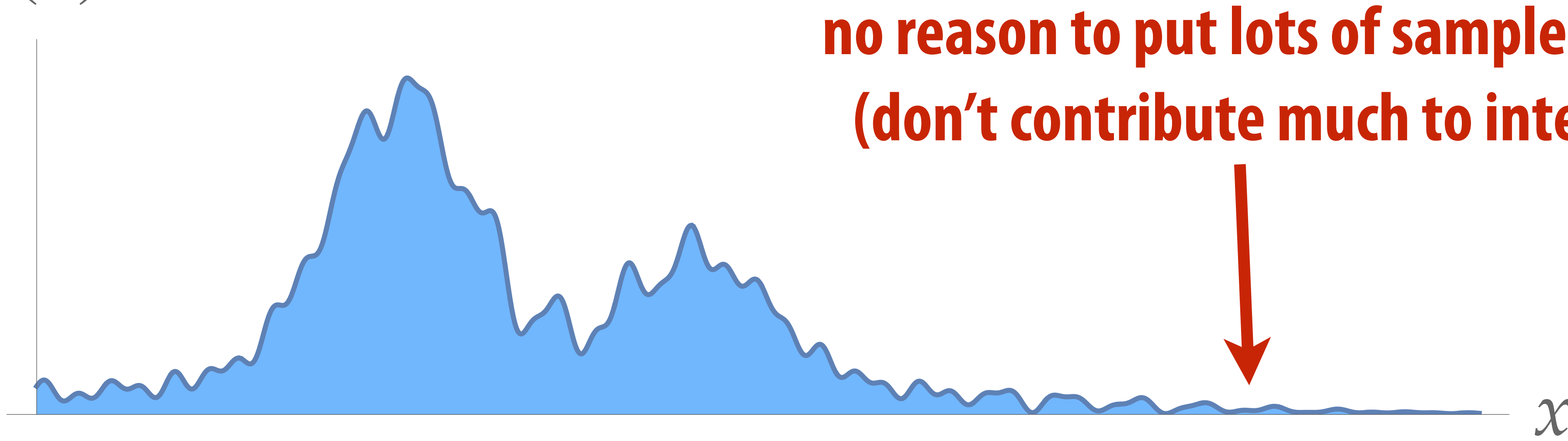
path tracing - 8192 samples/pixel

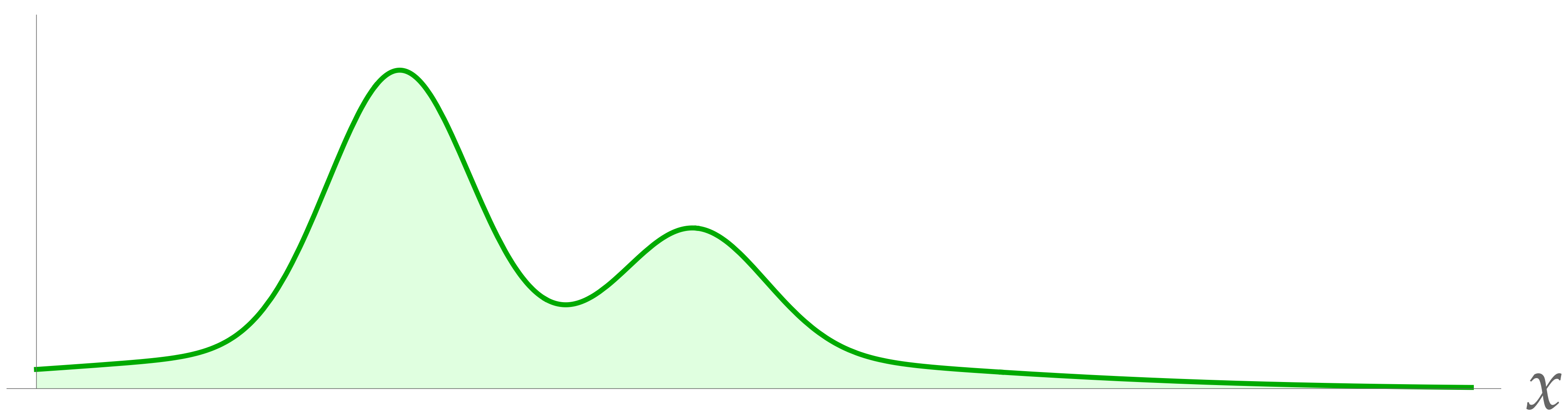
how do we get here? (photo)

We need better sampling strategies!

Review: Importance Sampling

- Simple idea: sample the integrand according to how much we expect it to contribute to the integral.





naïve Monte Carlo:

$$V(\Omega)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

(x_i are sampled uniformly)

importance sampled Monte Carlo:

$$\frac{1}{n}\sum_{i=1}^{n} \frac{f(x_i)}{p(x_i)}$$

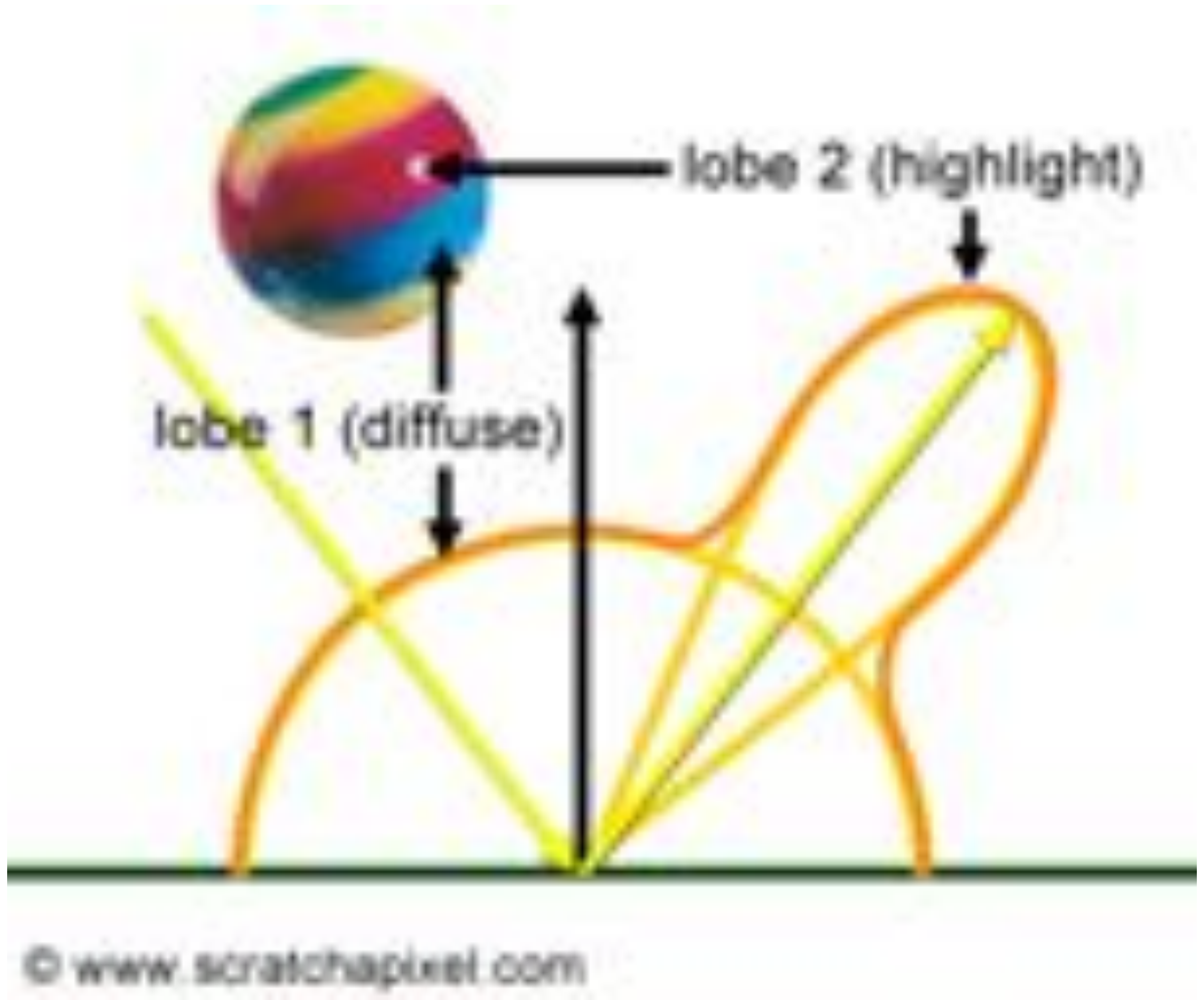
(x_i are sampled proportional to p)

"If I sample x more frequently, each sample should count for less; if I sample x less frequently, each sample should count for more."

Q: What happens when p is proportional to f (p = cf)?

