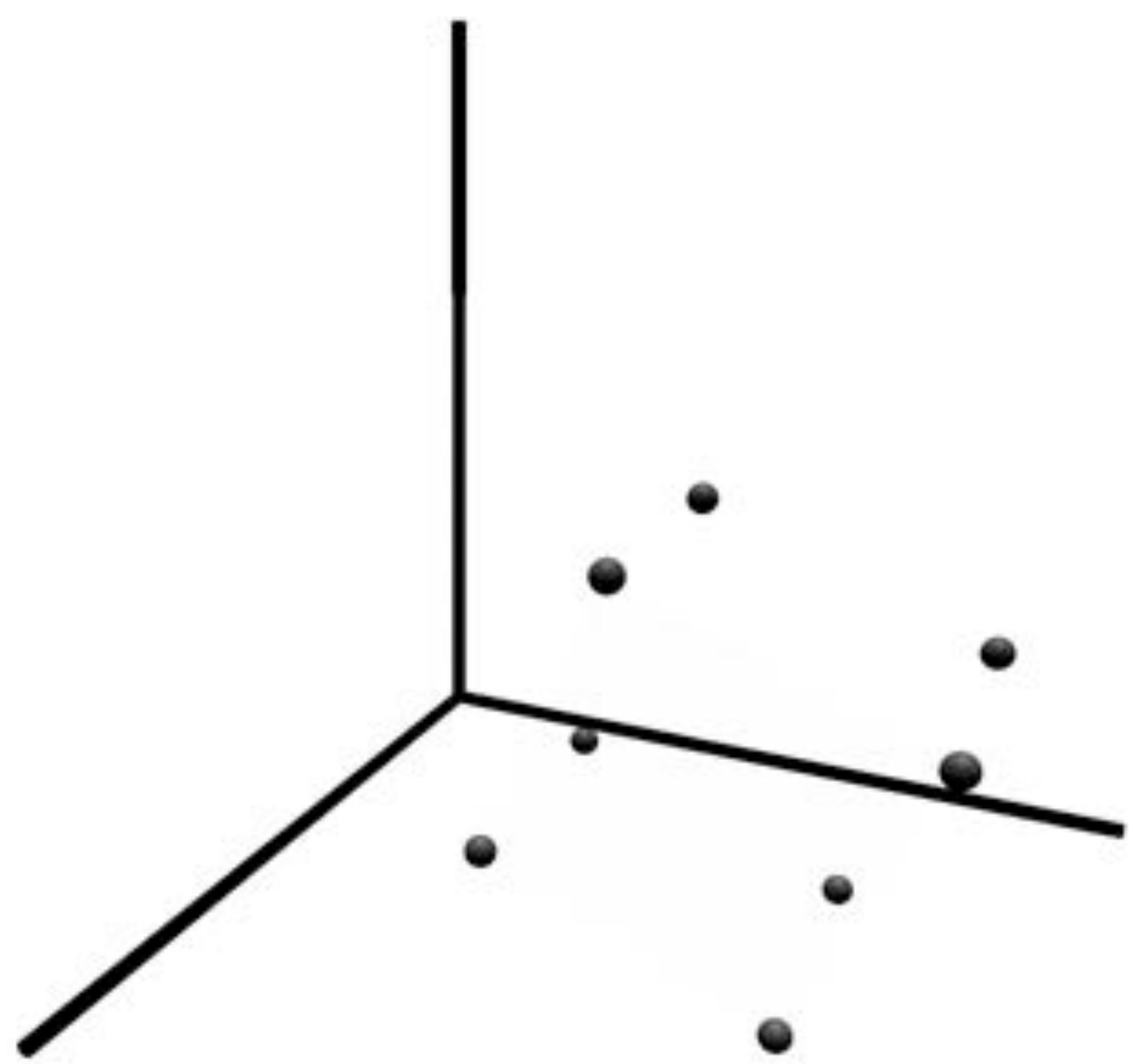
Spatial Transformations

Computer Graphics
CMU 15-462/15-662

Spatial Transformation

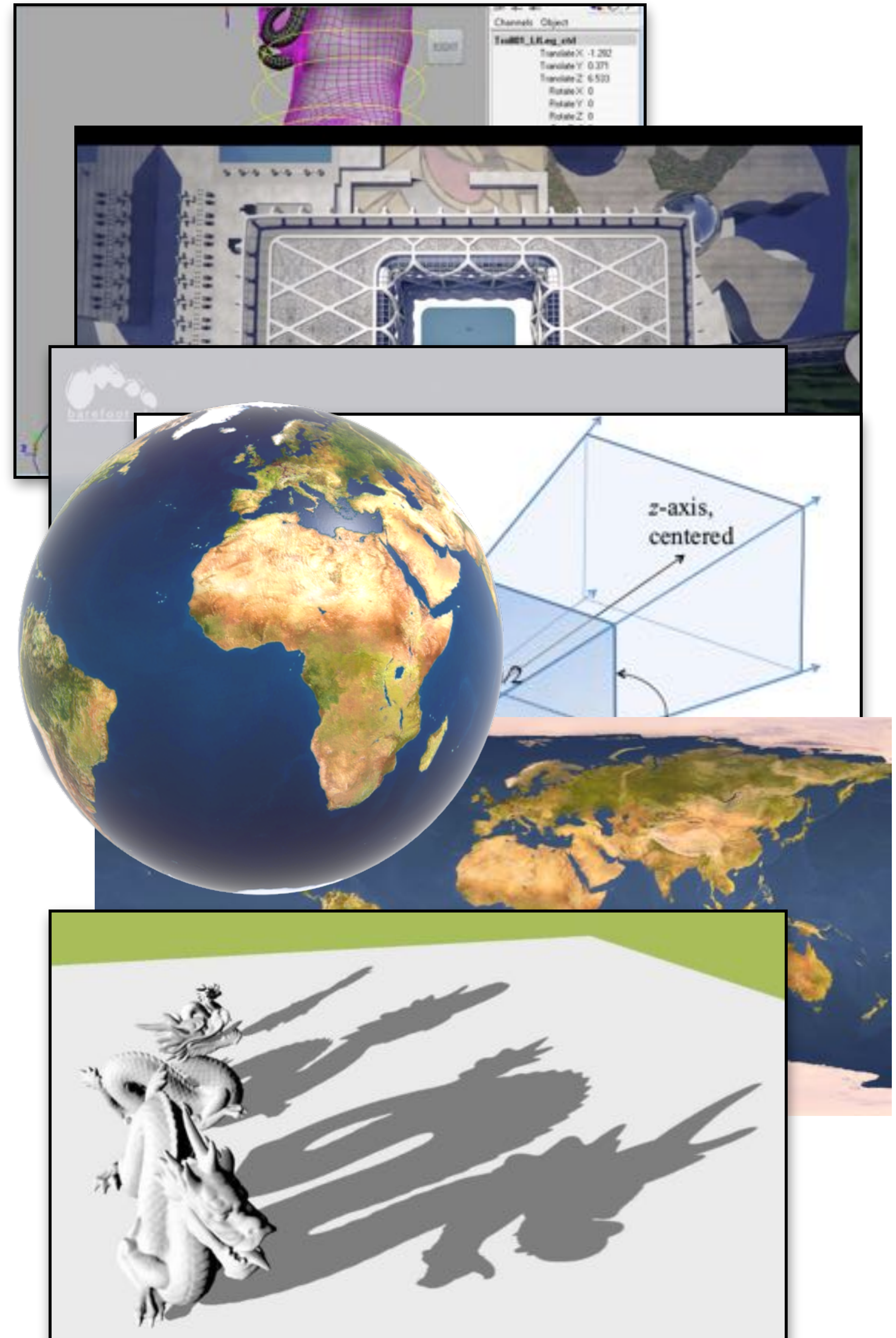
- **Basically any function that assigns each point a new location**
- **Today we'll focus on common transformations of space (rotation, scaling, etc.) encoded by <u>linear</u> maps**



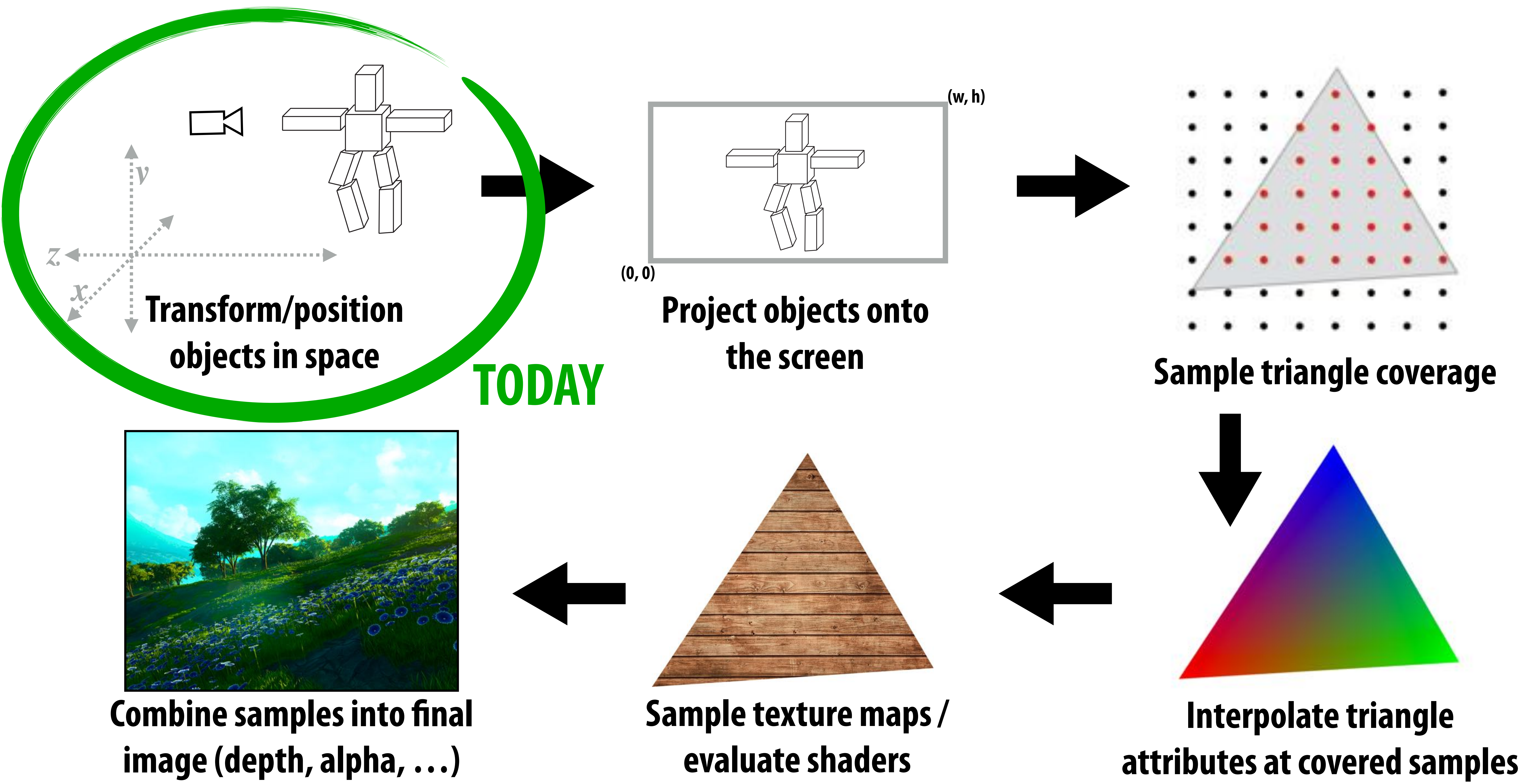
$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Transformations in Computer Graphics

- Where are linear transformations used in computer graphics?
- All over the place!
  - Position/deform objects in space
  - Move the camera
  - Animate objects over time
  - Project 3D objects onto 2D images
  - Map 2D textures onto 3D objects
  - Project shadows of 3D objects onto other 3D objects
  - …



The Rasterization Pipeline



Review: Linear Maps

Q: What does it mean for a map $f: \mathbb{R}^n \to \mathbb{R}^n$ to be <u>linear</u>?

Geometrically: it maps <u>lines</u> to <u>lines</u>, and preserves the origin

Algebraically: preserves vector space operations (addition & scaling)

$$\begin{tikzcd} \mathbf{x},\mathbf{y} \arrow[r, "\text{add first}"] \arrow[d, "\text{apply f first}"] & \mathbf{x}+\mathbf{y} \arrow[d, "\text{then apply f}"] \\ f(\mathbf{x}), f(\mathbf{y}) \arrow[r, "\text{then add}"] & \begin{array}{c} f(\mathbf{x})+f(\mathbf{y}) \\ = \\ f(\mathbf{x}+\mathbf{y}) \end{array} \end{tikzcd}$$

Why do we care about linear transformations?

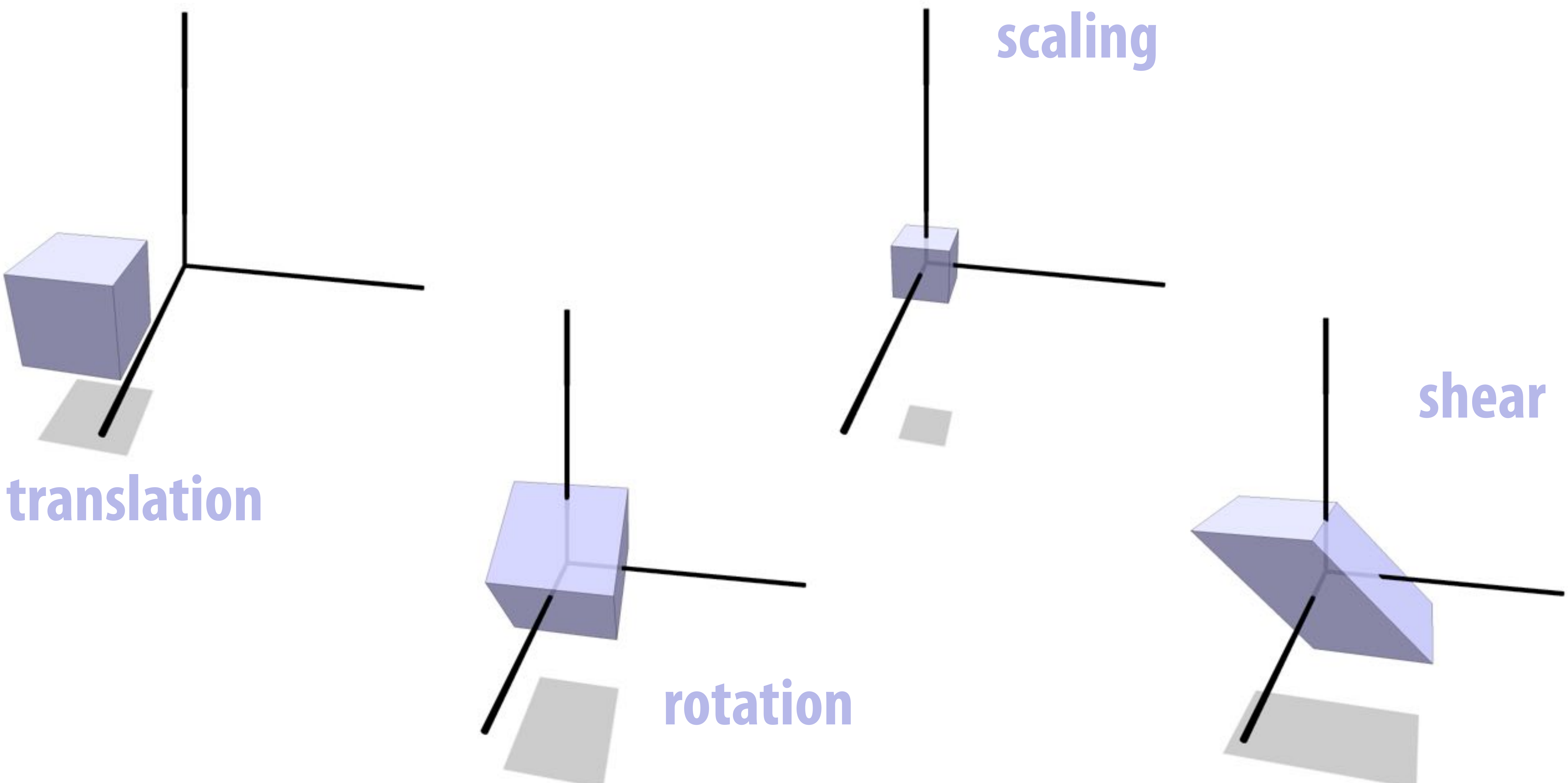
- Cheap to apply
- Usually pretty easy to solve for (linear systems)
- Composition of linear transformations is linear
  - product of <u>many</u> matrices is a <u>single</u> matrix
  - gives uniform representation of transformations
  - simplifies graphics algorithms, systems (e.g., GPUs & APIs)

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\textit{rotation}} \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\textit{scale}} \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{\textit{rotation}} \cdots = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\textit{composite transformation}}$$

What kinds of linear transformations can we compose?

Types of Transformations

What would you call each of these types of transformations?



Q: How did you know that? (Hint: you did <u>not</u> inspect a formula!)

Invariants of Transformation

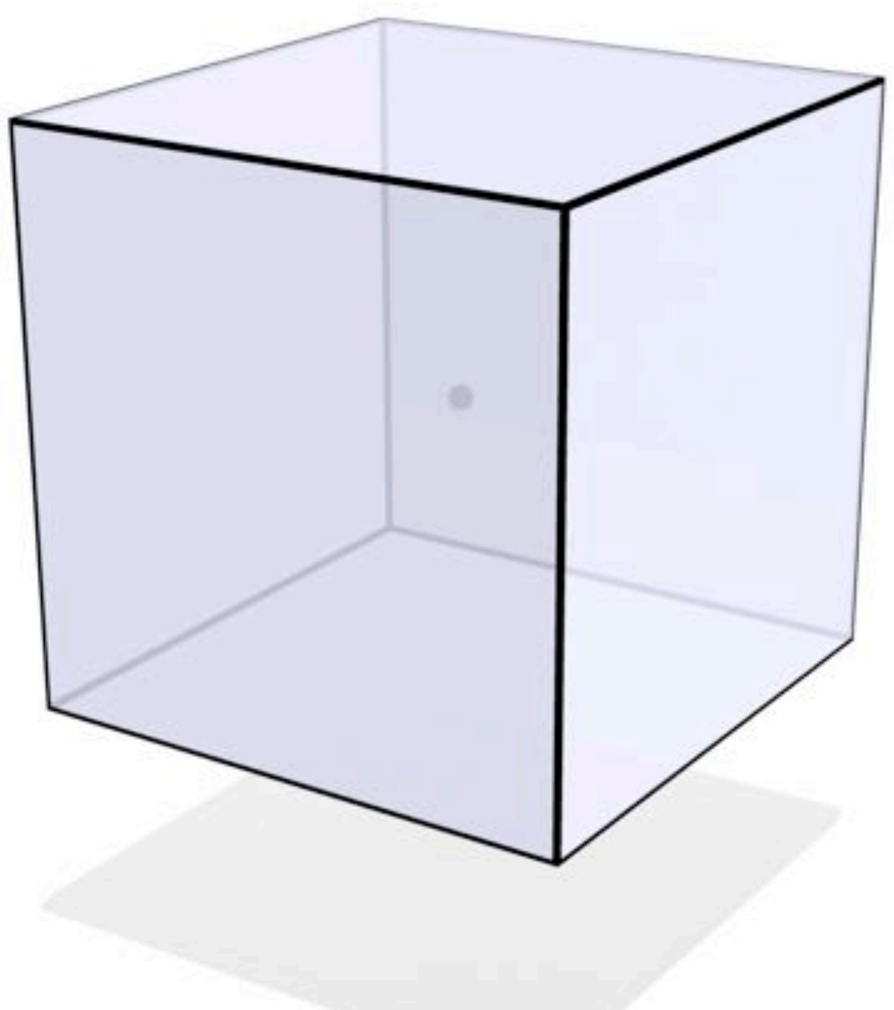
A transformation is determined by the <u>invariants</u> it preserves

transformation	**invariants**	**algebraic description**
linear	*straight lines / origin*	$f(a\mathbf{x}+\mathbf{y}) = af(\mathbf{x}) + f(\mathbf{y})$, $f(0) = 0$
translation	*differences between pairs of points*	$f(\mathbf{x}-\mathbf{y}) = \mathbf{x}-\mathbf{y}$
scaling	*lines through the origin / direction of vectors*	$f(\mathbf{x})/\lvert f(\mathbf{x})\rvert = \mathbf{x}/\lvert\mathbf{x}\rvert$
rotation	*origin / distances between points / orientation*	$\lvert f(\mathbf{x})-f(\mathbf{y})\rvert = \lvert\mathbf{x}-\mathbf{y}\rvert$, $\det(f) > 0$
…	…	…

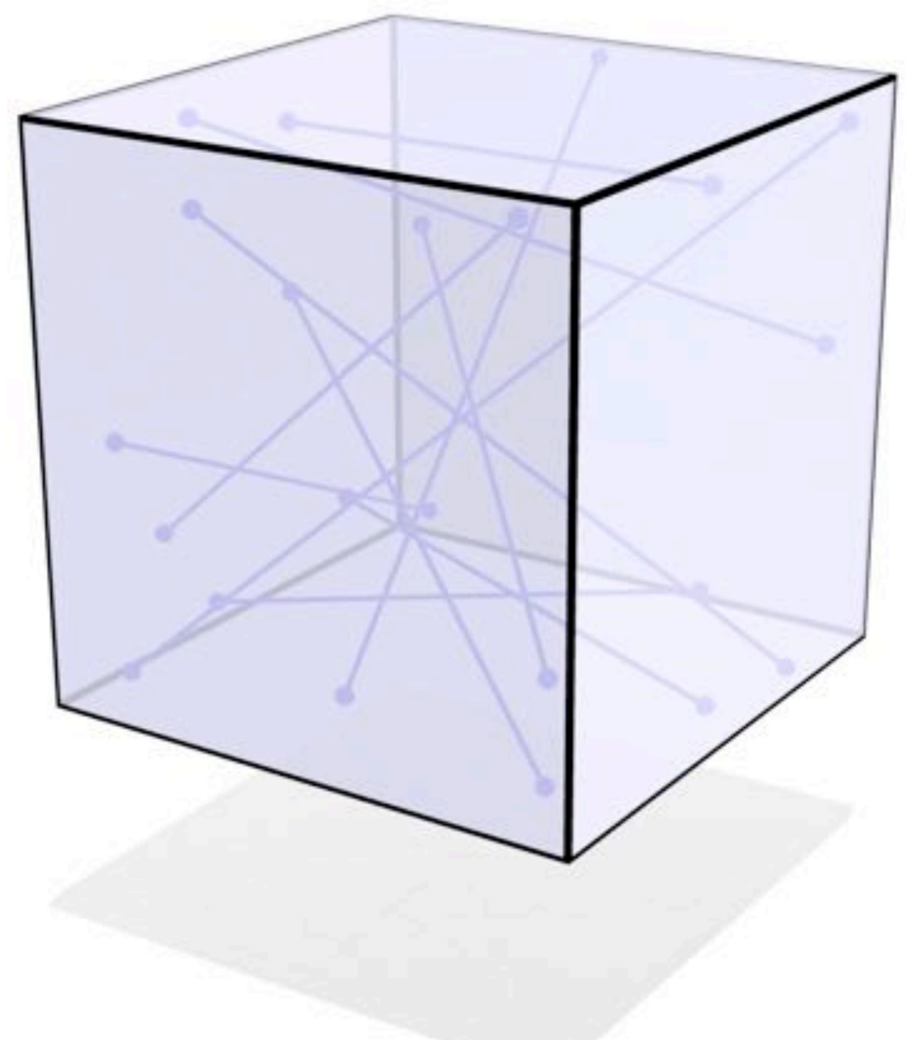
(Essentially how your brain "knows" what kind of transformation you're looking at…)

Rotation

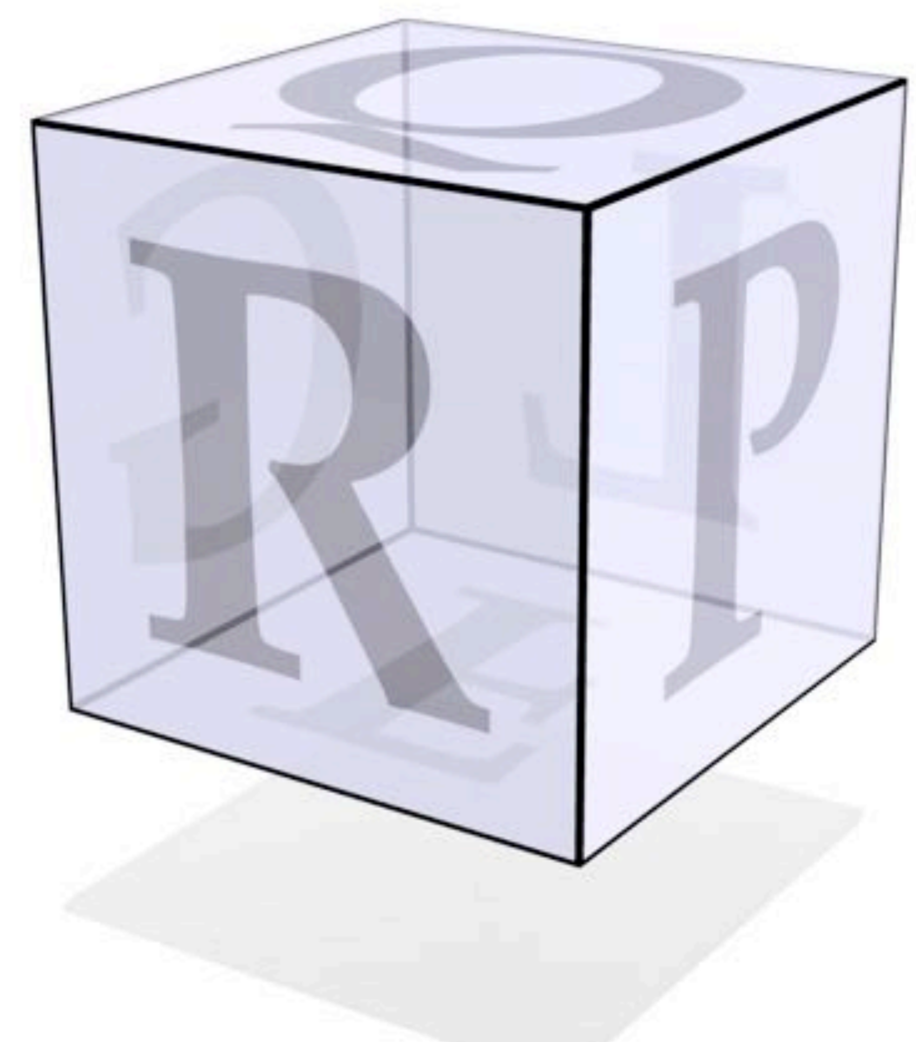
Rotations defined by three basic properties:



keeps origin fixed



preserves distances



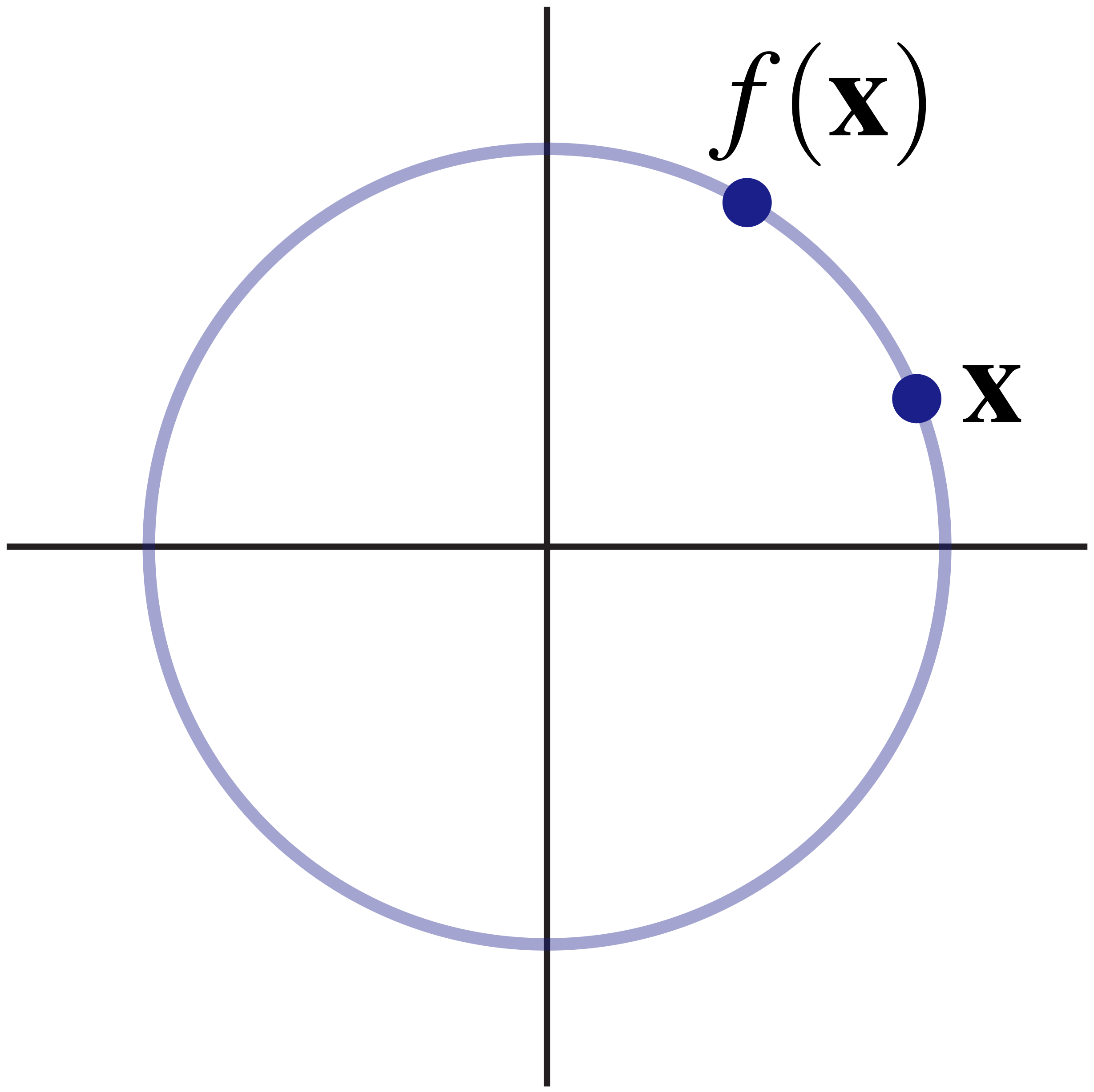
preserves orientation

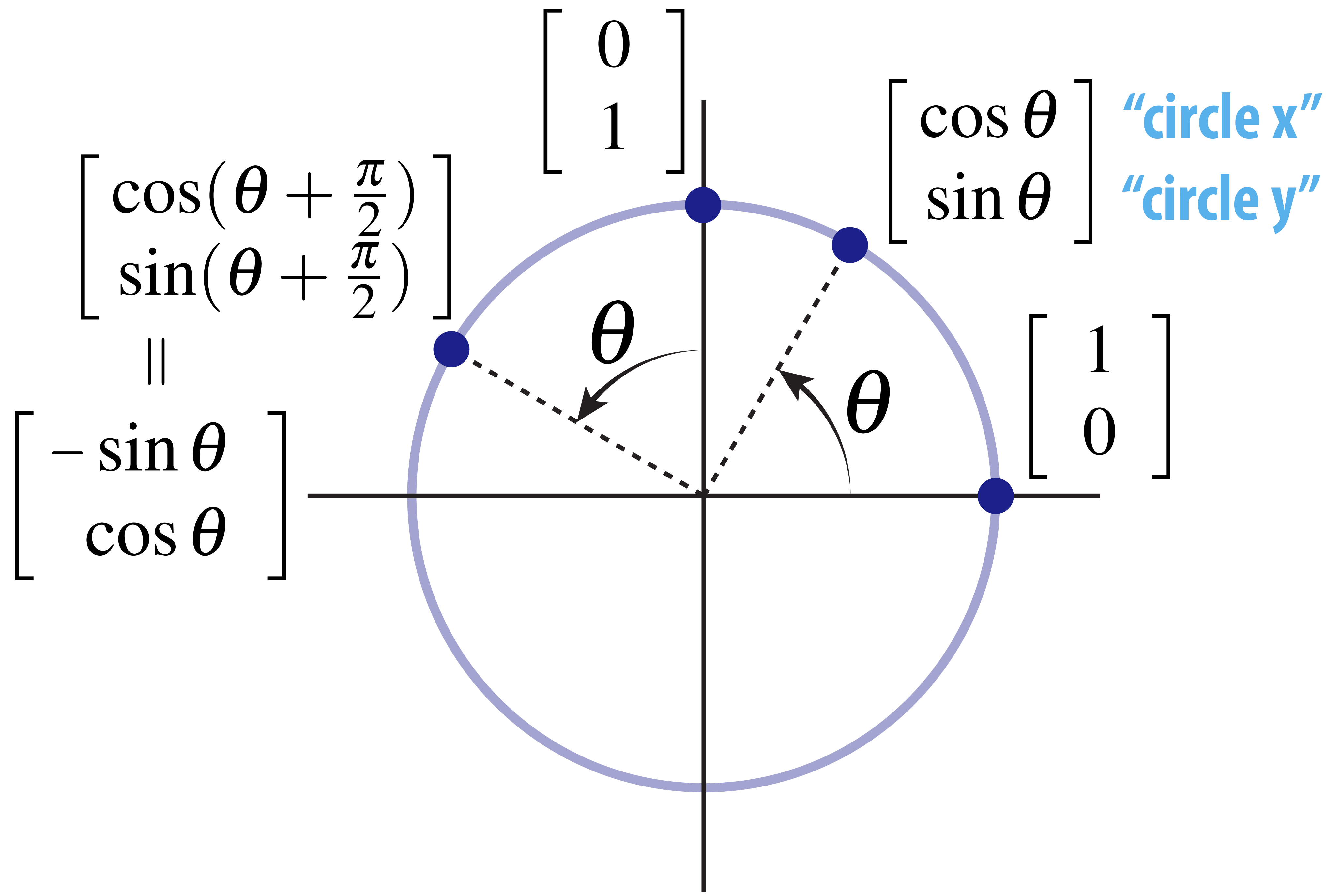
First two properties together imply that rotations are <u>linear</u>.

