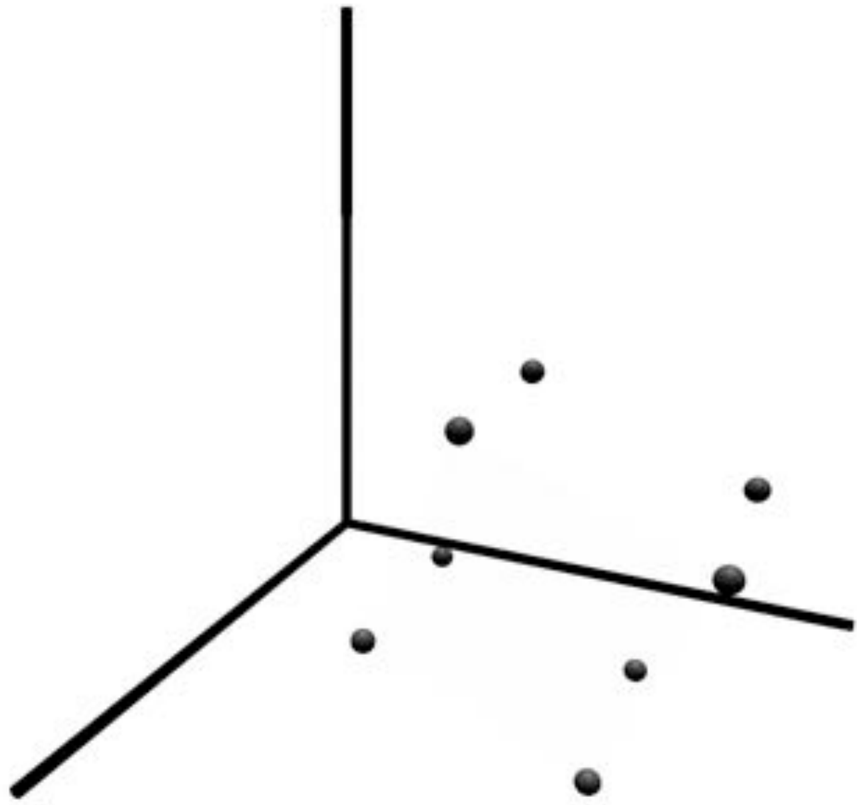
# Spatial Transformations

## Computer Graphics
## CMU 15-462/15-662

# Spatial Transformation

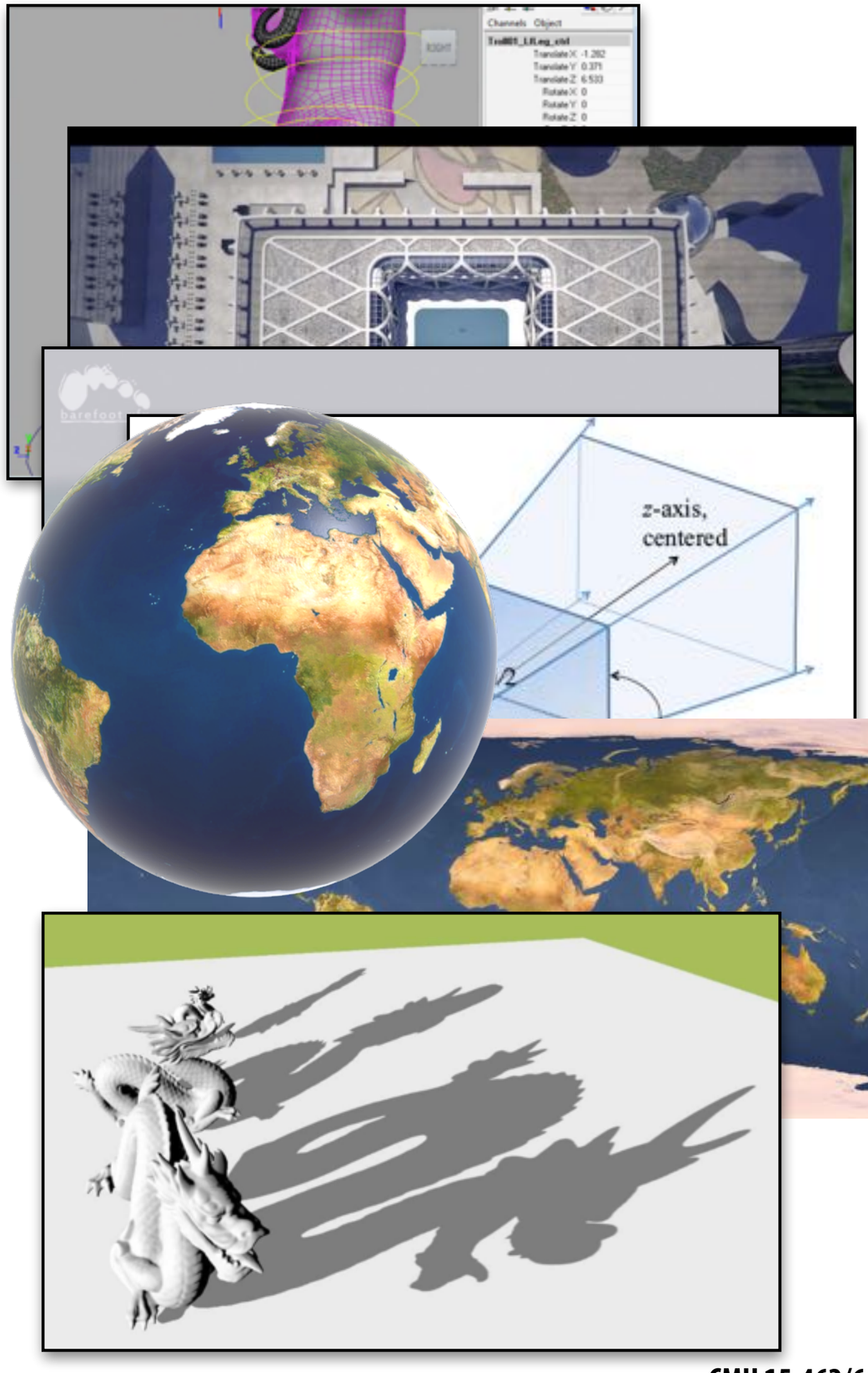
- Basically any function that assigns each point a new location
- Today we'll focus on common transformations of space (rotation, scaling, etc.) encoded by linear maps



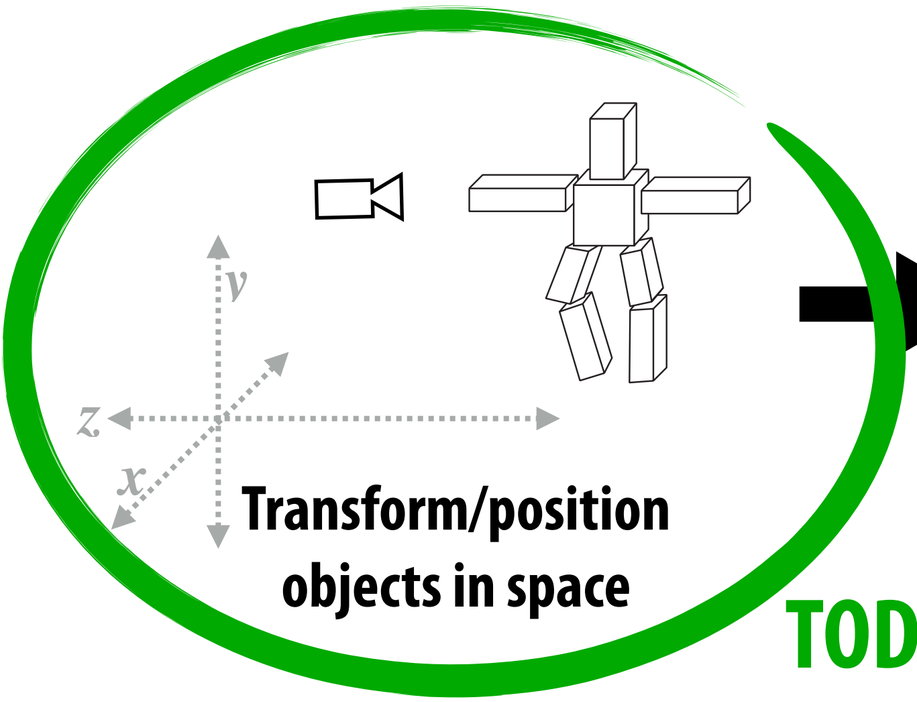
$$f : \mathbb{R}^n \to \mathbb{R}^n$$

# Transformations in Computer Graphics

- Where are linear transformations used in computer graphics?
- All over the place!
  - Position/deform objects in space
  - Move the camera
  - Animate objects over time
  - Project 3D objects onto 2D images
  - Map 2D textures onto 3D objects
  - Project shadows of 3D objects onto other 3D objects
  - …

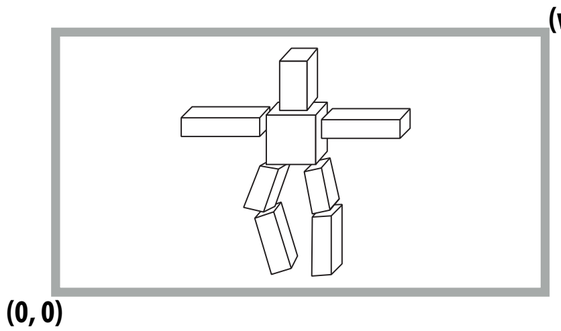


# The Rasterization Pipeline

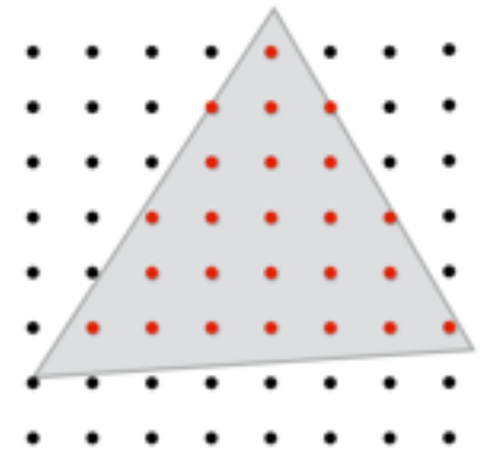



Transform/position objects in space

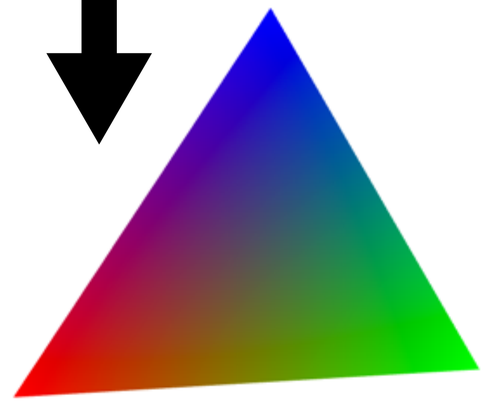
TODAY



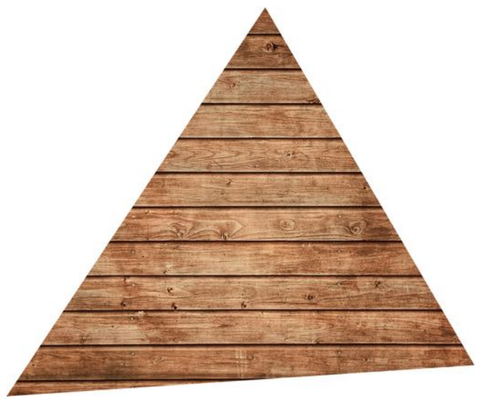

Project objects onto the screen



Sample triangle coverage



Interpolate triangle attributes at covered samples



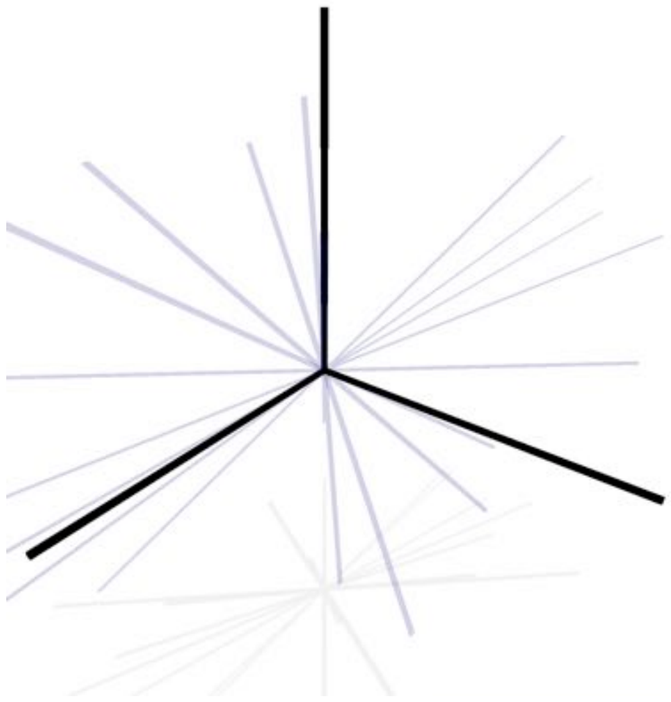
Sample texture maps / evaluate shaders

Combine samples into final image (depth, alpha, . . .)