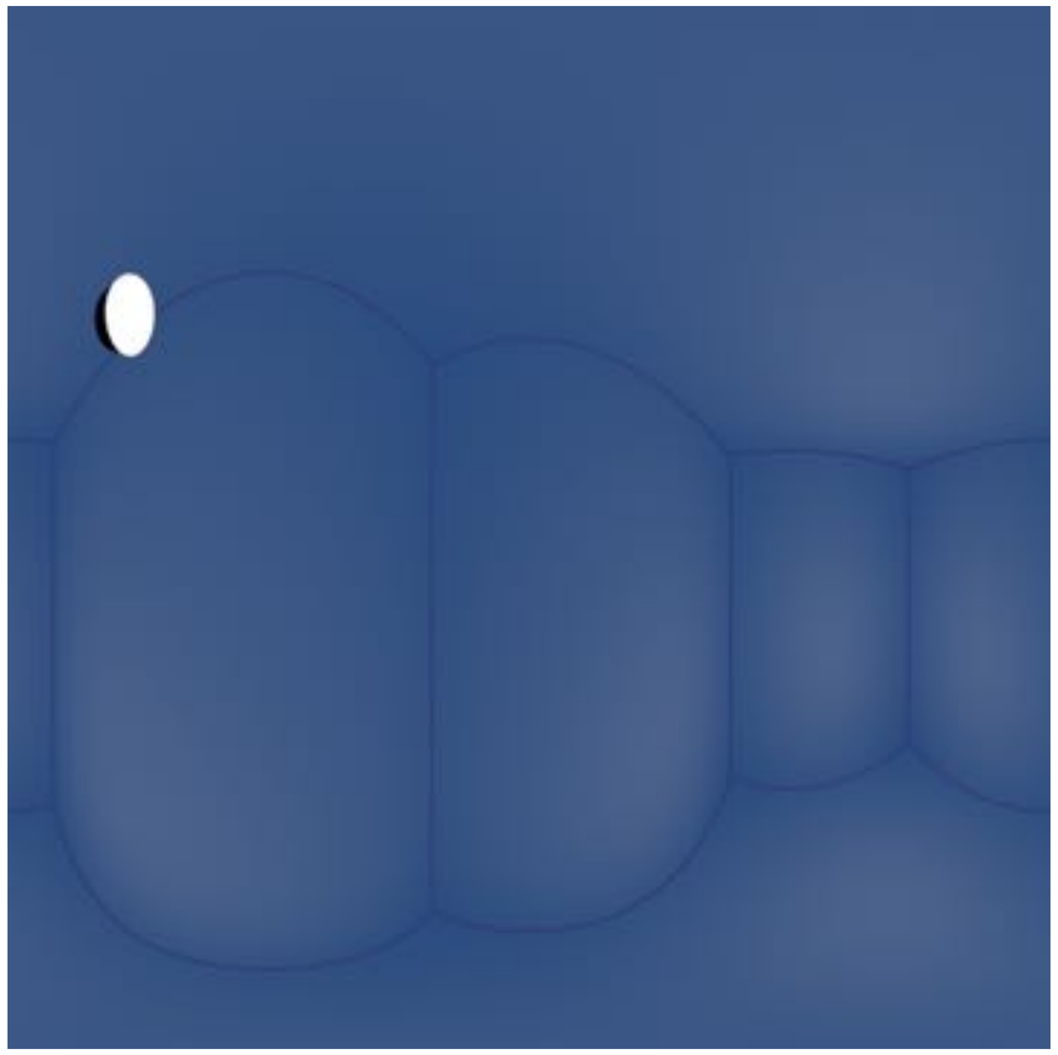
Importance Sampling in Rendering

materials: sample important "lobes"



(important special case: perfect mirror!)

illumination: sample bright lights



(important special case: point light!)

Q: How else can we re-weight our choice of samples?

Path Space Formulation of Light Transport

- **So far have been using recursive rendering equation:**

$$L_{\mathrm{o}}(\mathbf{x},\, \omega_{\mathrm{o}}) \;=\; L_e(\mathbf{x},\, \omega_{\mathrm{o}}) \;+\; \int_{\Omega} f_r(\mathbf{x},\, \omega_{\mathrm{i}},\, \omega_{\mathrm{o}})\, L_{\mathrm{i}}(\mathbf{x},\, \omega_{\mathrm{i}})\, (\omega_{\mathrm{i}} \,\cdot\, \mathbf{n})\; \mathrm{d}\,\omega_{\mathrm{i}}$$

- **Make intelligent "local" choices at each step (material/ lights)**
- **Alternatively, we can use a "path integral" formulation:**

how much "light" is carried by this path?

$$I = \int_{\Omega} f(\bar{x}) d\mu(\bar{x})$$

how much of path space does this path "cover"

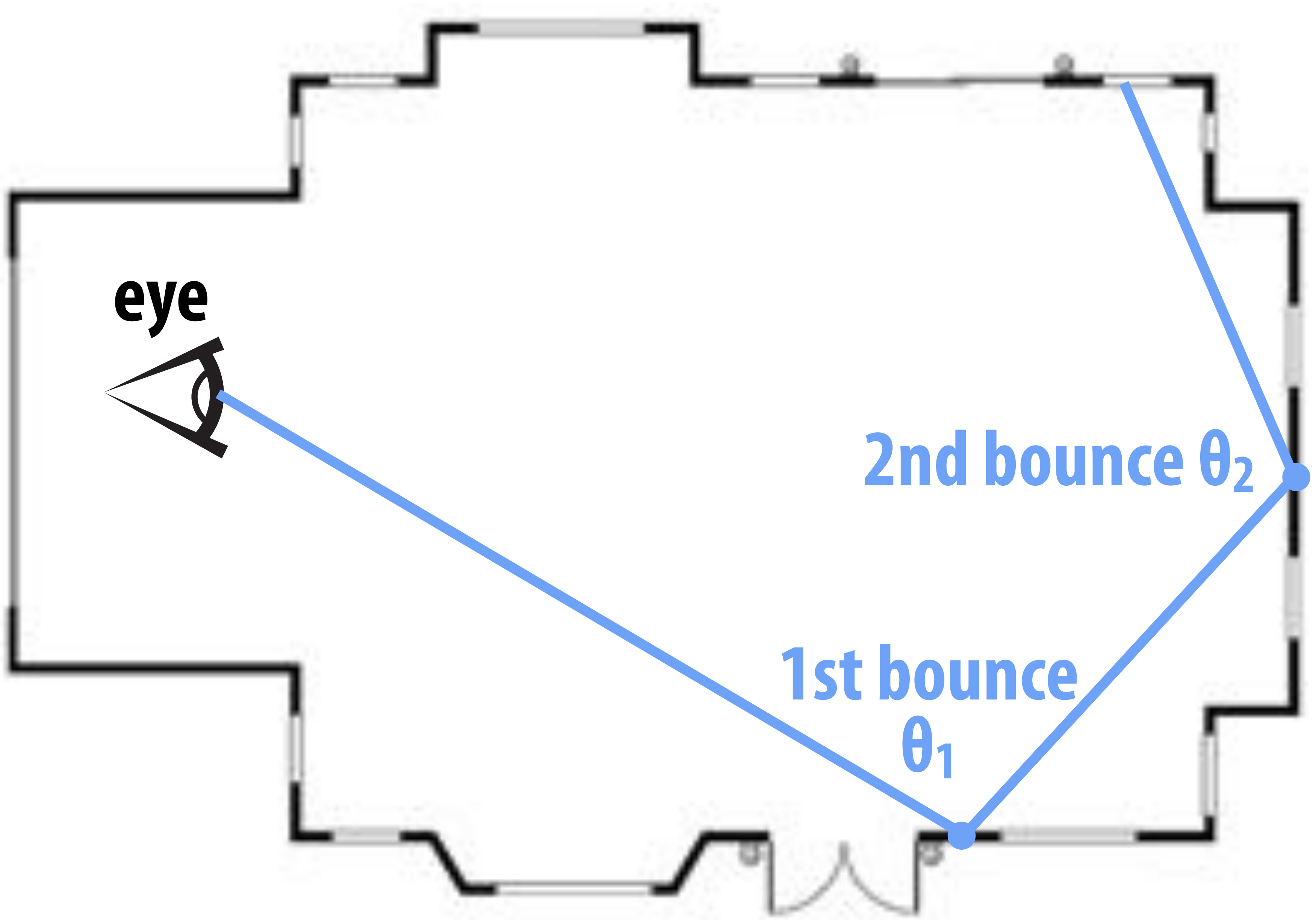
one particular path

all possible paths

- **Opens the door to intelligent "global" importance sampling. (But still hard!)**