Will have a lot more to say about rotations next lecture…

2D Rotations—Matrix Representation

Rotations preserve distances and the origin—hence, a 2D rotation by an angle θ maps each point $\mathbf{x}$ to a point $f_\theta(\mathbf{x})$ on the circle of radius $|\mathbf{x}|$:

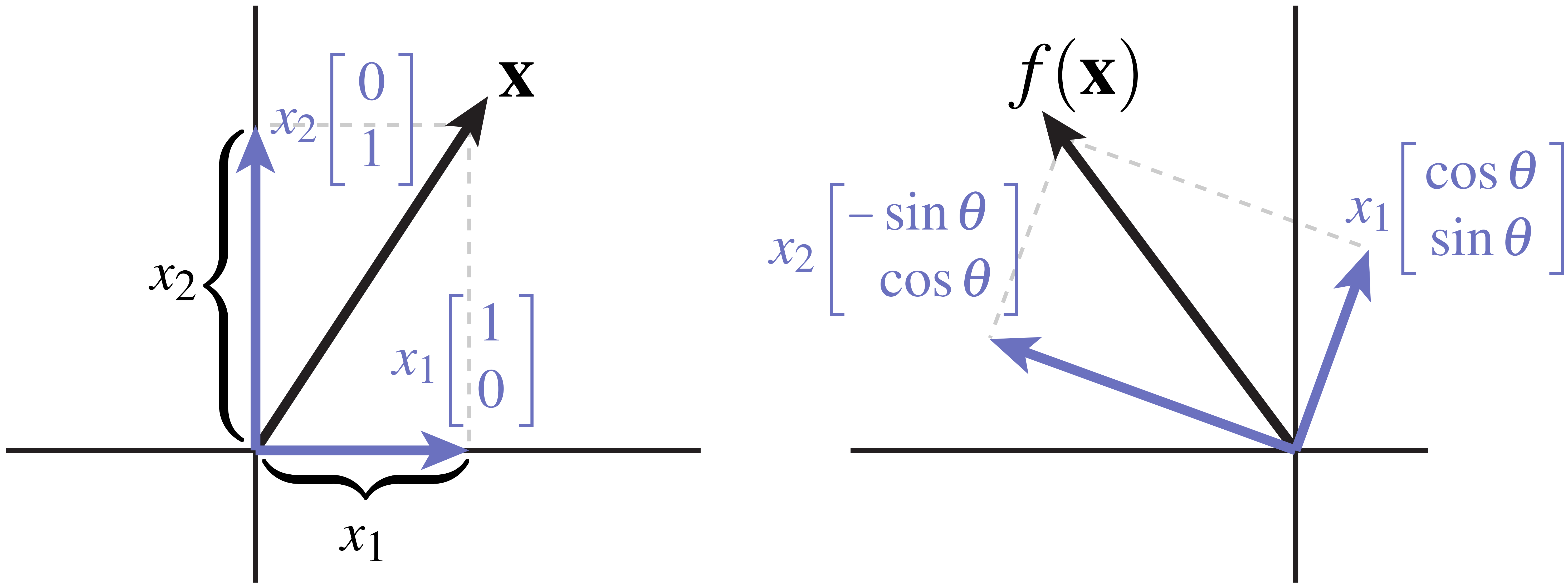




- **Where does $\mathbf{x} = (1,0)$ go if we rotate by θ (counter-clockwise)?**
- **How about $\mathbf{x} = (0,1)$?**

What about a general vector $\mathbf{x} = (x_1, x_2)$?

2D Rotations—Matrix Representation



$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} \qquad f(\mathbf{x}) = x_1\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + x_2\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

So, How do we represent the 2D rotation function $f_\theta(\mathbf{x})$ using a matrix?

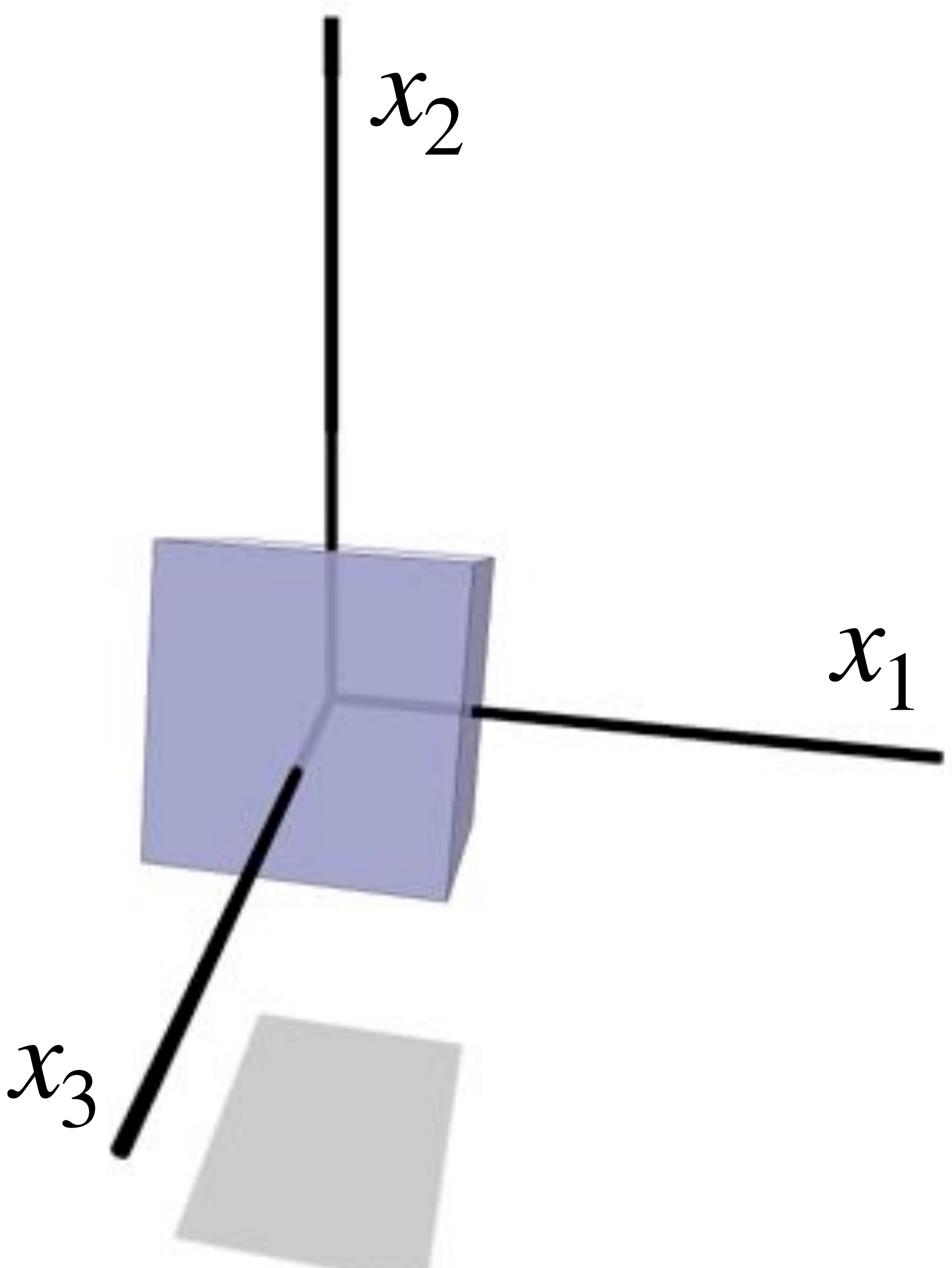
$$f_\theta(\mathbf{x}) = \begin{bmatrix}\cos\theta & -\sin(\theta)\\ \sin\theta & \cos(\theta)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

3D Rotations

- **Q: In 3D, how do we rotate around the x_3-axis?**
- **A: Just apply the same transformation of x_1, x_2; keep x_3 fixed**

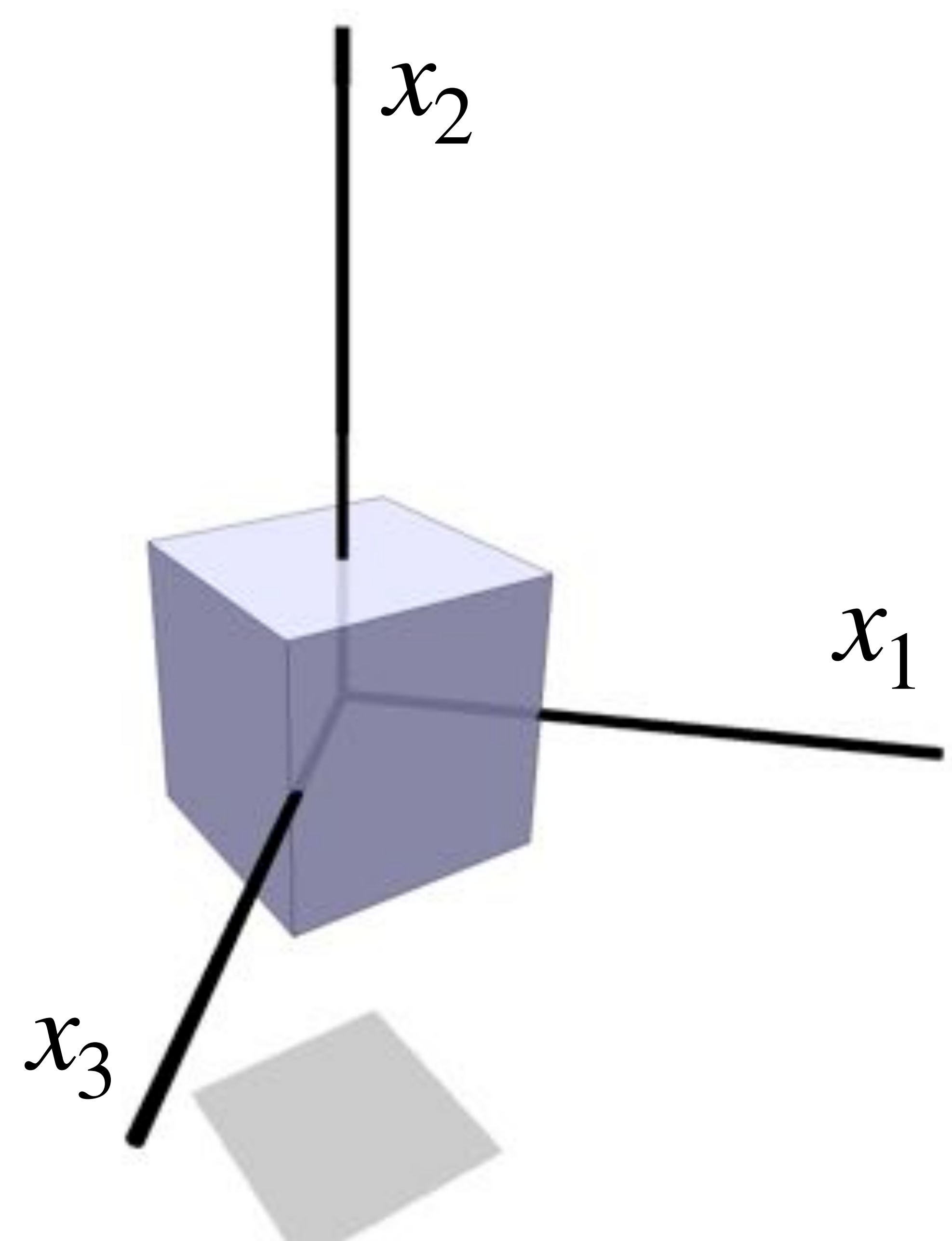
rotate around x_1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin(\theta) \\ 0 & \sin\theta & \cos(\theta) \end{bmatrix}$$



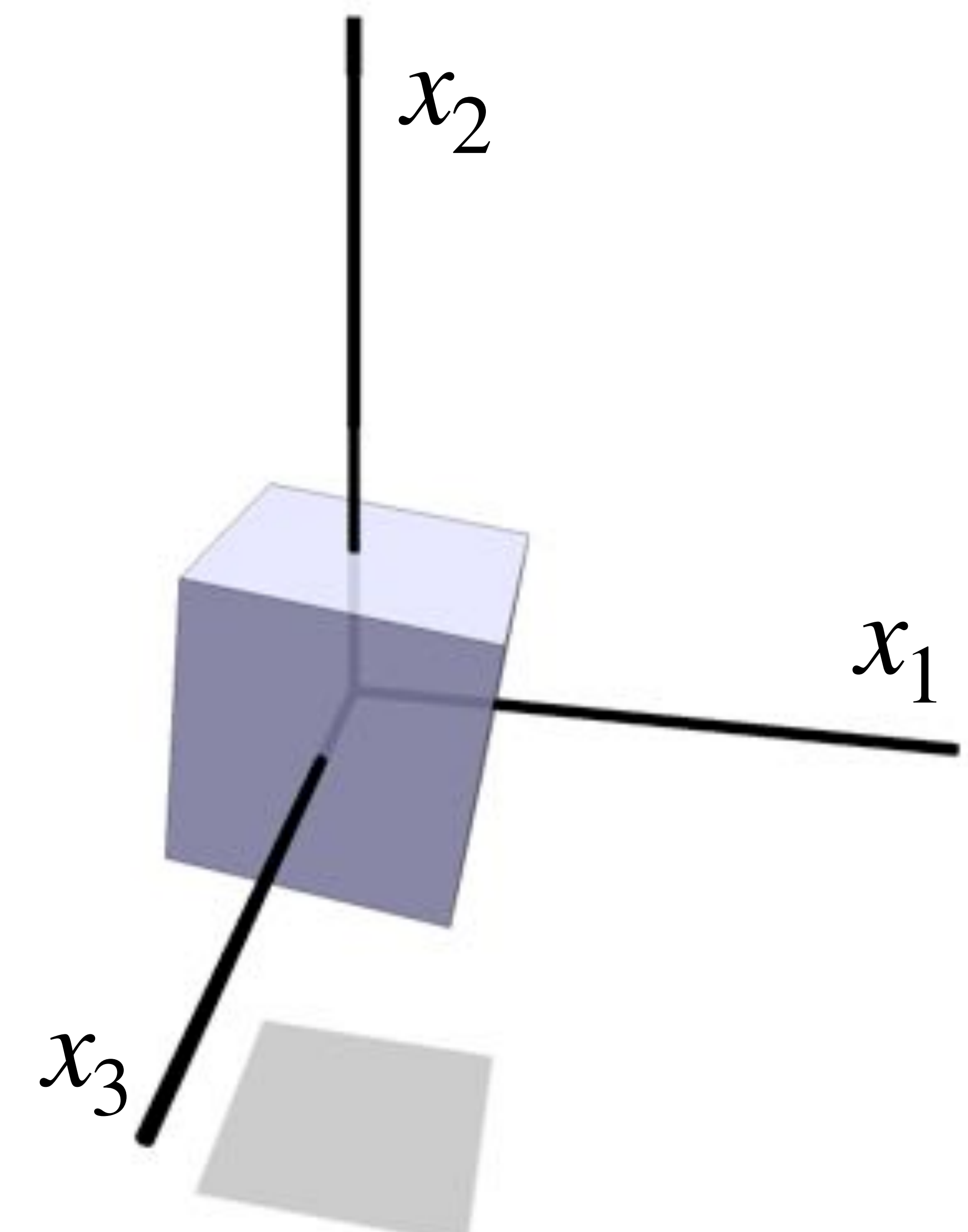
rotate around x_2

$$\begin{bmatrix} \cos\theta & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos(\theta) \end{bmatrix}$$



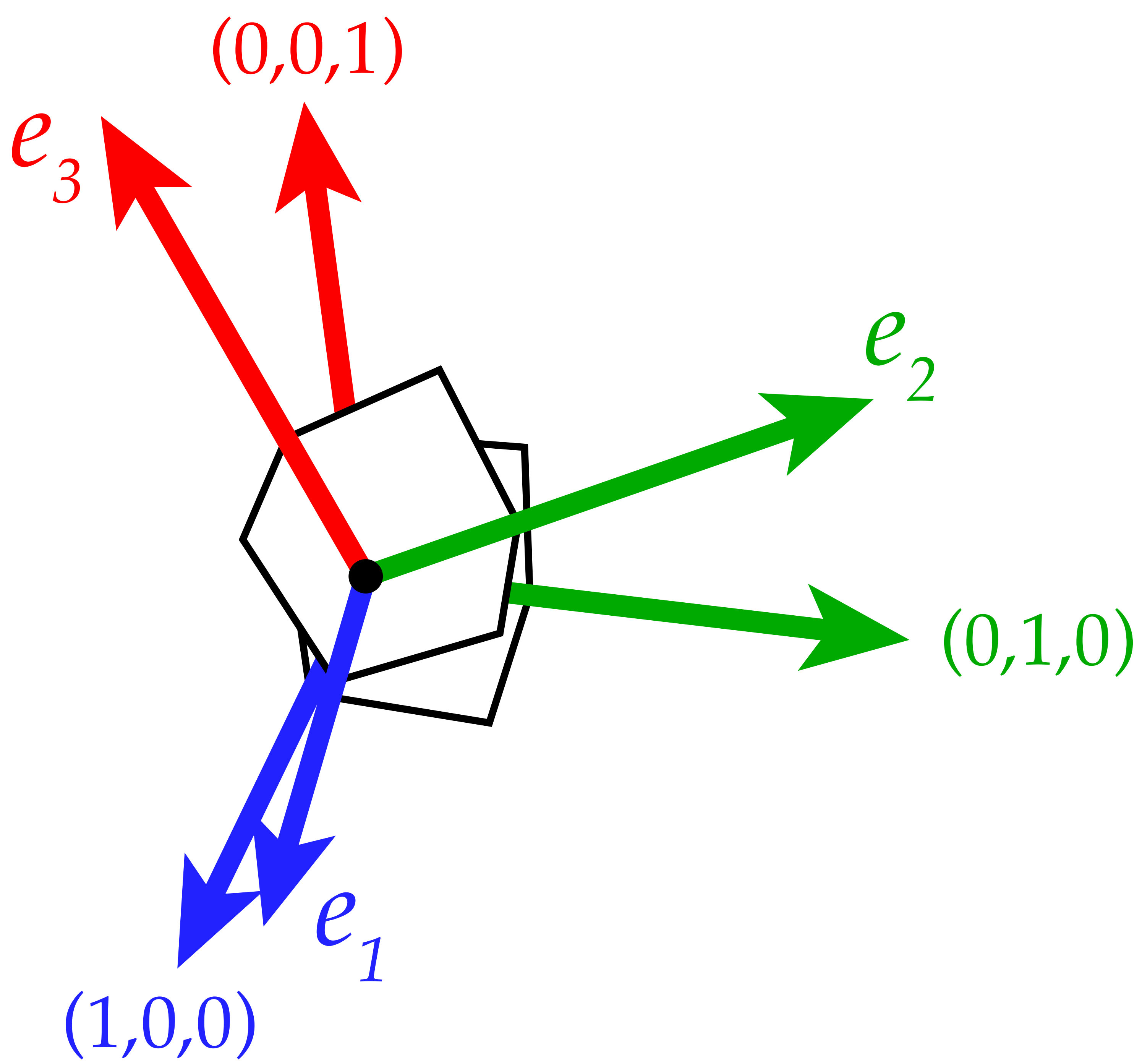
rotate around x_3

$$\begin{bmatrix} \cos\theta & -\sin(\theta) & 0 \\ \sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotations—Transpose as Inverse

Rotation will map standard basis to orthonormal basis e_1, e_2, e_3:



$$\overset{R^\top}{\begin{bmatrix} — & e_1^T & — \\ — & e_2^T & — \\ — & e_3^T & — \end{bmatrix}} \overset{R}{\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}}$$

$$= \underset{I}{\begin{bmatrix} e_1^T e_1 & e_1^T e_2 & e_1^T e_3 \\ e_2^T e_1 & e_2^T e_2 & e_2^T e_3 \\ e_3^T e_1 & e_3^T e_2 & e_3^T e_3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $R^\top R = I$, or equivalently, $R^\top = R^{-1}$.

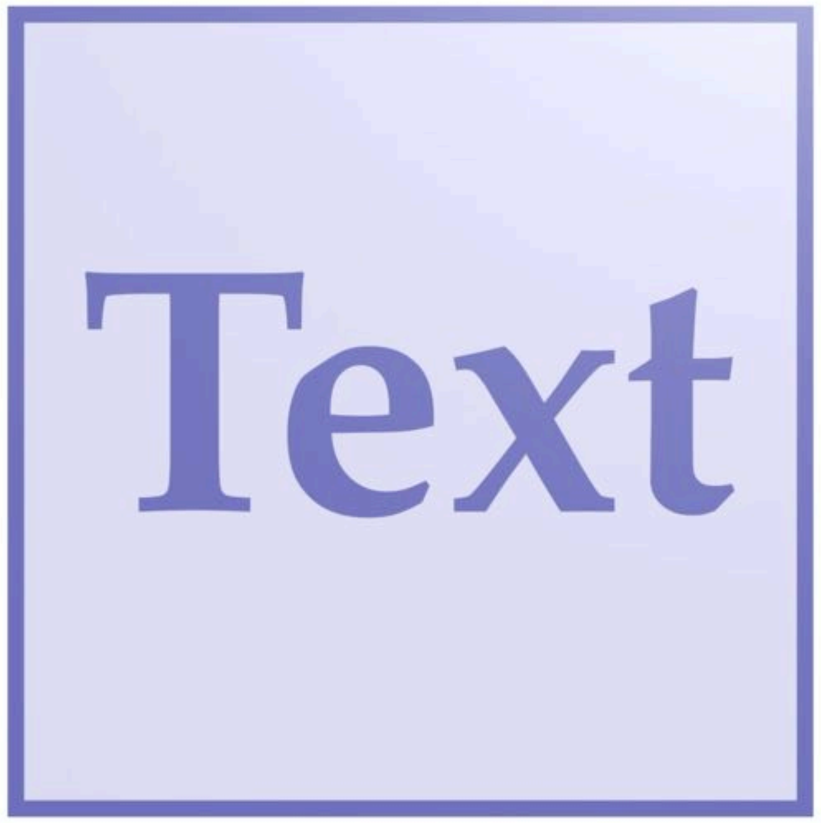
Reflections

- Q: Does every matrix $Q^{\mathsf{T}}Q = I$ describe a rotation?
- Remember that rotations must preserve the origin, preserve distances, and preserve orientation
- Consider for instance this matrix:

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q^{\mathsf{T}}Q = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Q: Does this matrix represent a rotation?
(If not, which invariant does it fail to preserve?)

A: No! It represents a reflection across the y-axis (and hence fails to preserve orientation)



Orthogonal Transformations

- In general, transformations that preserve <u>distances</u> and the <u>origin</u> are called orthogonal transformations
- Represented by matrices $Q^\top Q = I$
  - Rotations additionally <u>preserve</u> orientation: $\det(Q) > 0$
  - Reflections <u>reverse</u> orientation: $\det(Q) < 0$

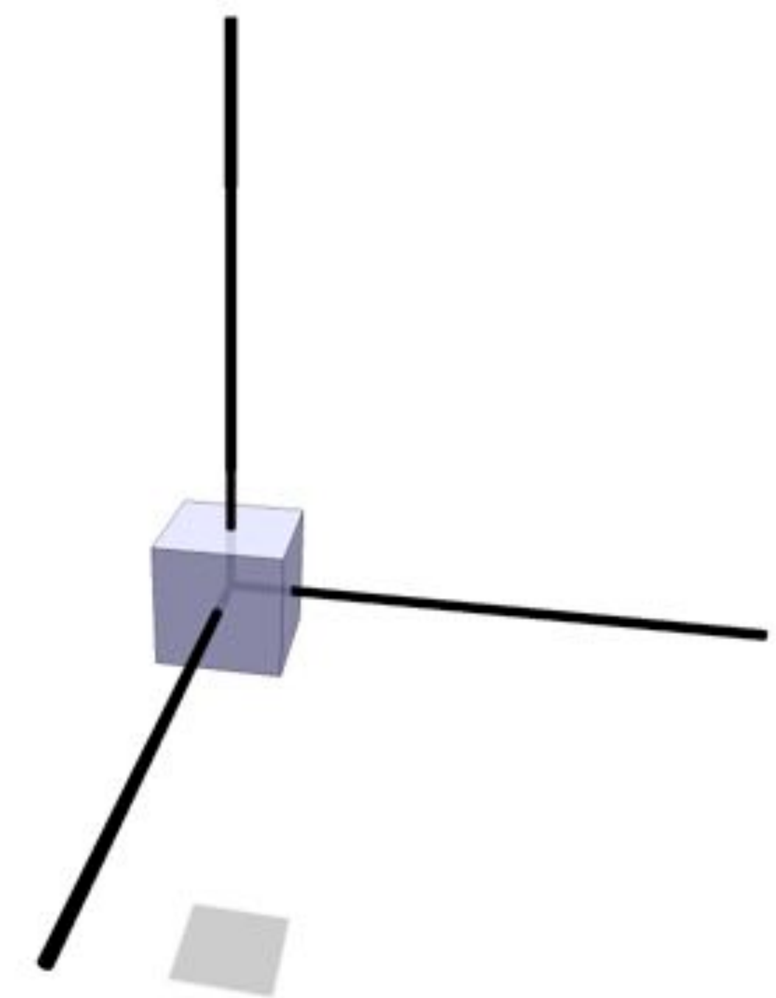
rotation

reflection

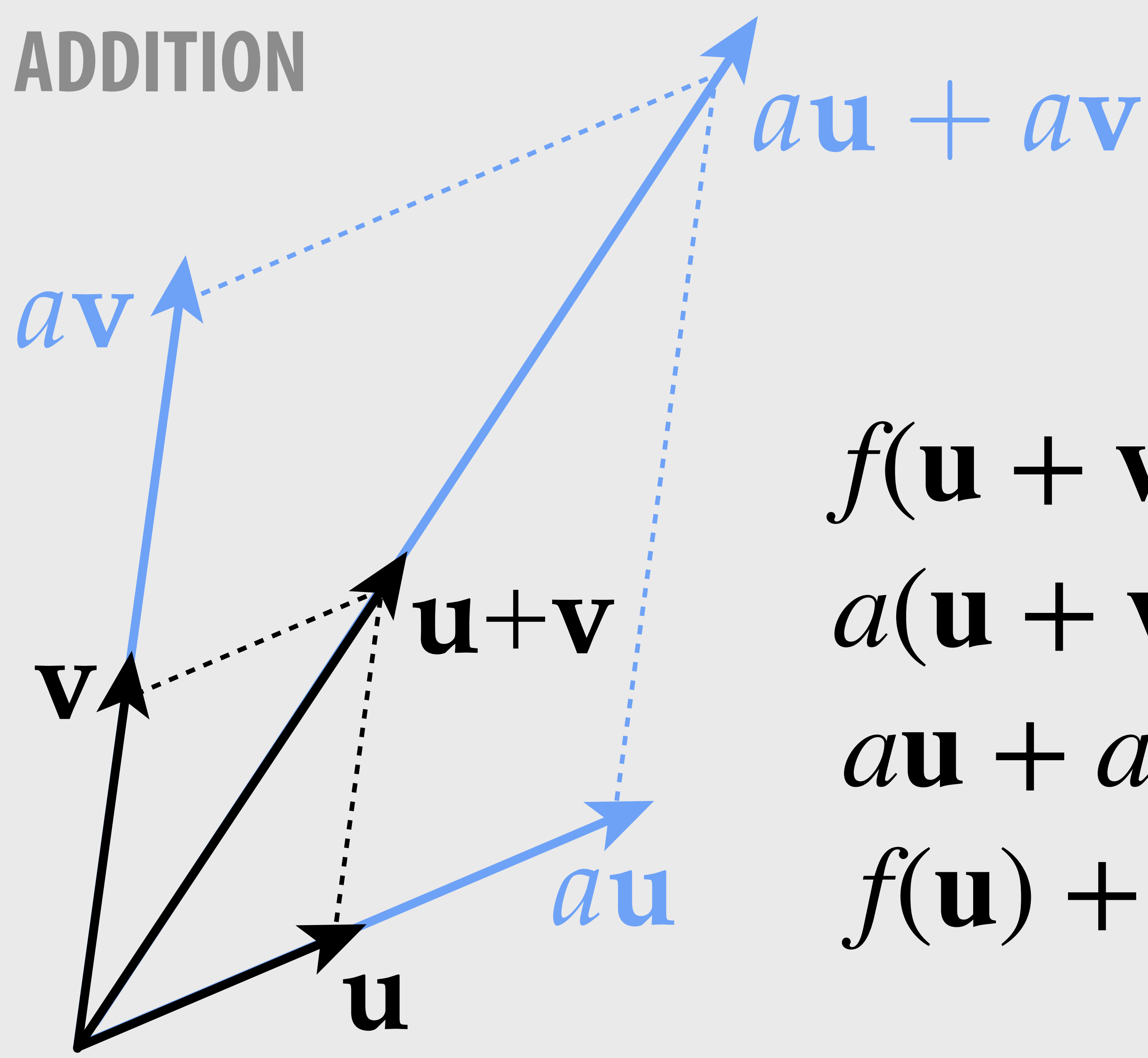
*assuming $a \neq 0,\ \mathbf{u} \neq 0$

Scaling

- Each vector $\mathbf{u}$ gets mapped to a scalar multiple
  - $f(\mathbf{u}) = a\mathbf{u}, \quad a \in \mathbb{R}$
- Preserves the <u>direction</u> of all vectors*
  - $$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{a\mathbf{u}}{|a\mathbf{u}|}$$
- Q: Is scaling a linear transformation? A: Yes!