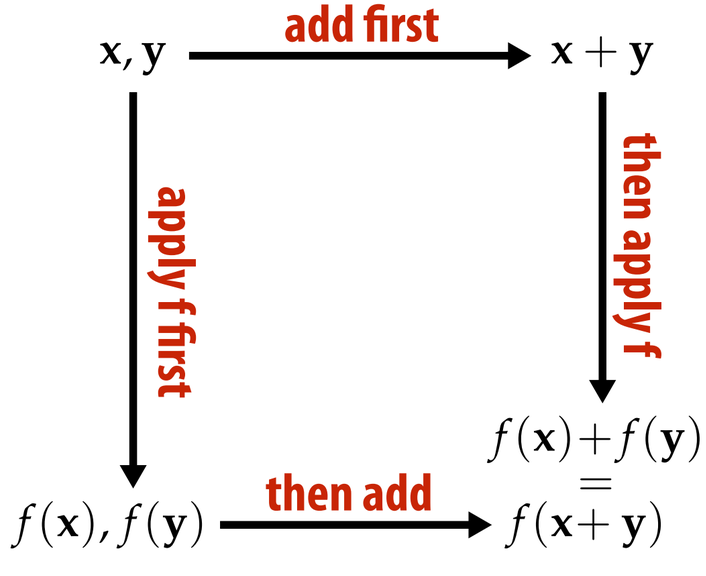
# Review: Linear Maps

**Q: What does it mean for a map $f: \mathbb{R}^n \to \mathbb{R}^n$ to be linear?**

**Geometrically: it maps lines to lines, and preserves the origin**



**Algebraically: preserves vector space operations (addition & scaling)**



# Why do we care about linear transformations?

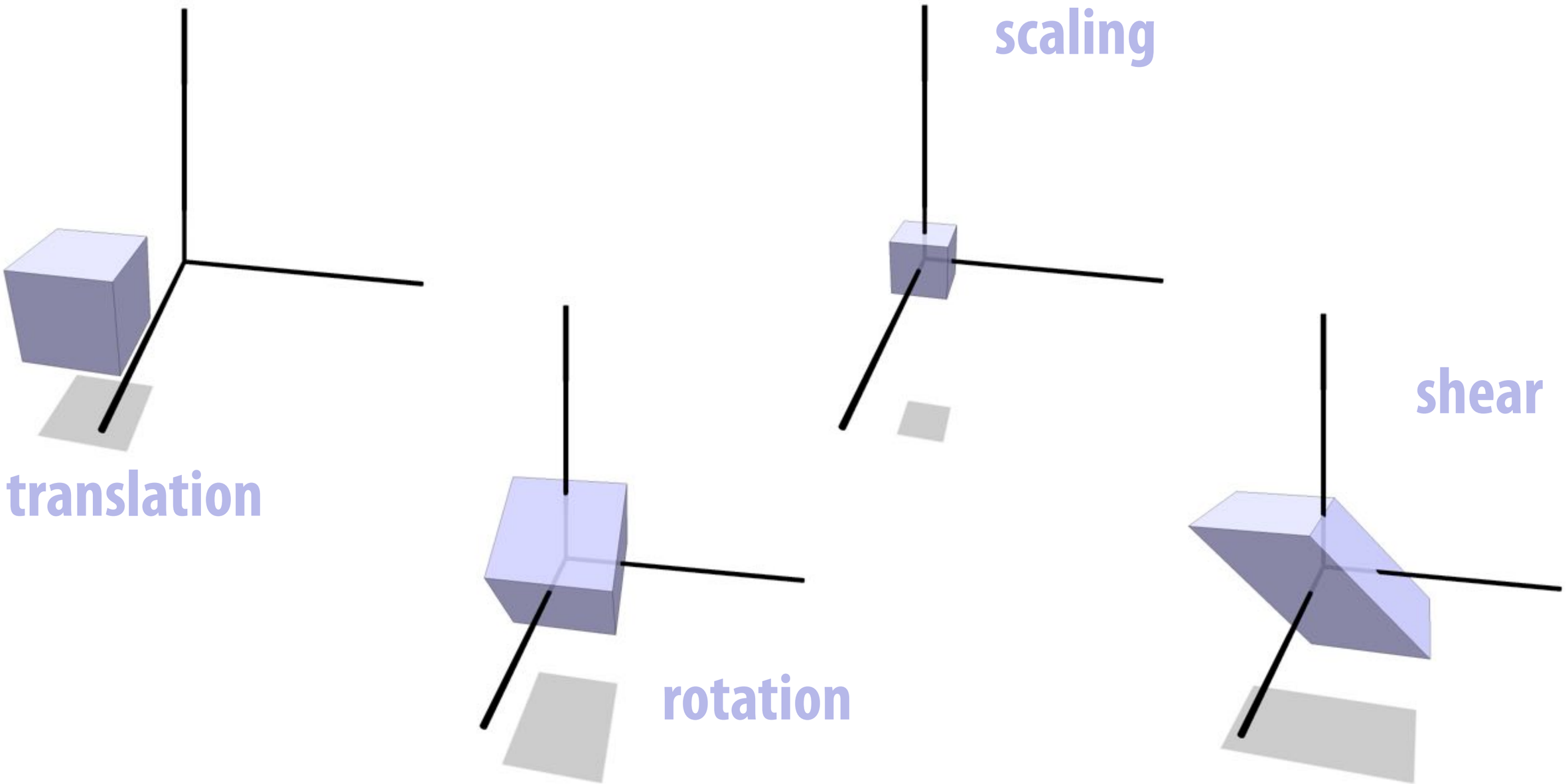
- Cheap to apply
- Usually pretty easy to solve for (linear systems)
- Composition of linear transformations is linear
  - product of many matrices is a single matrix
  - gives uniform representation of transformations
  - simplifies graphics algorithms, systems (e.g., GPUs & APIs)

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\textit{rotation}} \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\textit{scale}} \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{\textit{rotation}} \cdots = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\textit{composite transformation}}$$

# What kinds of linear transformations can we compose?

# Types of Transformations

## What would you call each of these types of transformations?




## Q: How did you know that? (Hint: you did not inspect a formula!)

# Invariants of Transformation

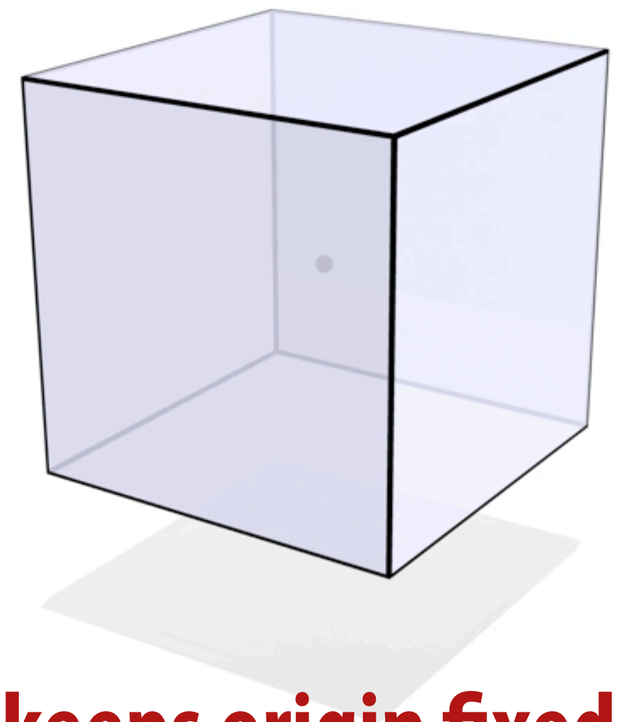
## A transformation is determined by the invariants it preserves

| transformation | invariants | algebraic description |
|---|---|---|
| linear | *straight lines / origin* | $f(a\mathbf{x}+\mathbf{y}) = af(\mathbf{x}) + f(\mathbf{y}),$ $f(0) = 0$ |
| translation | *differences between pairs of points* | $f(\mathbf{x}-\mathbf{y}) = \mathbf{x}-\mathbf{y}$ |
| scaling | *lines through the origin / direction of vectors* | $f(\mathbf{x})/\lvert f(\mathbf{x})\rvert = \mathbf{x}/\lvert\mathbf{x}\rvert$ |
| rotation | *origin / distances between points / orientation* | $\lvert f(\mathbf{x})-f(\mathbf{y})\rvert = \lvert\mathbf{x}-\mathbf{y}\rvert,$ $\det(f) > 0$ |
| … | … | … |

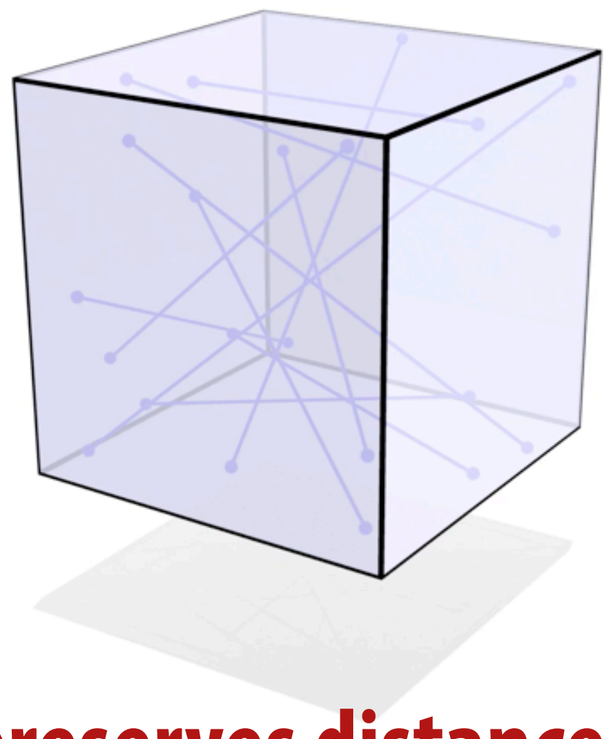
**(Essentially how your brain "knows" what kind of transformation you're looking at…)**

# Rotation

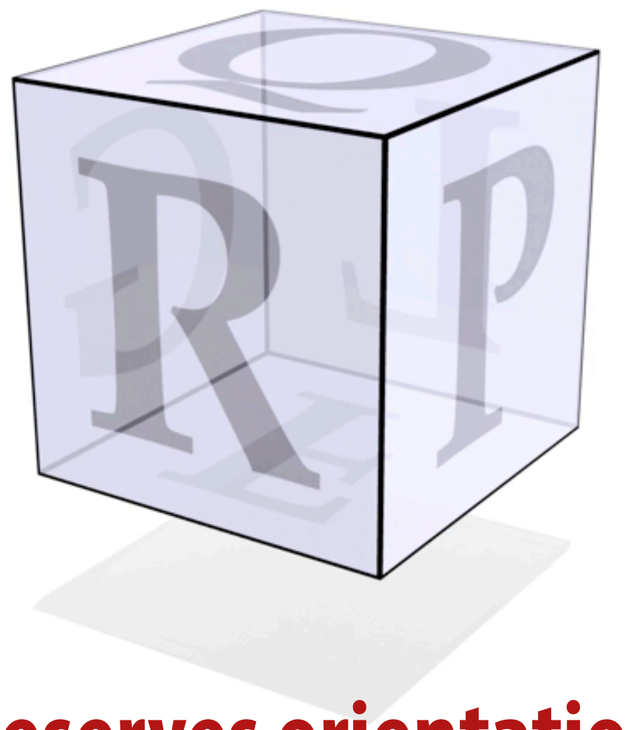
## Rotations defined by three basic properties:



keeps origin fixed



preserves distances



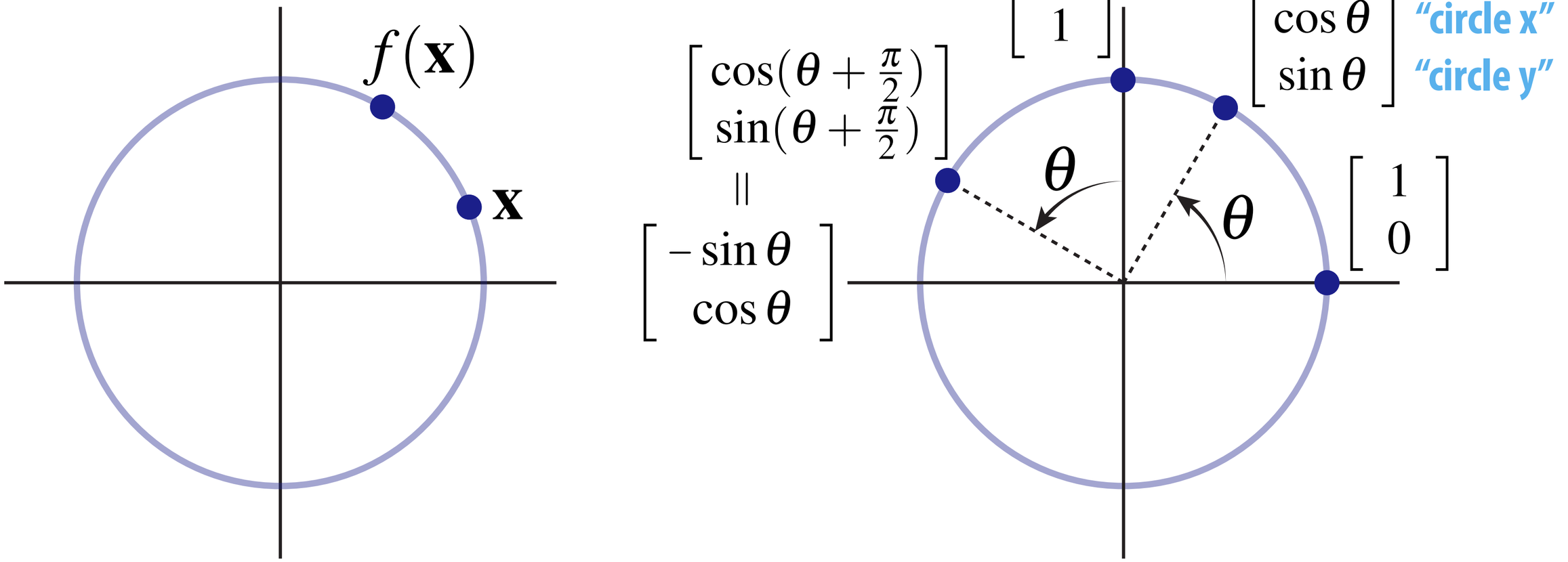
preserves orientation

**First two properties together imply that rotations are linear.**

**Will have a lot more to say about rotations next lecture…**

# 2D Rotations—Matrix Representation

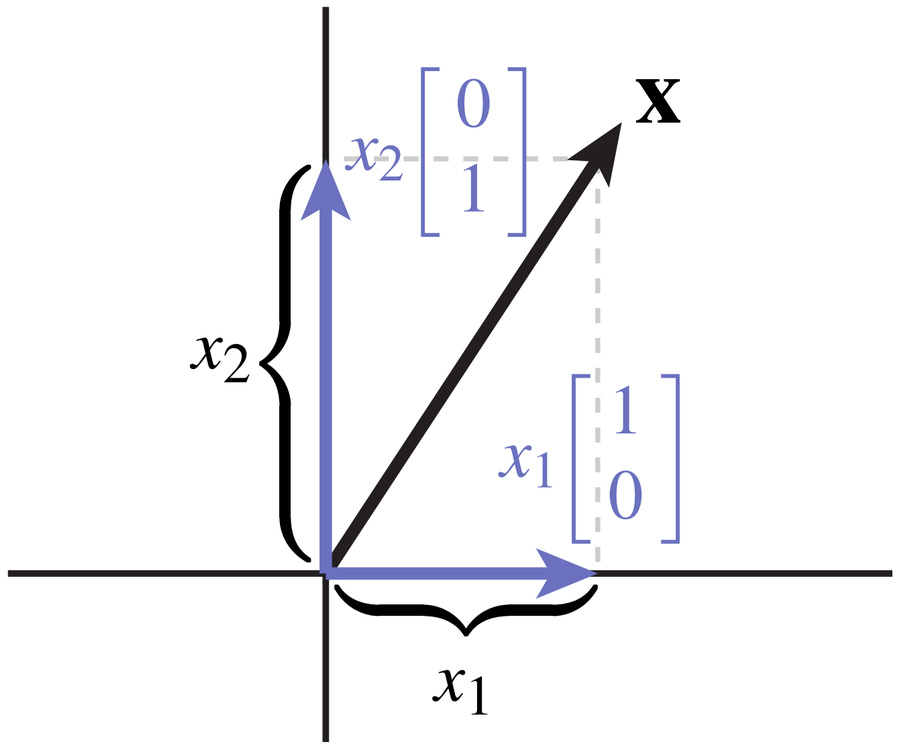
**Rotations preserve distances and the origin—hence, a 2D rotation by an angle $\theta$ maps each point $\mathbf{x}$ to a point $f_\theta(\mathbf{x})$ on the circle of radius $|\mathbf{x}|$:**