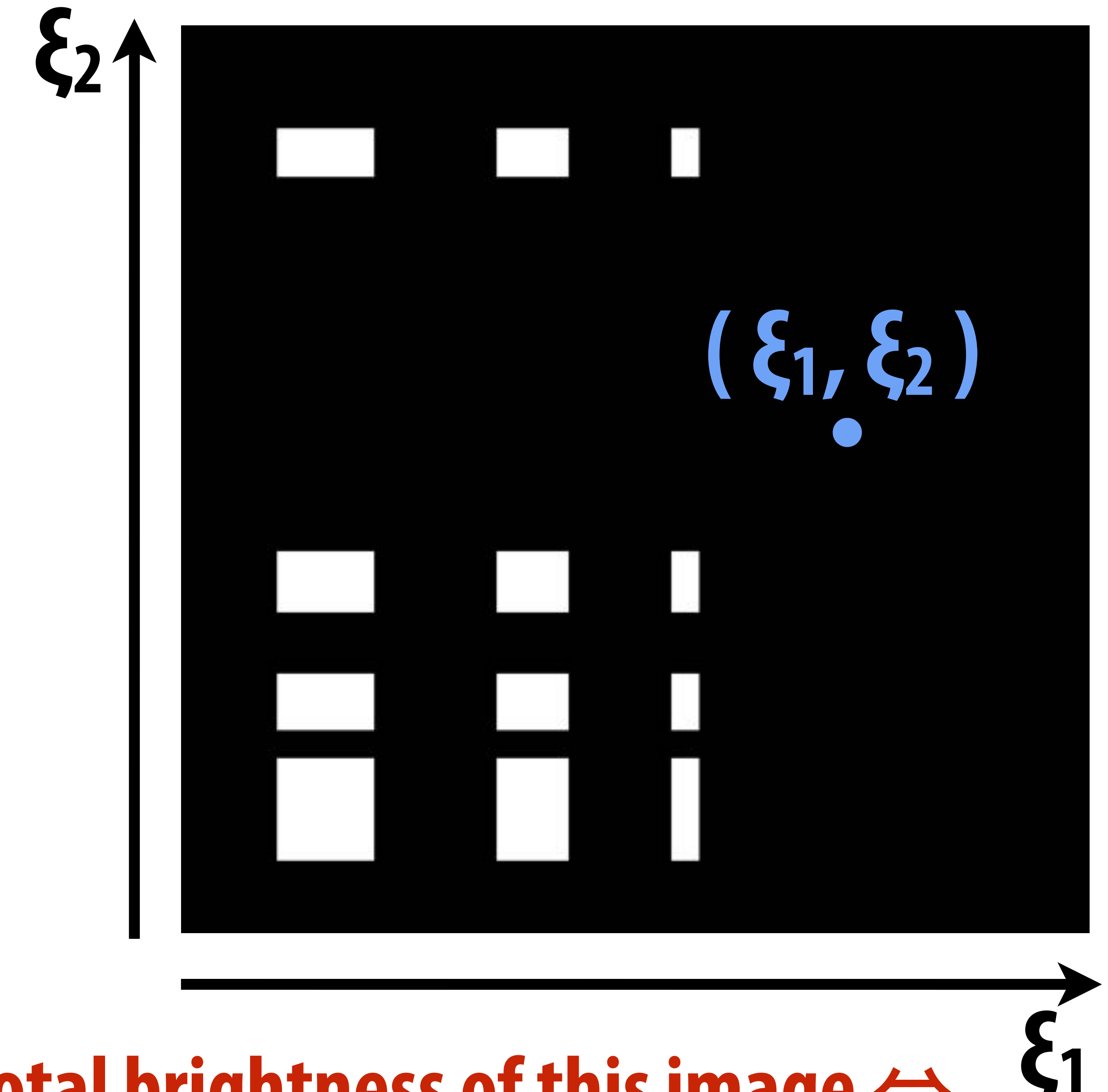
Unit Hypercube View of Path Space

- Paths determined by a sequence of random values ξ in [0,1]
- Hence, path of length k is a point in hypercube $[0,1]^k$
- "Just" integrate over cubes of each dimension k
- E.g., two bounces in a 2D scene:



each bounce: $\xi \in [0,1] \mapsto \theta \in [0,\pi]$

Each point is a path of length 2:

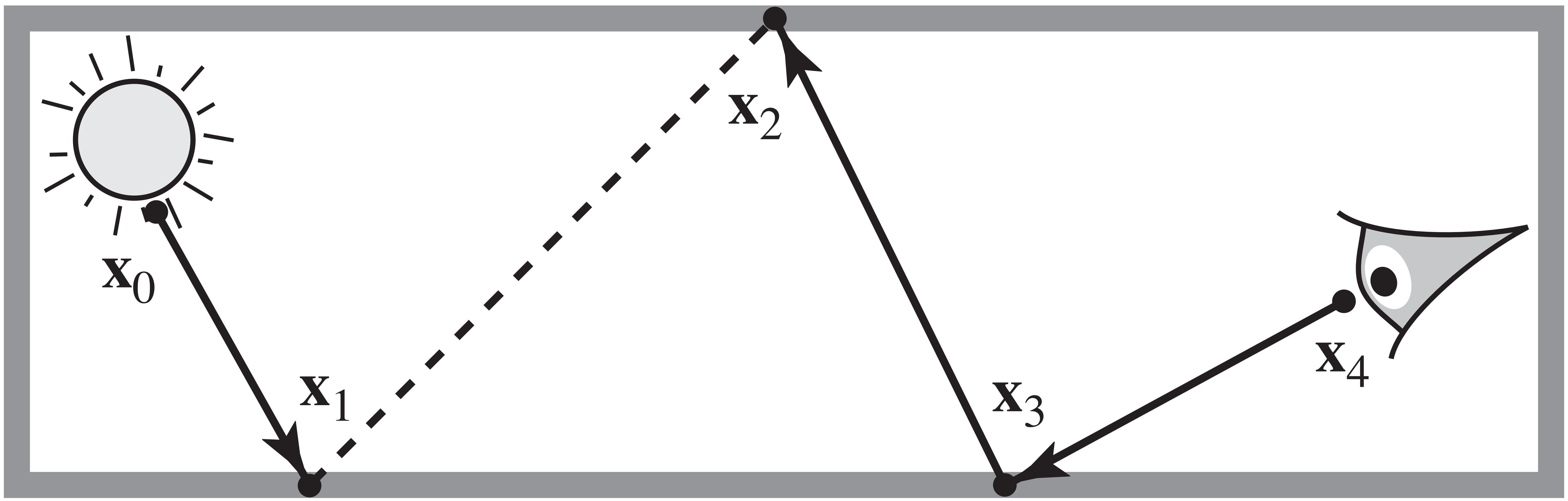


Total brightness of this image ⇔ total contribution of length-2 paths.

How do we choose paths—and path lengths?

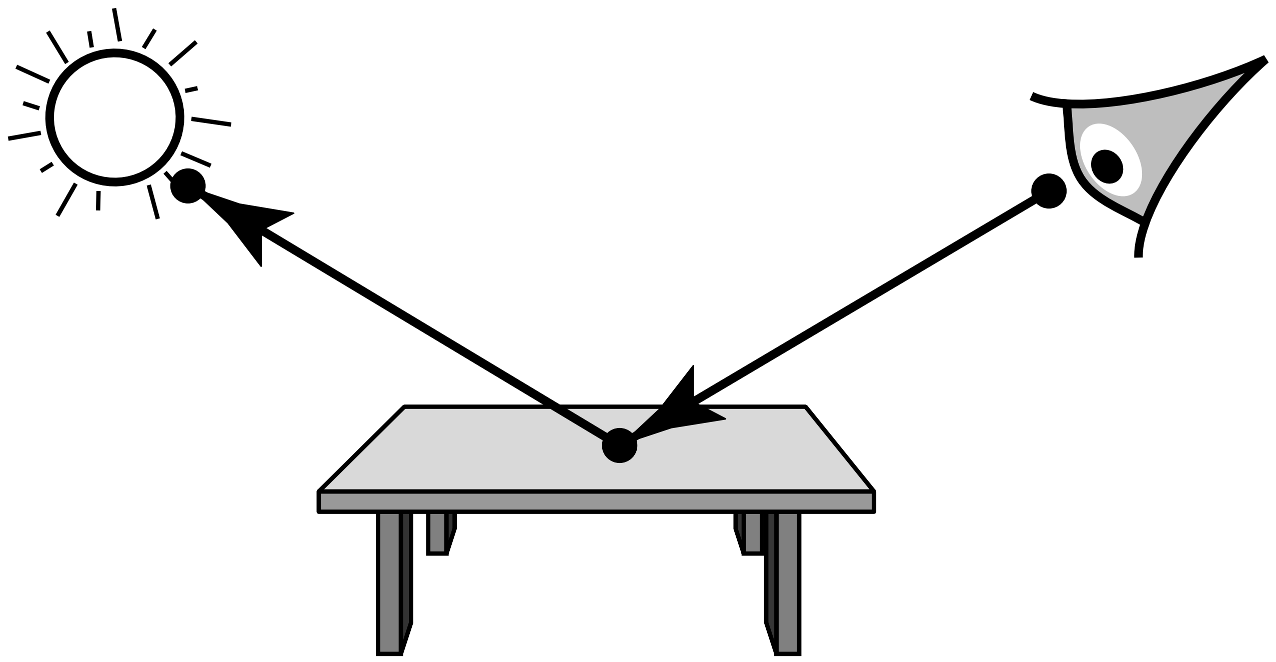
Bidirectional Path Tracing

- Forward path tracing: no control over path length (hits light after n bounces, or gets terminated by Russian Roulette)
- Idea: connect paths from light, eye ("bidirectional")