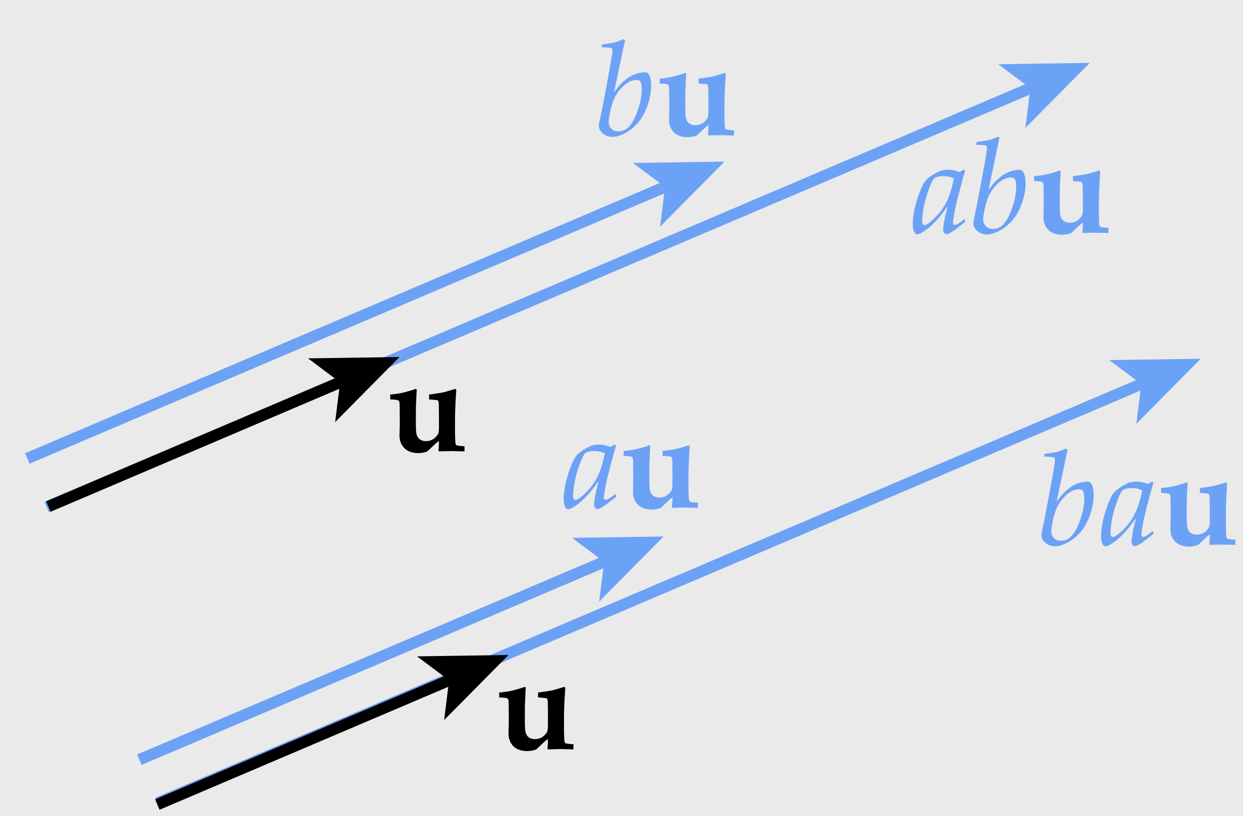
ADDITION



$$f(\mathbf{u} + \mathbf{v}) =$$
$$a(\mathbf{u} + \mathbf{v}) =$$
$$a\mathbf{u} + a\mathbf{v} =$$
$$f(\mathbf{u}) + f(\mathbf{v})$$

SCALAR MULTIPLICATION

$$f(b\mathbf{u}) = ab\mathbf{u} = ba\mathbf{u} = bf(\mathbf{u})$$



Scaling — Matrix Representation

Q: Suppose we want to scale a vector $\mathbf{u} = (u_1, u_2, u_3)$ by a .
How would we represent this operation via a <u>matrix</u>?

A: Just build a diagonal matrix D, with a along the diagonal:

$$\underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}}_{a\mathbf{u}}$$

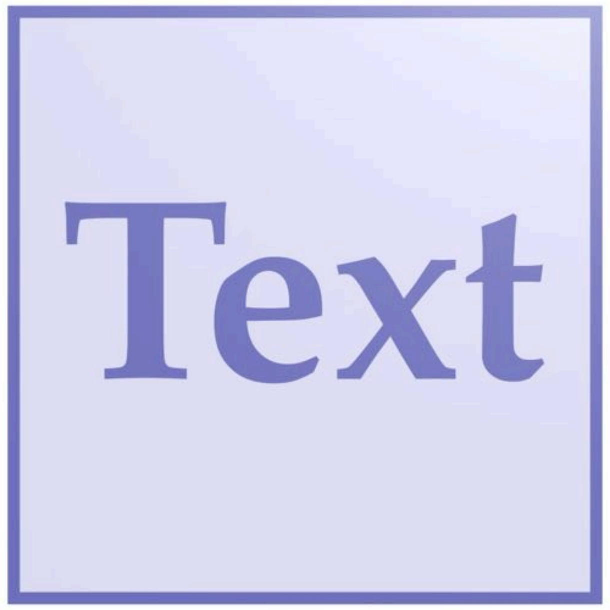
Q: What happens if a is <u>negative</u>?

Negative Scaling

For $a = -1$, can think of scaling by a as sequence of reflections.

E.g., in 2D:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

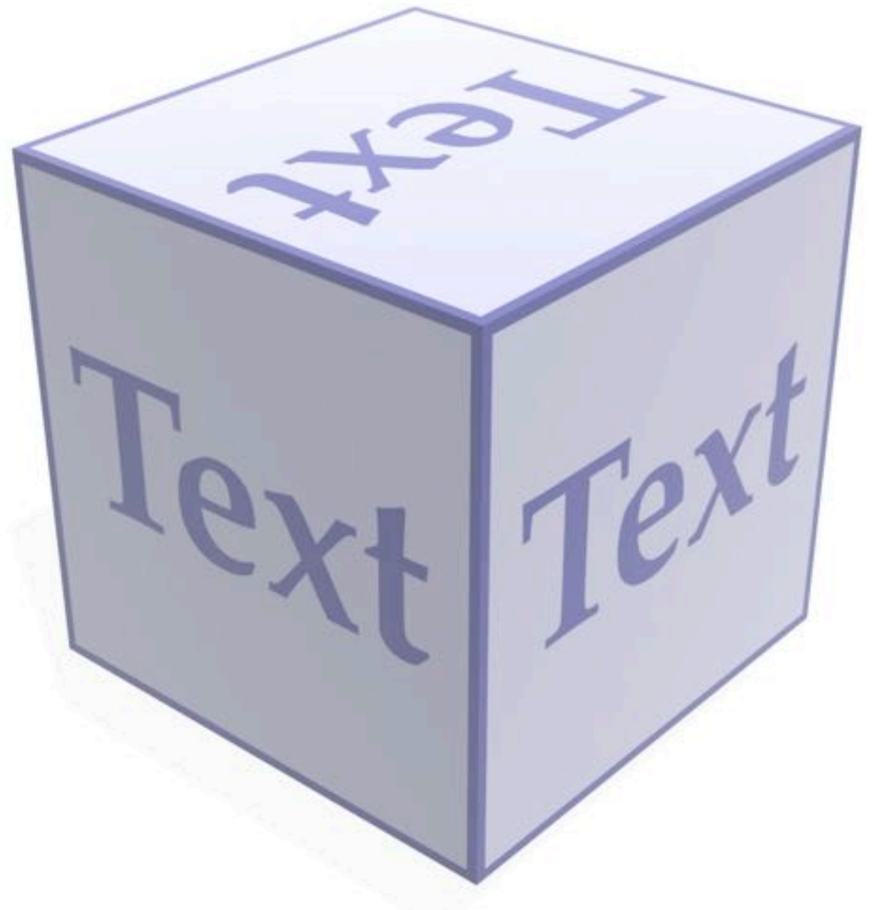


Since each reflection reverses orientation, orientation is preserved.

What about 3D?

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

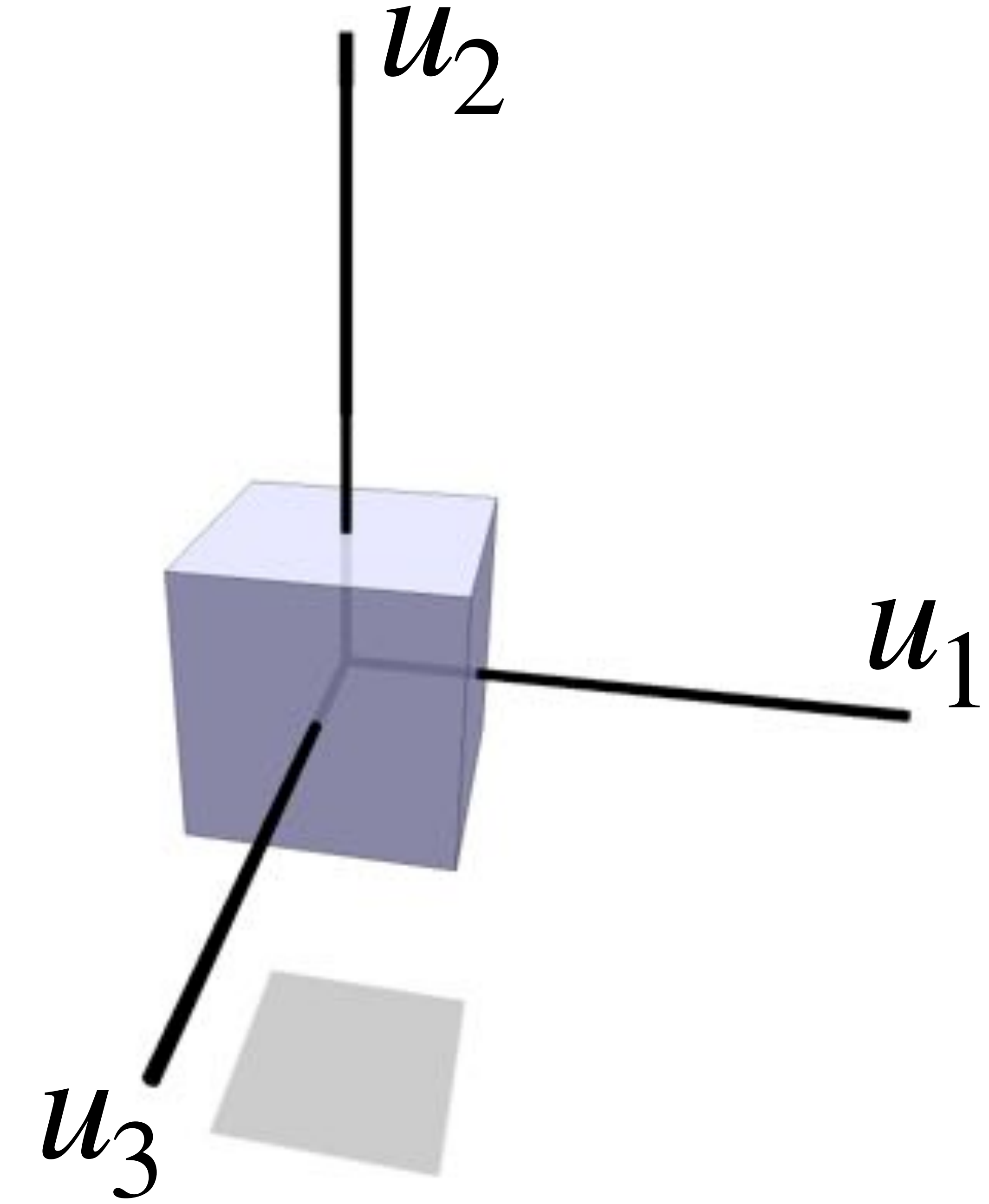


Now we have three reflections, and so orientation is reversed!

Nonuniform Scaling (Axis-Aligned)

- **We can also scale each axis by a different amount**
  - $f(u_1, u_2, u_3) = (au_1, bu_2, cu_3), \quad a, b, c \in \mathbb{R}$
- **Q: What's the matrix representation?**
- **A: Just put a, b, c on the diagonal:**

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ bu_2 \\ cu_3 \end{bmatrix}$$



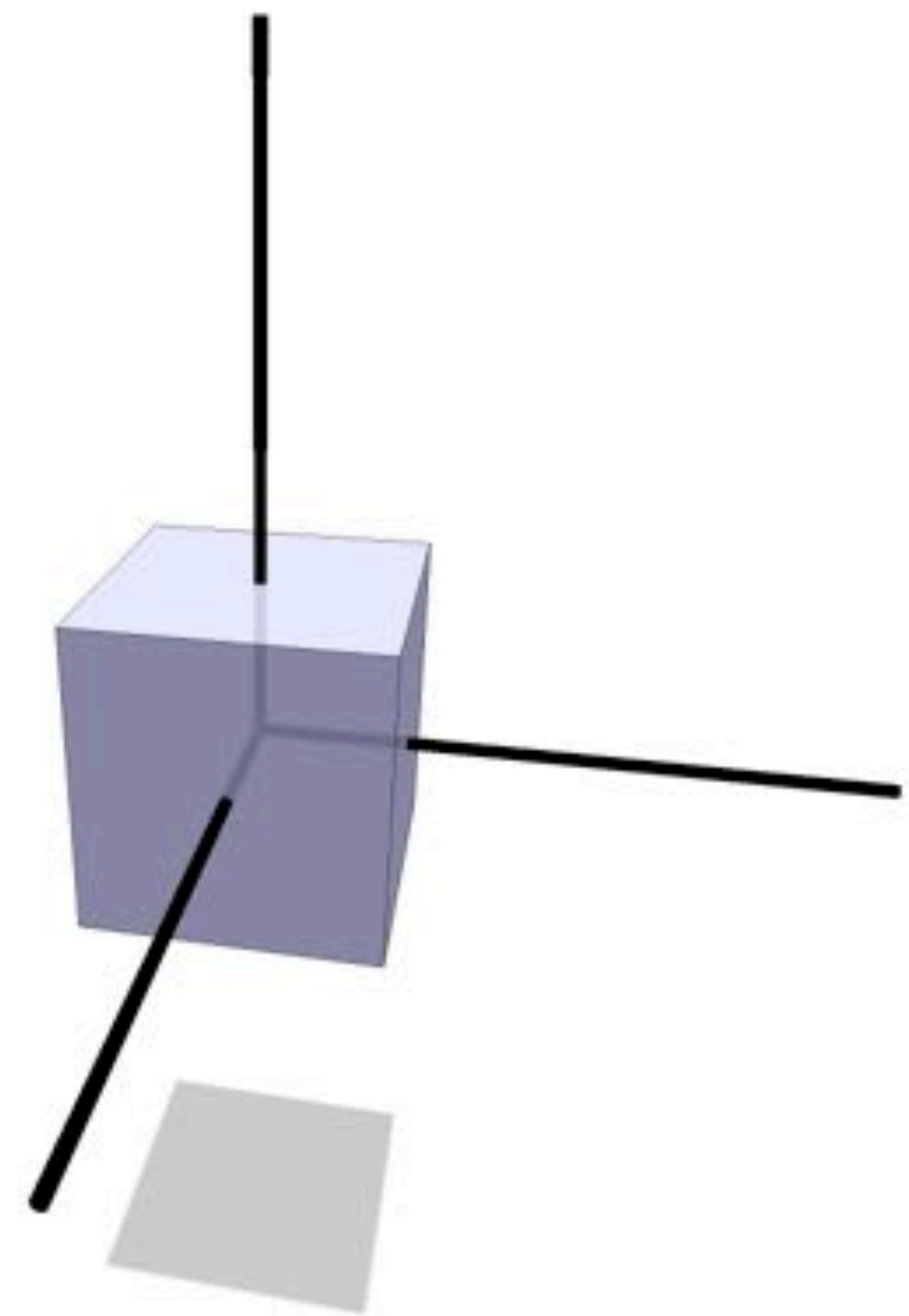
Ok, but what if we want to scale along some other axes?

Nonuniform Scaling

- **Idea.** We could:
  - rotate to the new axes (R)
  - apply a diagonal scaling (D)
  - rotate back* to the original axes ($R^\top$)

- Notice that the overall transformation is represented by a symmetric matrix

$$A := R^\top D R$$



$$f(\mathbf{x}) = R^\top D R \mathbf{x}$$

Q: Do all symmetric matrices represent nonuniform scaling (for some choice of axes)?

*Recall that for a rotation, the inverse equals the transpose: $R^{-1} = R^\top$

Spectral Theorem

- A: Yes! Spectral theorem says a symmetric matrix $A = A^\top$ has
  - orthonormal eigenvectors $e_1, \ldots, e_n \in \mathbb{R}^n$
  - real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

$$\boxed{Ae_i = \lambda_i e_i}$$

- Can also write this relationship as $AR = RD$, where

$$R = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- Equivalently, $A = RDR^\top$
- Hence, every symmetric matrix performs a non-uniform scaling along some set of orthogonal axes.
- If A is positive definite ($\lambda_i > 0$), this scaling is positive.

Shear

- A shear displaces each point $\mathbf{x}$ in a direction $\mathbf{u}$ according to its distance along a fixed vector $\mathbf{v}$:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

- Q: Is this transformation linear?
- A: Yes—for instance, can represent it via a matrix

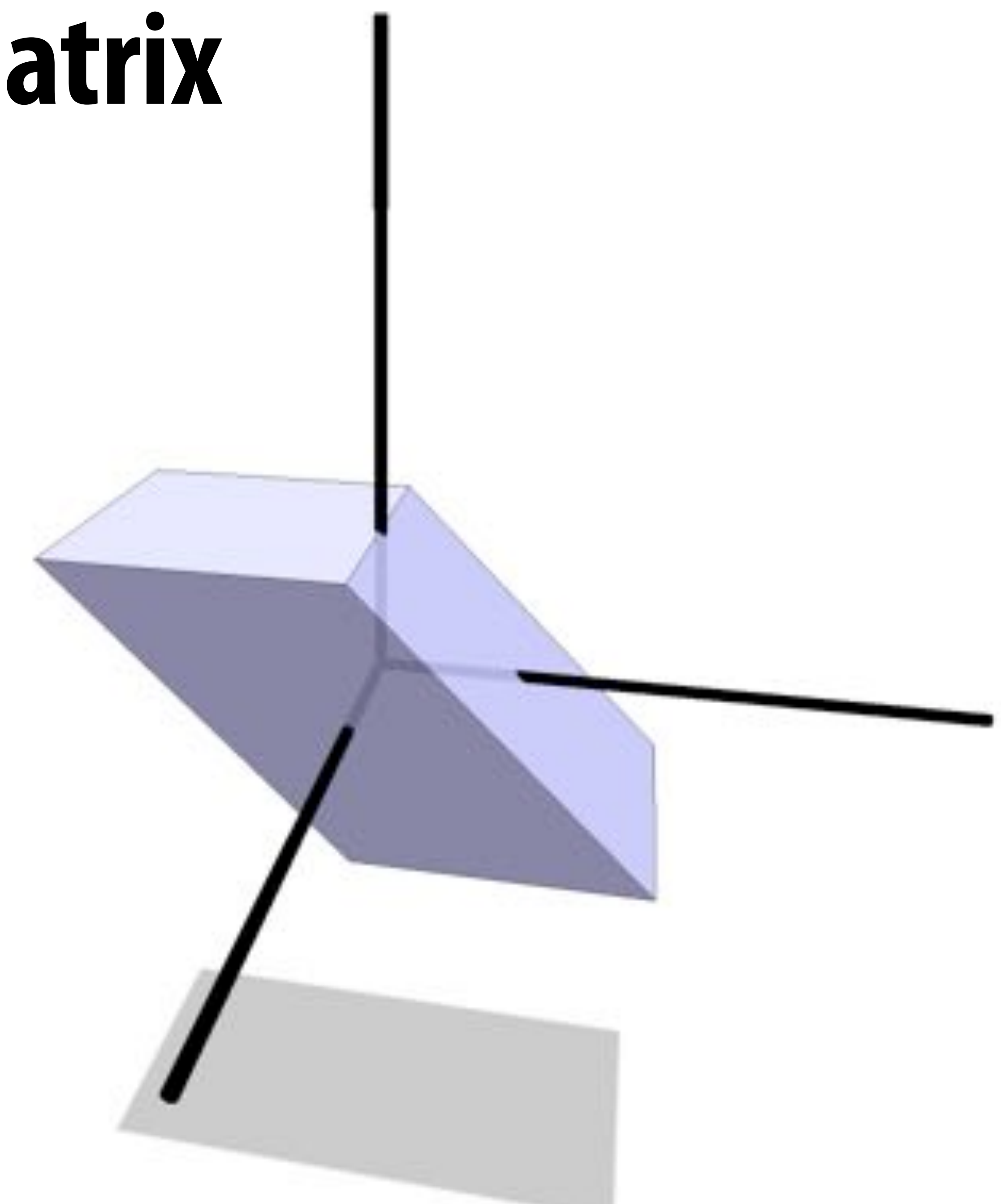
$$A_{\mathbf{u},\mathbf{v}} = I + \mathbf{u}\mathbf{v}^\mathsf{T}$$

<u>Example.</u>

$$\mathbf{u} = (\cos(t),0,0)$$

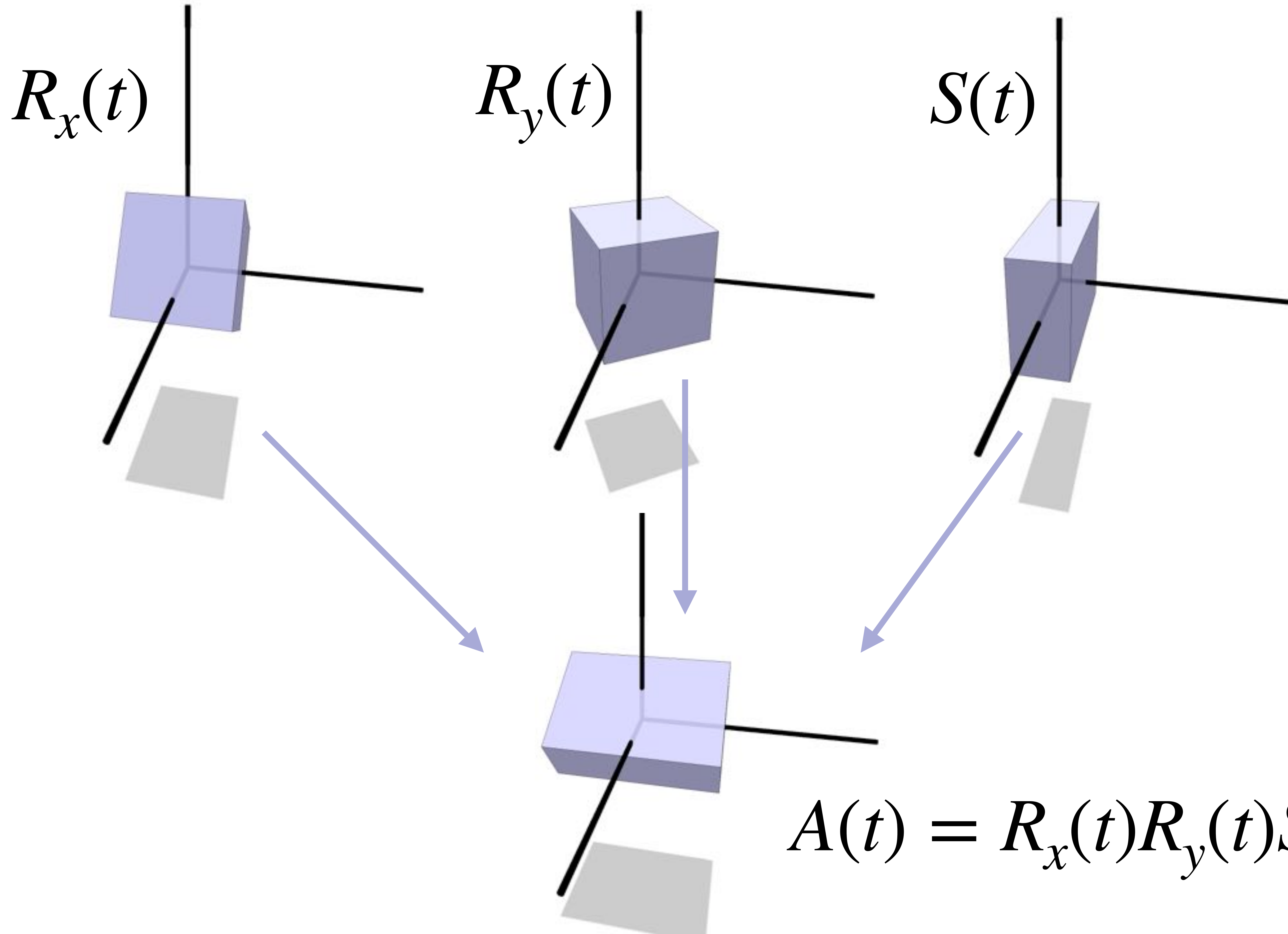
$$\mathbf{v} = (0,1,0)$$

$$A_{\mathbf{u},\mathbf{v}} = \begin{bmatrix} 1 & \cos(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composite Transformations

From these basic transformations (rotation, reflection, scaling, shear…) we can now build up composite transformations via matrix multiplication:



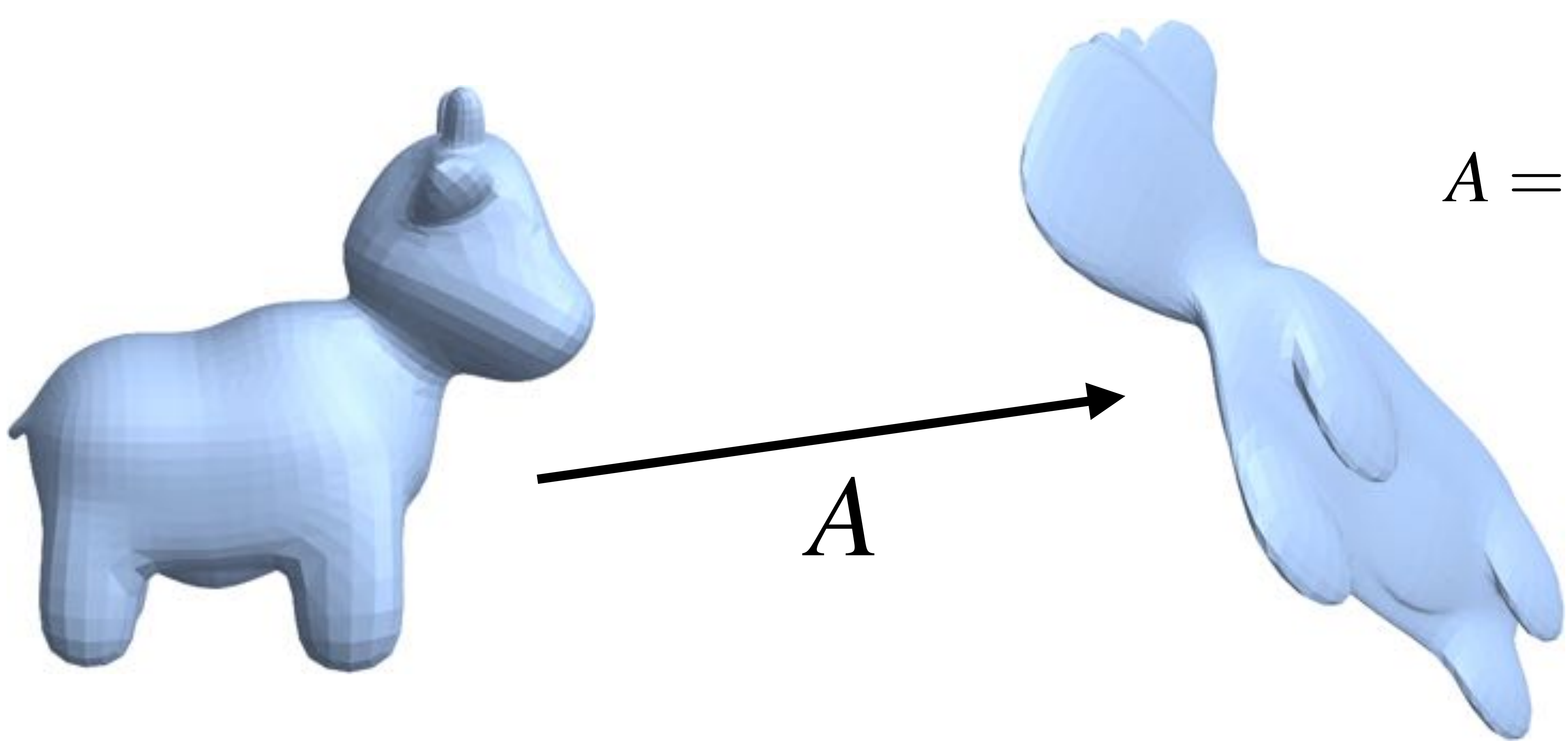
$$A(t) = R_x(t)R_y(t)S(t)$$

How do we decompose a linear transformation into pieces?