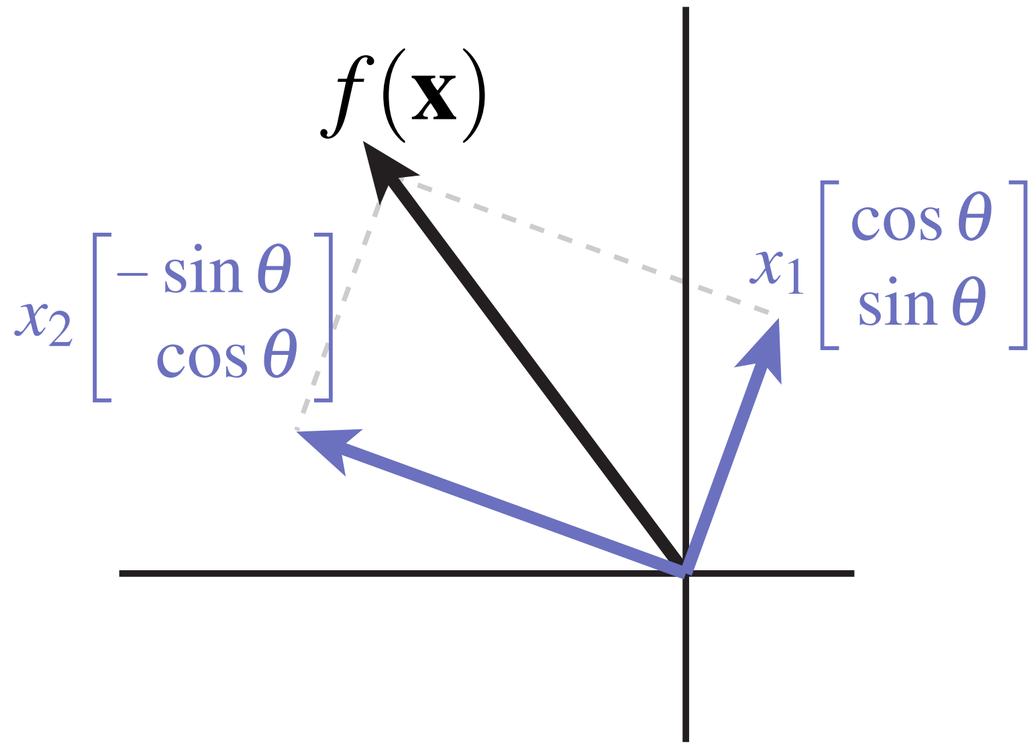



- **Where does $\mathbf{x} = (1,0)$ go if we rotate by $\theta$ (counter-clockwise)?**
- **How about $\mathbf{x} = (0,1)$?**

**What about a general vector $\mathbf{x} = (x_1, x_2)$?**

# 2D Rotations—Matrix Representation




$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad f(\mathbf{x}) = x_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

**So, How do we represent the 2D rotation function $f_\theta(\mathbf{x})$ using a matrix?**

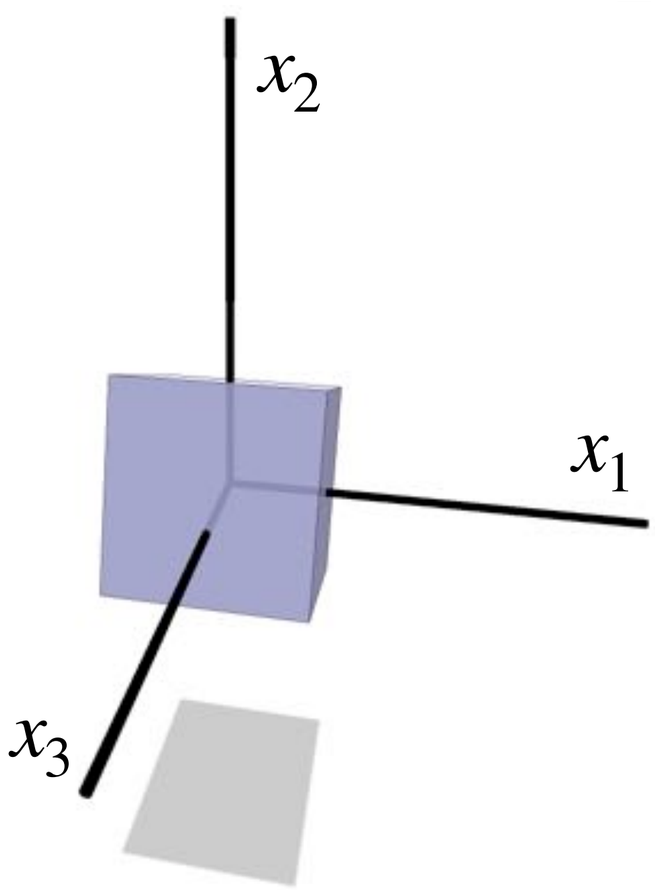
$$f_\theta(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

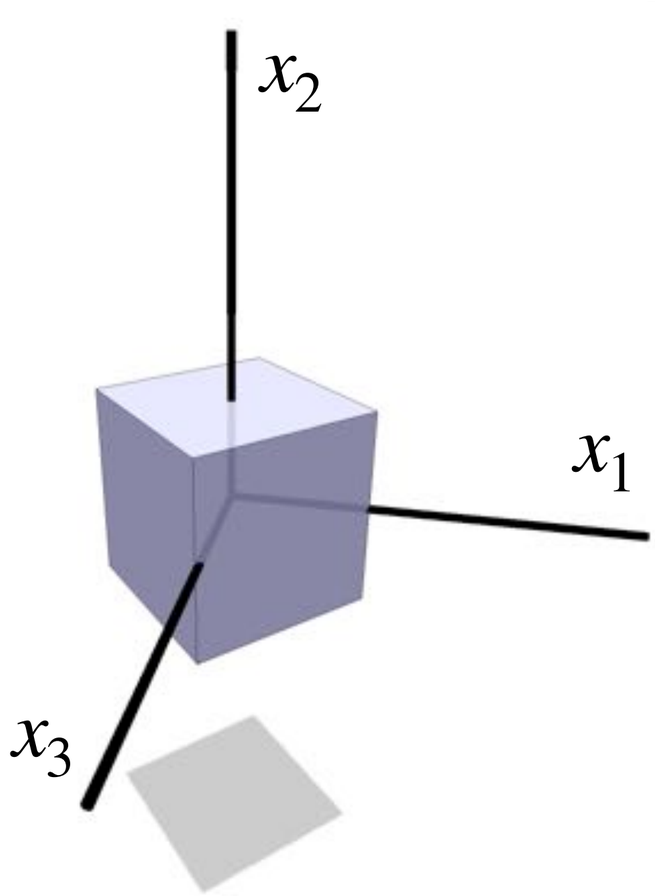
# 3D Rotations

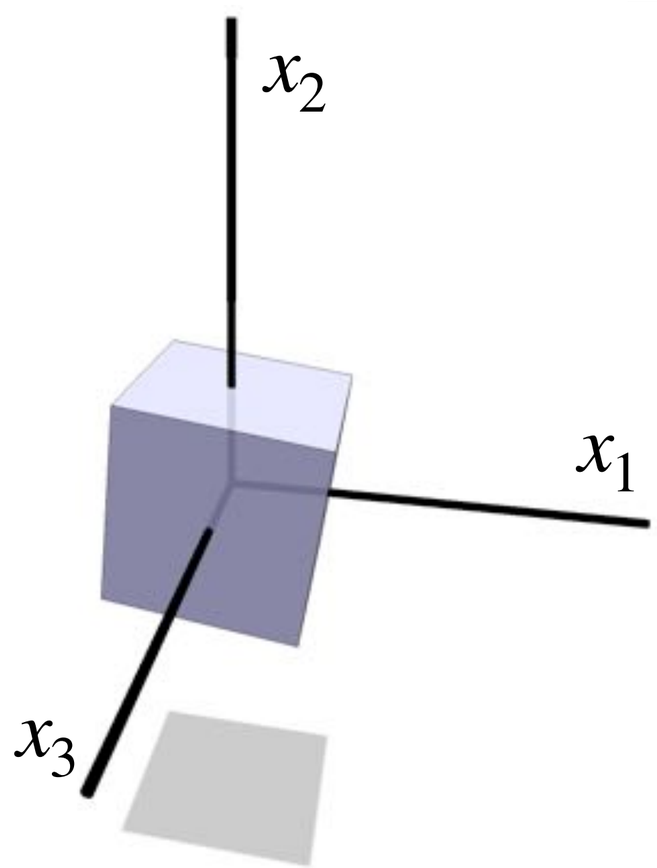
- **Q: In 3D, how do we rotate around the $x_3$-axis?**
- **A: Just apply the same transformation of $x_1$, $x_2$; keep $x_3$ fixed**

rotate around $x_1$ | rotate around $x_2$ | rotate around $x_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin(\theta) \\ 0 & \sin\theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin(\theta) & 0 \\ \sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

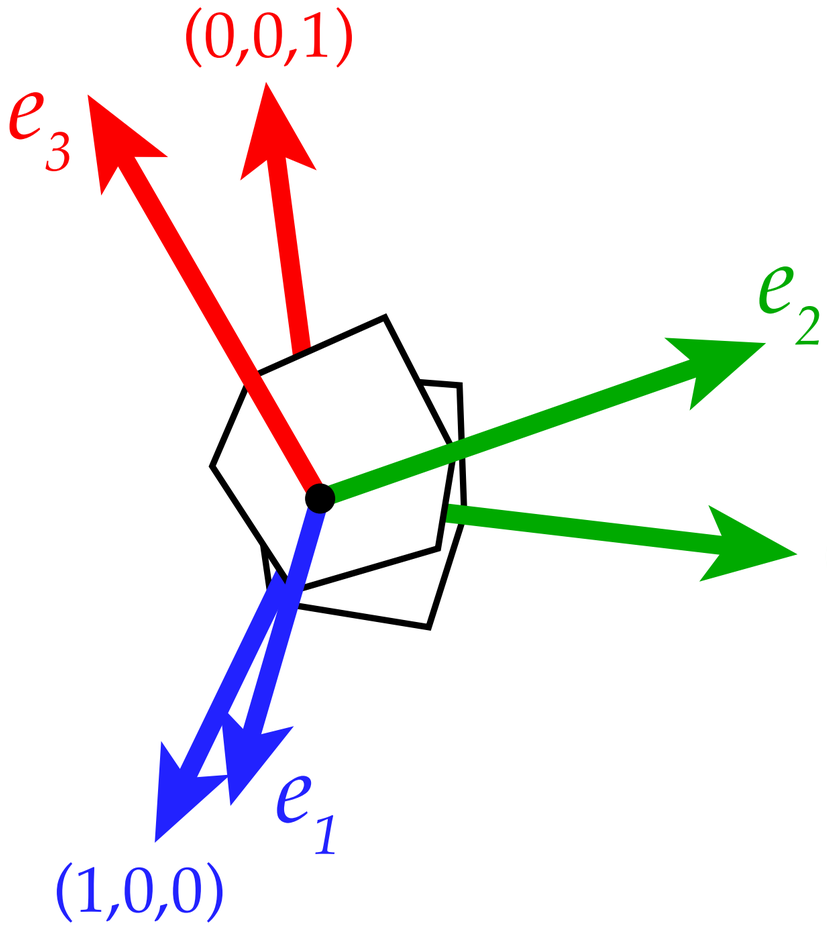


# Rotations—Transpose as Inverse

**Rotation will map standard basis to orthonormal basis $e_1, e_2, e_3$:**




$$R^\mathsf{T} \qquad R$$

$$\begin{bmatrix} — & e_1^T & — \\ — & e_2^T & — \\ — & e_3^T & — \end{bmatrix} \begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} e_1^T e_1 & e_1^T e_2 & e_1^T e_3 \\ e_2^T e_1 & e_2^T e_2 & e_2^T e_3 \\ e_3^T e_1 & e_3^T e_2 & e_3^T e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I$$

**Hence, $R^\mathsf{T} R = I$, or equivalently, $R^\mathsf{T} = R^{-1}$.**

# Reflections

- Q: Does <u>every</u> matrix $Q^\mathsf{T}Q = I$ describe a rotation?
- Remember that rotations must preserve the <u>origin</u>, preserve <u>distances</u>, and preserve <u>orientation</u>
- Consider for instance this matrix:

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q^\mathsf{T}Q = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 1 \end{bmatrix} = I$$

**Q: Does this matrix represent a rotation?**
**(If not, which invariant does it fail to preserve?)**

**A: No! It represents a reflection across the y-axis (and hence fails to preserve <u>orientation</u>)**

# Orthogonal Transformations

- In general, transformations that preserve distances and the origin are called orthogonal transformations
- Represented by matrices $Q^\mathsf{T}Q = I$
  - Rotations additionally preserve orientation: $\det(Q) > 0$
  - Reflections reverse orientation: $\det(Q) < 0$