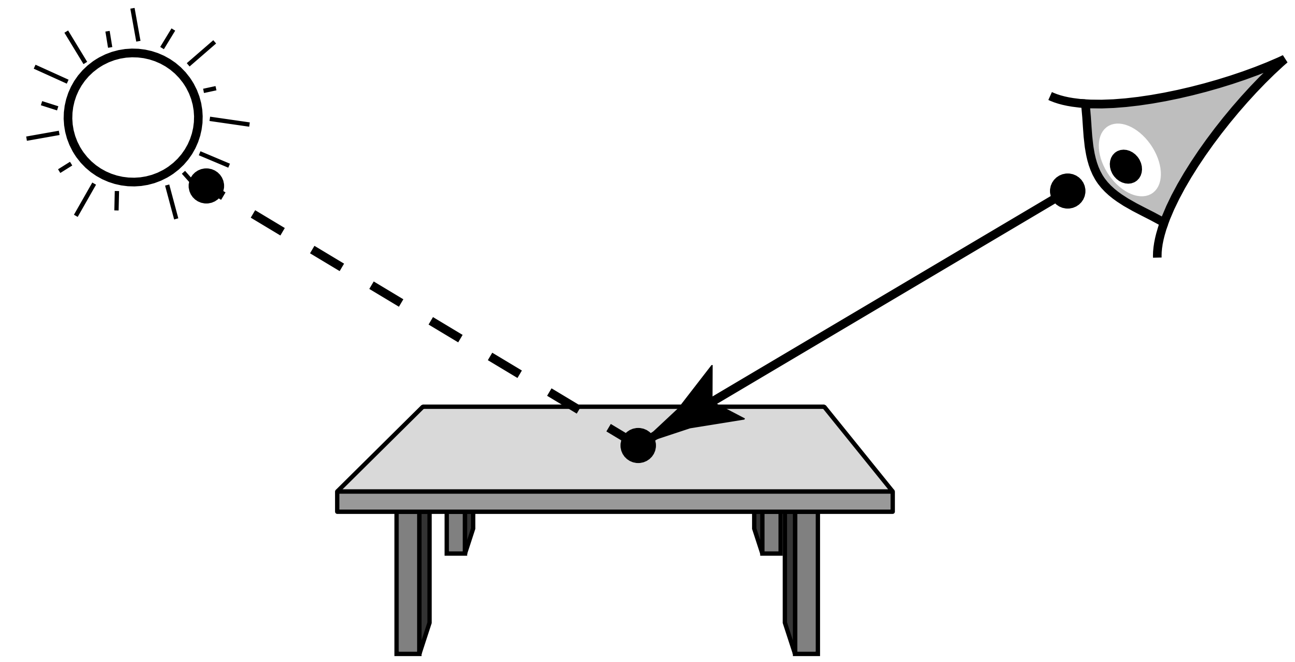


- Importance sampling? Need to carefully weight contributions of path according to sampling strategy.
- (Details in Veach & Guibas, "Bidirectional Estimators for Light Transport")

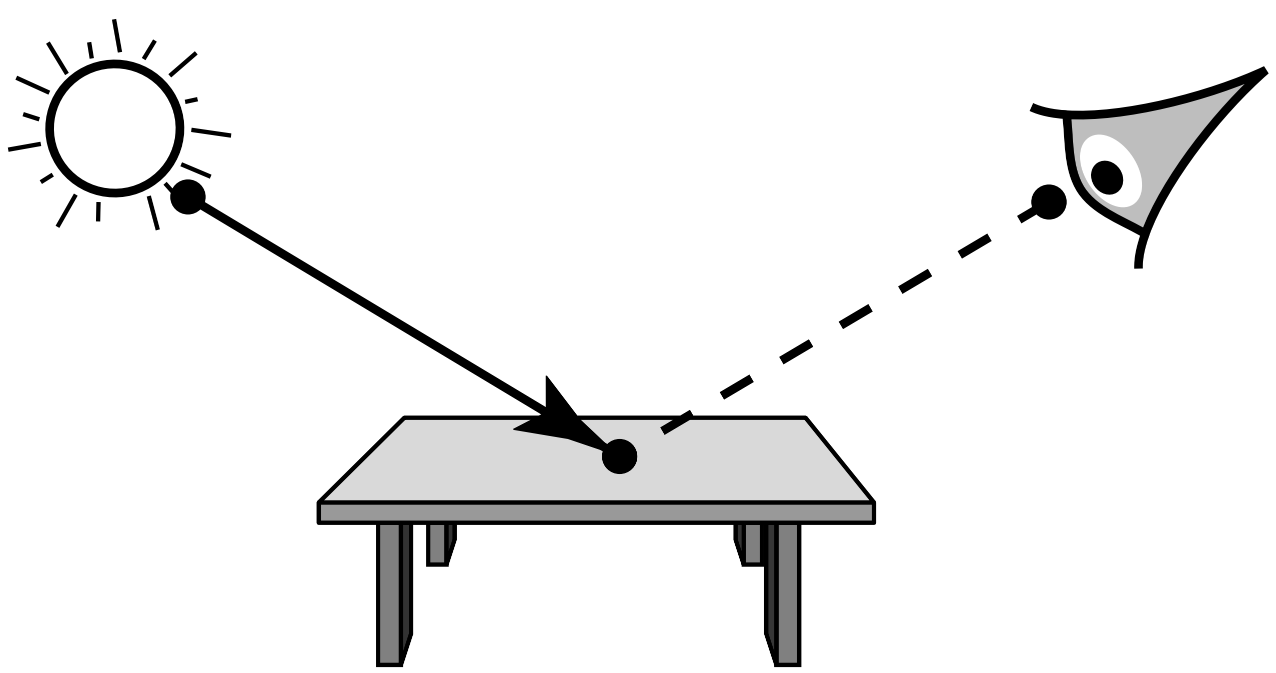
Bidirectional Path Tracing (Path Length=2)



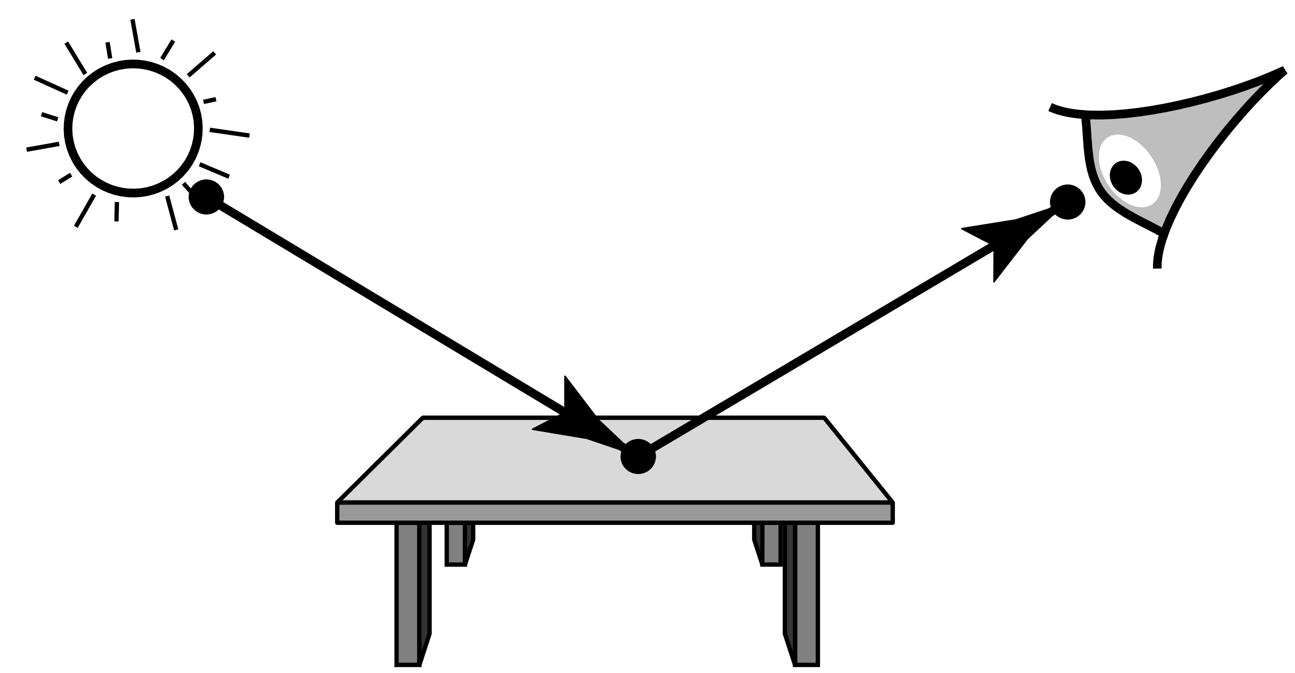
standard (forward) path tracing
fails for point light sources