(rotations, reflections, scaling, …)

Decomposition of Linear Transformations

- **In general, no unique way to write a given linear transformation as a composition of basic transformations!**
- **However, there are many useful decompositions:**
  - singular value decomposition (good for signal processing)
  - LU factorization (good for solving linear systems)
  - polar decomposition (good for spatial transformations)
  - ...
- **Consider for instance this linear transformation:**



$$A = \begin{bmatrix} .34 & -.11 & -.89 \\ -.65 & .52 & -.70 \\ .25 & .23 & -.69 \end{bmatrix}$$

Polar & Singular Value Decomposition

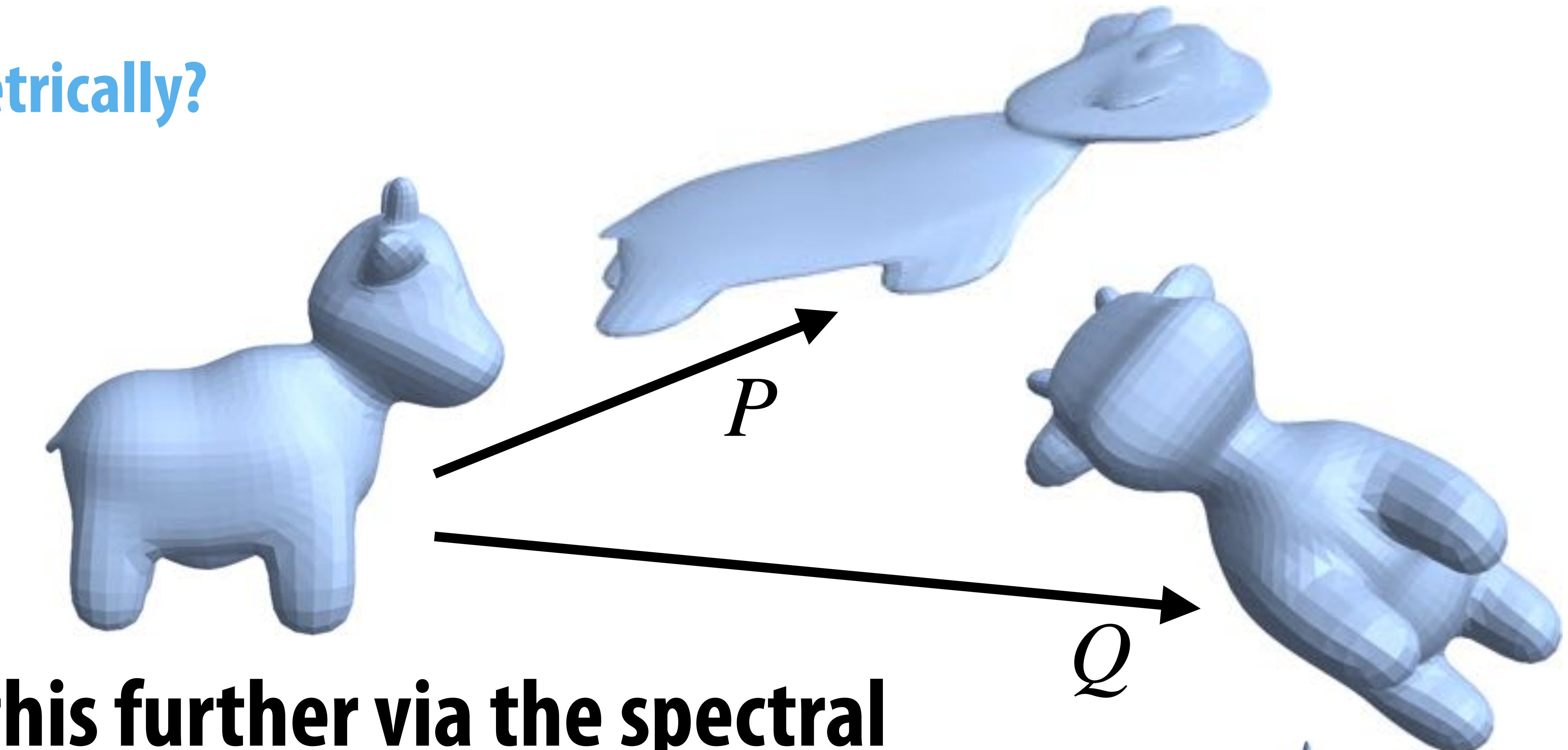
For example, polar decomposition decomposes any matrix A into orthogonal matrix Q and symmetric positive-semidefinite matrix P:

Q: What do each of the parts mean geometrically?

rotation/reflection

nonnegative, nonuniform scaling

$$A = QP$$



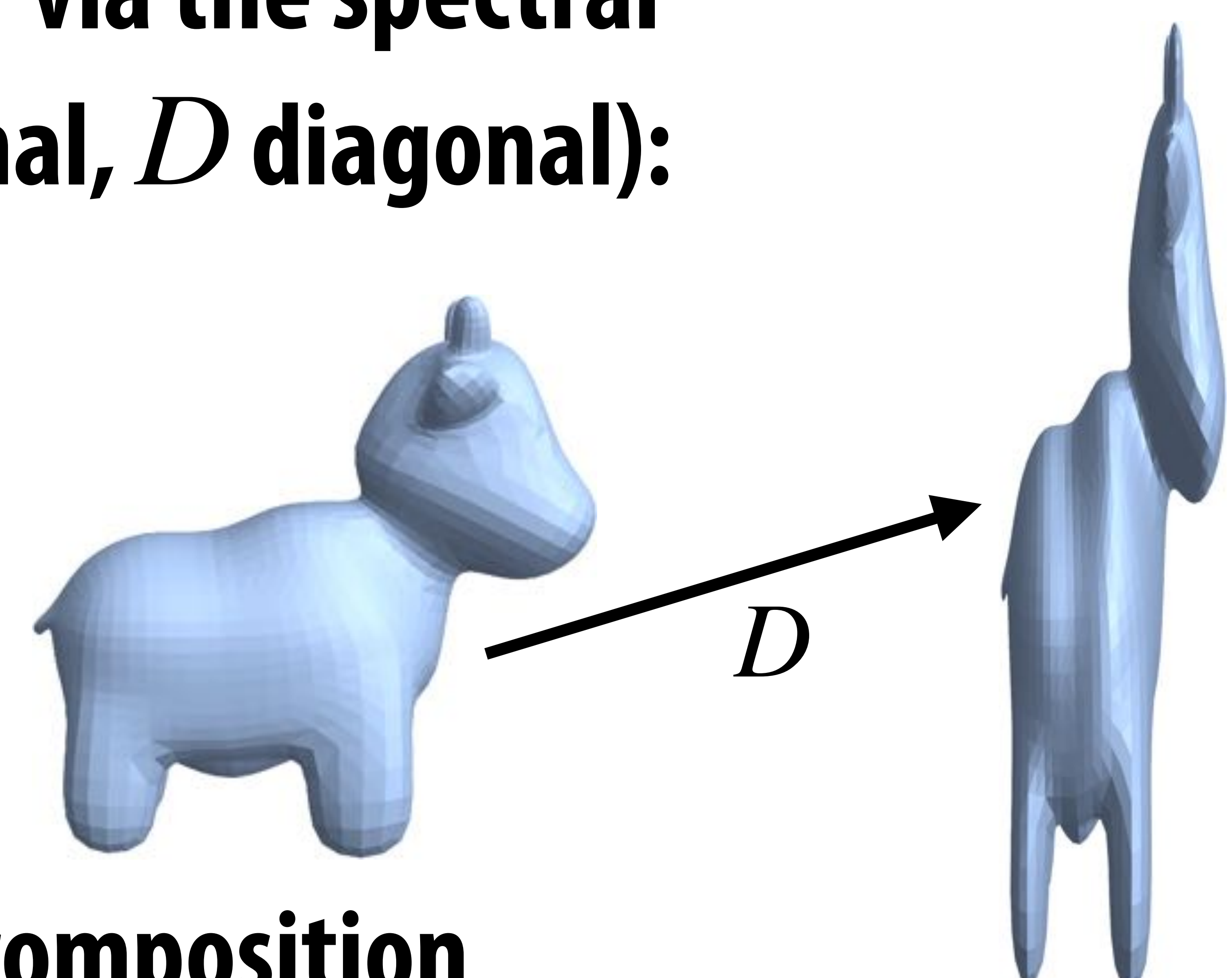
Since P is symmetric, can take this further via the spectral decomposition $P = VDV^{\mathsf{T}}$ (V orthogonal, D diagonal):

$$A = \underbrace{QV}_{U}DV^{\mathsf{T}} = UDV^{\mathsf{T}}$$

rotation

axis-aligned scaling

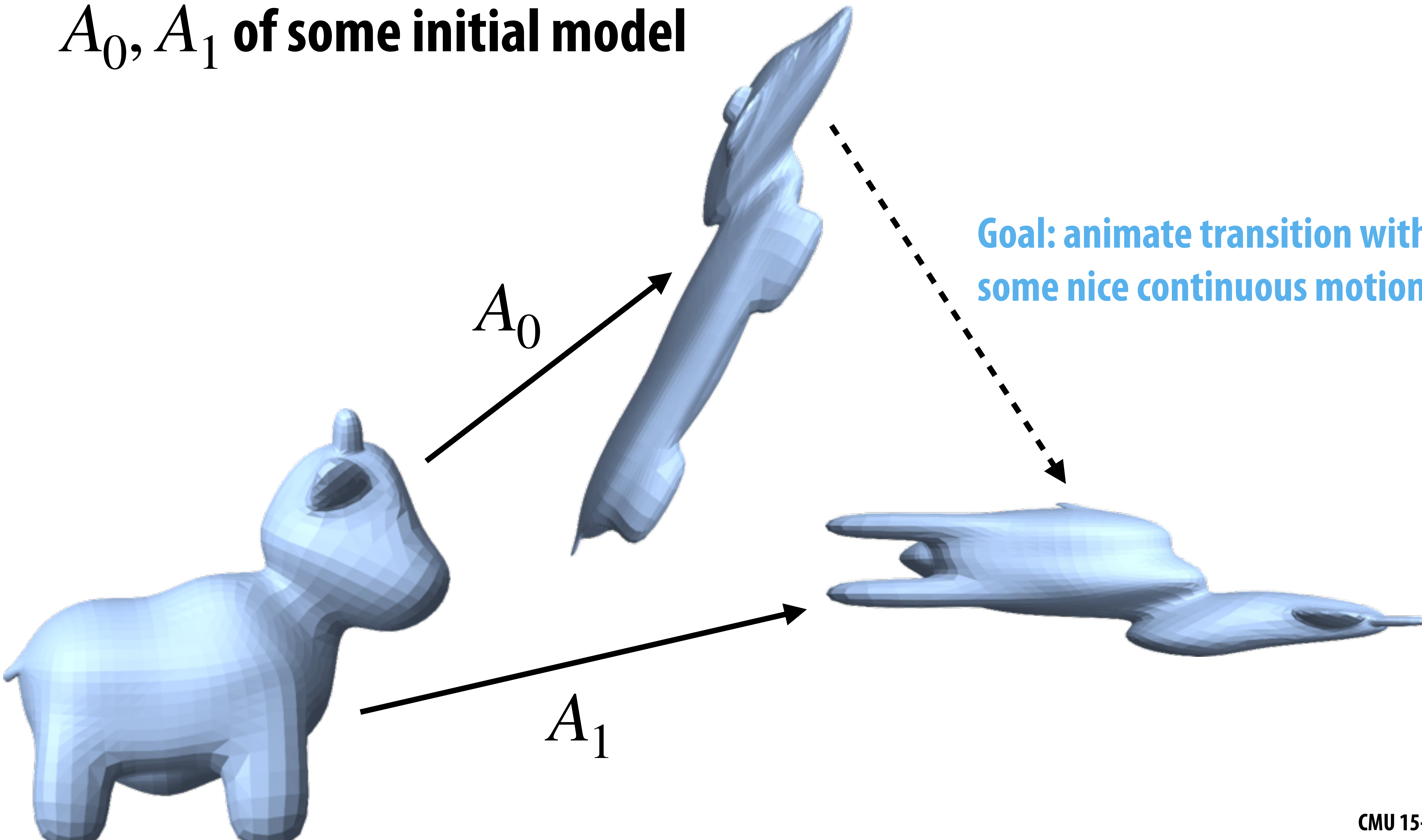
rotation



Result UDV^{T} is called the singular value decomposition

Interpolating Transformations

- How are these decompositions useful for graphics?
- Consider interpolating between two linear transformations A_0, A_1 of some initial model



Interpolating Transformations—Linear

One idea: just take a linear combination of the two matrices, weighted by the current time $t \in [0,1]$

$$A(t) = (1-t)A_0 + tA_1$$

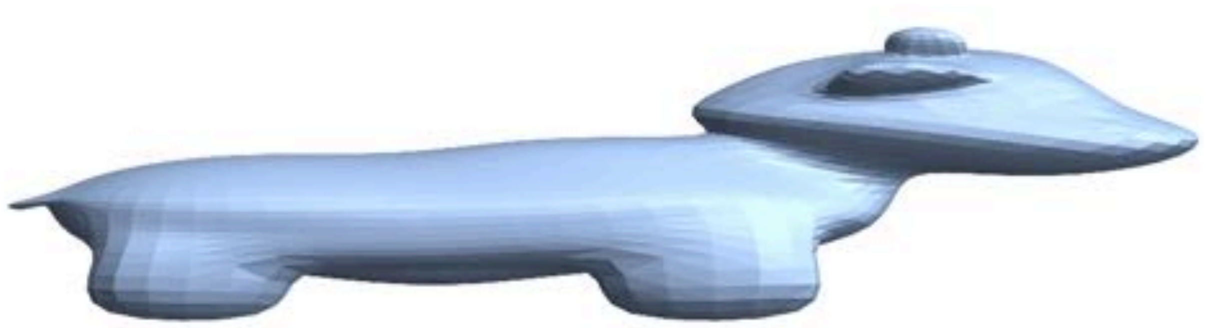
Hits the right start/endpoints… but looks awful in between!

Interpolating Transformations—Polar

Better idea: separately interpolate components of polar decomposition.

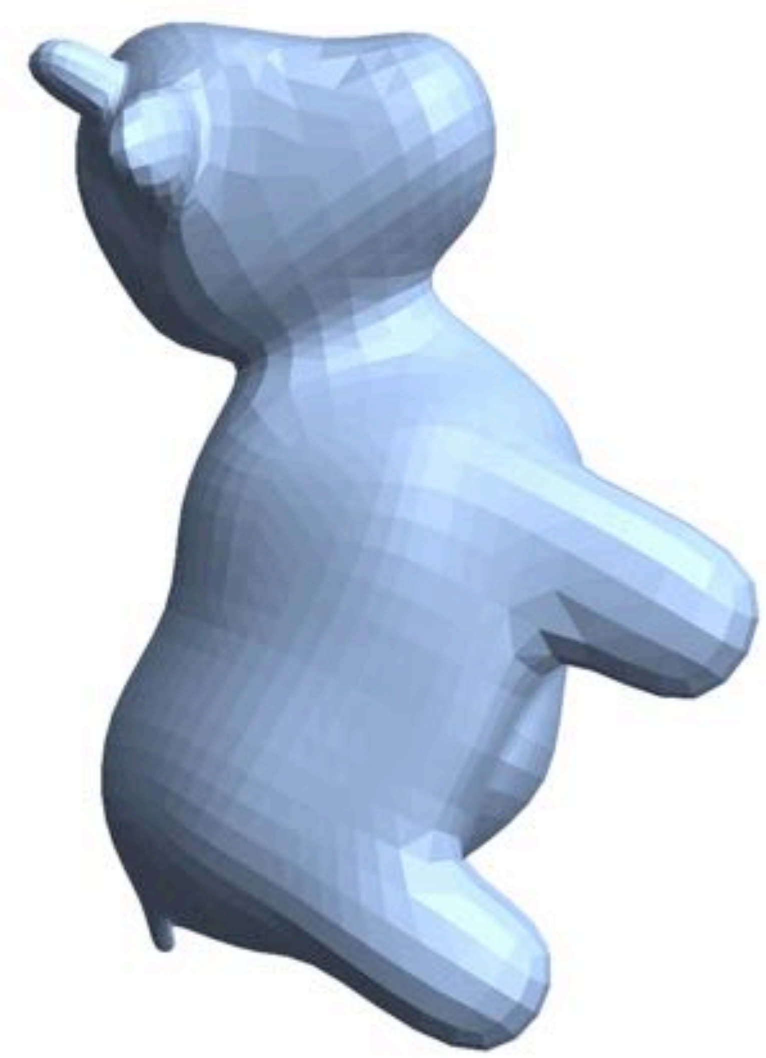
$$A_0 = Q_0 P_0, \quad A_1 = Q_1 P_1$$

scaling



$$P(t) = (1-t)P_0 + tP_1$$

rotation



$$\widetilde{Q}(t) = (1-t)Q_0 + tQ_1$$
$$\widetilde{Q}(t) = Q(t)X(t)$$

final interpolation

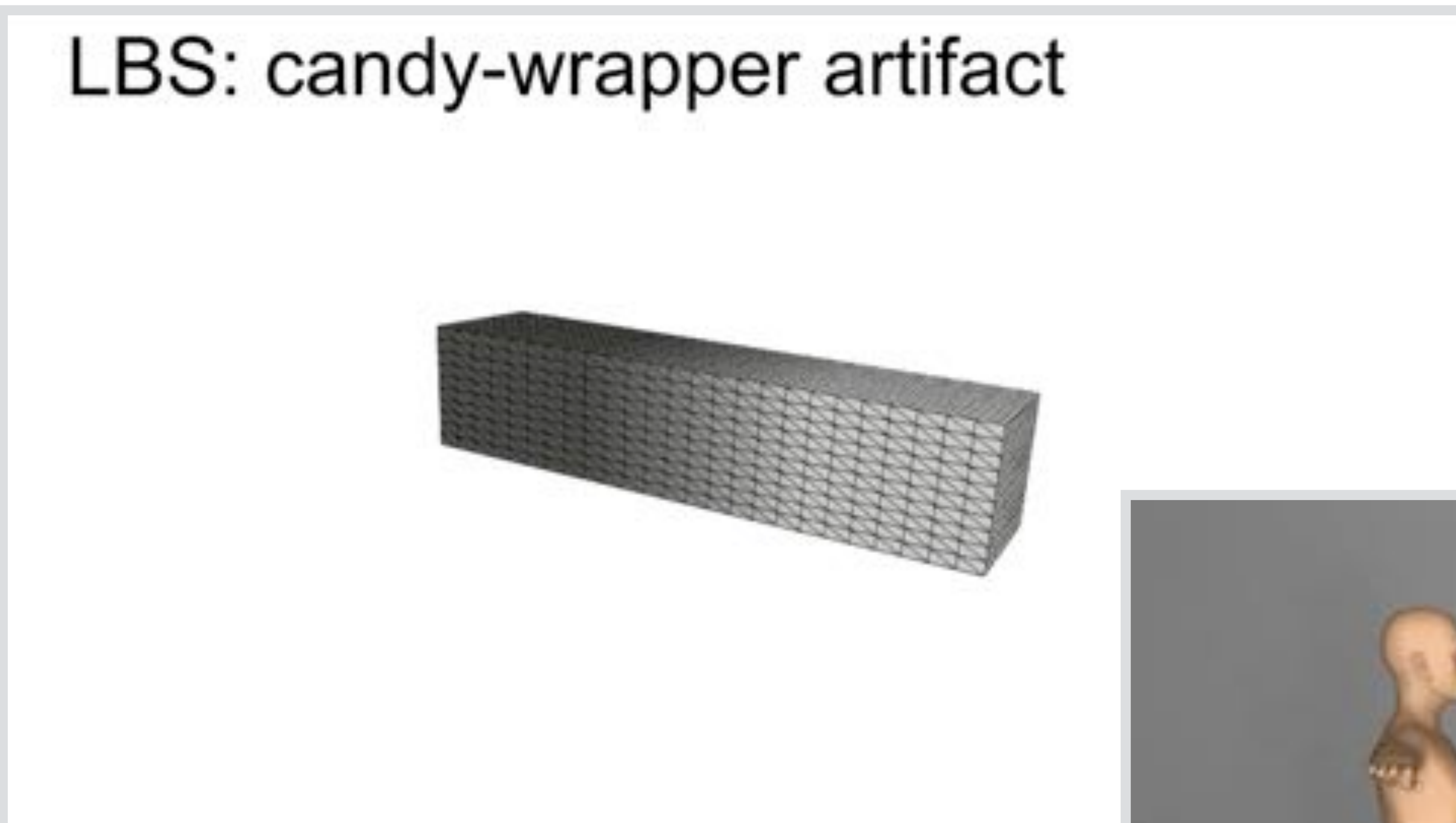
$$A(t) = Q(t)P(t)$$

…looks better!

See: Shoemake & Duff, "Matrix Animation and Polar Decomposition"

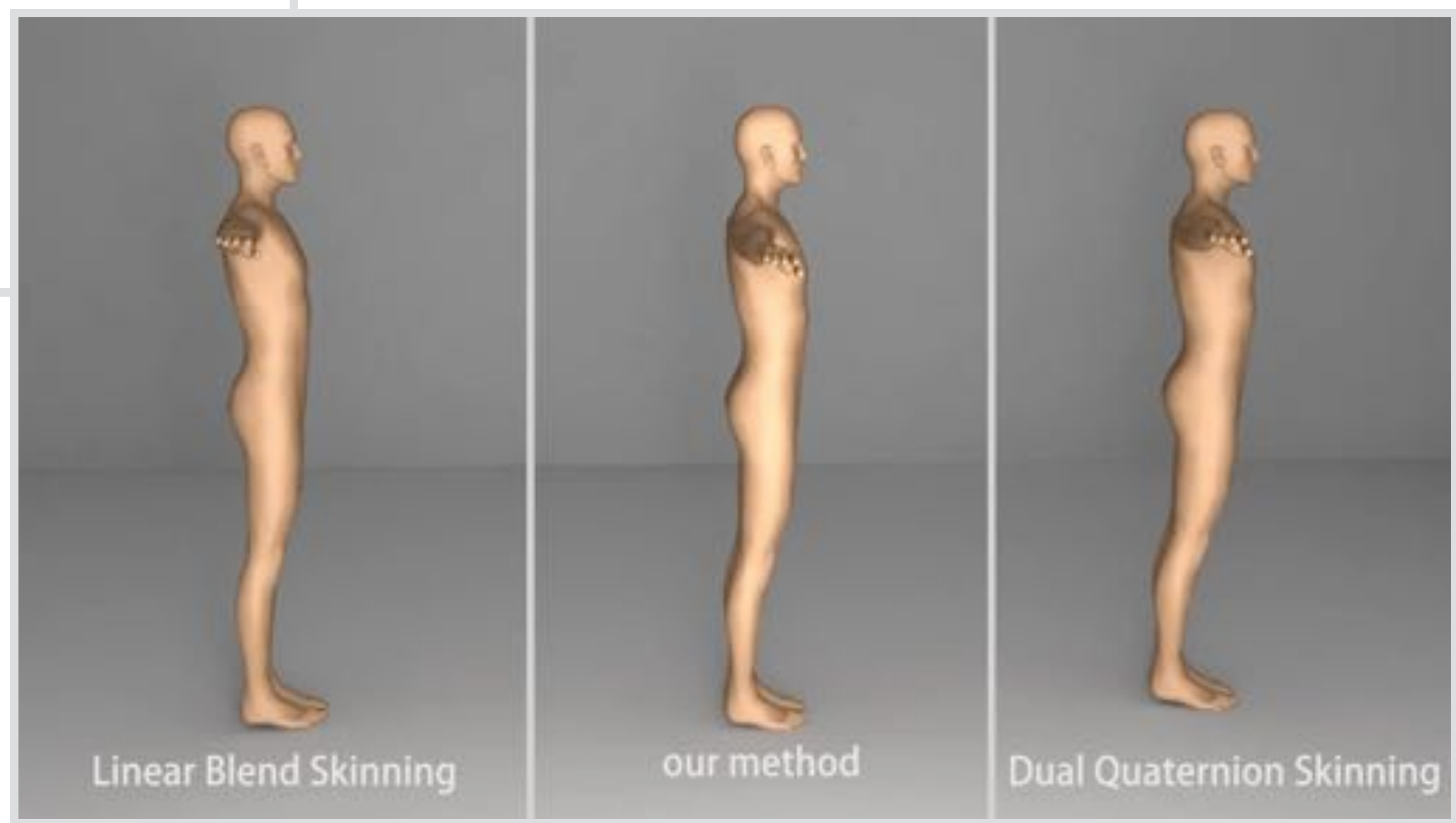
Example: Linear Blend Skinning

- Naïve linear interpolation also causes artifacts when blending between transformations on a character ("candy wrapper effect")
- Lots of research on alternative ways to blend transformations…



Jacobson, Deng, Kavan, & Lewis (2014)
"Skinning: Real-time Shape Deformation"

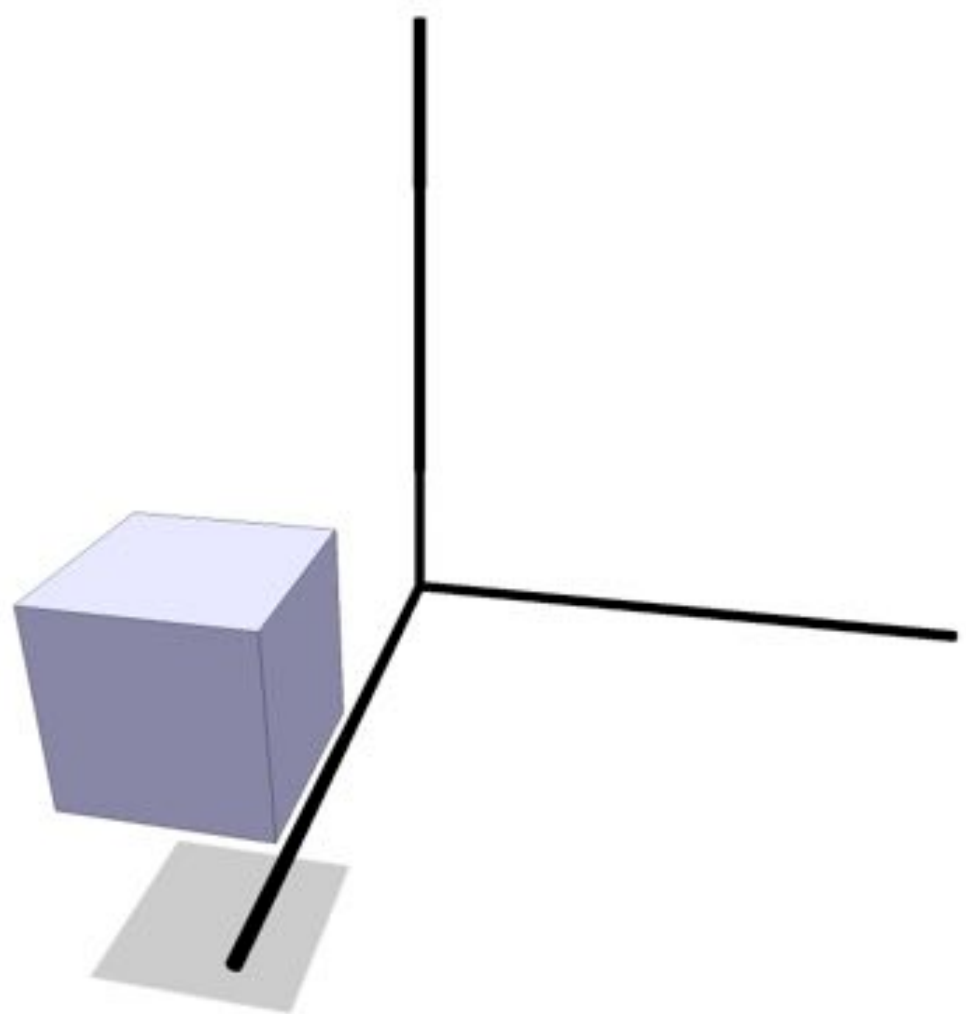
Rumman & Fratarcangeli (2015)
"Position-based Skinning for Soft Articulated Characters"



Translations

- So far we've ignored a basic transformation—translations
- A translation simply adds an offset $\mathbf{u}$ to the given point $\mathbf{x}$:

$$f_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}$$



Q: Is this transformation <u>linear</u>?
(Certainly seems to move us along a line…)

Let's carefully check the definition…

<u>additivity</u>

$$f_{\mathbf{u}}(\mathbf{x}+\mathbf{y}) = \mathbf{x}+\mathbf{y}+\mathbf{u}$$

$$f_{\mathbf{u}}(\mathbf{x}) + f_{\mathbf{u}}(\mathbf{y}) = \mathbf{x}+\mathbf{y}+2\mathbf{u}$$

<u>homogeneity</u>

$$f_{\mathbf{u}}(a\mathbf{x}) = a\mathbf{x}+\mathbf{u}$$

$$af_{\mathbf{u}}(\mathbf{x}) = a\mathbf{x}+a\mathbf{u}$$

A: No! Translation is <u>affine</u>, not linear!

Composition of Transformations

- **Recall we can compose <u>linear</u> transformations via matrix multiplication:**

$$A_3(A_2(A_1\mathbf{x})) = (A_3A_2A_1)\mathbf{x}$$

- **It's easy enough to compose translations—just add vectors:**

$$f_{\mathbf{u}_3}(f_{\mathbf{u}_2}(f_{\mathbf{u}_1}(\mathbf{x}))) = f_{\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3}(\mathbf{x})$$

- **What if we want to intermingle translations and linear transformations (rotation, scale, shear, etc.)?**

$$A_2(A_1\mathbf{x}+\mathbf{b}_1)+\mathbf{b}_2 = (A_2A_1)\mathbf{x} + (A_2\mathbf{b}_1+\mathbf{b}_2)$$

- **Now we have to keep track of a matrix and a vector**
- **Moreover, we'll see (later) that this encoding won't work for other important cases, such as perspective transformations**

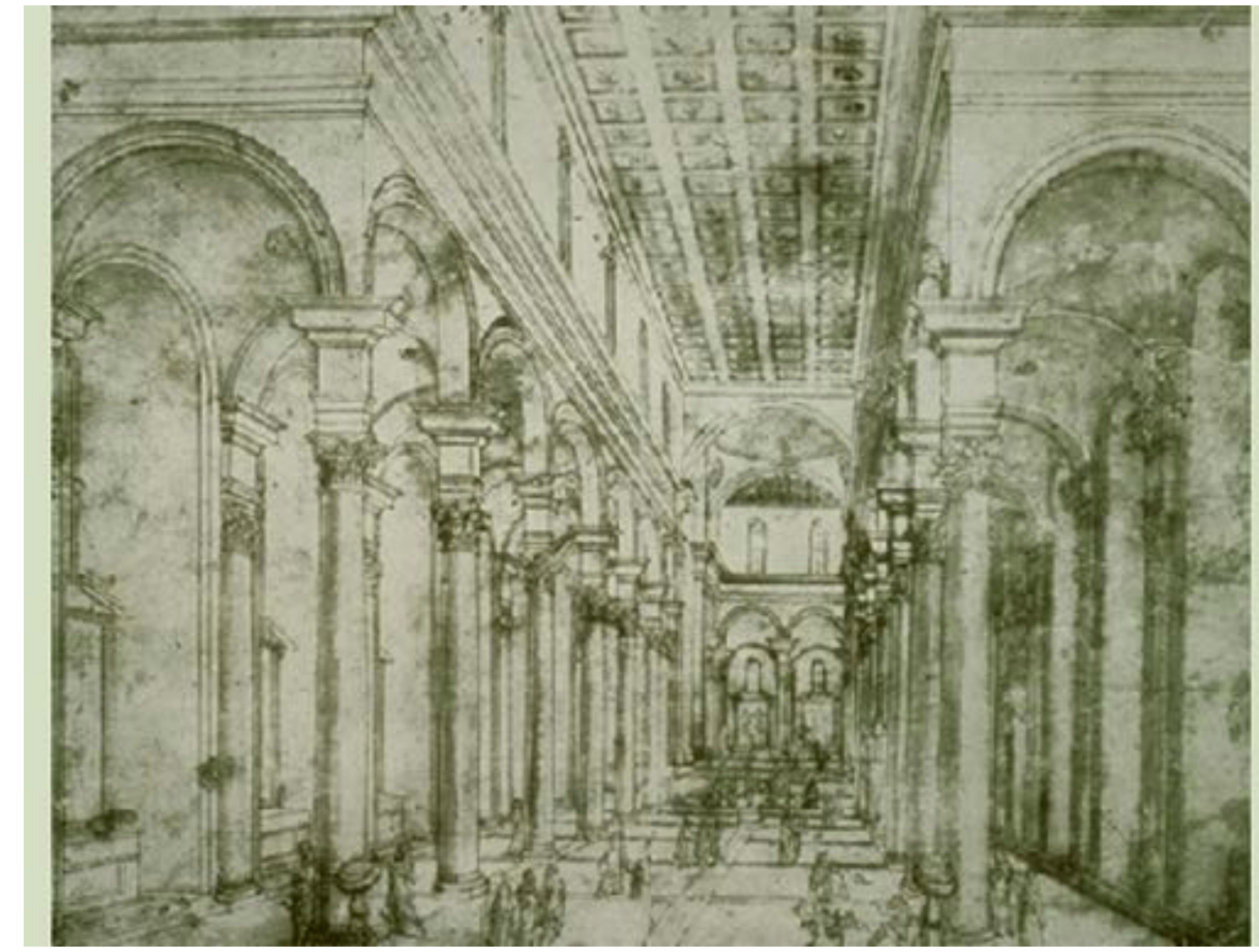
But there is a better way…

Strange idea:

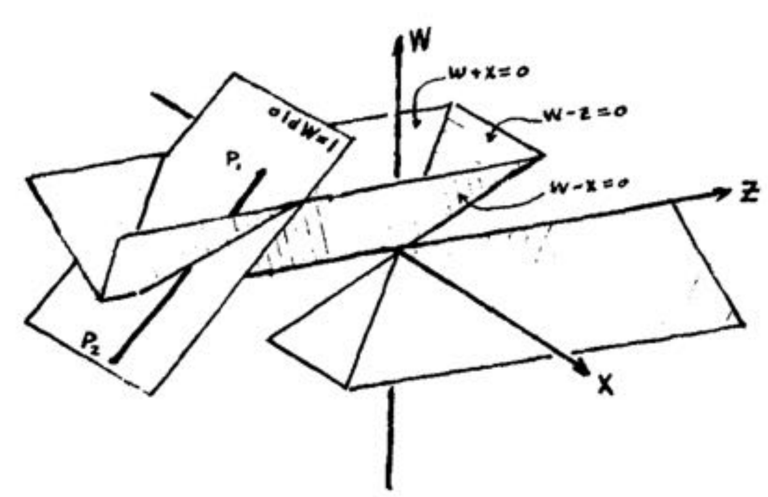
Maybe translations turn into <u>linear</u> transformations if we go into the 4th dimension…!