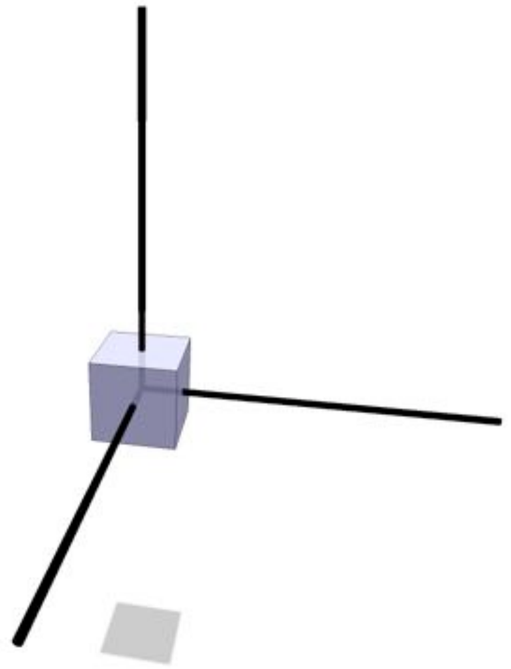
rotation

reflection

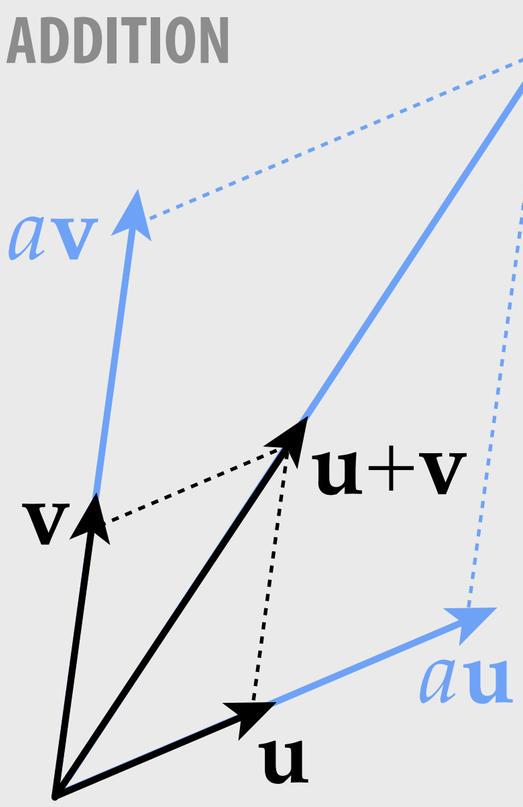
# Scaling

*assuming $a \neq 0,\ \mathbf{u} \neq 0$

- Each vector $\mathbf{u}$ gets mapped to a scalar multiple
  - $f(\mathbf{u}) = a\mathbf{u}, \quad a \in \mathbb{R}$
- Preserves the <u>direction</u> of all vectors*
  - $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \dfrac{a\mathbf{u}}{|a\mathbf{u}|}$
- Q: Is scaling a linear transformation? A: Yes!



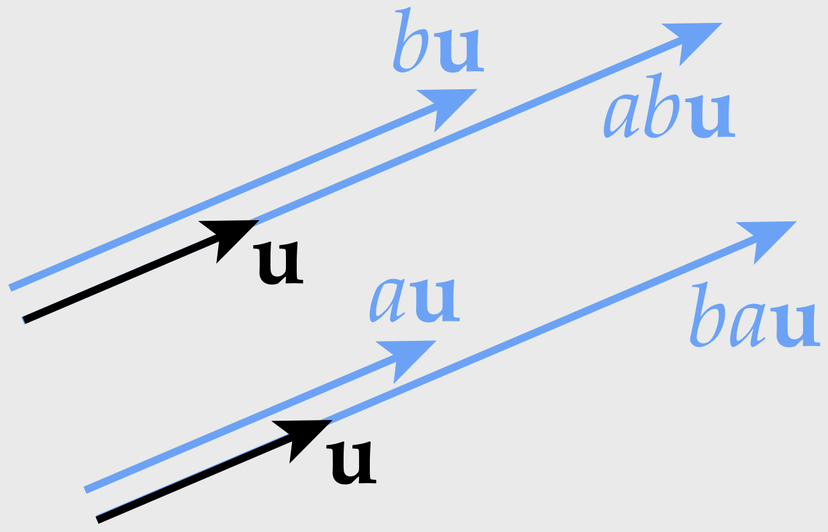
## ADDITION




$$f(\mathbf{u}+\mathbf{v}) =$$
$$a(\mathbf{u}+\mathbf{v}) =$$
$$a\mathbf{u} + a\mathbf{v} =$$
$$f(\mathbf{u}) + f(\mathbf{v})$$

## SCALAR MULTIPLICATION

$$f(b\mathbf{u}) = ab\mathbf{u} = ba\mathbf{u} = bf(\mathbf{u})$$



# Scaling — Matrix Representation

**Q: Suppose we want to scale a vector $\mathbf{u} = (u_1, u_2, u_3)$ by $a$ .**
**How would we represent this operation via a matrix?**

**A: Just build a diagonal matrix $D$, with $a$ along the diagonal:**

$$\underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}}_{a\mathbf{u}}$$

**Q: What happens if $a$ is negative?**

# Negative Scaling

**For $a = -1$, can think of scaling by $a$ as sequence of reflections.**

**E.g., in 2D:**

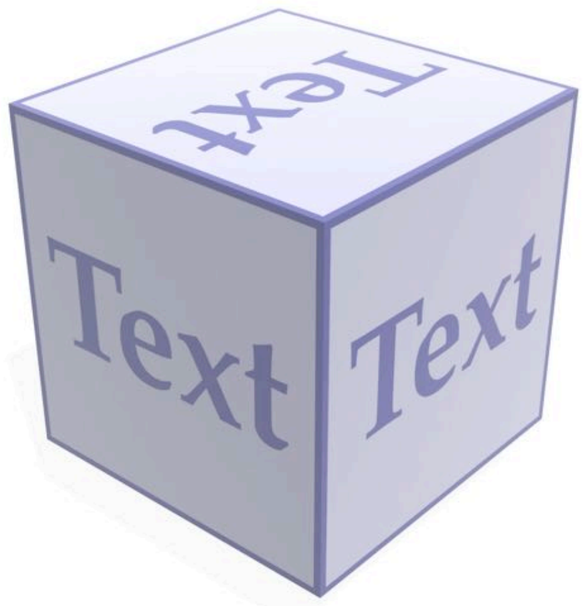
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Since each reflection reverses orientation, orientation is <u>preserved.</u>**

## What about 3D?

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

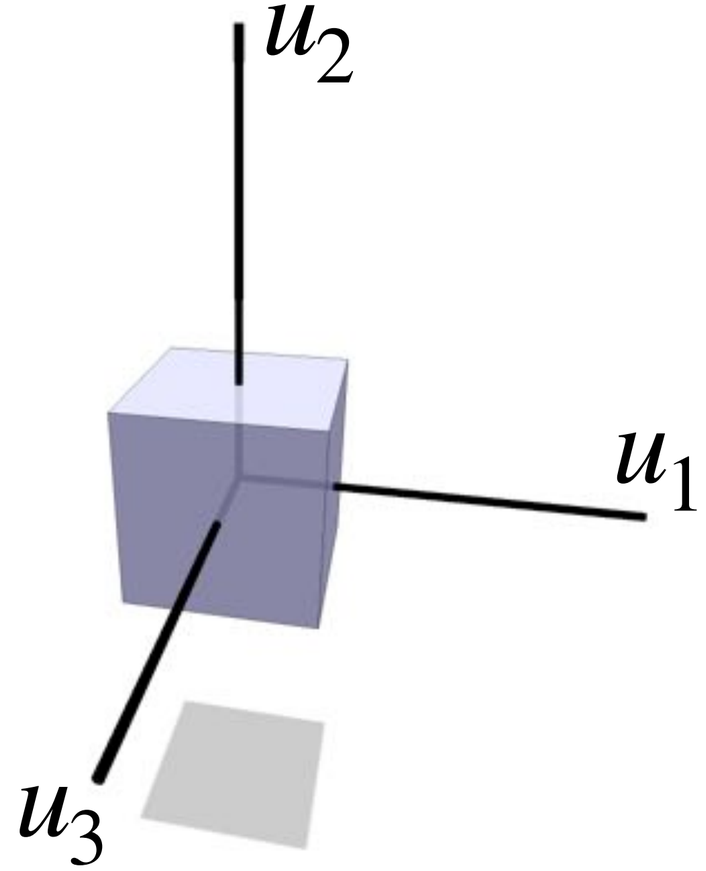



**Now we have three reflections, and so orientation is <u>reversed!</u>**

# Nonuniform Scaling (Axis-Aligned)

- We can also scale each axis by a different amount
  - $f(u_1, u_2, u_3) = (au_1, bu_2, cu_3), \quad a, b, c \in \mathbb{R}$
- Q: What's the matrix representation?
- A: Just put $a, b, c$ on the diagonal:

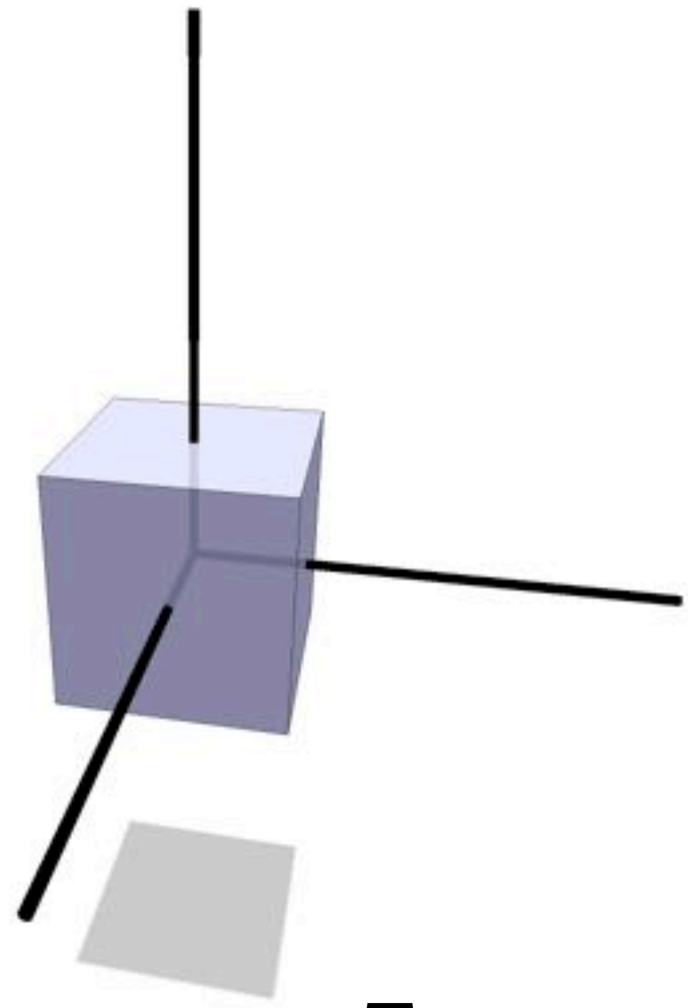
$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ bu_2 \\ cu_3 \end{bmatrix}$$




**Ok, but what if we want to scale along some other axes?**

# Nonuniform Scaling

- **Idea.** We could:
  - rotate to the new axes ($R$)
  - apply a diagonal scaling ($D$)
  - rotate back* to the original axes ($R^\mathsf{T}$)



$$f(\mathbf{x}) = R^\mathsf{T} D R \mathbf{x}$$

- Notice that the overall transformation is represented by a symmetric matrix

$$A := R^\mathsf{T} D R$$

**Q: Do all symmetric matrices represent nonuniform scaling (for some choice of axes)?**

*Recall that for a rotation, the inverse equals the transpose: $R^{-1} = R^\mathsf{T}$

# Spectral Theorem

- **A: Yes! Spectral theorem says a symmetric matrix $A = A^\mathsf{T}$ has**
  - **orthonormal eigenvectors $e_1, \ldots, e_n \in \mathbb{R}^n$**
  - **real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$**

$$Ae_i = \lambda_i e_i$$

- **Can also write this relationship as $AR = RD$, where**

$$R = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- **Equivalently, $A = RDR^\mathsf{T}$**
- **Hence, every symmetric matrix performs a non-uniform scaling along some set of orthogonal axes.**
- **If $A$ is positive definite ($\lambda_i > 0$), this scaling is positive.**

# Shear

- A shear displaces each point $\mathbf{x}$ in a direction $\mathbf{u}$ according to its distance along a fixed vector $\mathbf{v}$:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

- Q: Is this transformation linear?
- A: Yes—for instance, can represent it via a matrix

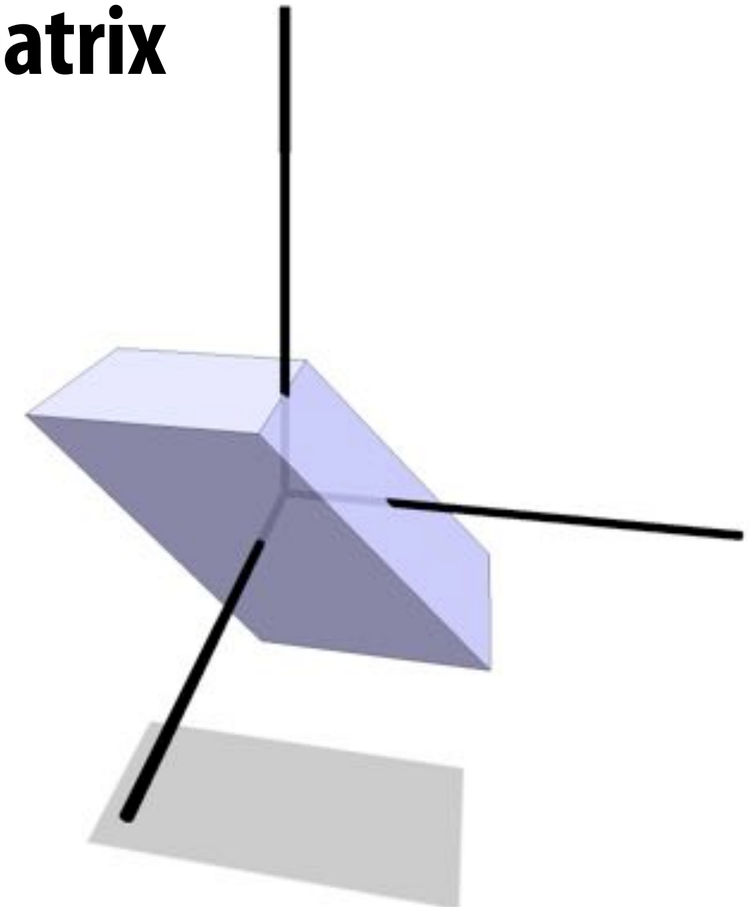
$$A_{\mathbf{u},\mathbf{v}} = I + \mathbf{u}\mathbf{v}^\mathsf{T}$$