direct lighting



visualize particles from light

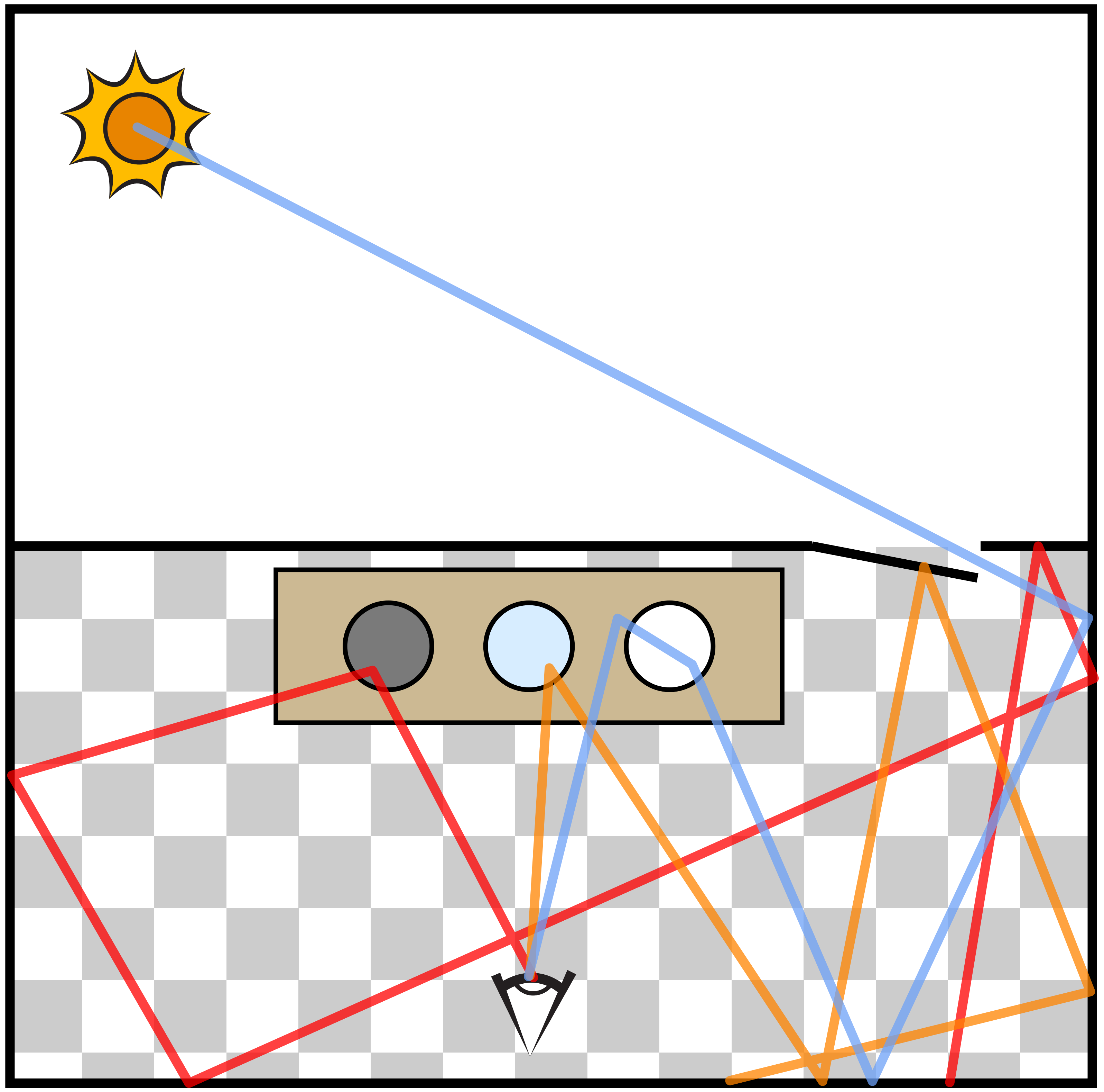


backward path tracing
fails for a pinhole camera

Contributions of Different Path Lengths

final image

Good paths can be hard to find!



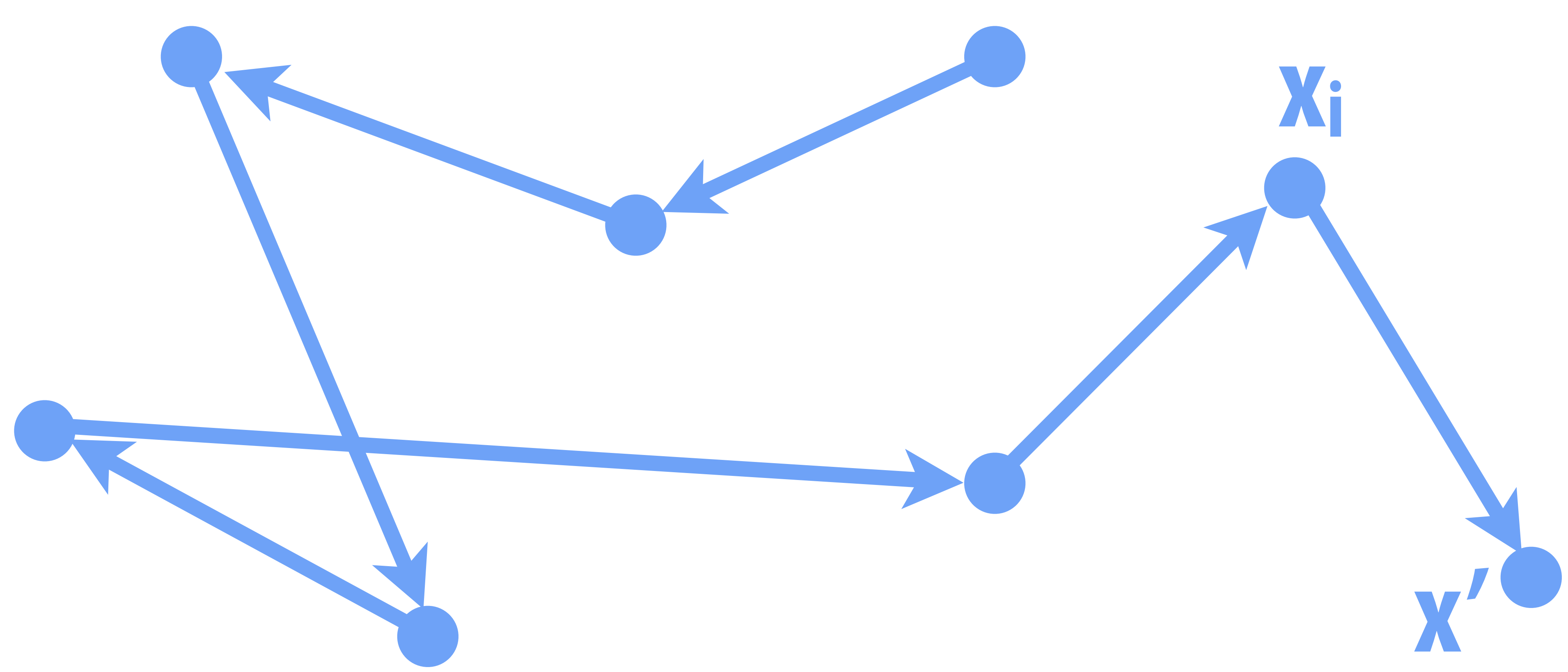
bidirectional path tracing

Metropolis light transport (MLT)

Idea:
Once we find a good path, perturb it to find nearby "good" paths.

Metropolis-Hastings Algorithm (MH)

- Standard Monte Carlo: sum up independent samples
- MH: take random walk of dependent samples ("mutations")
- Basic idea: prefer to take steps that increase sample value



$\alpha := f(x') / f(x_i)$ "transition probability"

if random # in [0,1] $< \alpha$:

  $x_{i+1} = x'$

else:

  $x_{i+1} = x_i$

- If careful, sample distribution will be proportional to integrand
  - make sure mutations are "ergodic" (reach whole space)
  - need to take a long walk, so initial point doesn't matter ("mixing")

Metropolis-Hastings: Sampling an Image

- Want to take samples proportional to image density f
- Start at random point; take steps in (normal) random direction
- Occasionally jump to random point (ergodicity)
- Transition probability is "relative darkness" $f(x')/f(x_i)$