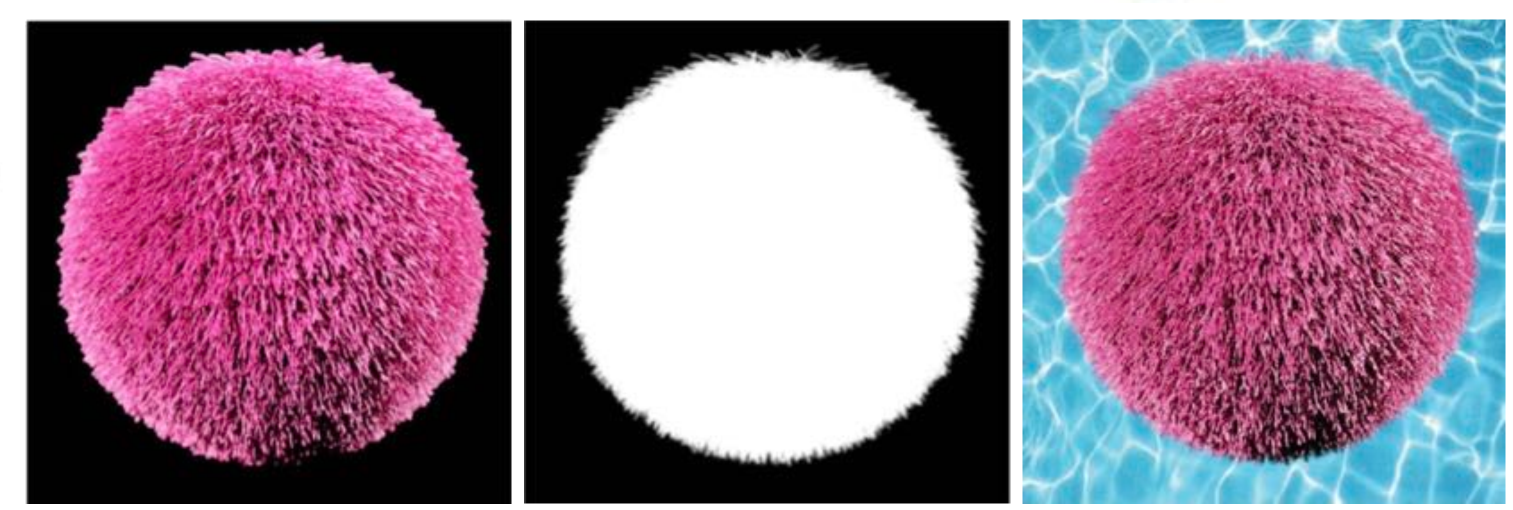
Homogeneous Coordinates

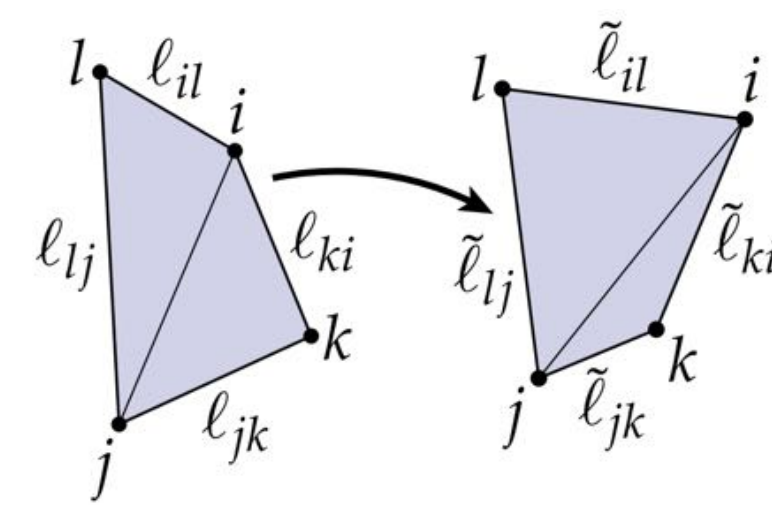
Filippo Brunelleschi, 1428



- Came from efforts to study <u>perspective</u>
- Introduced by Möbius as a natural way of assigning coordinates to <u>lines</u>
- Show up naturally in a surprising large number of places in computer graphics:
  - 3D transformations
  - perspective projection
  - quadric error simplification
  - premultiplied alpha
  - shadow mapping
  - projective texture mapping
  - discrete conformal geometry
  - hyperbolic geometry
  - clipping
  - directional lights
  - …



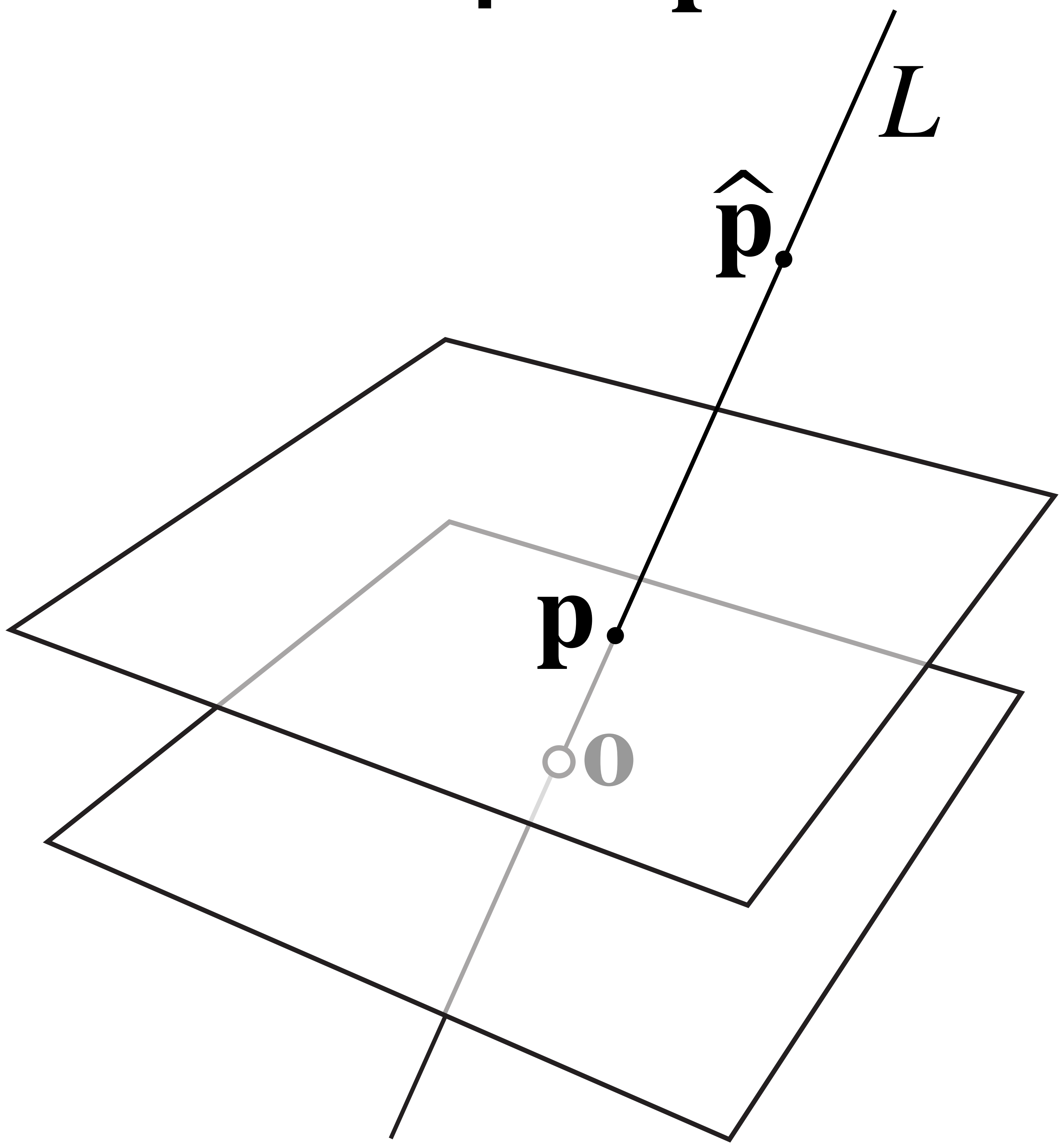


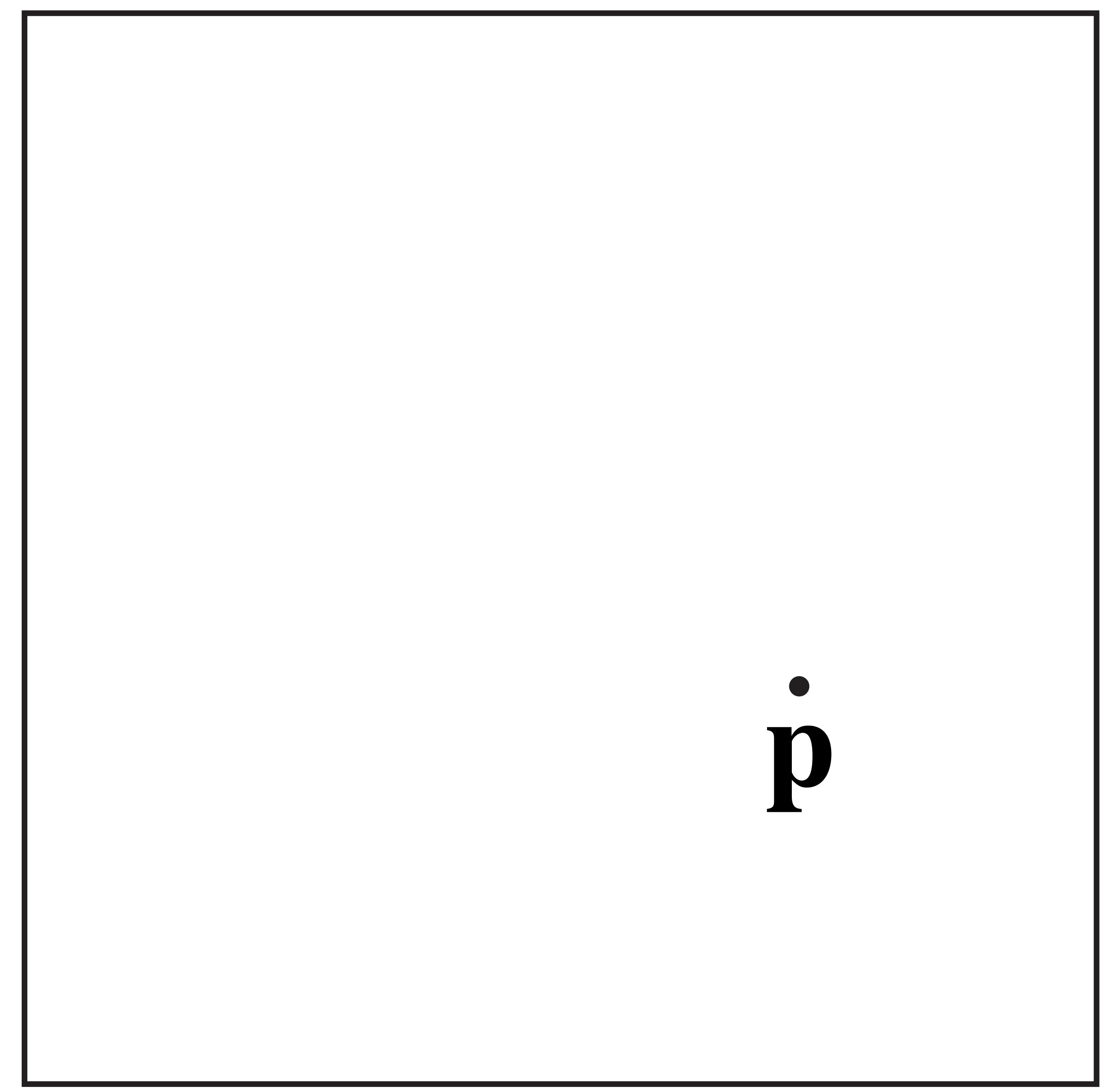


Probably worth understanding!

Homogeneous Coordinates—Basic Idea

- **Consider any 2D plane that does not pass through the origin $\mathbf{o}$ in 3D**
- **Every <u>line</u> through the origin in 3D corresponds to a <u>point</u> in the 2D plane**
  - **Just find the point $\mathbf{p}$ where the line L pierces the plane**



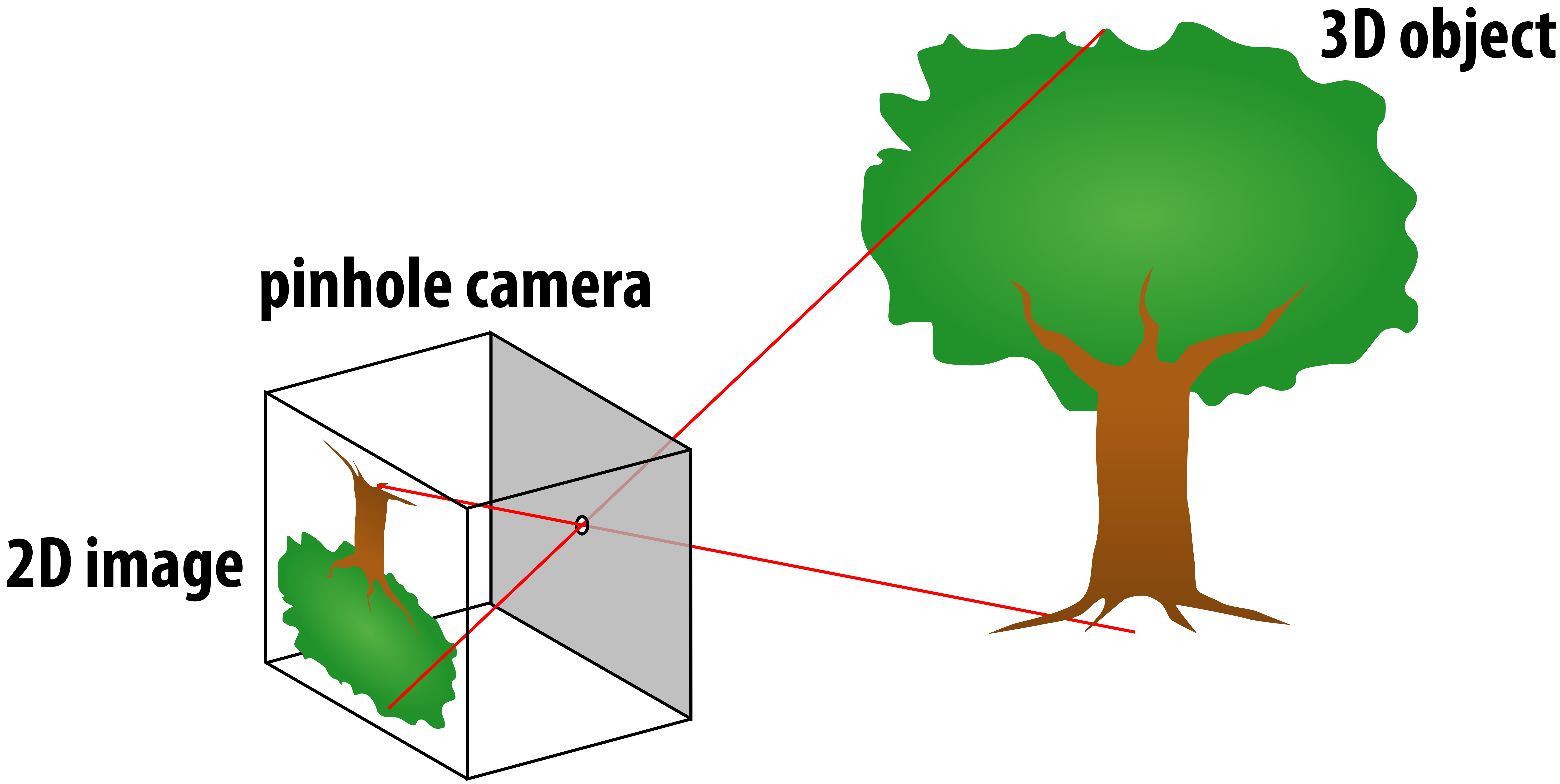


Hence, <u>any</u> point $\hat{\mathbf{p}}$ on the line L can be used to represent the point $\mathbf{p}$.

Q: What does this story remind you of?

Review: Perspective projection

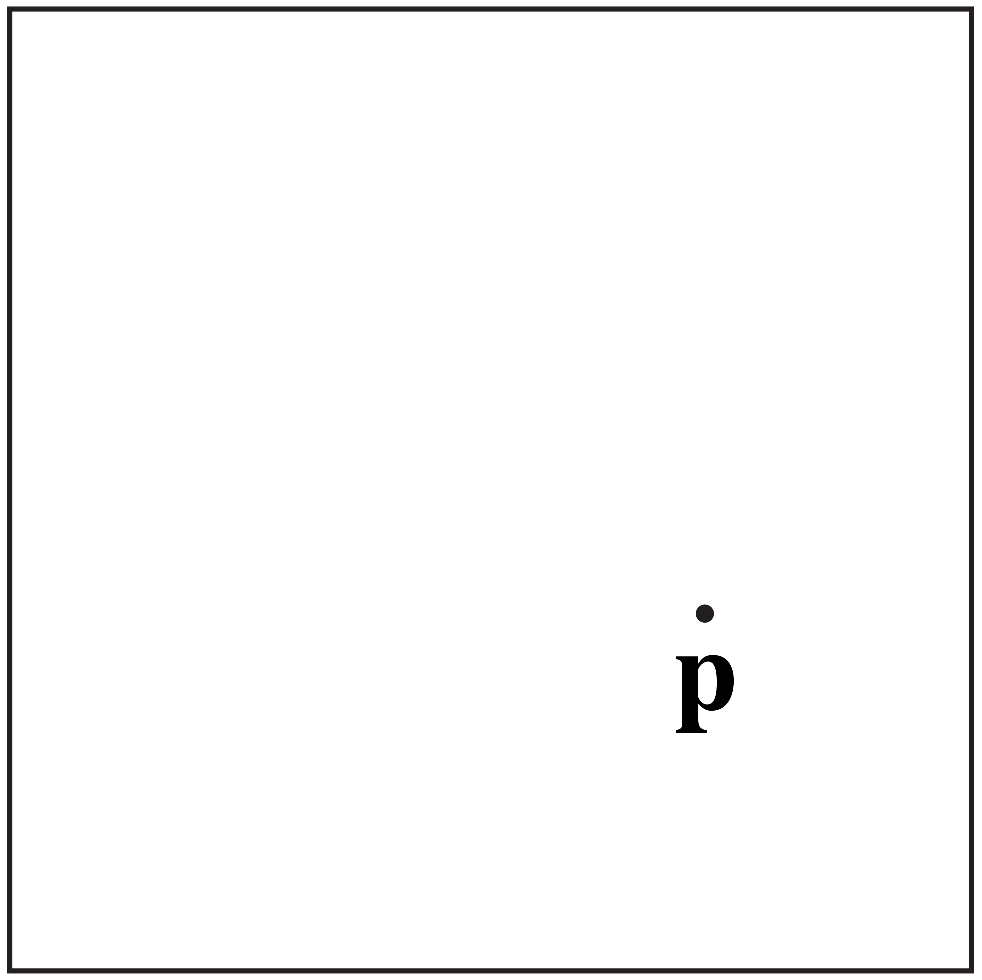
- Hopefully it reminds you of our "pinhole camera"
- Objects along the same line project to the same point

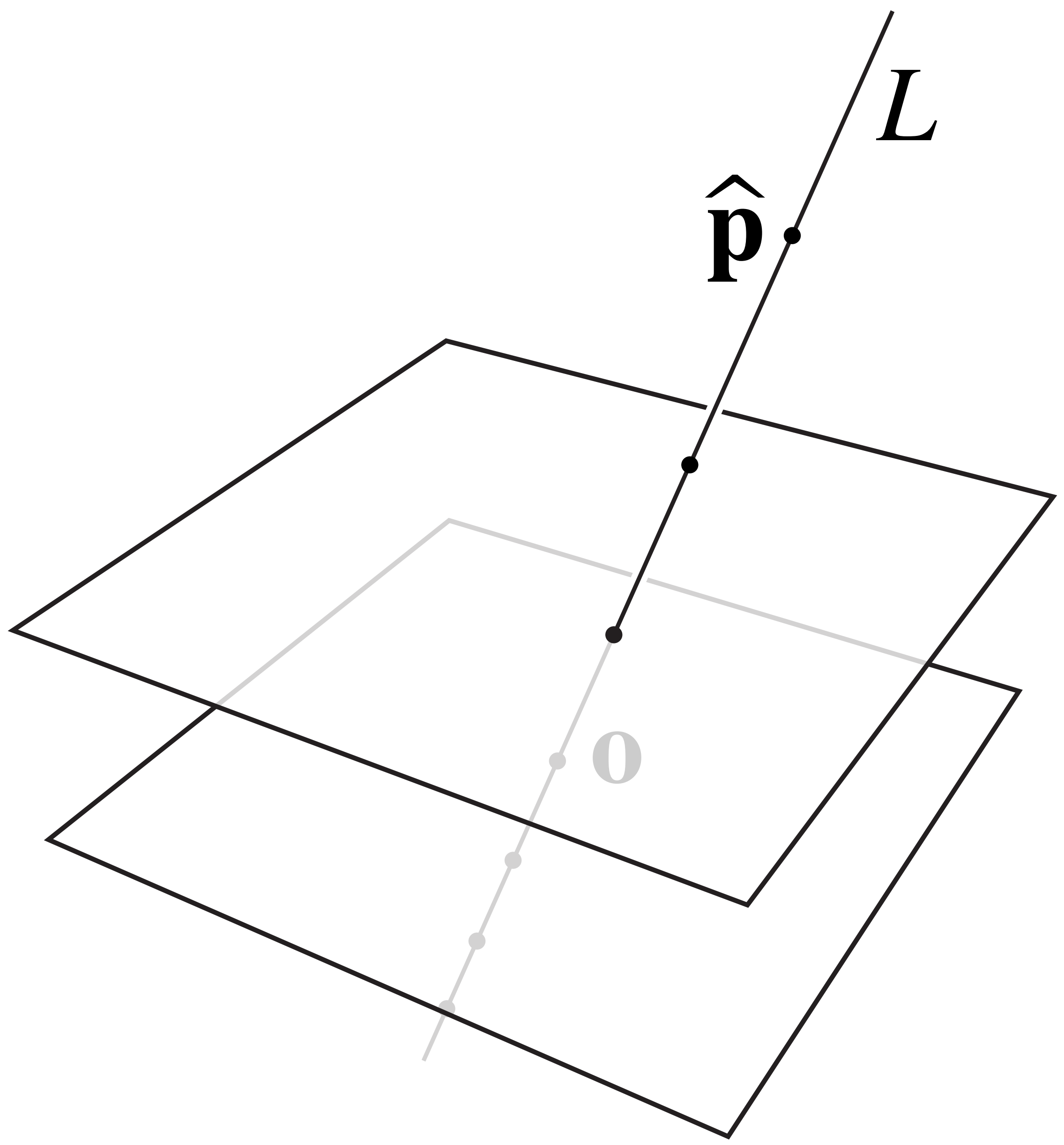


If you have an image of a single dot, can't know where it is!
Only which line it belongs to.

Homogeneous Coordinates (2D)

- More explicitly, consider a point $\mathbf{p} = (x, y)$, and the plane $z = 1$ in 3D
- Any three numbers $\widehat{\mathbf{p}} = (a, b, c)$ such that $(a/c, b/c) = (x, y)$ are <u>homogeneous coordinates</u> for $\mathbf{p}$
  - E.g., $(x, y, 1)$
  - In general: (cx, cy, c) for $c \neq 0$
- Hence, two points $\widehat{\mathbf{p}}, \widehat{\mathbf{q}} \in \mathbb{R}^3 \backslash \{O\}$ describe the same point in 2D (and line in 3D) if $\widehat{\mathbf{p}} = \lambda \widehat{\mathbf{q}}$ for some $\lambda \neq 0$

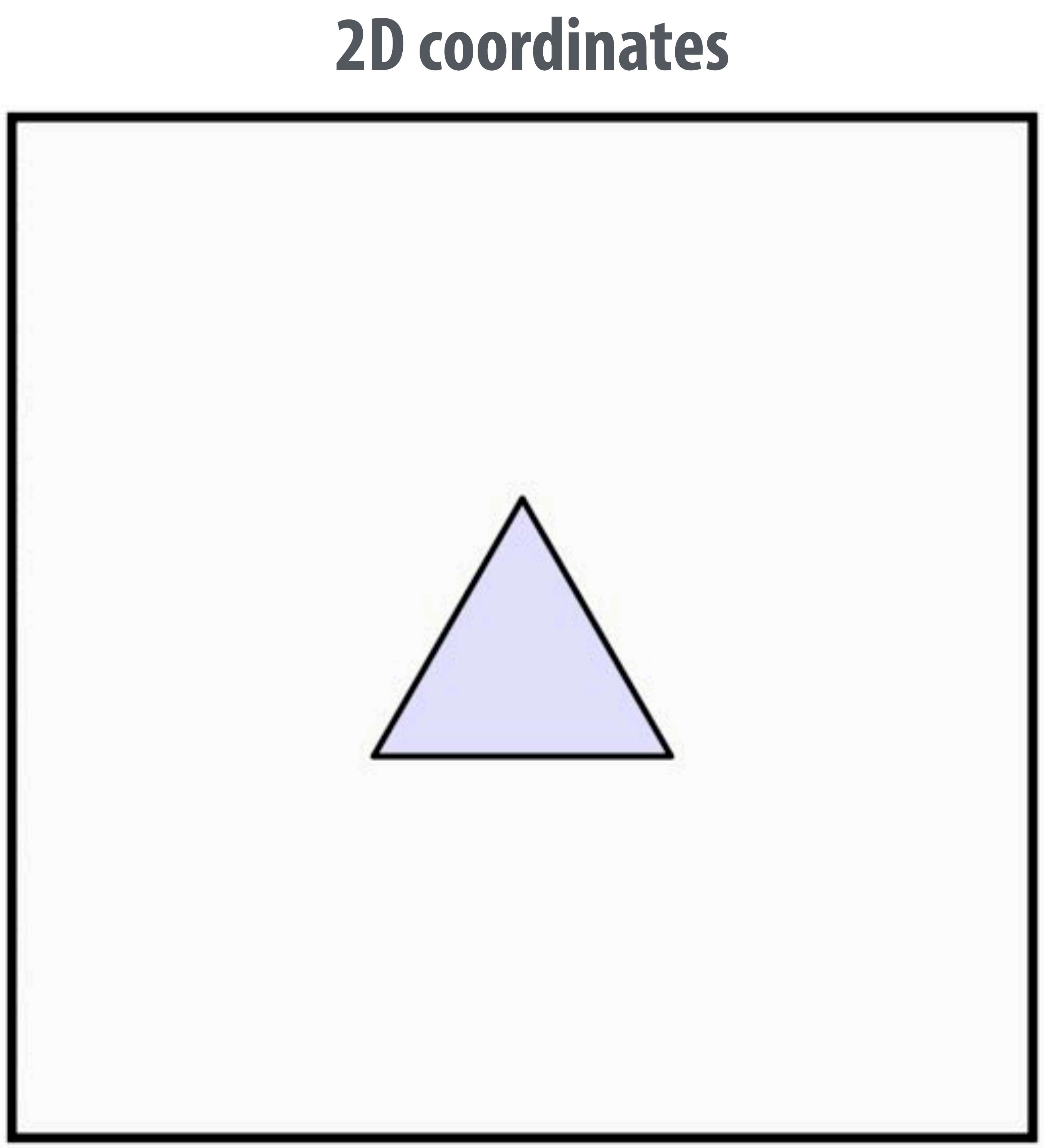




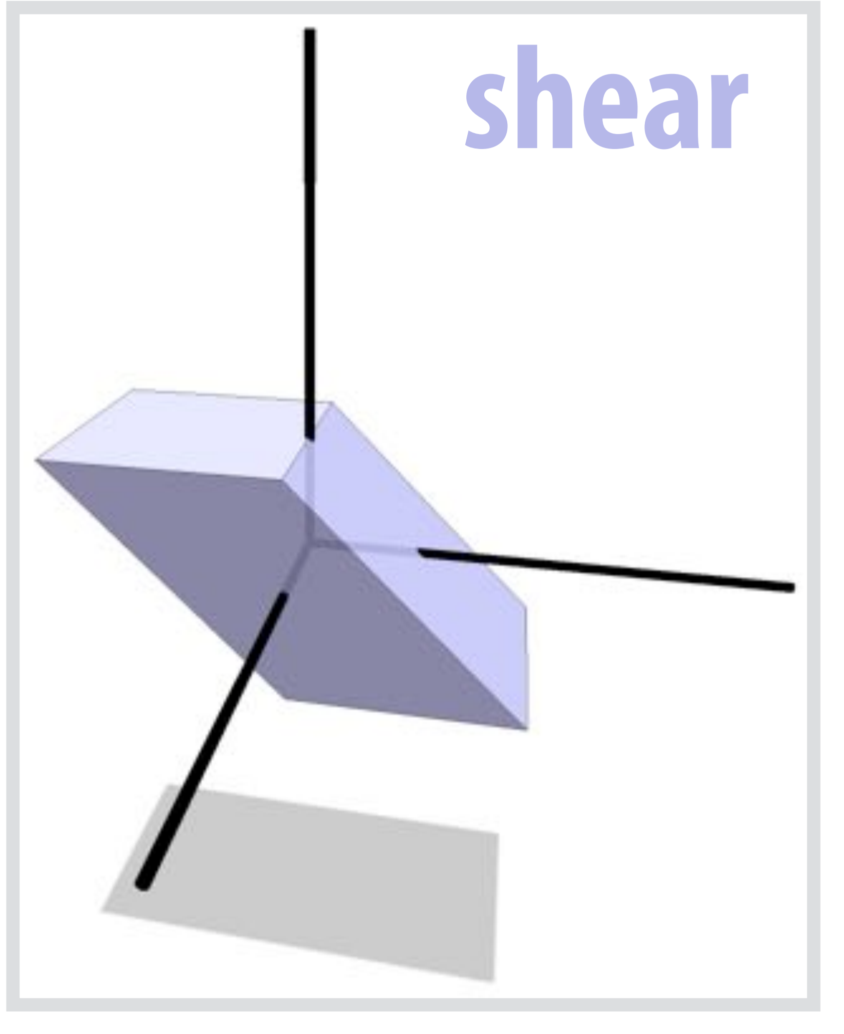
Great… but how does this help us with transformations?

Translation in Homogeneous Coordinates

Let's think about what happens to our homogeneous coordinates $\hat{\mathbf{p}}$ if we apply a translation to our 2D coordinates $\mathbf{p}$



Q: What kind of transformation does this look like?