**Example.**

$$\mathbf{u} = (\cos(t),0,0)$$

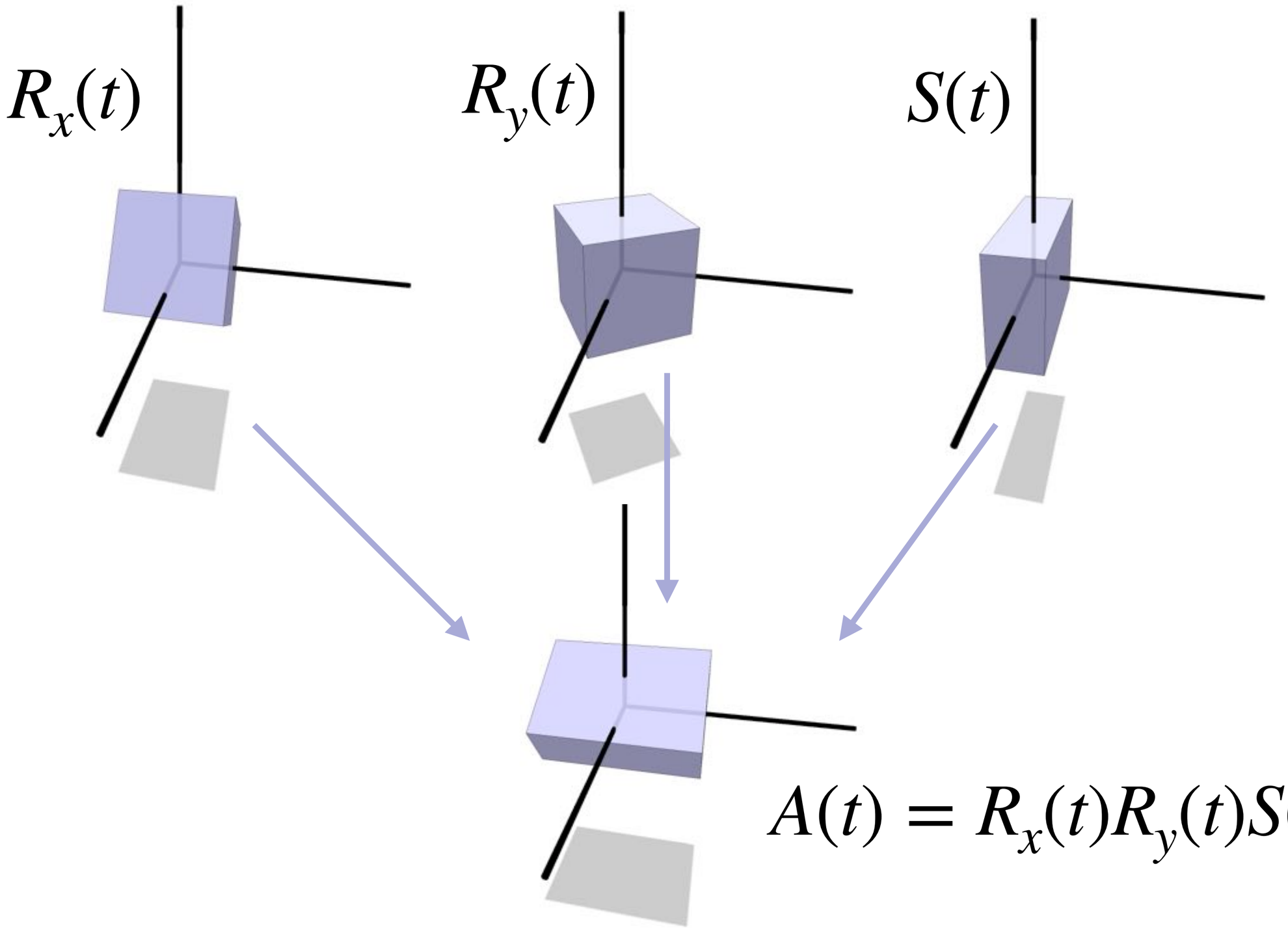
$$\mathbf{v} = (0,1,0)$$

$$A_{\mathbf{u},\mathbf{v}} = \begin{bmatrix} 1 & \cos(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Composite Transformations

**From these basic transformations (rotation, reflection, scaling, shear…) we can now build up composite transformations via matrix multiplication:**



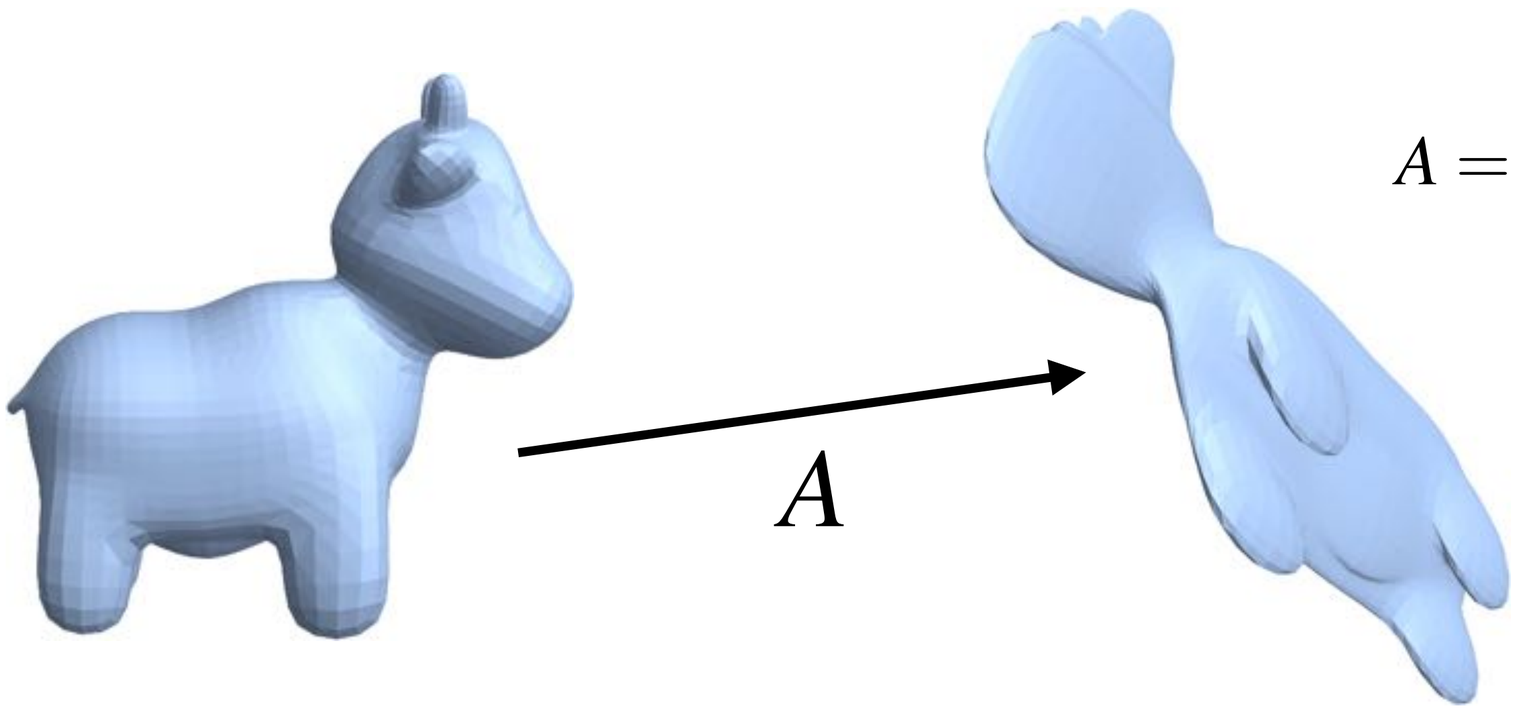

$$A(t) = R_x(t)R_y(t)S(t)$$

# How do we decompose a linear transformation into pieces?

(rotations, reflections, scaling, …)

# Decomposition of Linear Transformations

- In general, no **unique** way to write a given linear transformation as a composition of basic transformations!
- However, there are many useful decompositions:
  - singular value decomposition (good for signal processing)
  - LU factorization (good for solving linear systems)
  - polar decomposition (good for spatial transformations)
  - …
- Consider for instance this linear transformation:




$$A = \begin{bmatrix} .34 & -.11 & -.89 \\ -.65 & .52 & -.70 \\ .25 & .23 & -.69 \end{bmatrix}$$

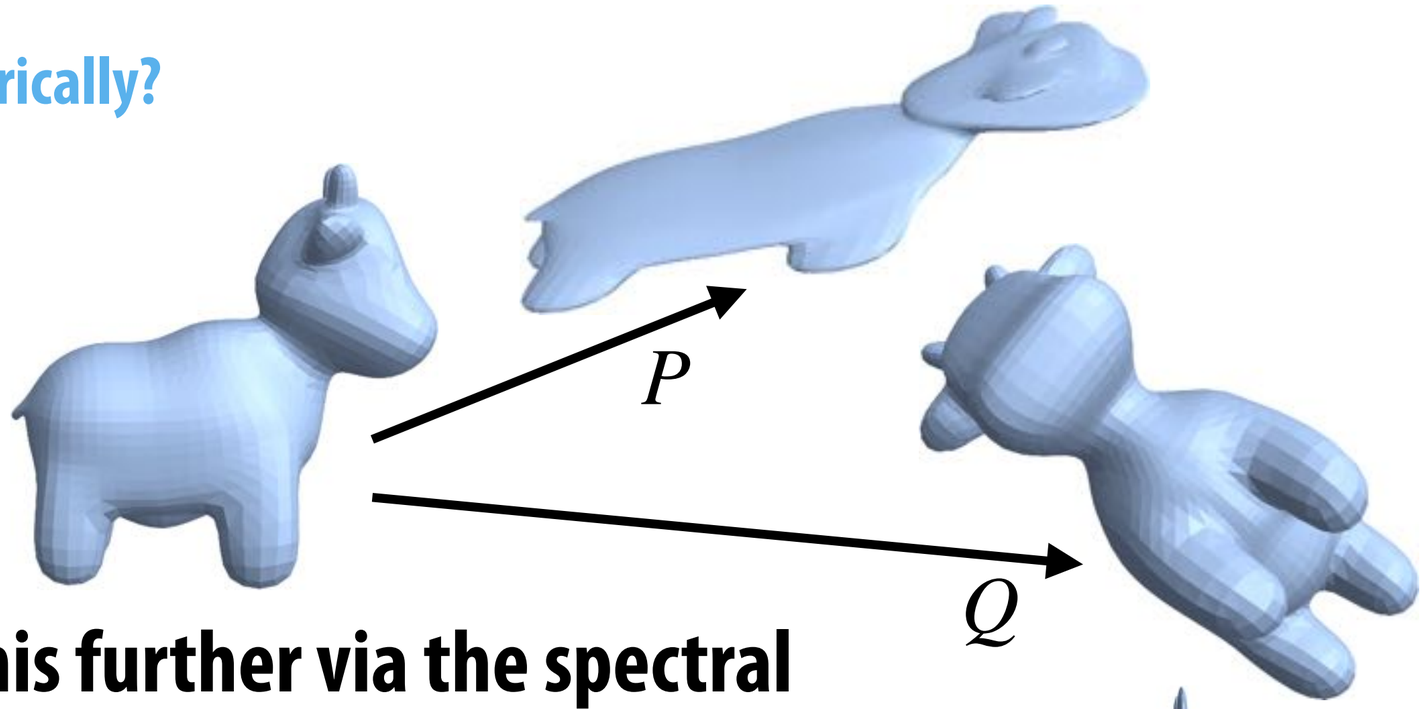
# Polar & Singular Value Decomposition

**For example, polar decomposition decomposes any matrix $A$ into orthogonal matrix $Q$ and symmetric positive-semidefinite matrix $P$:**

**Q: What do each of the parts mean geometrically?**

**rotation/reflection** **nonnegative, nonuniform scaling**

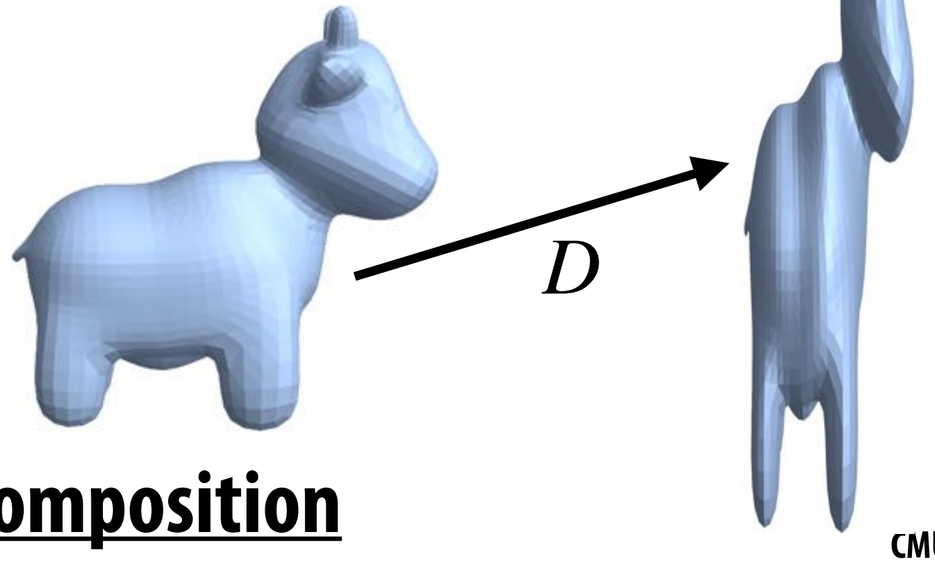
$$A = QP$$




**Since $P$ is symmetric, can take this further via the spectral decomposition $P = VDV^\mathsf{T}$ ($V$ orthogonal, $D$ diagonal):**