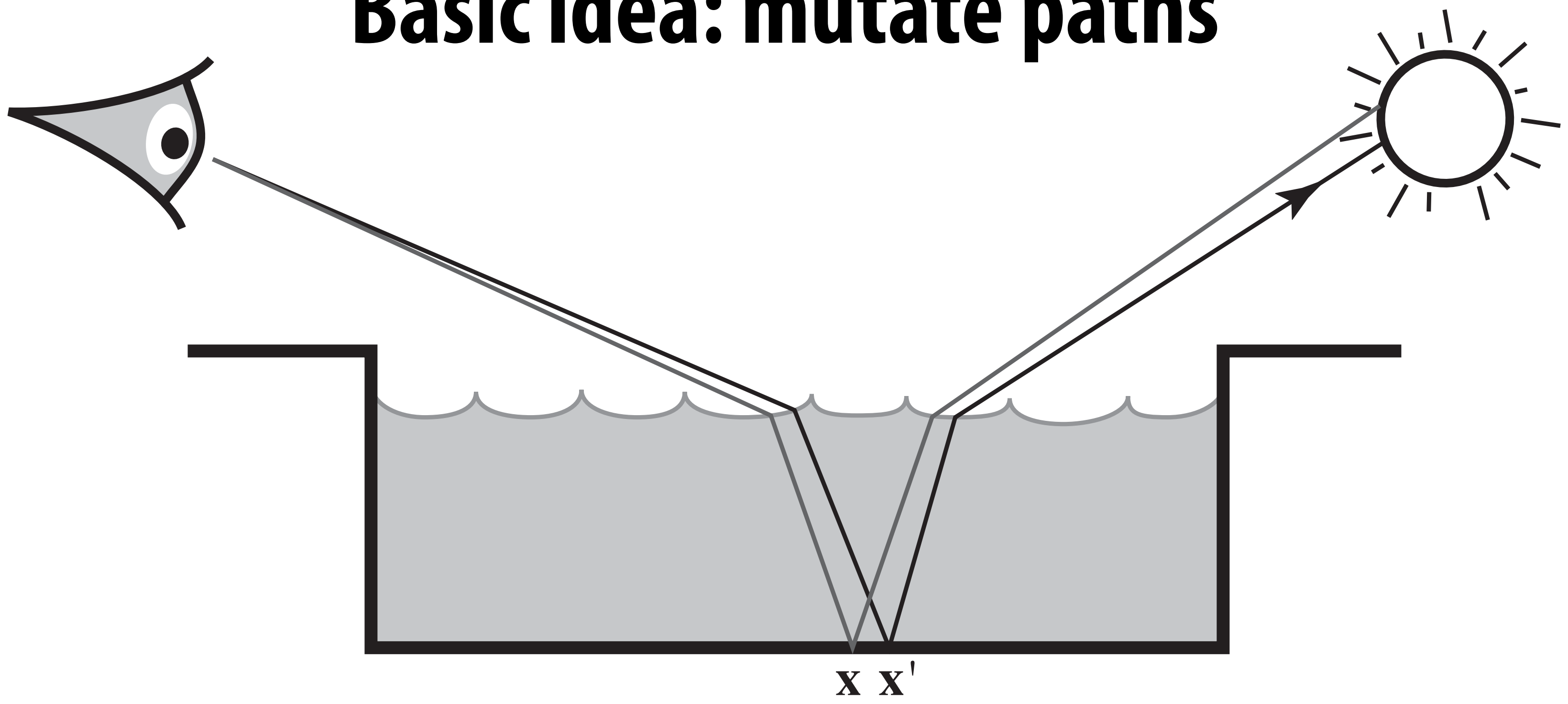
short walk

long walk

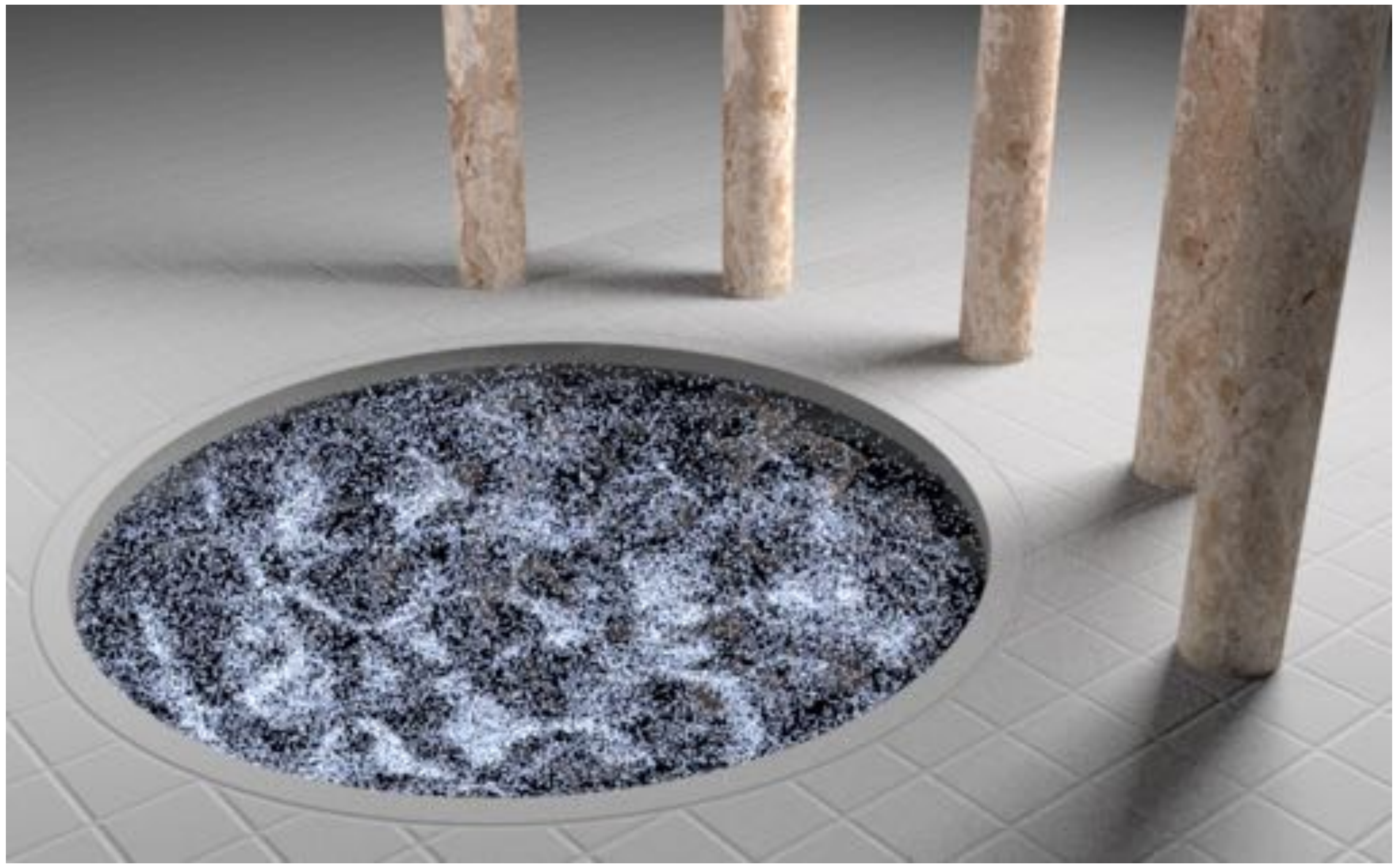
(original image)

Metropolis Light Transport

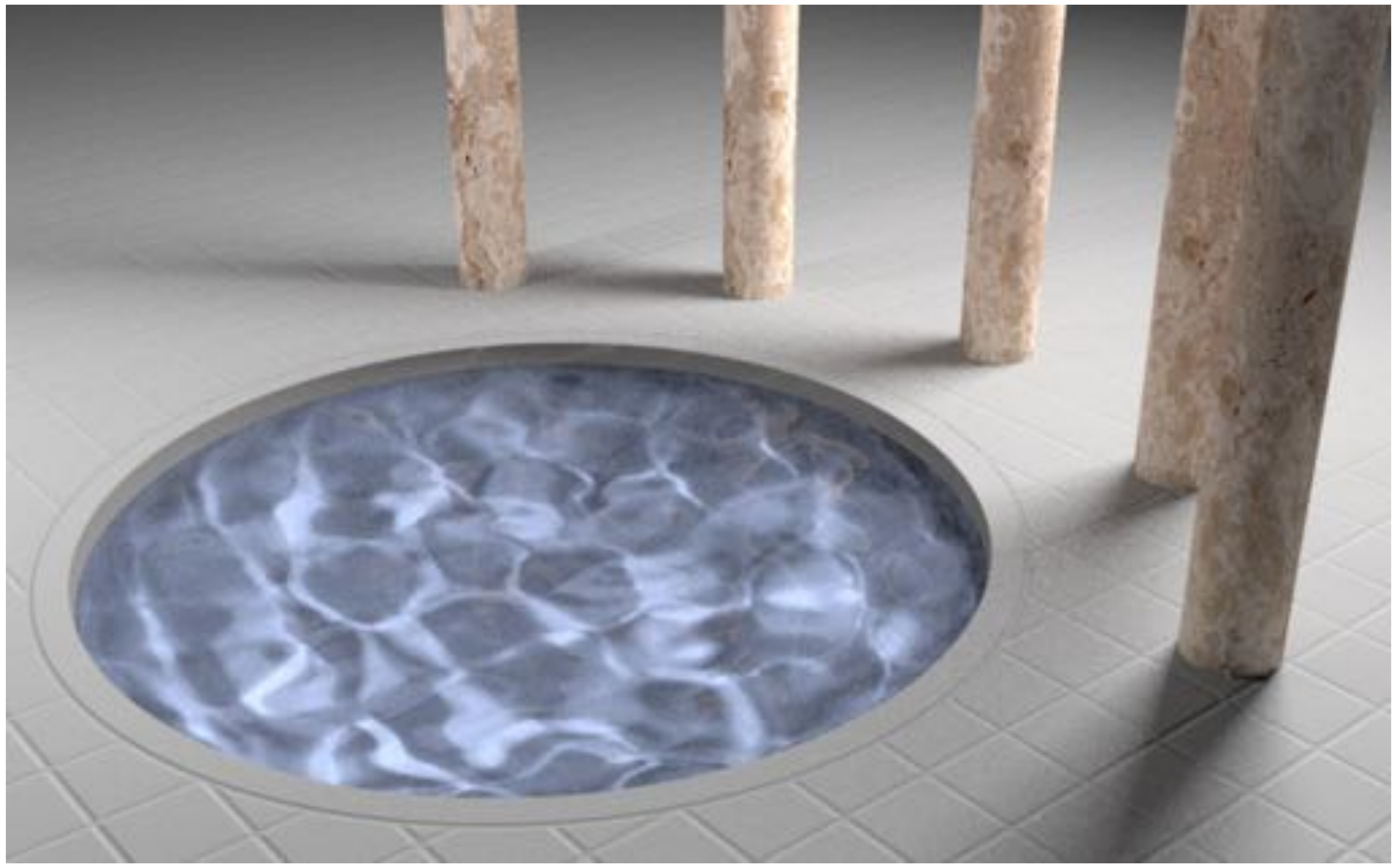
Basic idea: mutate paths



(For details see Veach, "Robust Monte Carlo Methods for Light Transport Simulation")