Translation in Homogeneous Coordinates

- But wait a minute—shear is a linear transformation!
- Can this be right? Let's check in coordinates…
- Suppose we translate a point $\mathbf{p} = (p_1, p_2)$ by a vector $\mathbf{u} = (u_1, u_2)$ to get $\mathbf{p}' = (p_1 + u_1, p_2 + u_2)$
- The homogeneous coordinates $\hat{\mathbf{p}} = (cp_1, cp_2, c)$ then become $\hat{\mathbf{p}}' = (cp_1 + cu_1, cp_2 + cu_2, c)$
- Notice that we're shifting $\hat{\mathbf{p}}$ by an amount $c\mathbf{u}$ that's proportional to the distance c along the third axis—a shear

Using homogeneous coordinates, we can represent an affine transformation in 2D as a linear transformation in 3D

Homogeneous Translation—Matrix Representation

- To write as a matrix, recall that a shear in the direction $\mathbf{u} = (u_1, u_2)$ according to the distance along a direction $\mathbf{v}$ is

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

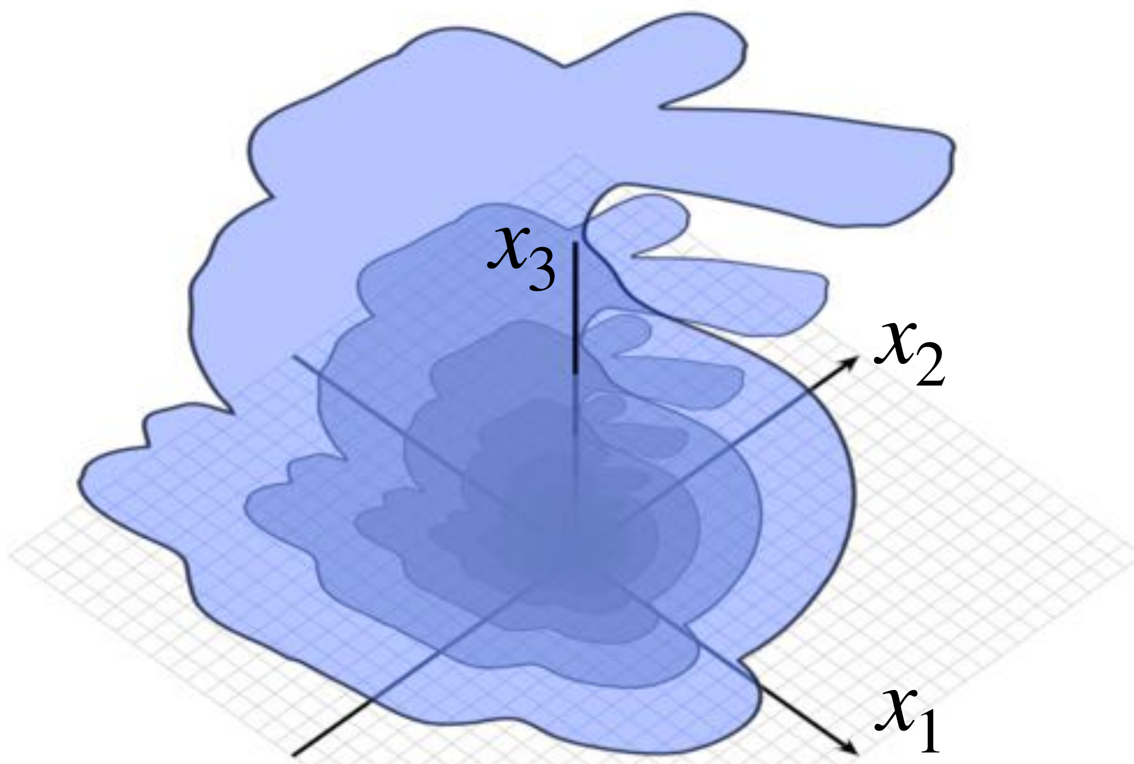
- In matrix form:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \left(I + \mathbf{u}\mathbf{v}^\top \right) \mathbf{x}$$

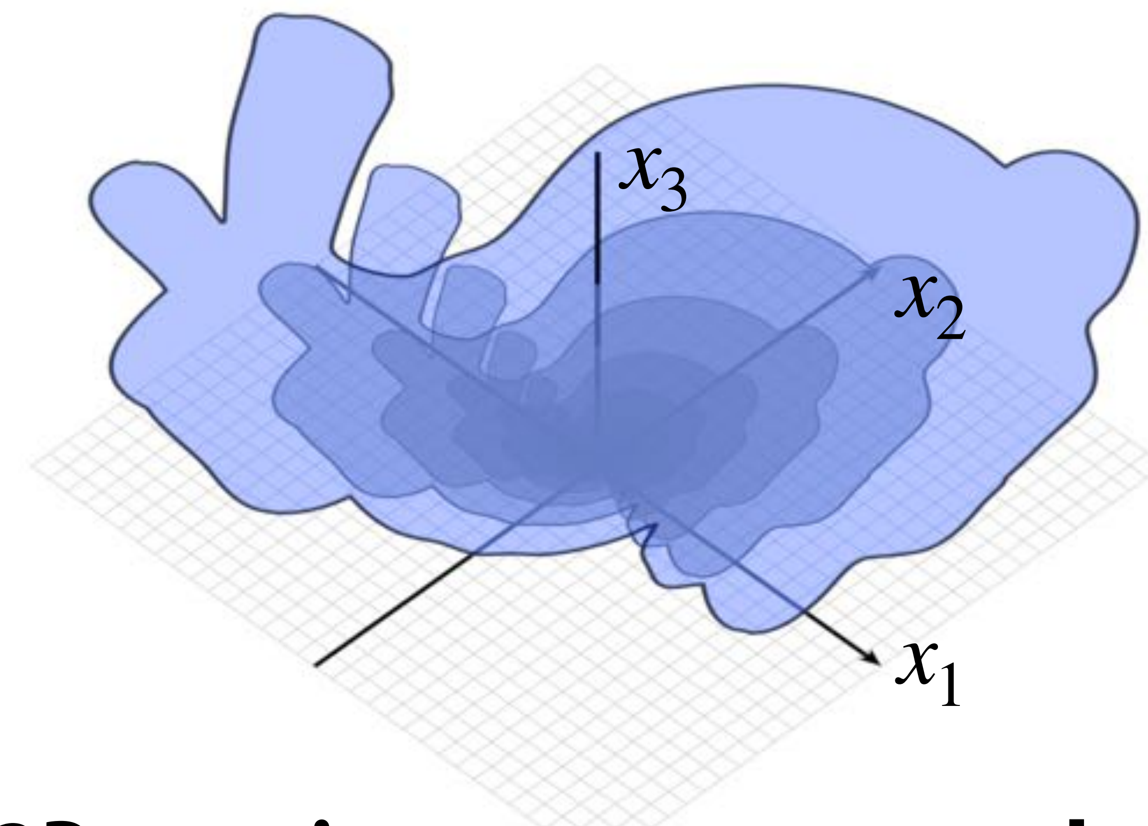
- In our case, $\mathbf{v} = (0,0,1)$ and so we get a matrix

$$\begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cp_1 \\ cp_2 \\ c \end{bmatrix} = \begin{bmatrix} c(p_1 + u_1) \\ c(p_2 + u_2) \\ c \end{bmatrix} \overset{1/c}{\Longrightarrow} \begin{bmatrix} p_1 + u_1 \\ p_2 + u_2 \end{bmatrix}$$

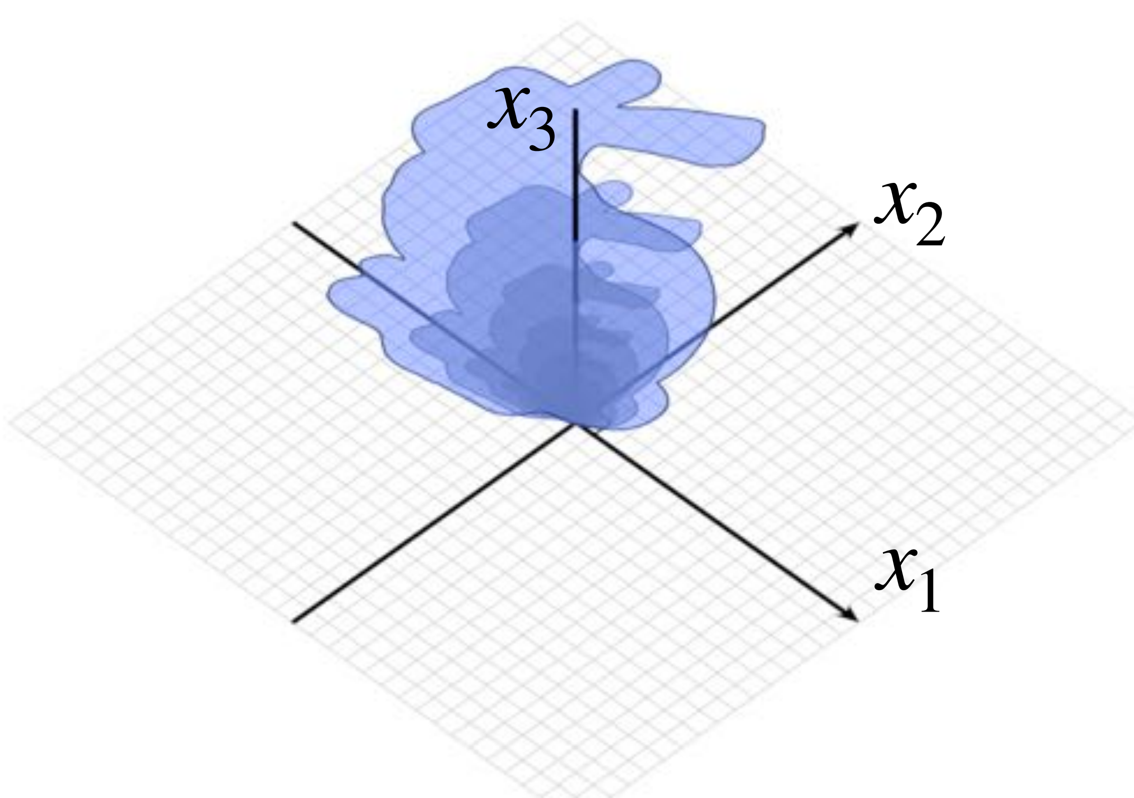
Other 2D Transformations in Homogeneous Coordinates



Original shape in 2D can be viewed as many copies, uniformly scaled by x_3

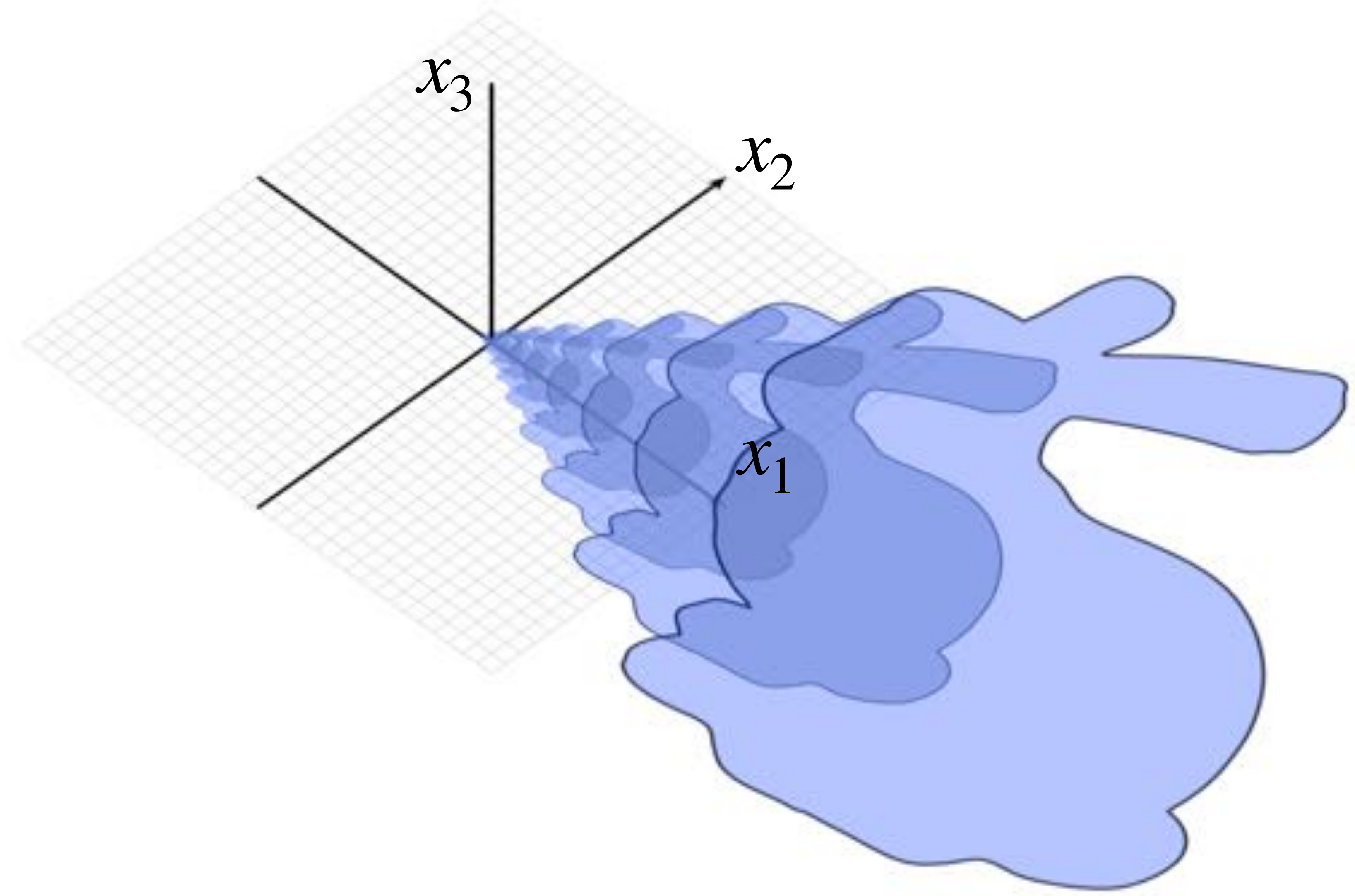


2D rotation ↔ rotate around x_3



2D scale ↔ scale x_1 and x_2; preserve x_3

(Q: what happens to 2D shape if you scale x_1, x_2, and x_3 uniformly?)



2D translate ↔ shear

Now easy to compose all these transformations

3D Transformations in Homogeneous Coordinates

- **Not much changes in three (or more) dimensions: just append one "homogeneous coordinate" to the first three**
- **Matrix representations of 3D linear transformations just get an additional identity row/column; translation is again a shear**

point in 3D

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

rotate (x, y, z) around y by θ

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

shear (x, y) by z in (s, t) direction

$$\begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

scale x, y, z by a, b, c

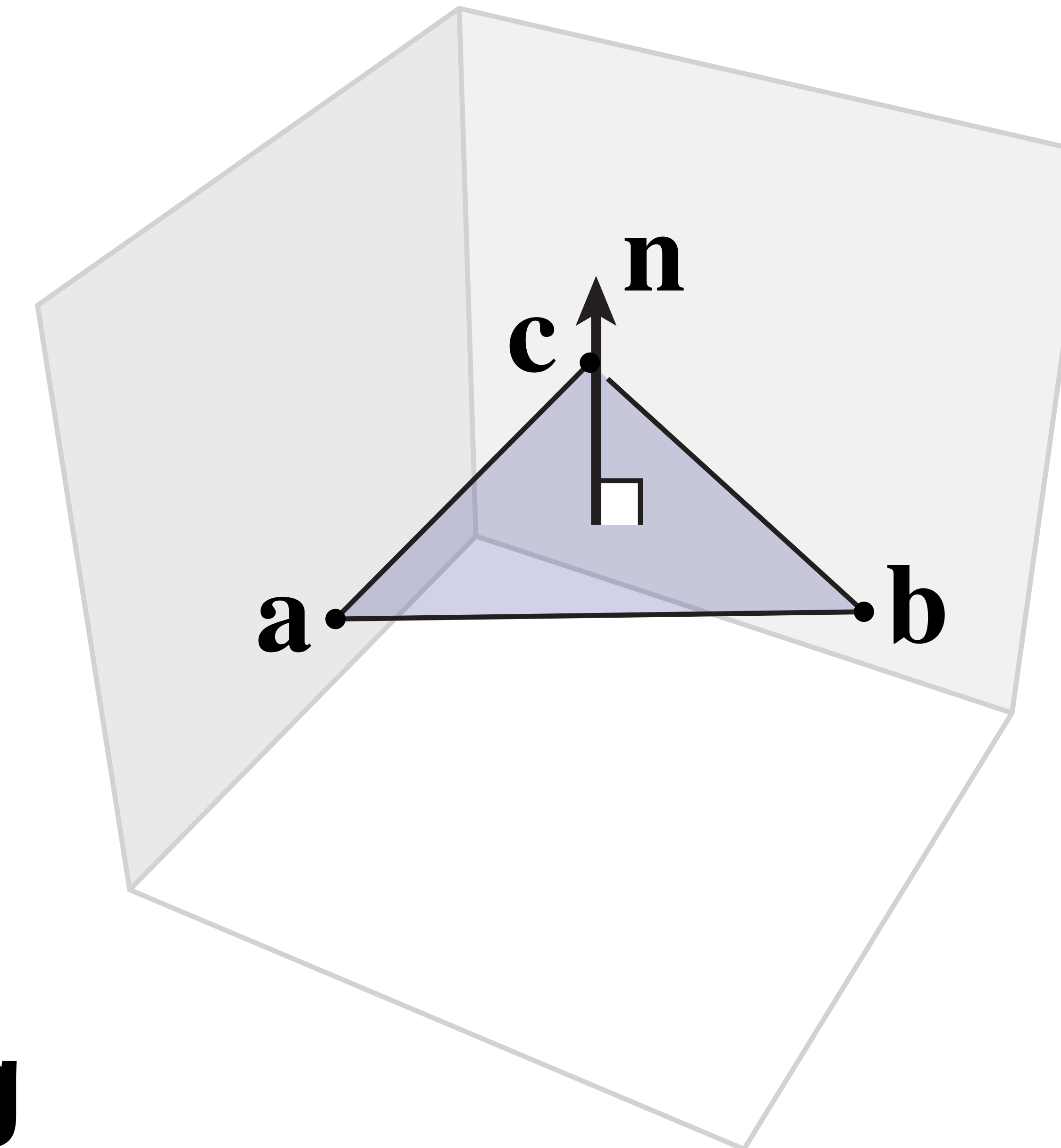
$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

translate (x, y, z) by (u, v, w)

$$\begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

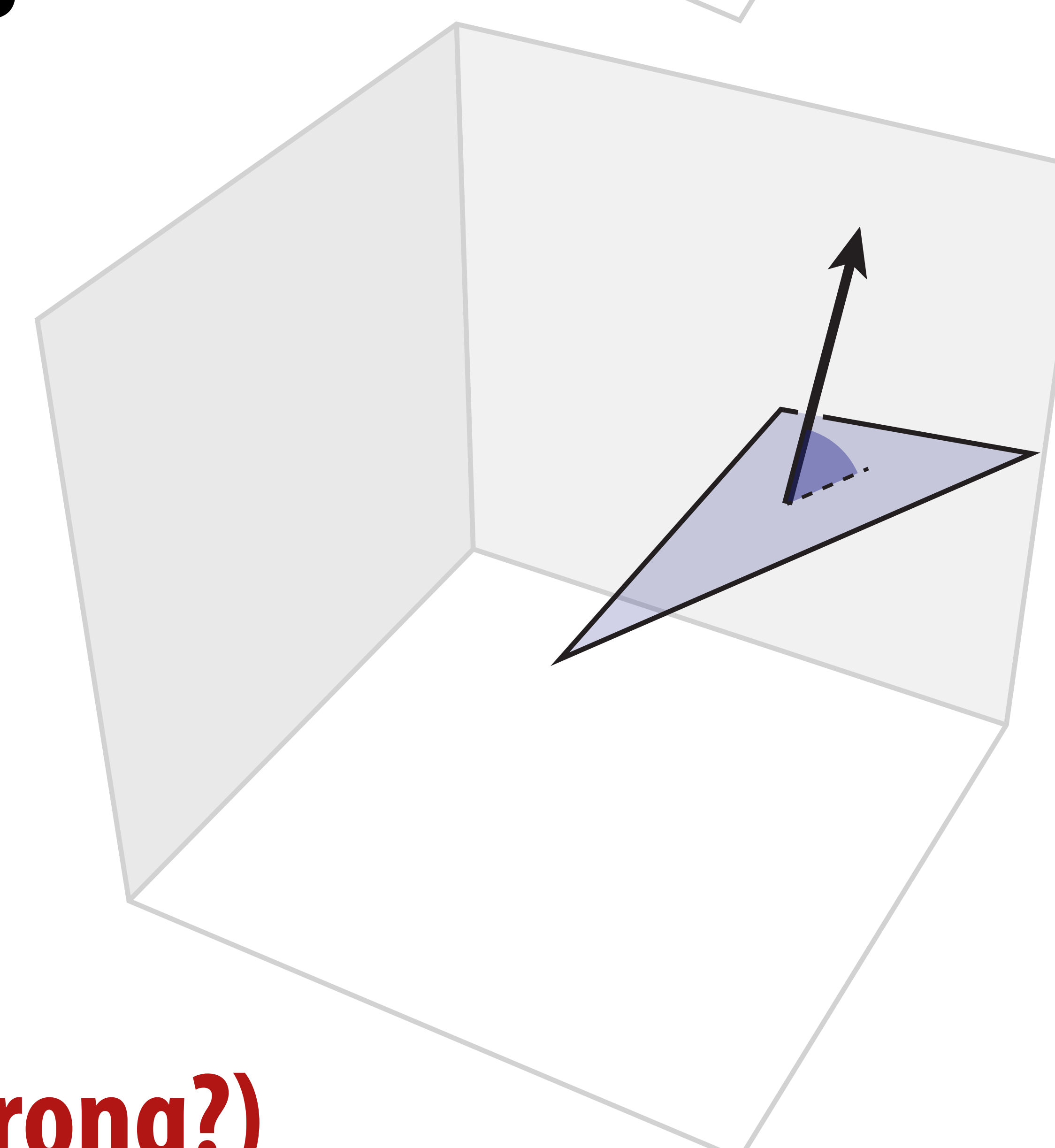
Points vs. Vectors

- Homogeneous coordinates have another useful feature: distinguish between <u>points</u> and <u>vectors</u>



- Consider for instance a triangle with:
  - vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$
  - normal vector $\mathbf{n} \in \mathbb{R}^3$
- Suppose we transform the triangle by appending "1" to $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{n}$ and multiplying by this matrix:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Normal is not orthogonal to triangle! (What went wrong?)

Points vs. Vectors (continued)

- Let's think about what happens when we multiply the normal vector $\mathbf{n}$ by our matrix:

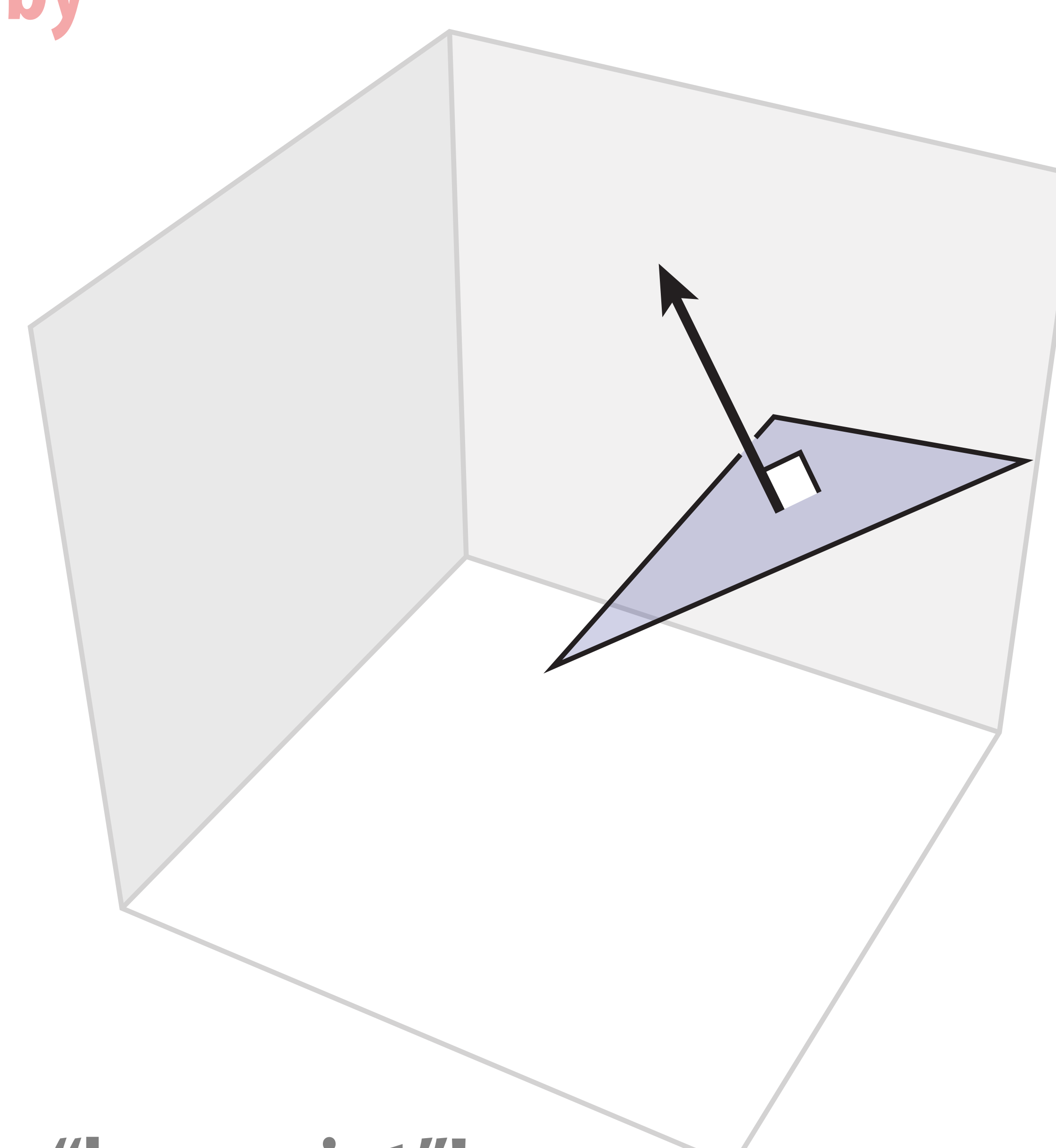
rotate normal around y by θ

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{bmatrix}$$

translate normal by (u, v, w)

- But when we rotate/translate a triangle, its normal should <u>just rotate!*</u>
- Solution? Just set homogeneous coordinate to zero!
- Translation now gets ignored; normal is orthogonal to triangle

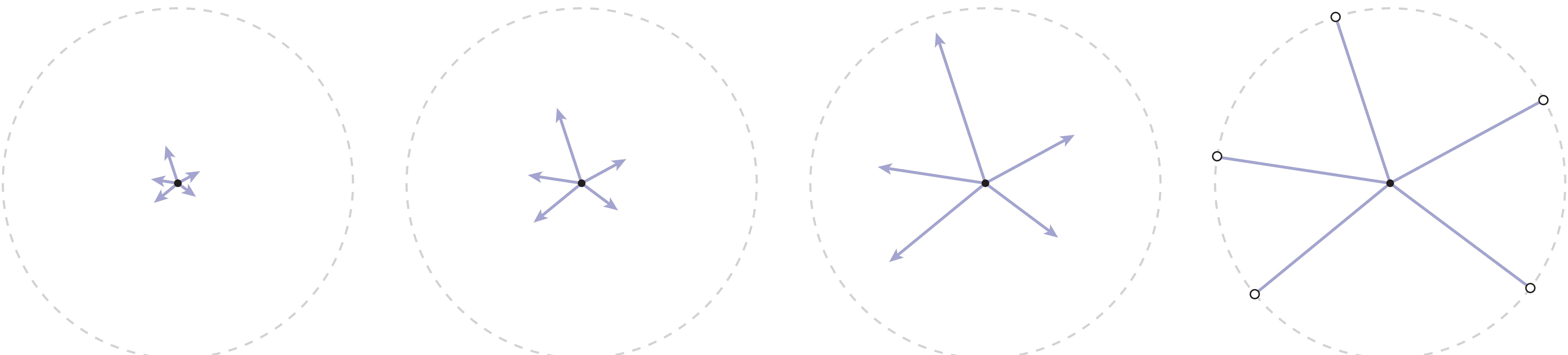
$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}$$



*Recall that <u>vectors</u> just have direction and magnitude—they don't have a "basepoint"!

Points vs. Vectors in Homogeneous Coordinates

- **In general:**
  - A point has a <u>nonzero</u> homogeneous coordinate $(c = 1)$
  - A vector has a <u>zero</u> homogeneous coordinate $(c = 0)$
- **But wait… what division by c mean when it's equal to zero?**
- **Well consider what happens as $c \to 0$…**



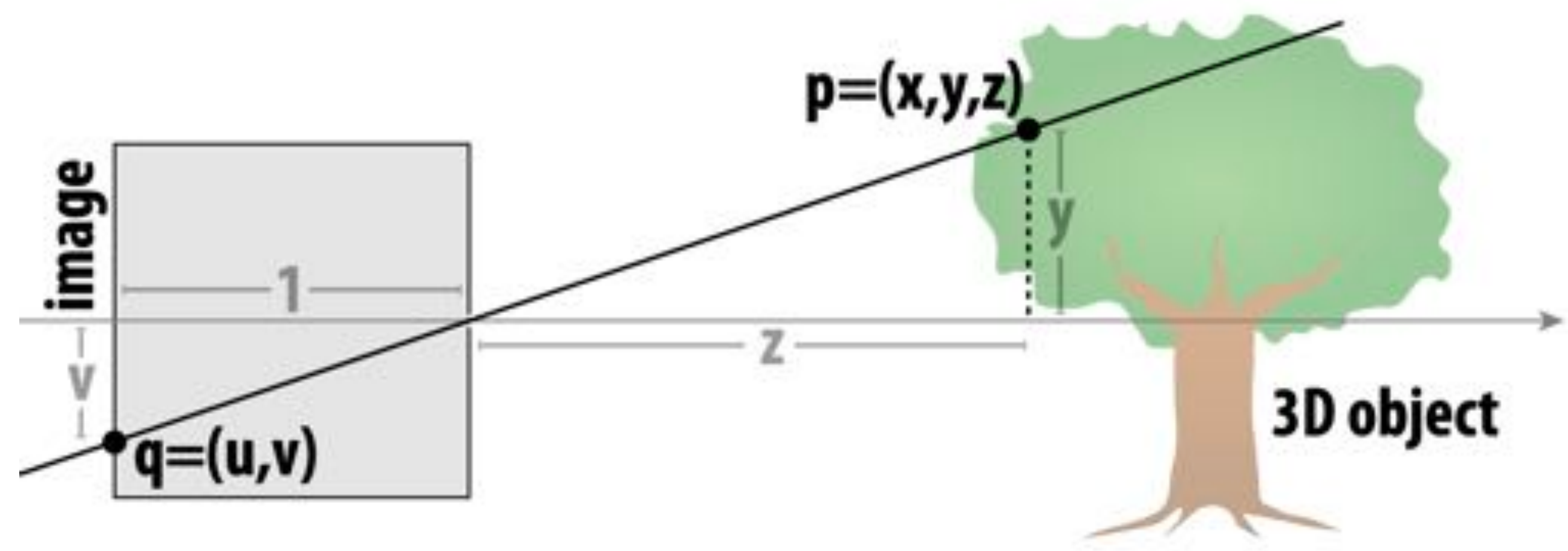
$(x, y)/1$ $(x, y)/0.5$ $(x, y)/0.25$ $(x, y)/0.001$

Can think of vectors as "points at infinity" (sometimes called "ideal points")

(In practice: still need to check for divide by zero!)