$$A = \underbrace{QV}_{U} DV^\mathsf{T} = UDV^\mathsf{T}$$

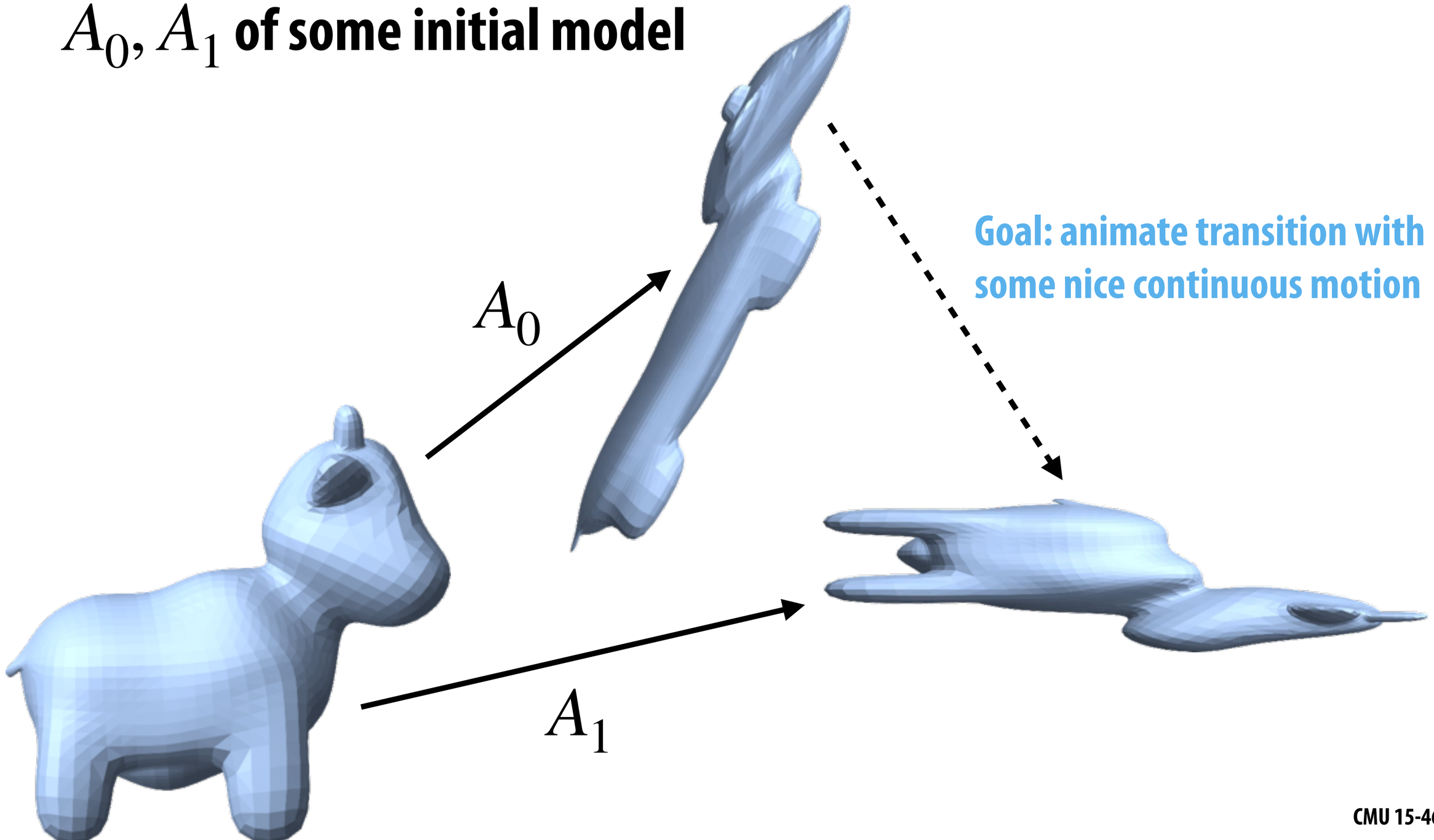
**rotation** **axis-aligned scaling** **rotation**




**Result $UDV^\mathsf{T}$ is called the singular value decomposition**

# Interpolating Transformations

- How are these decompositions useful for graphics?
- Consider interpolating between two linear transformations $A_0, A_1$ of some initial model



# Interpolating Transformations—Linear

**One idea: just take a linear combination of the two matrices, weighted by the current time $t \in [0,1]$**

$$A(t) = (1-t)A_0 + tA_1$$

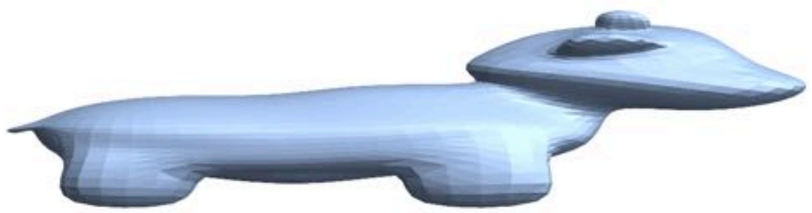
**Hits the right start/endpoints… but looks awful in between!**

# Interpolating Transformations—Polar

**Better idea: separately interpolate components of polar decomposition.**

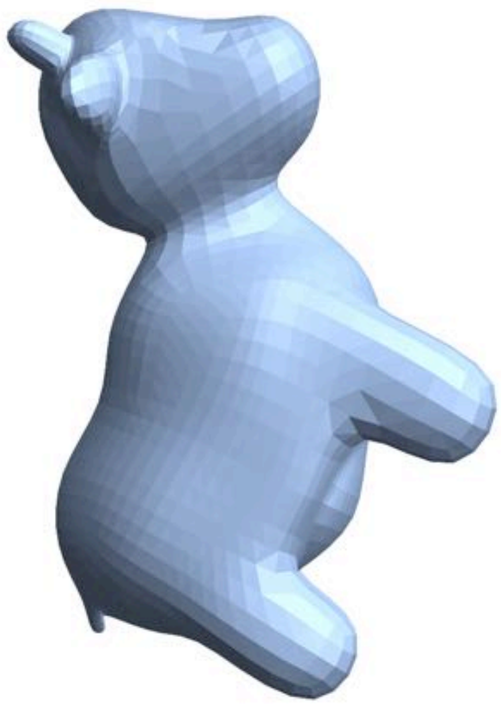
$$A_0 = Q_0 P_0, \quad A_1 = Q_1 P_1$$

**scaling**



$$P(t) = (1-t)P_0 + tP_1$$

**rotation**



$$\widetilde{Q}(t) = (1-t)Q_0 + tQ_1$$

$$\widetilde{Q}(t) = Q(t)X(t)$$

**final interpolation**

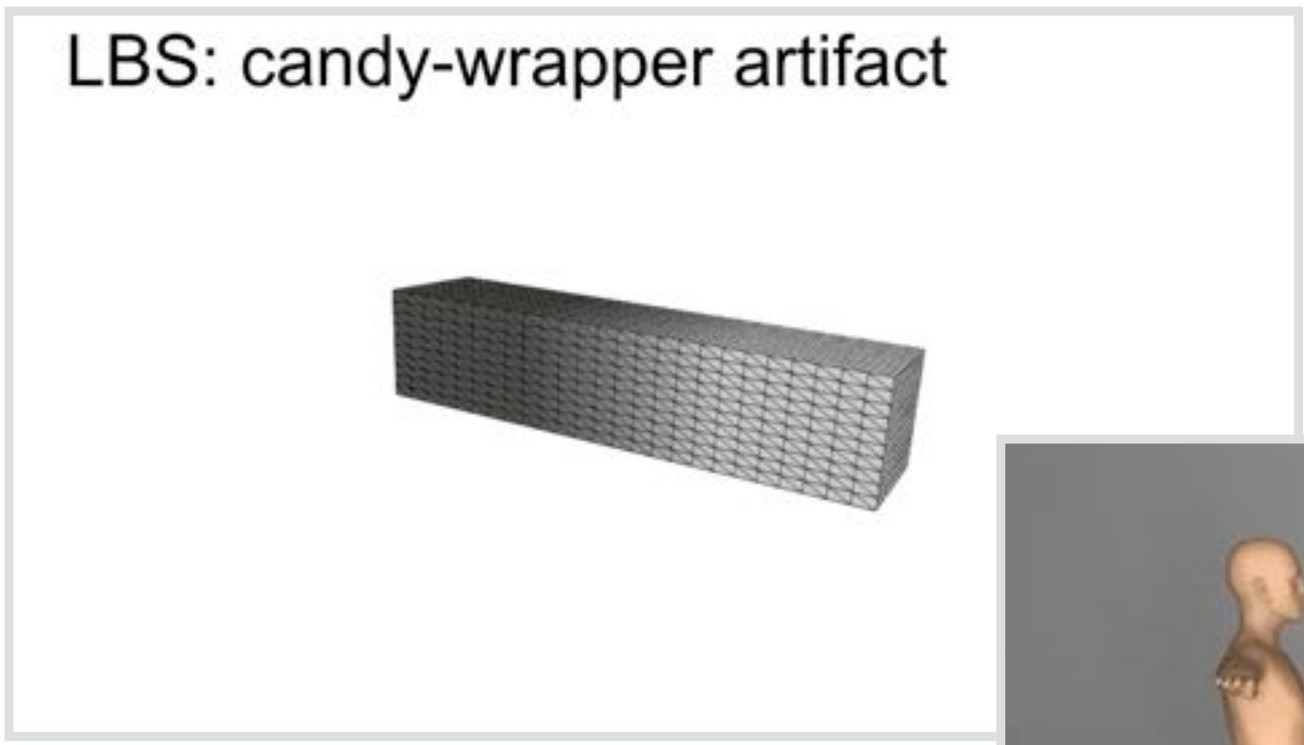
$$A(t) = Q(t)P(t)$$

**…looks better!**

See: Shoemake & Duff, "Matrix Animation and Polar Decomposition"

# Example: Linear Blend Skinning

- Naïve linear interpolation also causes artifacts when blending between transformations on a character ("candy wrapper effect")
- Lots of research on alternative ways to blend transformations…



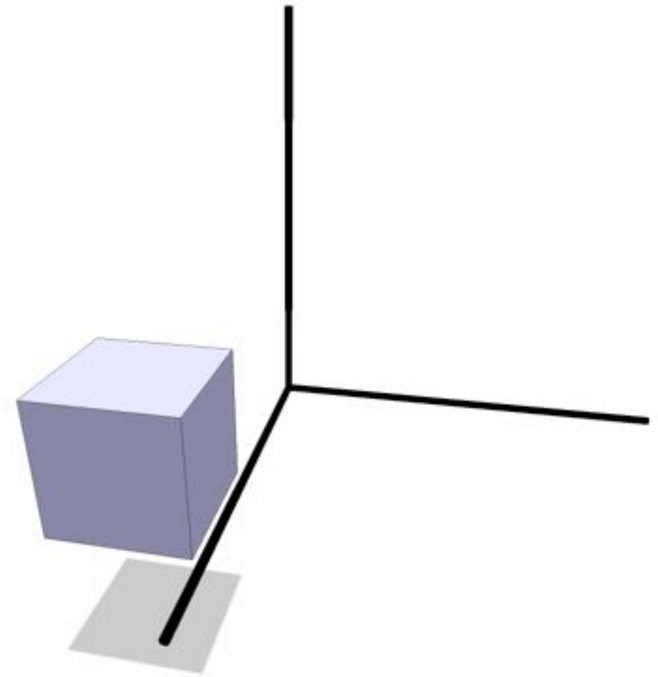

Jacobson, Deng, Kavan, & Lewis (2014)
"Skinning: Real-time Shape Deformation"

Rumman & Fratarcangeli (2015)
"Position-based Skinning for Soft Articulated Characters"

# Translations

- So far we've ignored a basic transformation—translations
- A translation simply adds an offset $\mathbf{u}$ to the given point $\mathbf{x}$:

$$f_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}$$



**Q: Is this transformation linear?**
**(Certainly seems to move us along a line…)**

**Let's carefully check the definition…**

additivity

$$f_{\mathbf{u}}(\mathbf{x}+\mathbf{y}) = \mathbf{x}+\mathbf{y}+\mathbf{u}$$

$$f_{\mathbf{u}}(\mathbf{x}) + f_{\mathbf{u}}(\mathbf{y}) = \mathbf{x}+\mathbf{y}+2\mathbf{u}$$

homogeneity

$$f_{\mathbf{u}}(a\mathbf{x}) = a\mathbf{x}+\mathbf{u}$$

$$af_{\mathbf{u}}(\mathbf{x}) = a\mathbf{x}+a\mathbf{u}$$

**A: No! Translation is affine, not linear!**

# Composition of Transformations

- Recall we can compose linear transformations via matrix multiplication:

$$A_3(A_2(A_1\mathbf{x})))=(A_3A_2A_1)\mathbf{x}$$

- It's easy enough to compose translations—just add vectors:

$$f_{\mathbf{u}_3}(f_{\mathbf{u}_2}(f_{\mathbf{u}_1}(\mathbf{x})))=f_{\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3}(\mathbf{x})$$

- What if we want to intermingle translations and linear transformations (rotation, scale, shear, etc.)?

$$A_2(A_1\mathbf{x}+\mathbf{b}_1)+\mathbf{b}_2=(A_2A_1)\mathbf{x}+(A_2\mathbf{b}_1+\mathbf{b}_2)$$

- Now we have to keep track of a matrix and a vector
- Moreover, we'll see (later) that this encoding won't work for other important cases, such as perspective transformations