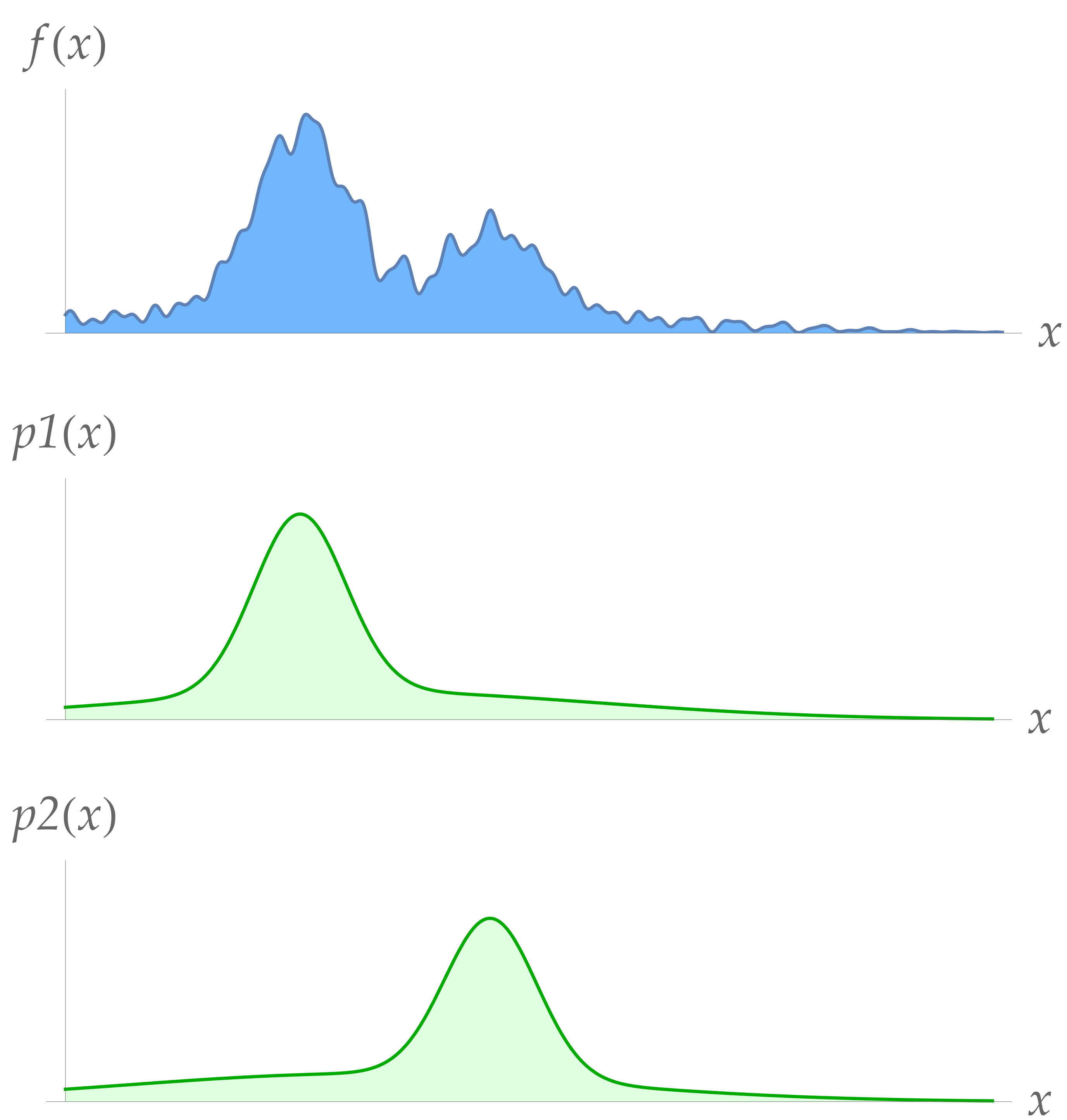
path tracing



Metropolis light transport (same time)

Multiple Importance Sampling (MIS)

- **Many possible importance sampling strategies**
- **Which one should we use for a given integrand?**
- **MIS: combine strategies to preserve strengths of all of them**



sum over strategies

sum over samples

jth sample taken with ith strategy

$$\frac{1}{N}\sum_{i=1}^{n}\sum_{j=1}^{n_i}\frac{f(x_{ij})}{\sum_k c_k p_k(x_{ij})}$$