Perspective Projection in Homogeneous Coordinates

- **Q: How can we perform perspective projection* using homogeneous coordinates?**
- **Remember from our pinhole camera model that the basic idea was to "divide by z"**
- **So, we can build a matrix that "copies" the z coordinate into the homogeneous coordinate**
- **Division by the homogeneous coordinate now gives us perspective projection onto the plane $z = 1$**



$$(x, y, z) \mapsto (x/z, y/z)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$$

*Assuming a pinhole camera at (0,0,0) looking down the z-axis

Screen Transformation

- One last transformation is needed in the rasterization pipeline: transform from viewing plane to pixel coordinates
- E.g., suppose we want to draw all points that fall inside the square [-1,1] x [-1,1] on the z = 1 plane, into a W x H pixel image

“normalized device coordinates”

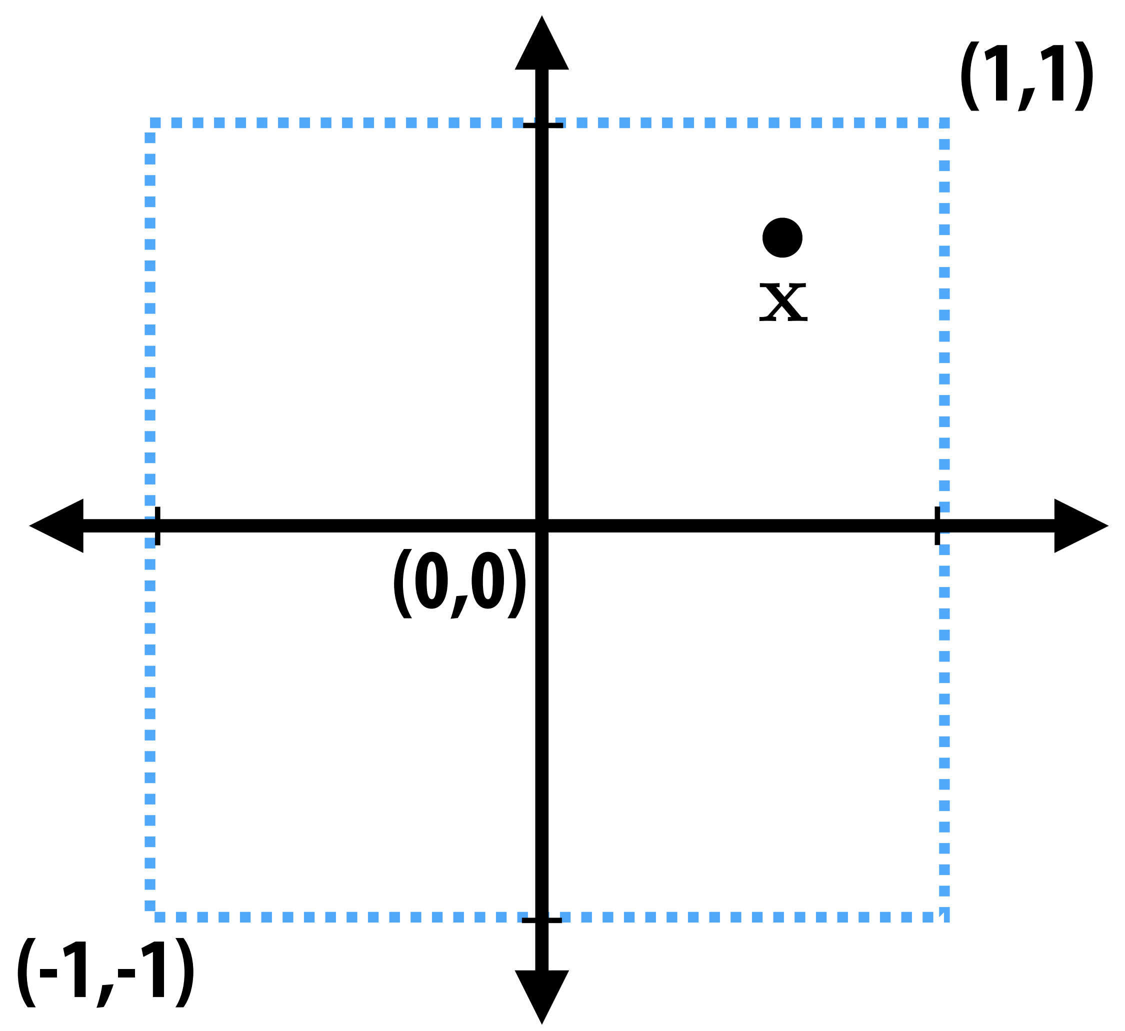
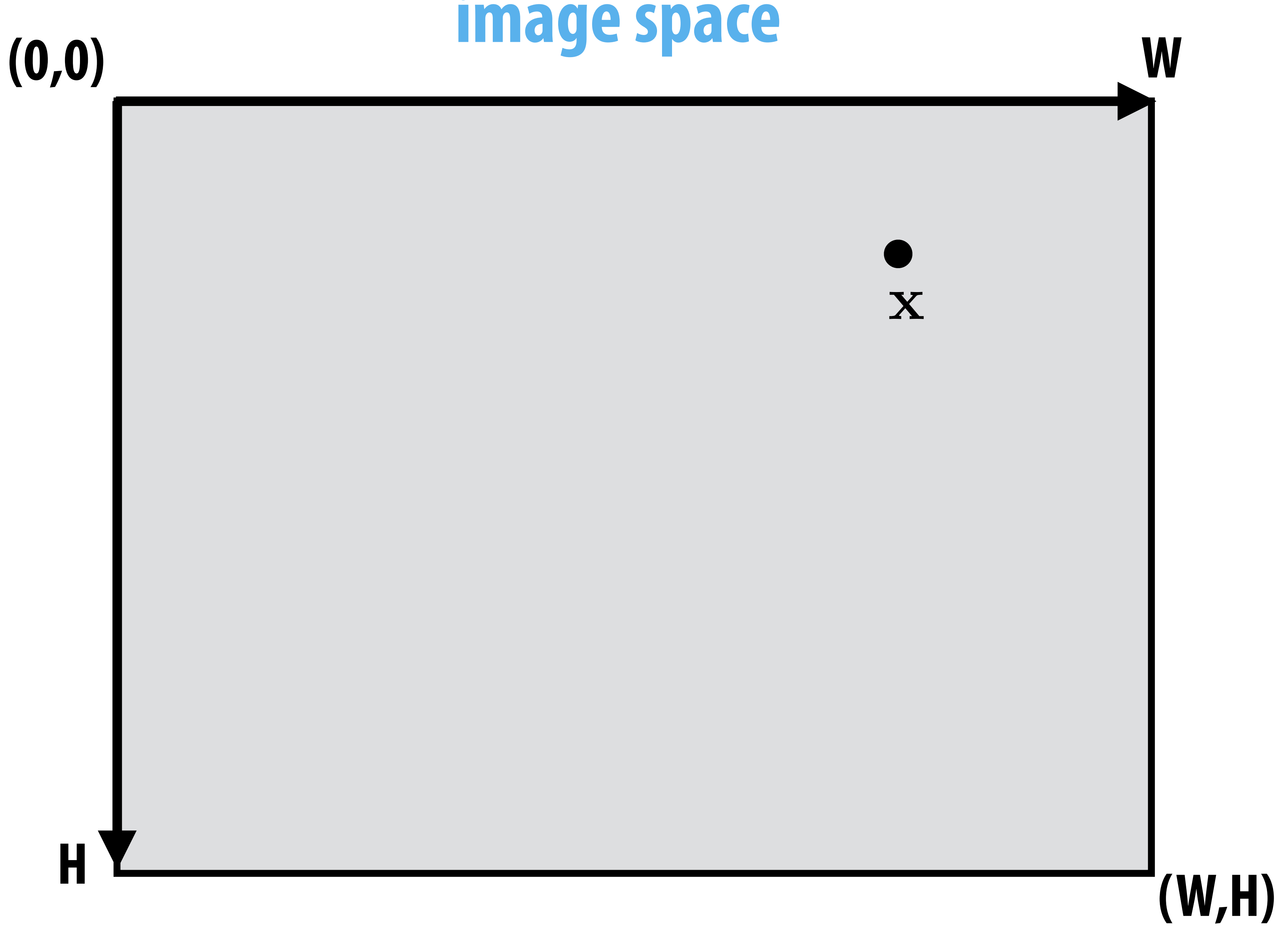
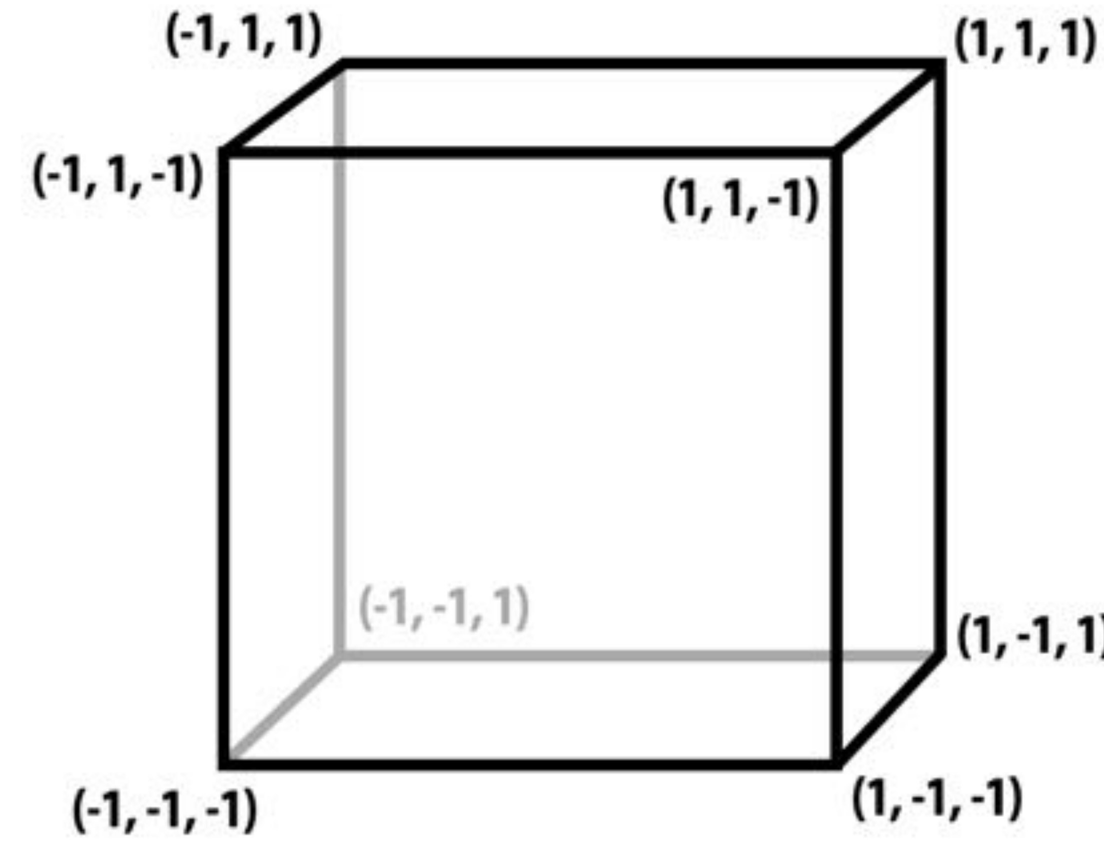


image space

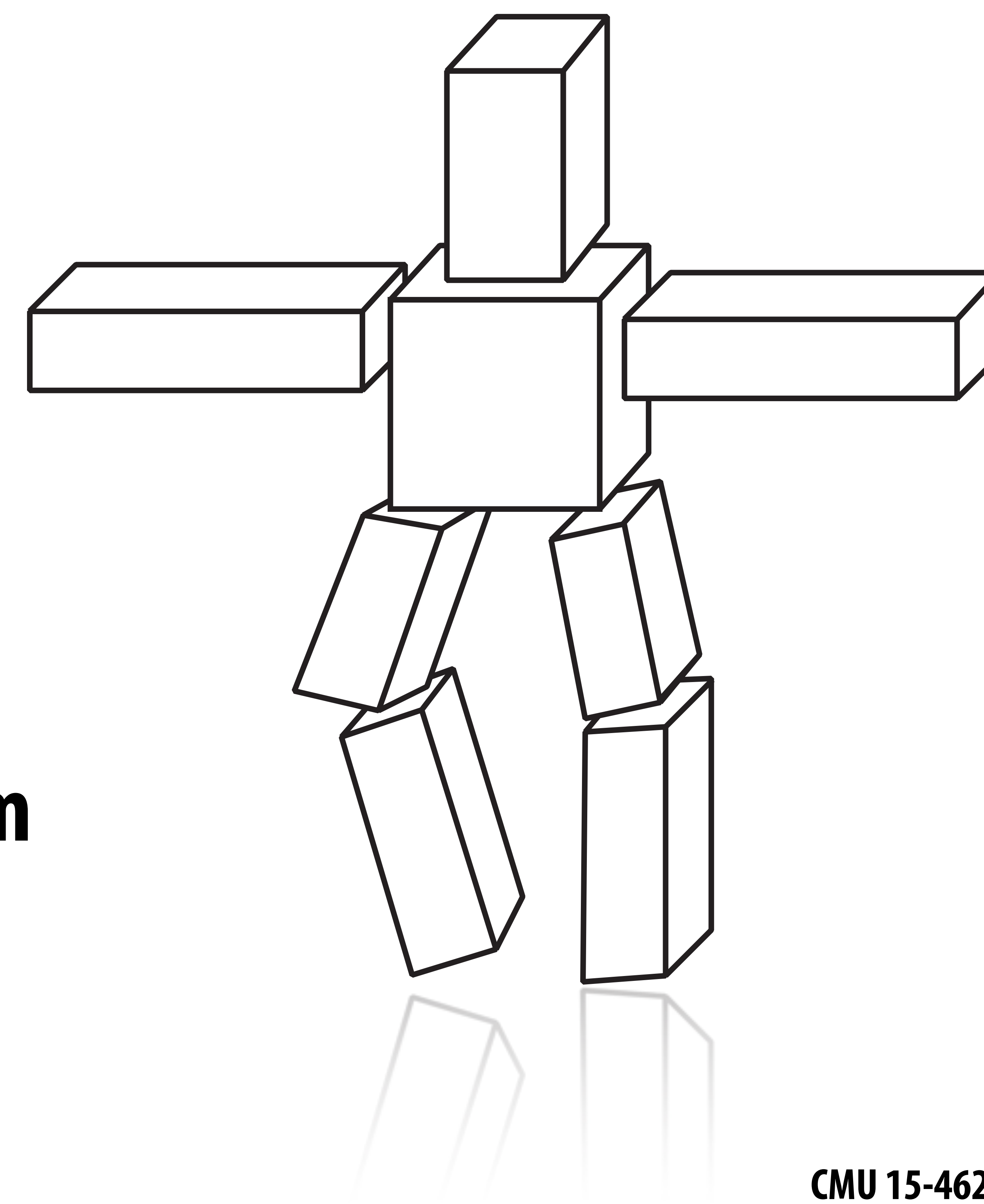


Q: What transformation(s) would you apply? (Careful: y is now down!)

Scene Graph

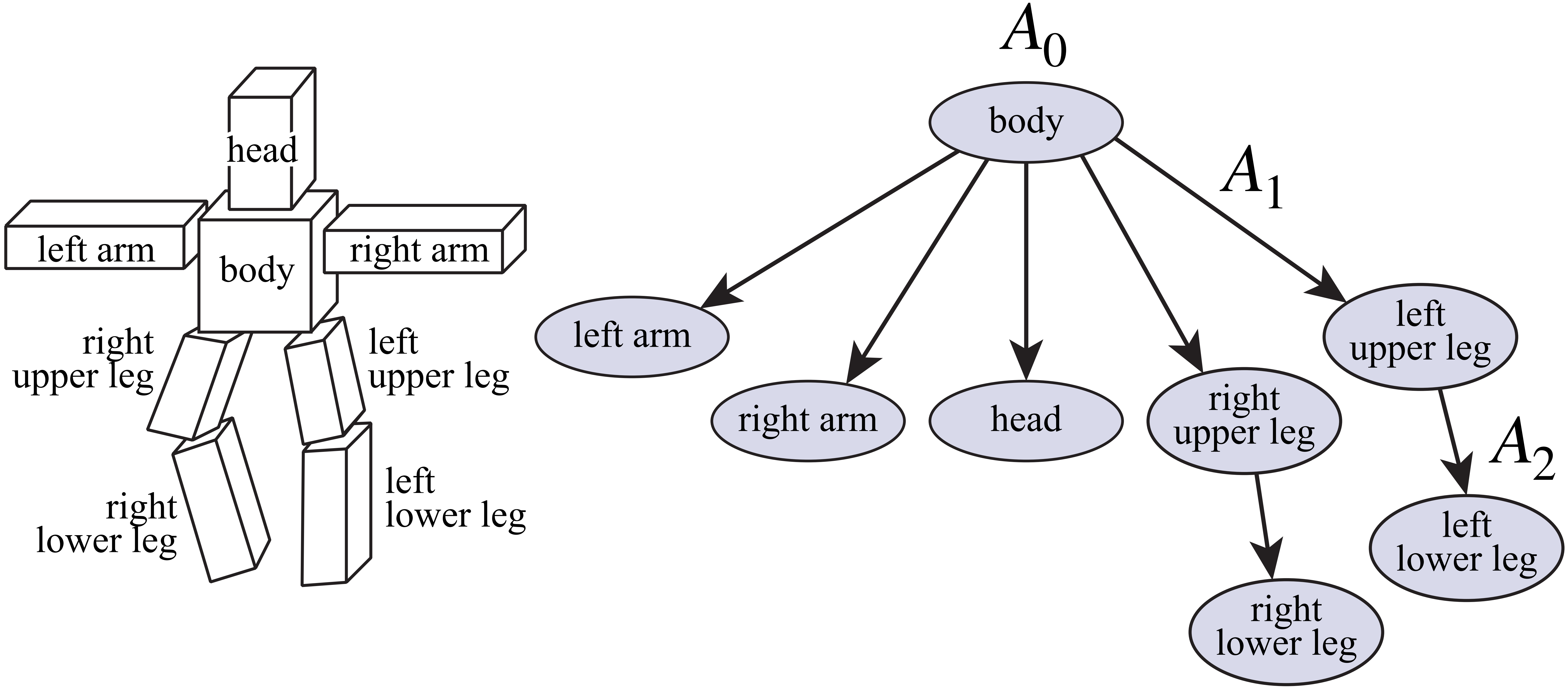


- For complex scenes (e.g., more than just a cube!) <u>scene graph</u> can help organize transformations
- Motivation: suppose we want to build a "cube creature" by transforming copies of the unit cube
- Difficult to specify each transformation directly
- Instead, build up transformations of "lower" parts from transformations of "upper" parts
  - E.g., first position the body
  - Then transform upper arm <u>relative to</u> the body
  - Then transform lower arm relative to upper arm
  - …



Scene Graph (continued)

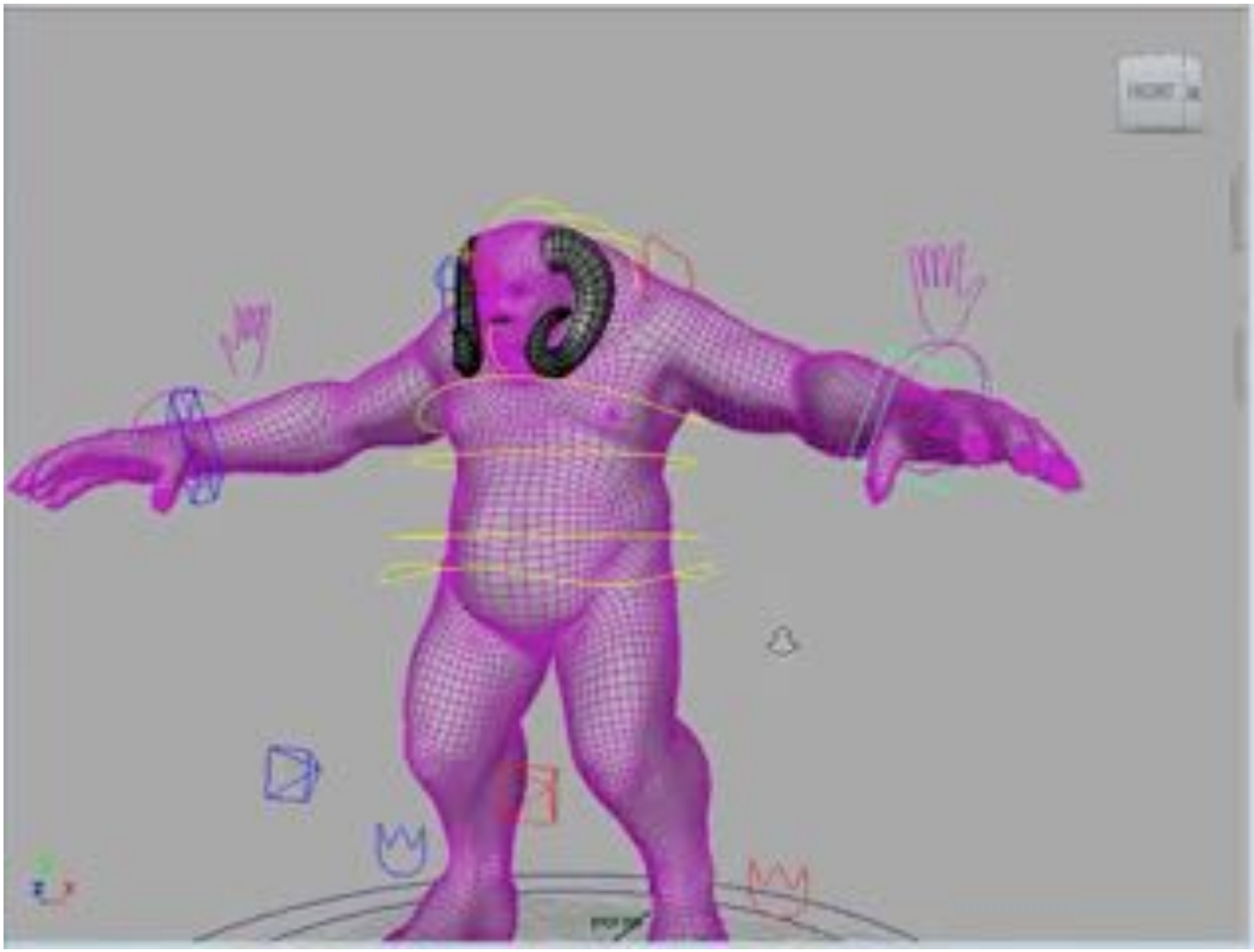
- **Scene graph stores relative transformations in directed graph**
- **Each edge (+root) stores a linear transformation (e.g., a 4x4 matrix)**
- **Composition of transformations gets applied to nodes**



- **E.g., $A_1 A_0$ gets applied to left upper leg; $A_2 A_1 A_0$ to left lower leg**
- **Keep transformations on a stack to reduce redundant multiplication**

Scene Graph—Example

Often used to build up complex "rig":

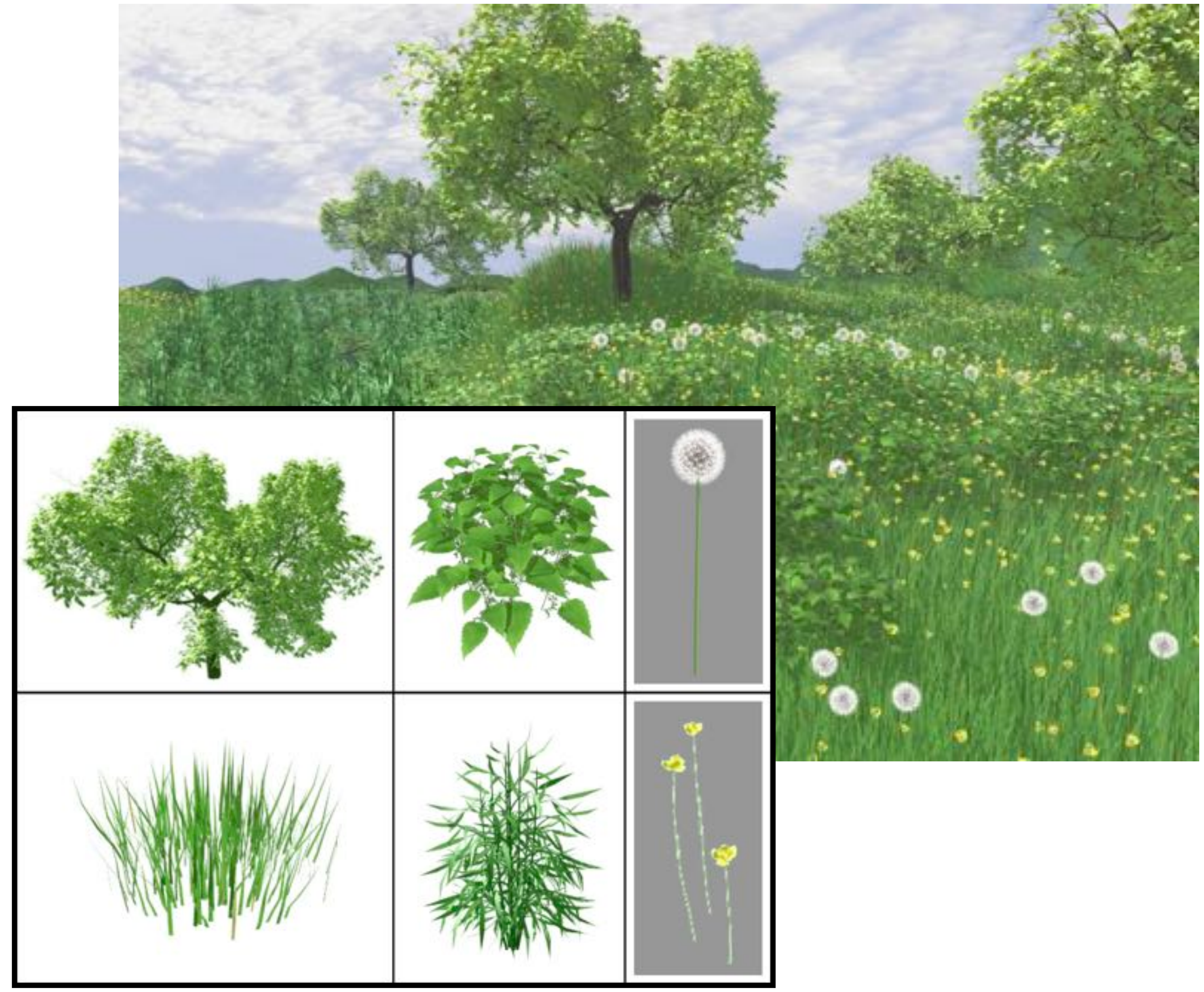


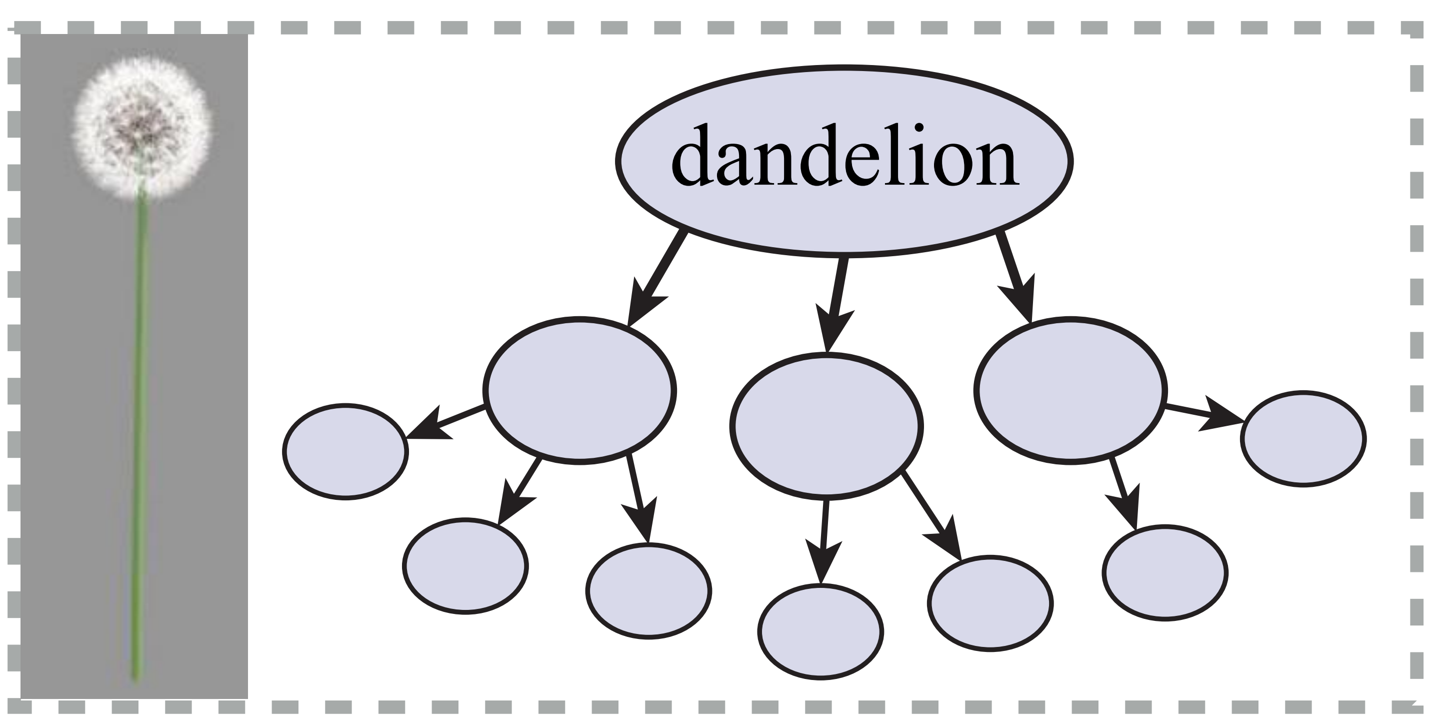
In general, scene graph also includes other models, lights, cameras, …

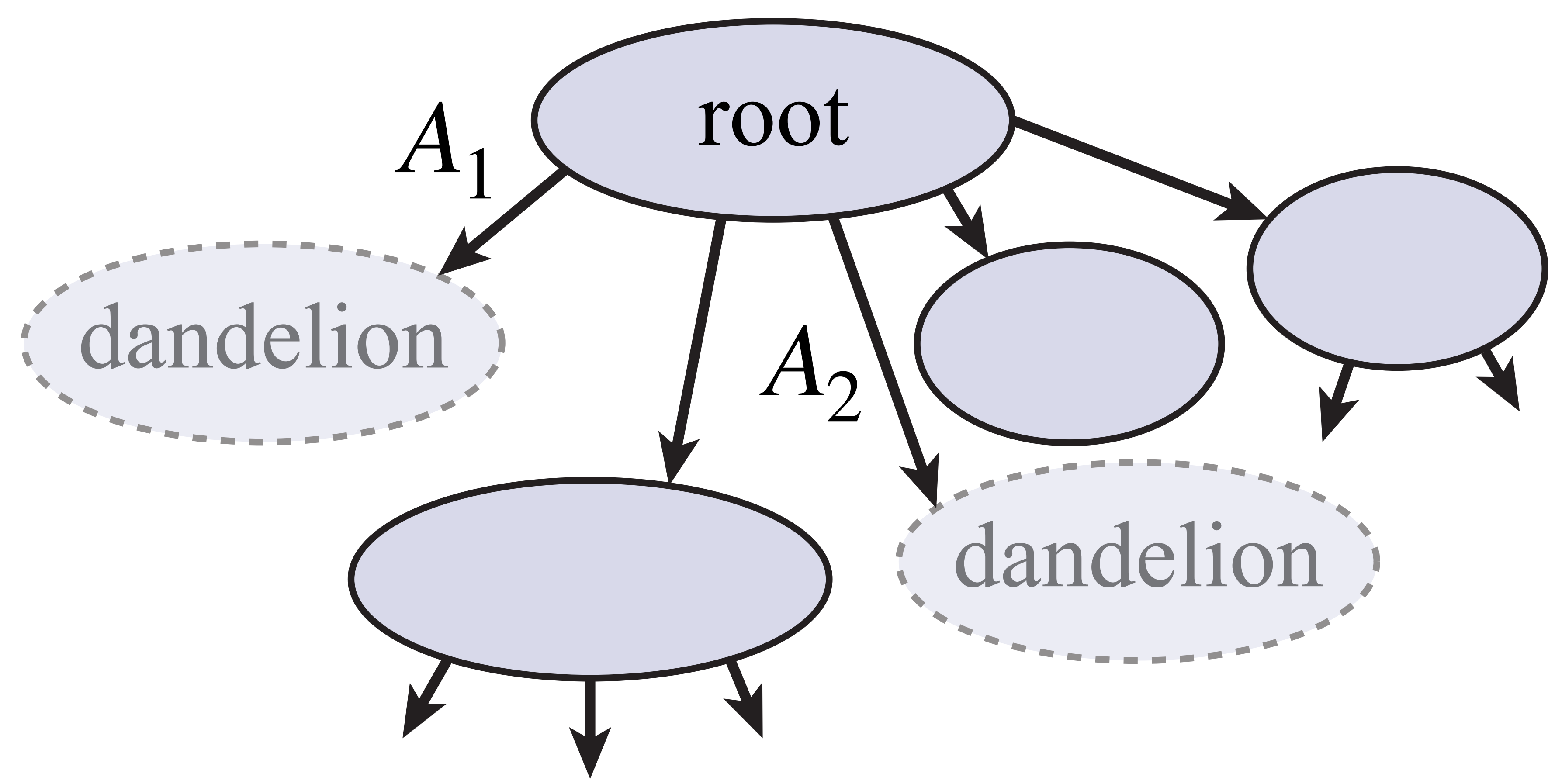
Deussen et al, "Realistic modeling and rendering of plant ecosystems"