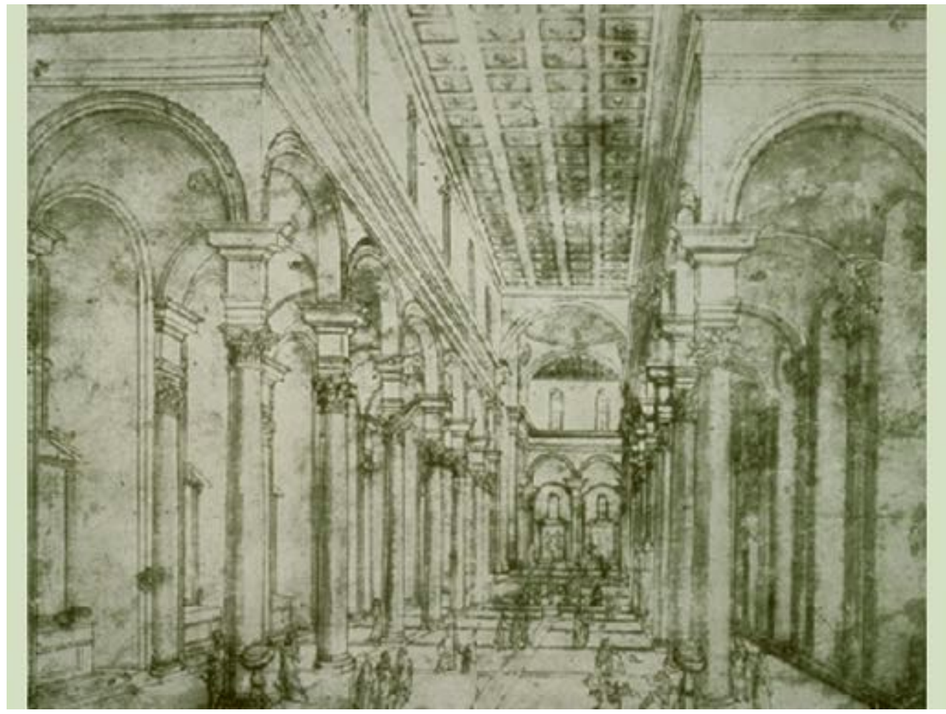
**But there is a better way…**

**Strange idea:**

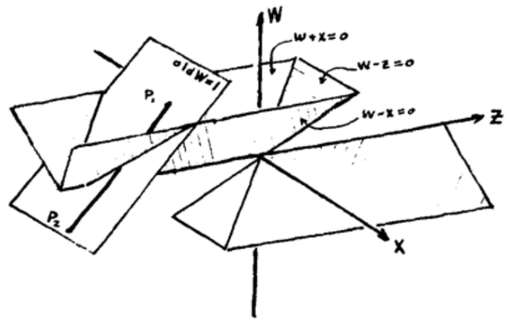
**Maybe translations turn into linear transformations if we go into the 4th dimension…!**

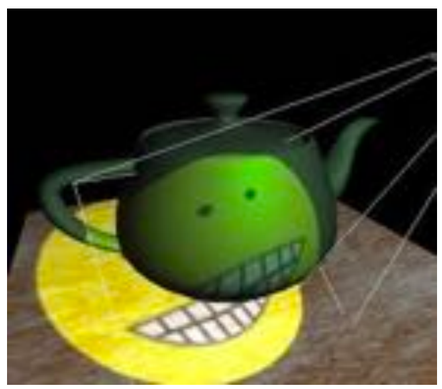
# Homogeneous Coordinates

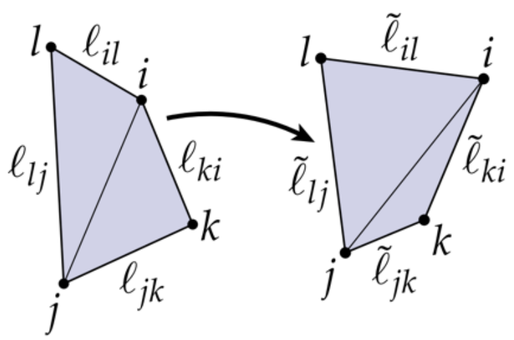
Filippo Brunelleschi, 1428



- Came from efforts to study <u>perspective</u>
- Introduced by Möbius as a natural way of assigning coordinates to <u>lines</u>
- Show up naturally in a surprising large number of places in computer graphics:
  - 3D transformations
  - perspective projection
  - quadric error simplification
  - premultiplied alpha
  - shadow mapping
  - projective texture mapping
  - discrete conformal geometry
  - hyperbolic geometry
  - clipping
  - directional lights
  - …



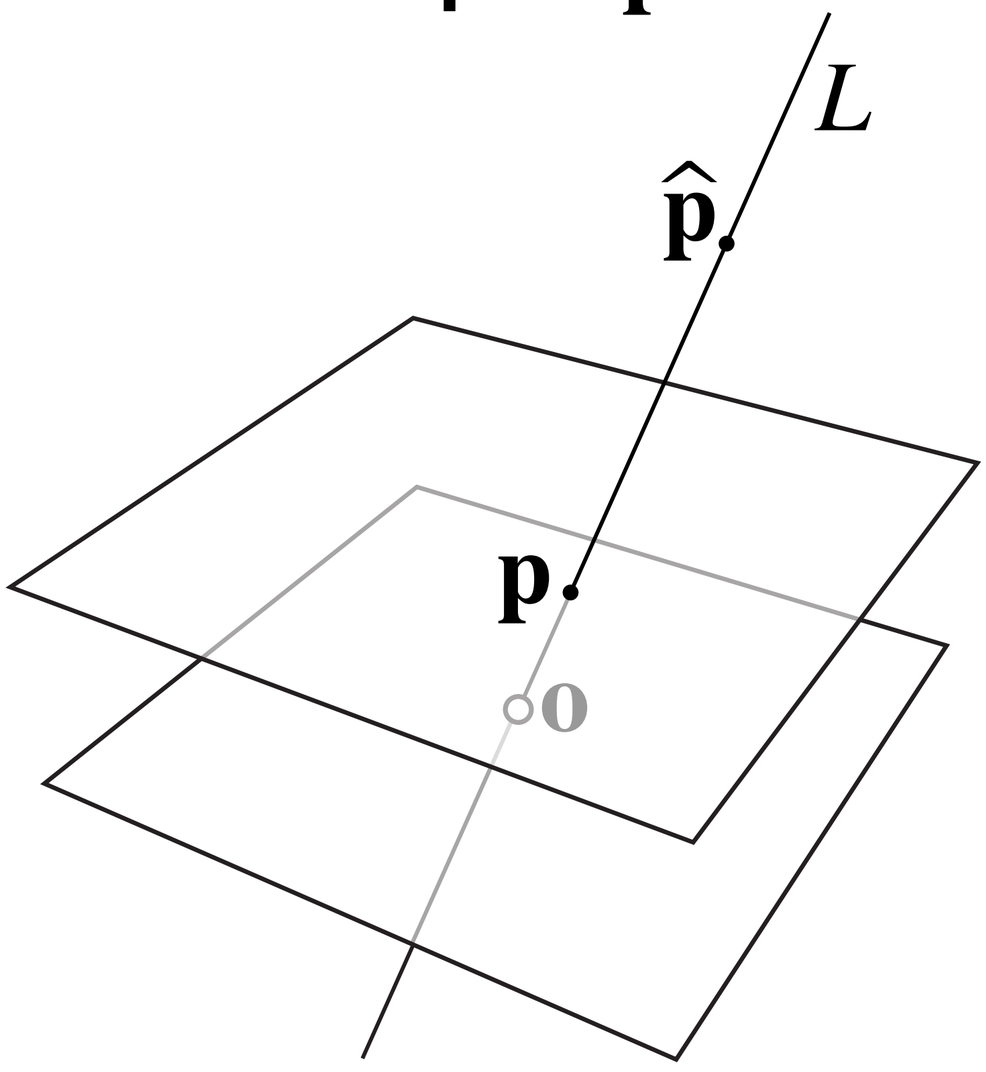


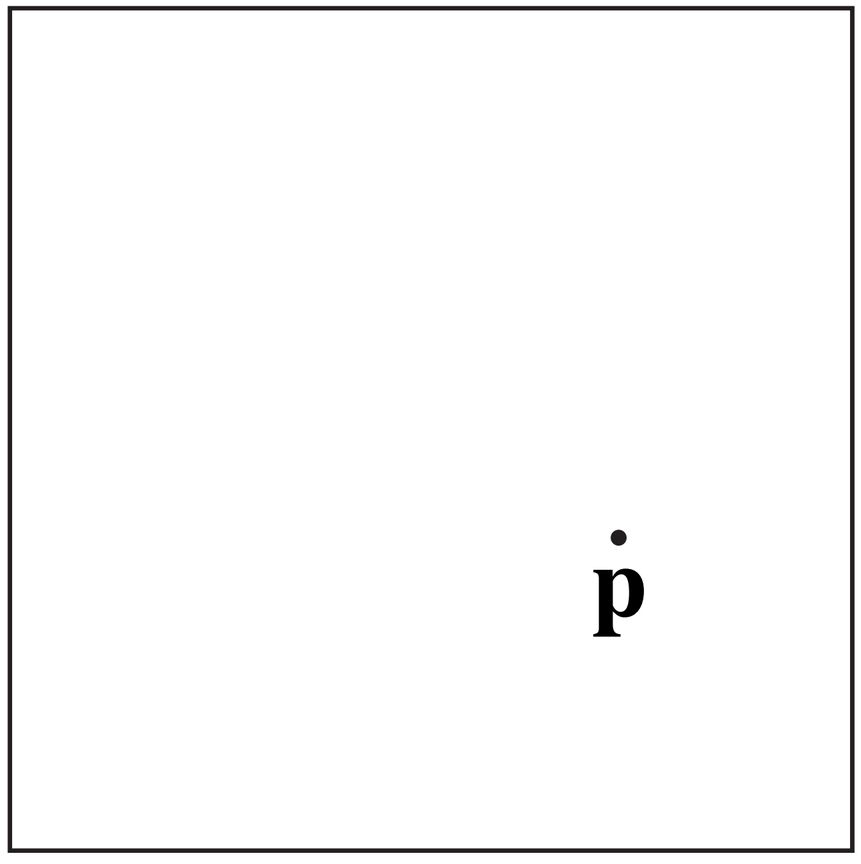


**Probably worth understanding!**

# Homogeneous Coordinates—Basic Idea

- Consider any 2D plane that does not pass through the origin $\mathbf{o}$ in 3D
- Every line through the origin in 3D corresponds to a point in the 2D plane
  - Just find the point $\mathbf{p}$ where the line $L$ pierces the plane



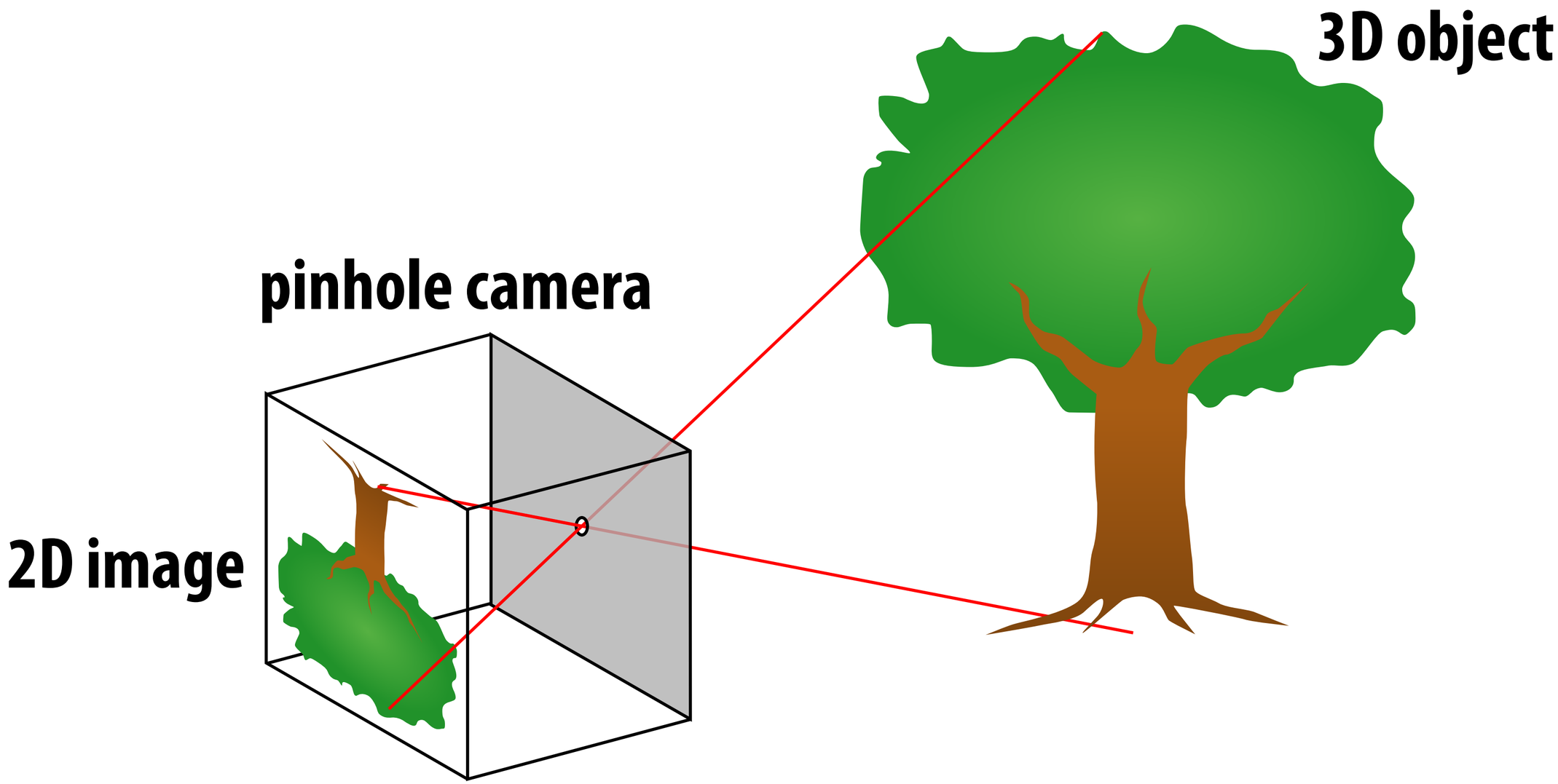




Hence, any point $\hat{\mathbf{p}}$ on the line $L$ can be used to represent the point $\mathbf{p}$.

Q: What does this story remind you of?