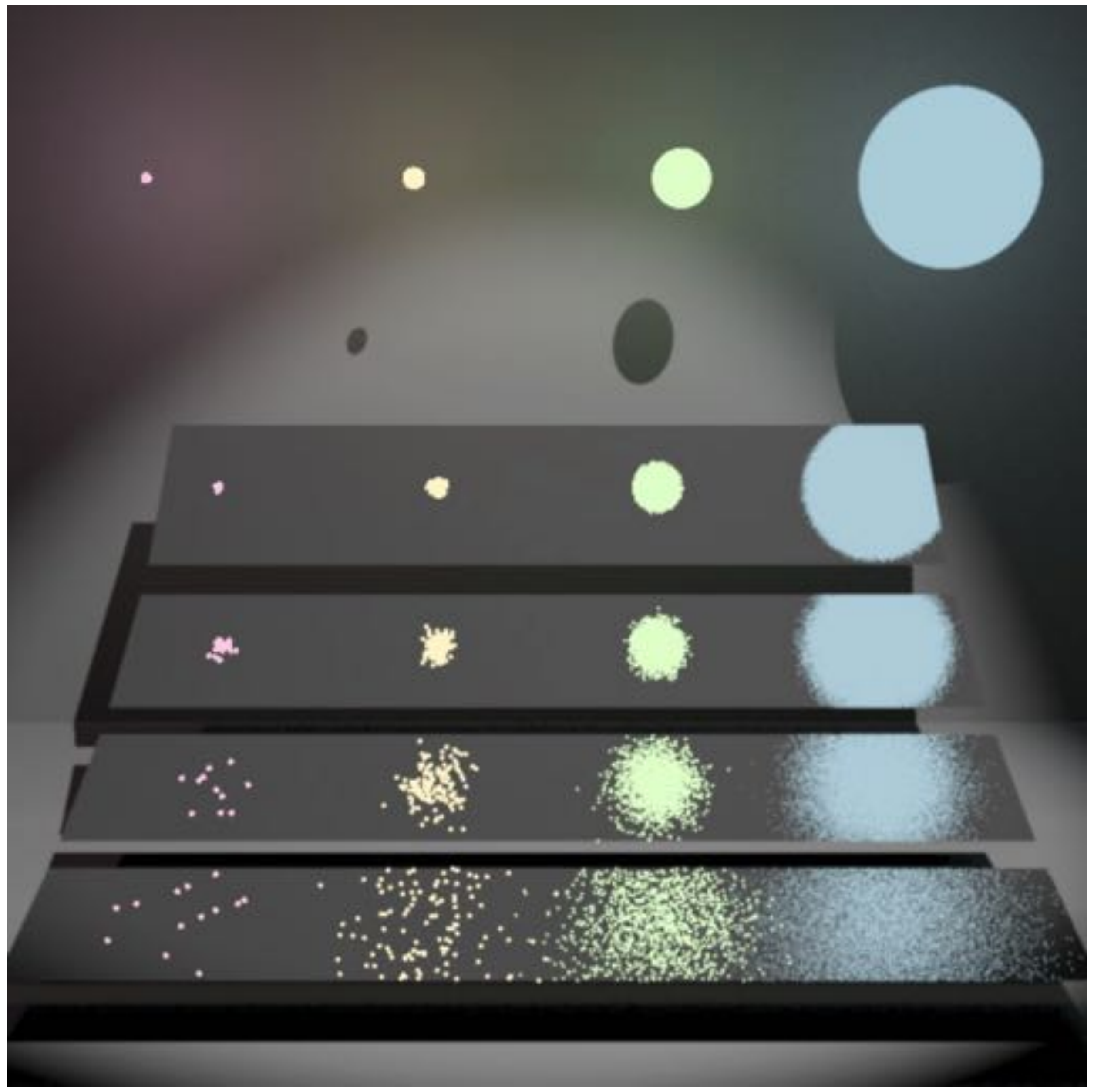
total # of samples

fraction of samples taken w/ kth strategy

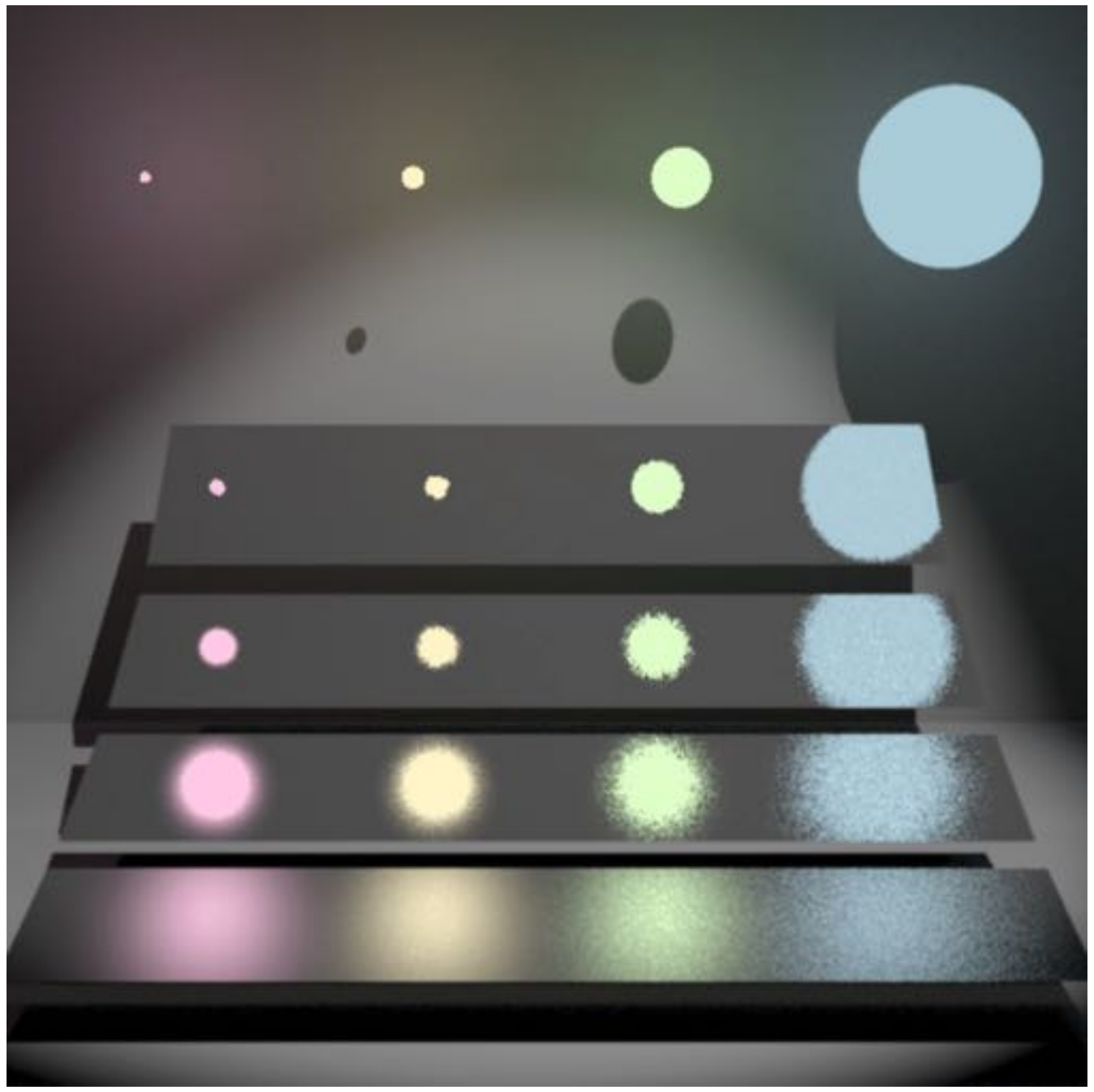
kth importance density

Still, several improvements possible (cutoff, power, max)—see Veach & Guibas.

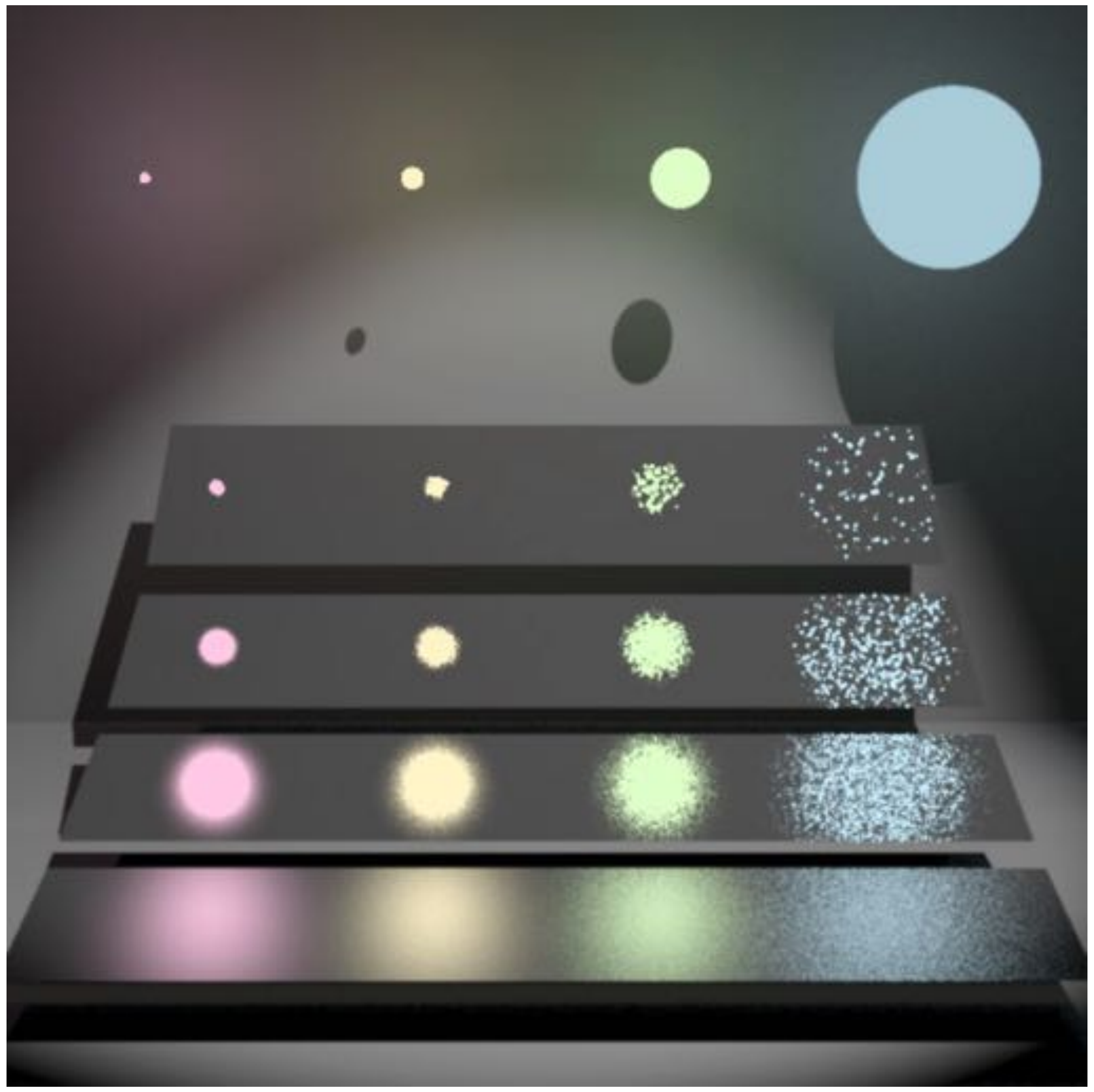
Multiple Importance Sampling: Example



sample materials



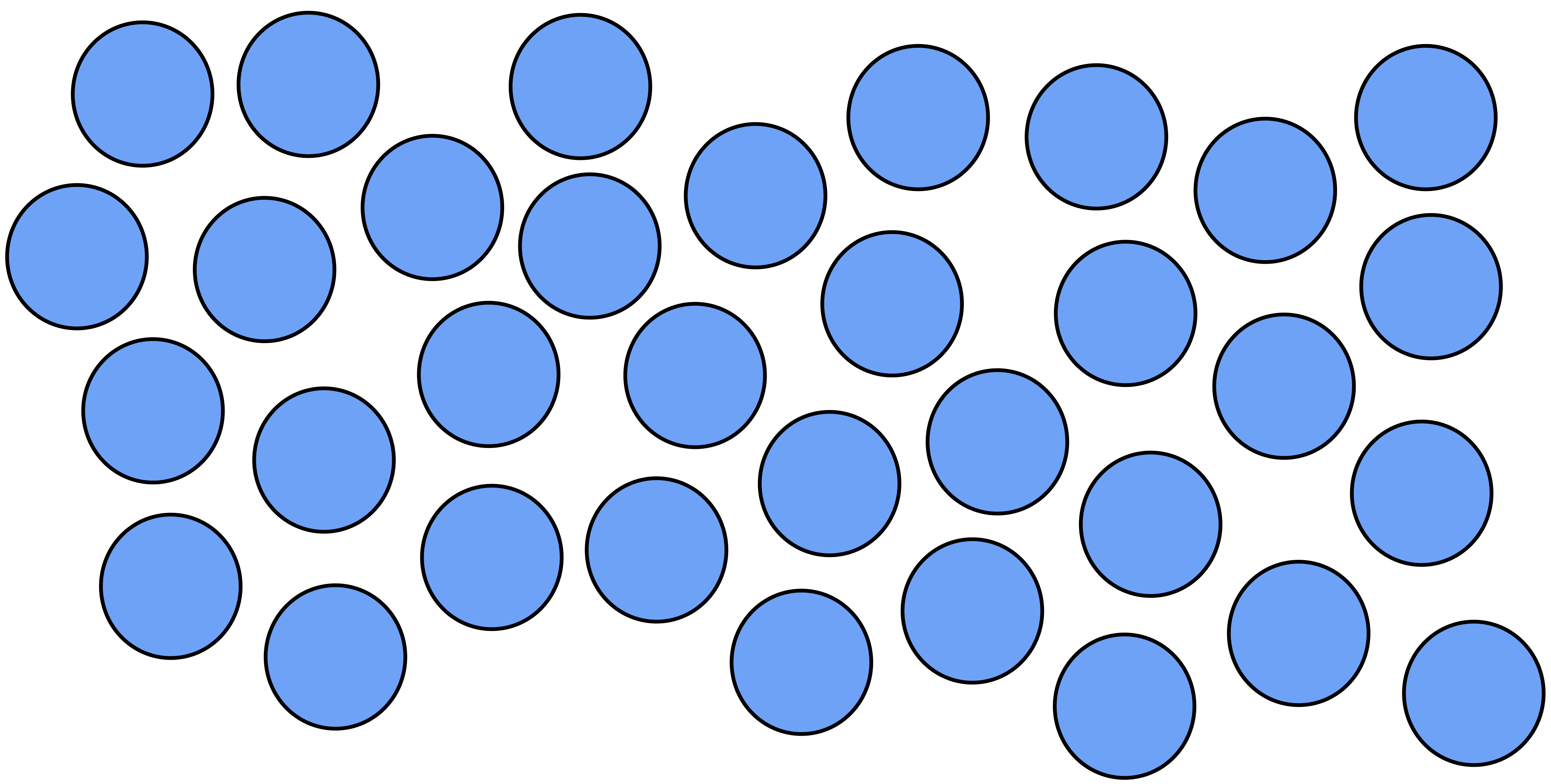
multiple importance sampling
(power heuristic)



sample lights

Ok, so importance is important.

But how do we sample our function in the first place?



Next time!