Instancing

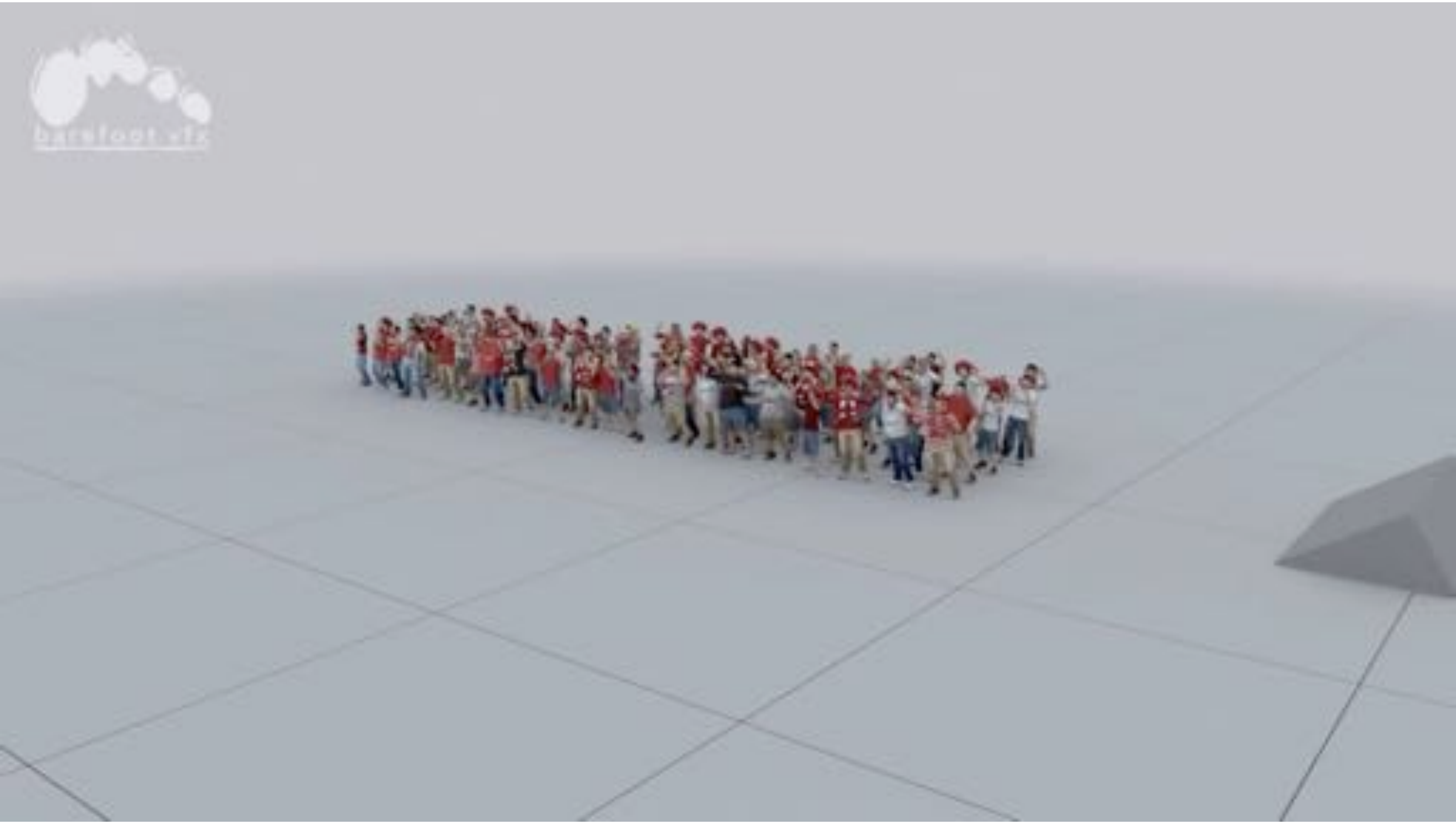
- What if we want many copies of the same object in a scene?
- Rather than have many copies of the geometry, scene graph, etc., can just put a "pointer" node in our scene graph
- Like any other node, can specify a different transformation on each incoming edge



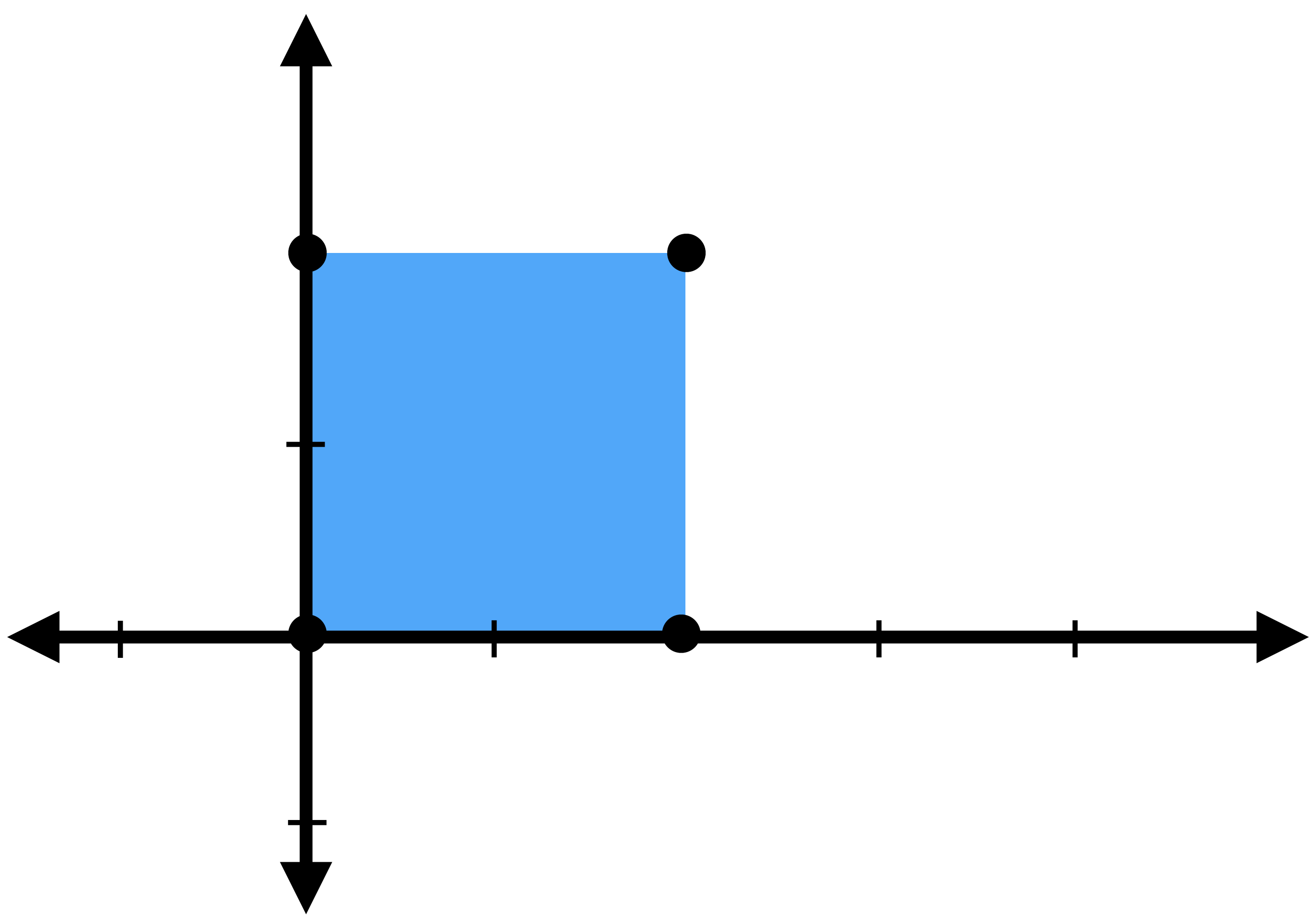




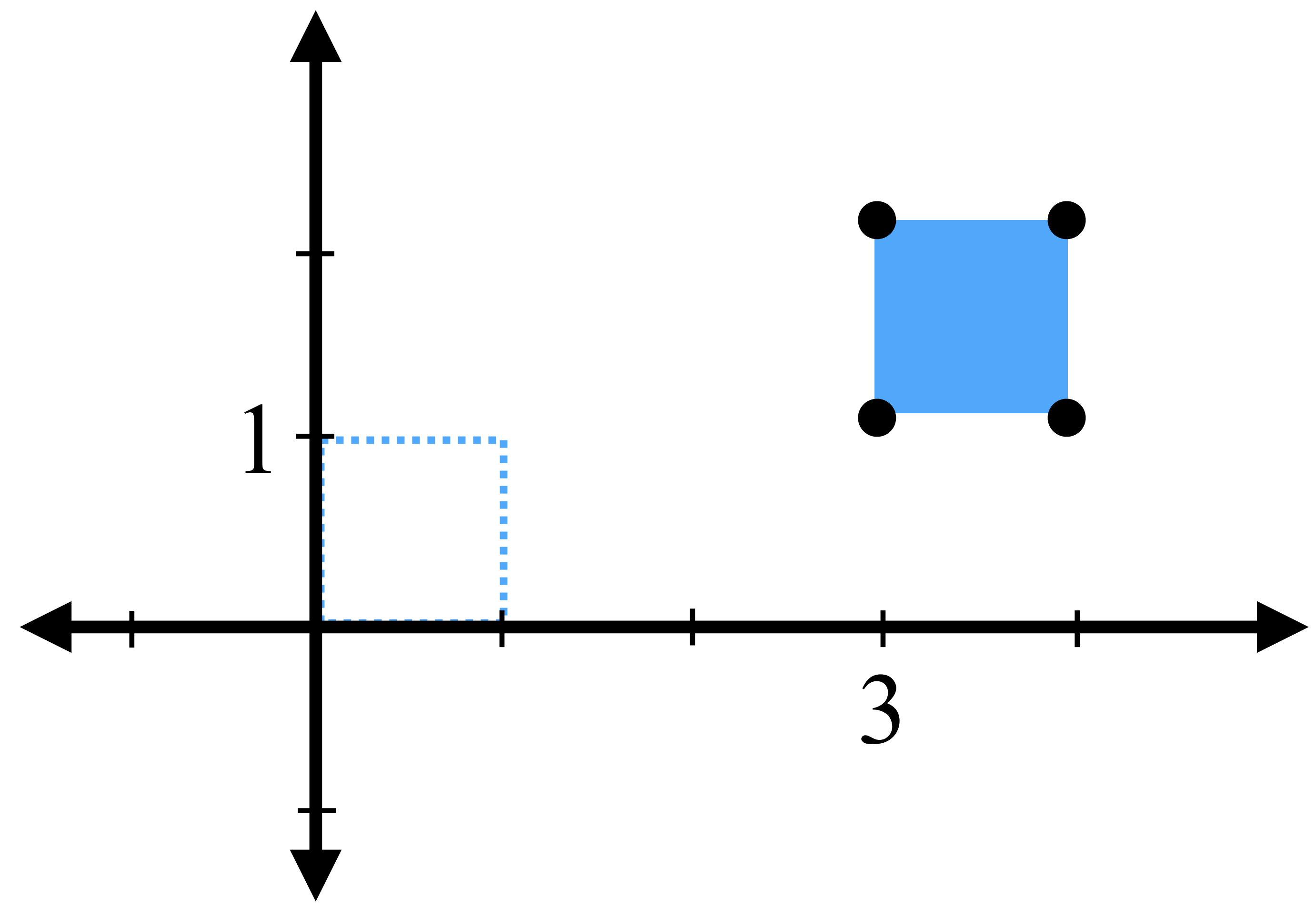
Instancing—Example



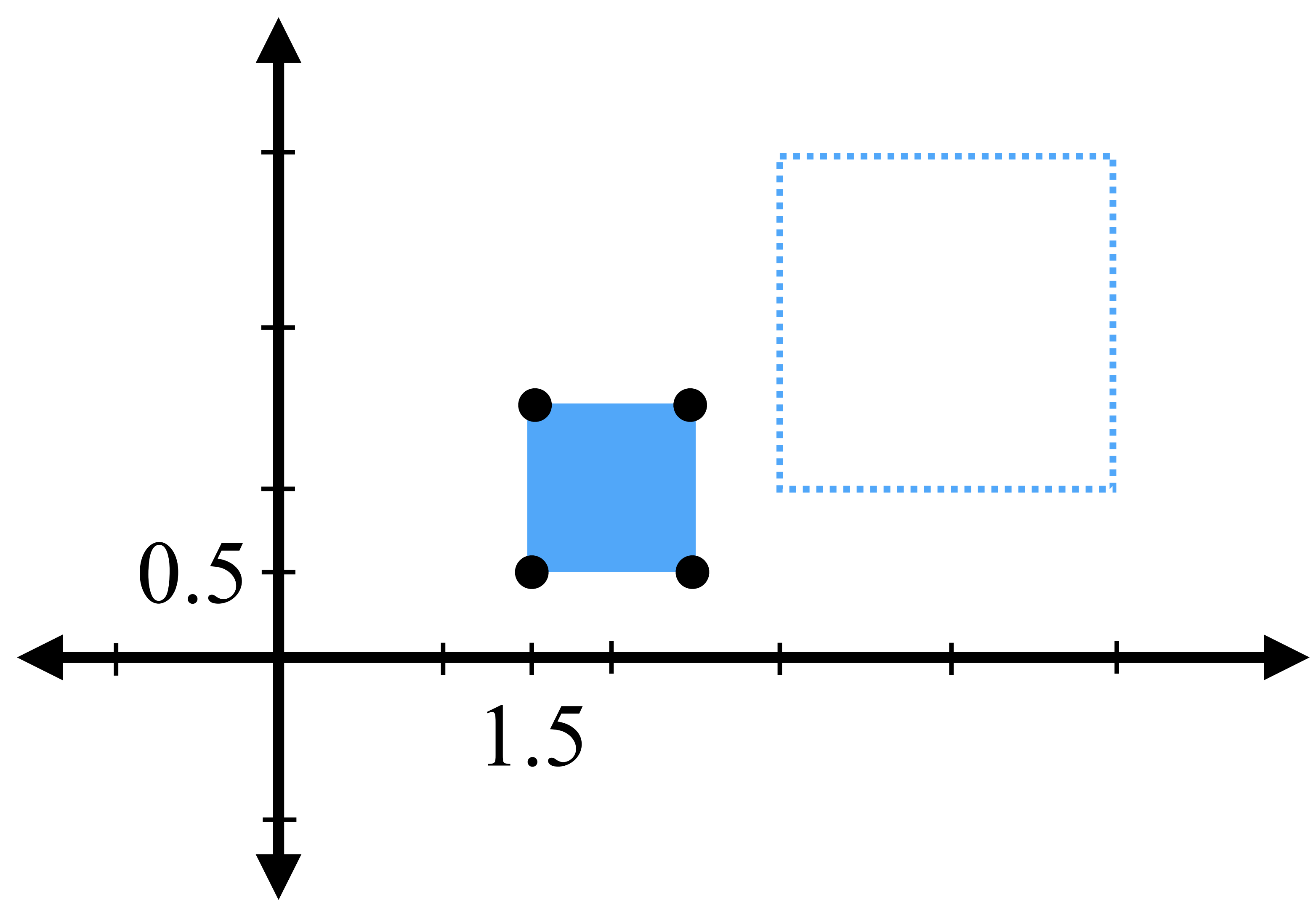
Order matters when composing transformations!



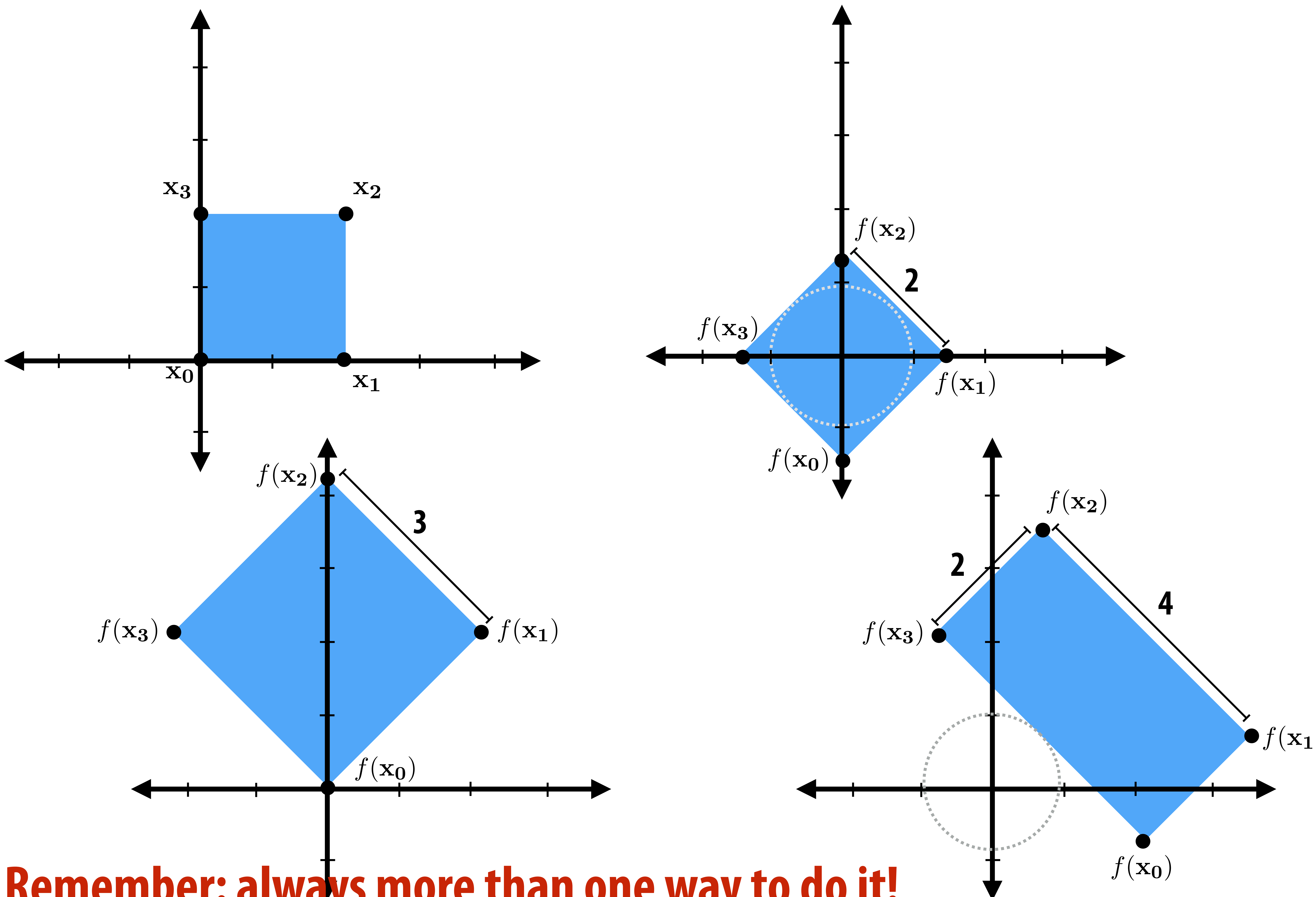
scale by 1/2, then translate by (3,1)



translate by (3,1), then scale by 1/2

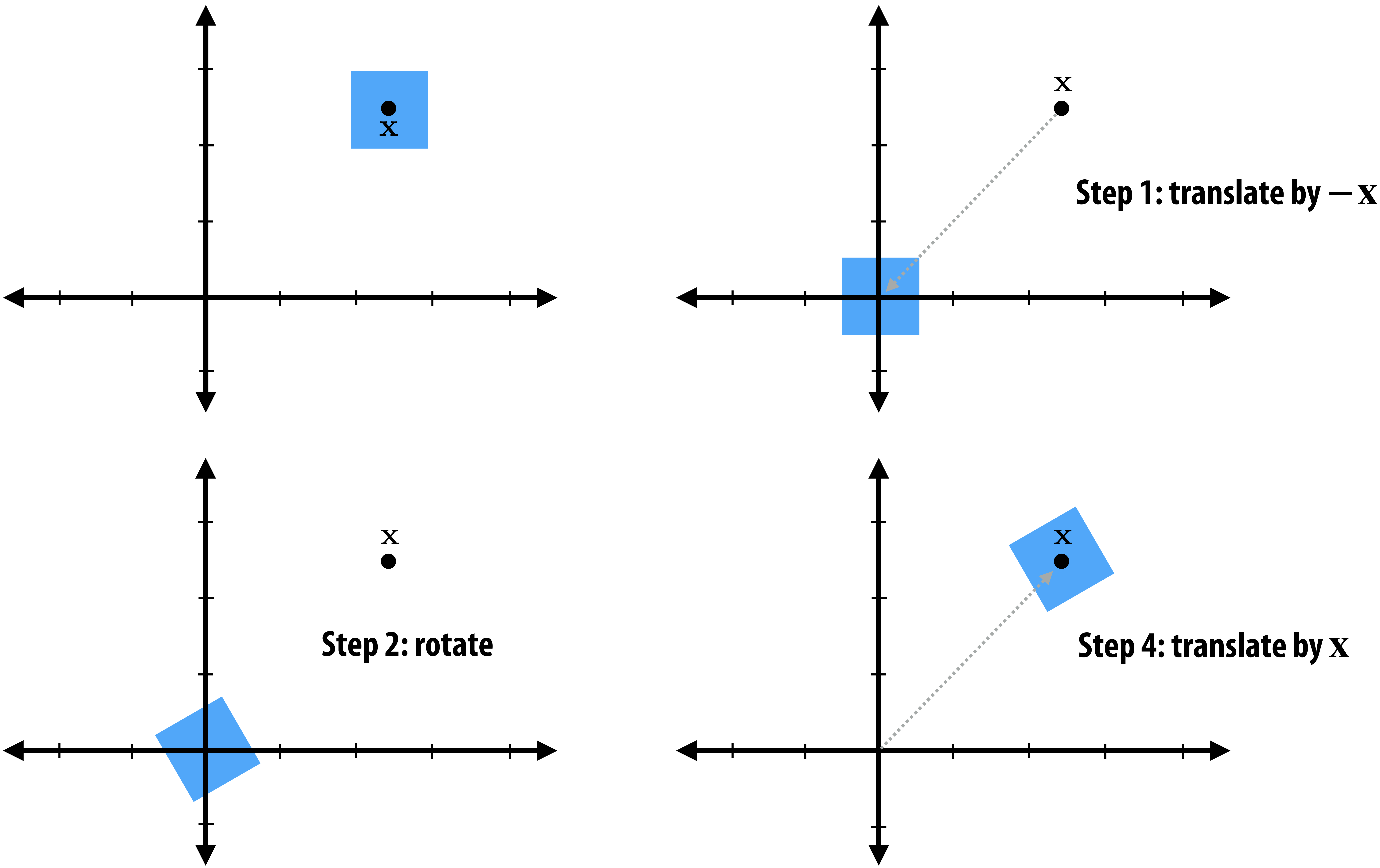


How would you perform these transformations?



Remember: always more than one way to do it!

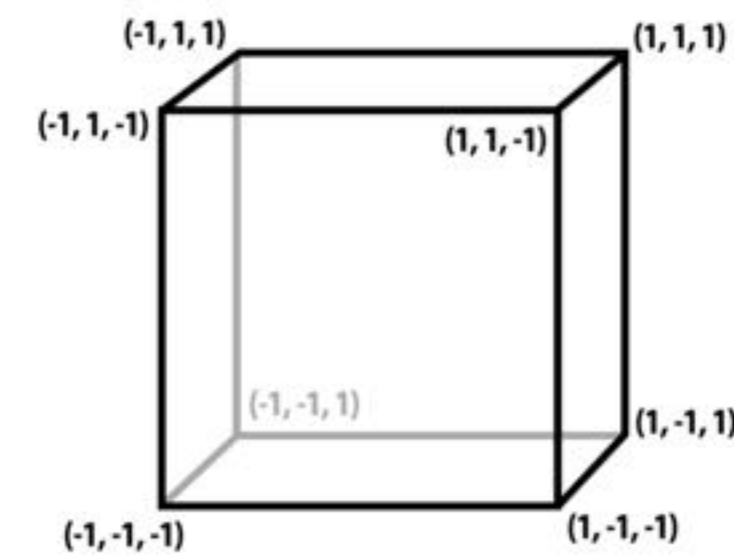
Common task: rotate about a point x

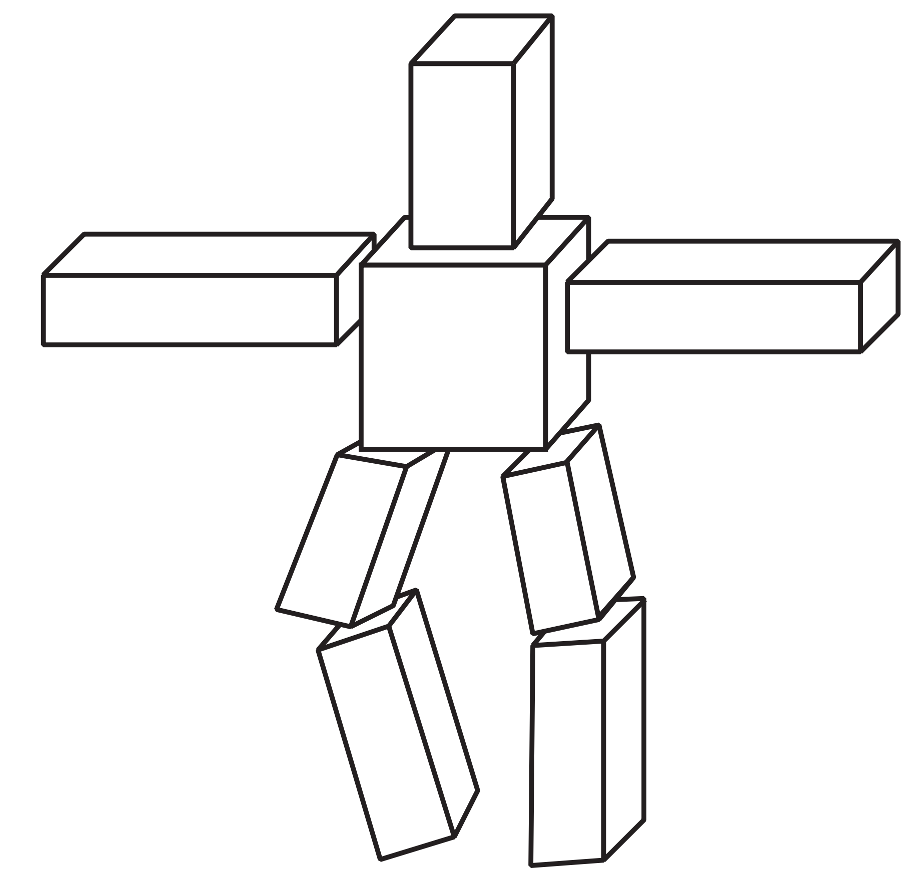


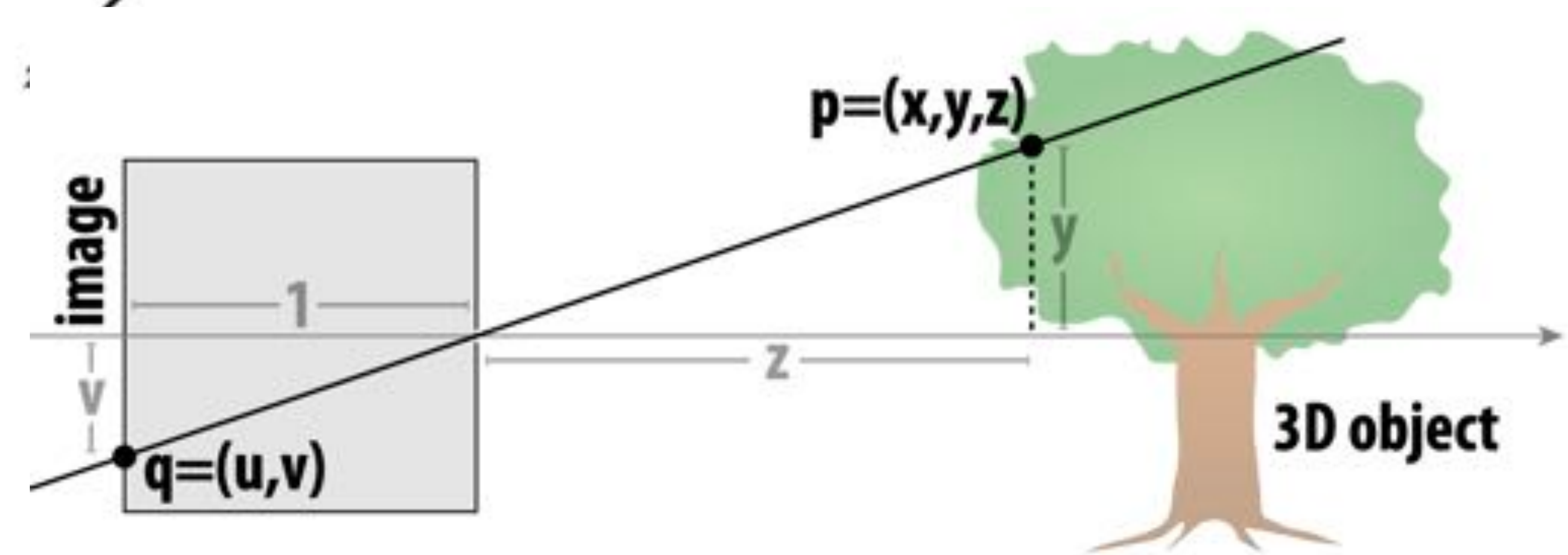
Q: What happens if we just rotate without translating first?

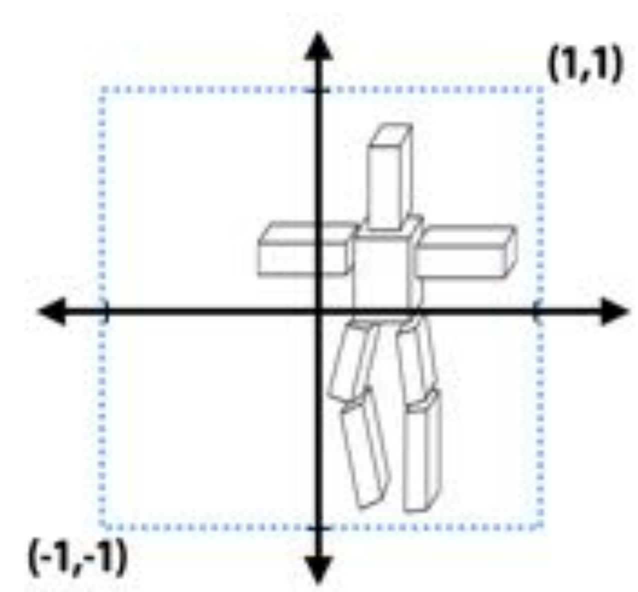
Drawing a Cube Creature

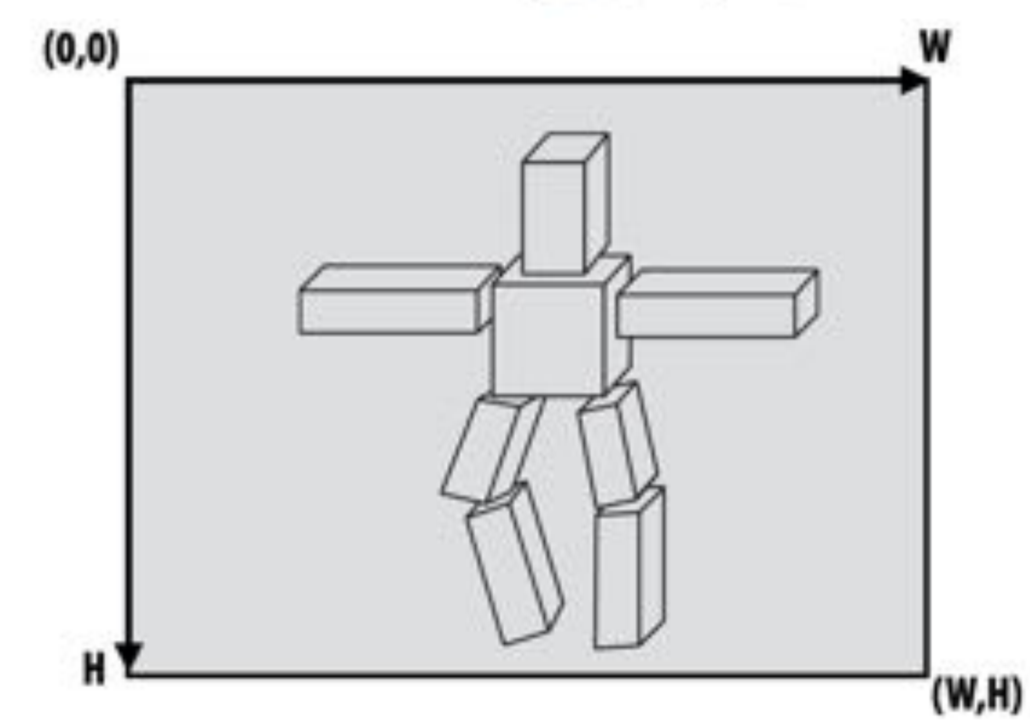
- Let’s put this all together: starting with our 3D cube, we want to make a 2D, perspective-correct image of a “cube creature”
- First we use our scene graph to apply 3D transformations to several copies of our cube
- Then we apply a 3D transformation to position our camera
- Then a perspective projection
- Finally we convert to image coordinates (and rasterize)
- …Easy, right? :-)











Spatial Transformations—Summary

transformation defined by its invariants

basic linear transformations

scaling
rotation
reflection
shear

basic nonlinear transformations

translation
perspective projection

linear when represented via homogeneous coords

homogeneous coords also distinguish points & vectors

composite transformations

- compose basic transformations to get more interesting ones
- always reduces to a single 4x4 matrix (in homogeneous coordinates)
  - simple, unified representation, efficient implementation
- order of composition matters!
- many ways to decompose a given transformation (polar, SVD, ...)
- use scene graph to organize transformations
  - use instancing to eliminate redundancy

Next time: 3D Rotations