# Review: Perspective projection

- Hopefully it reminds you of our "pinhole camera"
- Objects along the same line project to the same point



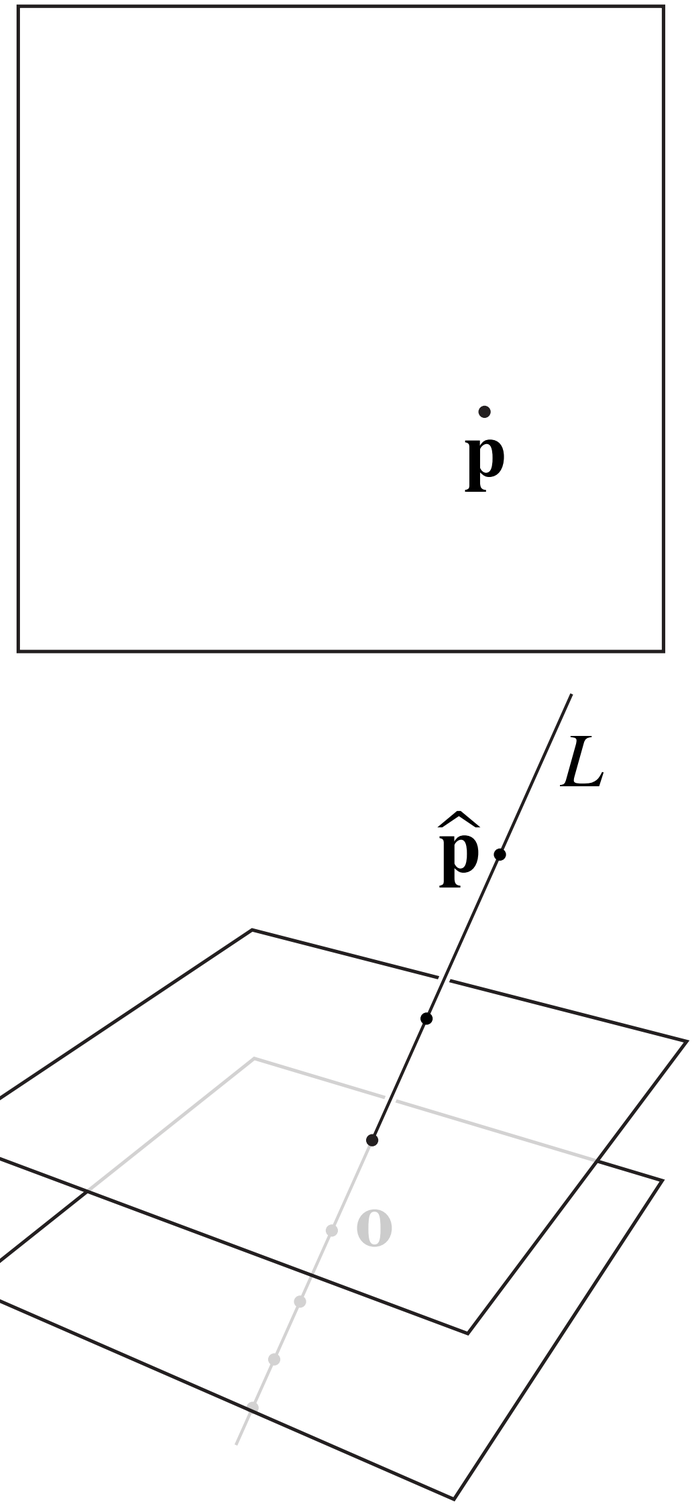

**If you have an image of a single dot, can't know where it is!**
**Only which line it belongs to.**

# Homogeneous Coordinates (2D)

- More explicitly, consider a point $\mathbf{p} = (x, y)$, and the plane $z = 1$ in 3D

- Any three numbers $\hat{\mathbf{p}} = (a, b, c)$ such that $(a/c, b/c) = (x, y)$ are homogeneous coordinates for $\mathbf{p}$
  - E.g., $(x, y, 1)$
  - In general: $(cx, cy, c)$ for $c \neq 0$

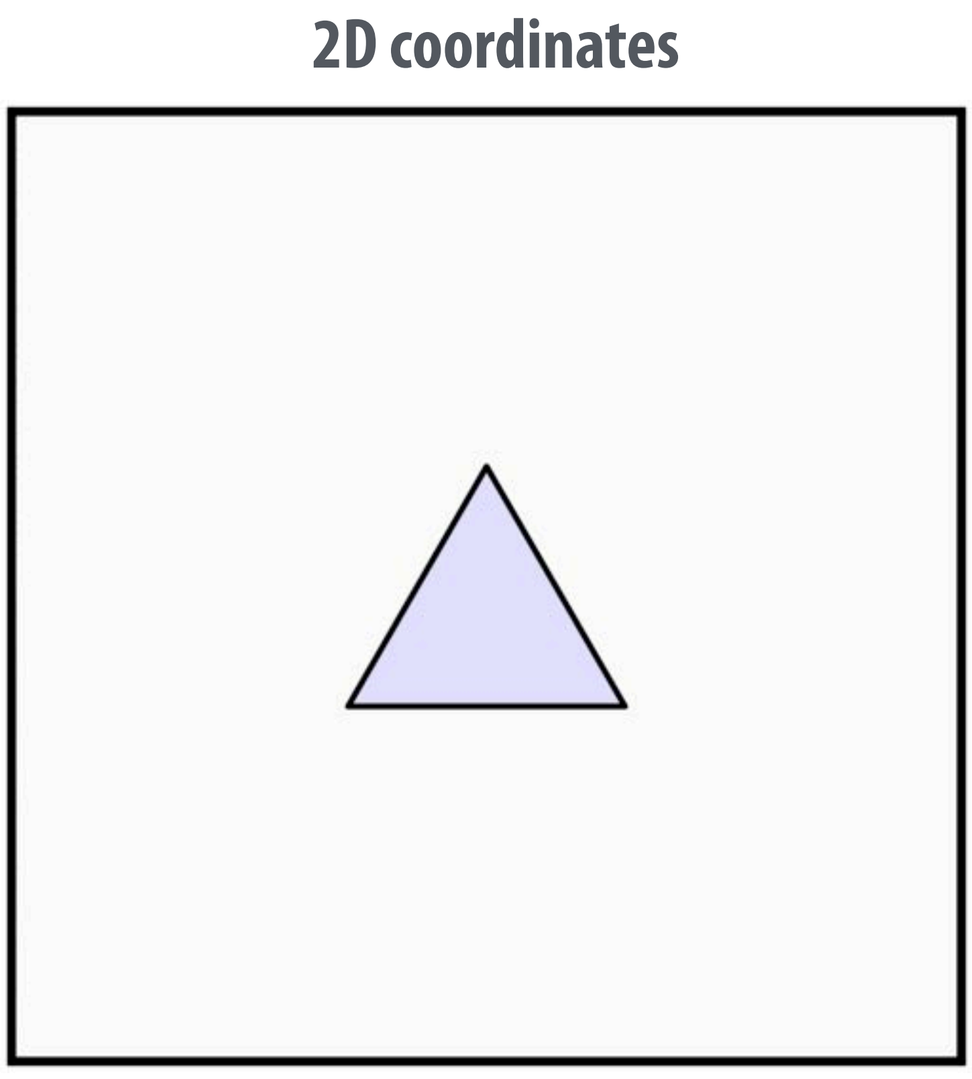
- Hence, two points $\hat{\mathbf{p}}, \hat{\mathbf{q}} \in \mathbb{R}^3 \backslash \{O\}$ describe the same point in 2D (and line in 3D) if $\hat{\mathbf{p}} = \lambda \hat{\mathbf{q}}$ for some $\lambda \neq 0$



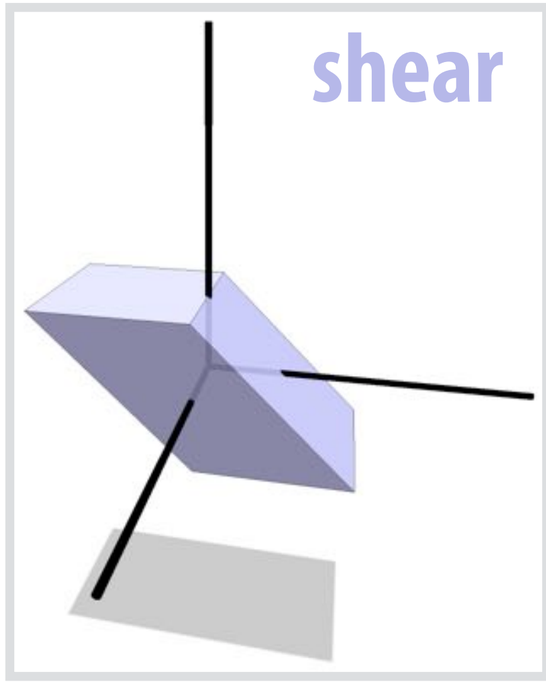

**Great… but how does this help us with transformations?**

# Translation in Homogeneous Coordinates

## Let's think about what happens to our homogeneous coordinates $\hat{\mathbf{p}}$ if we apply a translation to our 2D coordinates $\mathbf{p}$




**Q: What kind of transformation does this look like?**



# Translation in Homogeneous Coordinates

- But wait a minute—shear is a linear transformation!
- Can this be right? Let's check in coordinates…
- Suppose we translate a point $\mathbf{p} = (p_1, p_2)$ by a vector $\mathbf{u} = (u_1, u_2)$ to get $\mathbf{p}' = (p_1 + u_1, p_2 + u_2)$
- The homogeneous coordinates $\hat{\mathbf{p}} = (cp_1, cp_2, c)$ then become $\hat{\mathbf{p}}' = (cp_1 + cu_1, cp_2 + cu_2, c)$
- Notice that we're shifting $\hat{\mathbf{p}}$ by an amount $c\mathbf{u}$ that's proportional to the distance $c$ along the third axis—a shear

**Using homogeneous coordinates, we can represent an affine transformation in 2D as a linear transformation in 3D**

# Homogeneous Translation—Matrix Representation

- To write as a matrix, recall that a shear in the direction $\mathbf{u} = (u_1, u_2)$ according to the distance along a direction $\mathbf{v}$ is

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

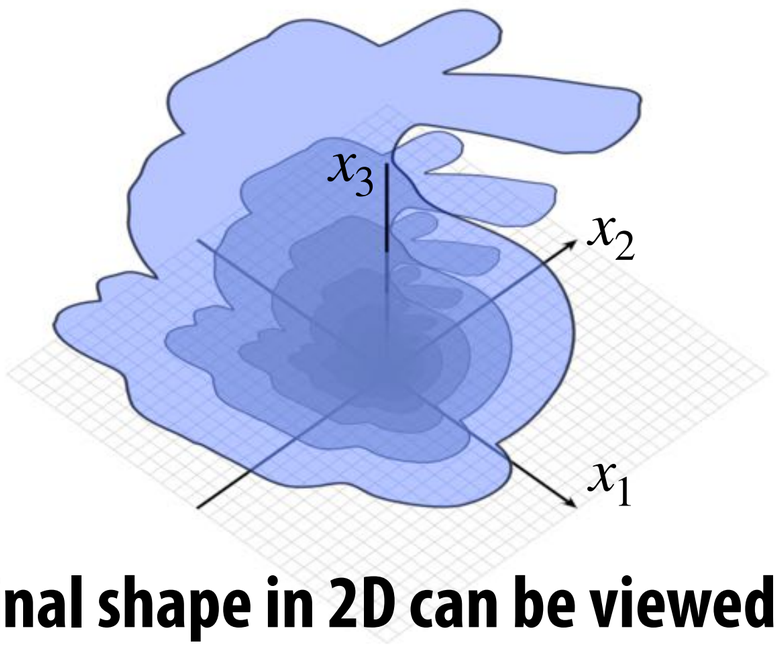
- In matrix form:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \left( I + \mathbf{u}\mathbf{v}^{\mathsf{T}} \right) \mathbf{x}$$

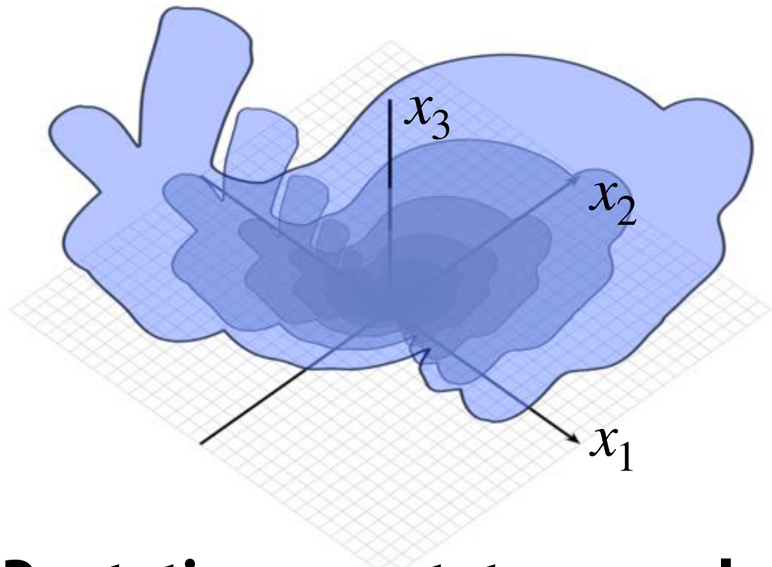
- In our case, $\mathbf{v} = (0,0,1)$ and so we get a matrix

$$\begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cp_1 \\ cp_2 \\ c \end{bmatrix} = \begin{bmatrix} c(p_1 + u_1) \\ c(p_2 + u_2) \\ c \end{bmatrix} \xrightarrow{1/c} \begin{bmatrix} p_1 + u_1 \\ p_2 + u_2 \end{bmatrix}$$

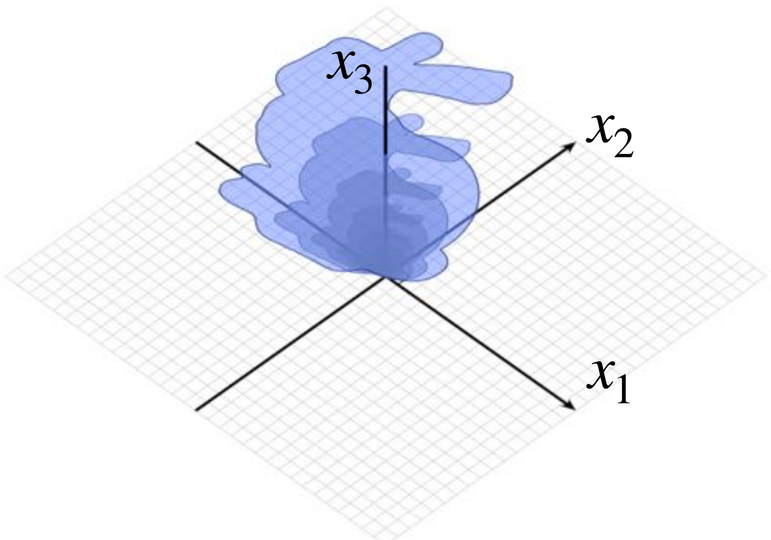
# Other 2D Transformations in Homogeneous Coordinates




**Original shape in 2D can be viewed as many copies, uniformly scaled by $x_3$**

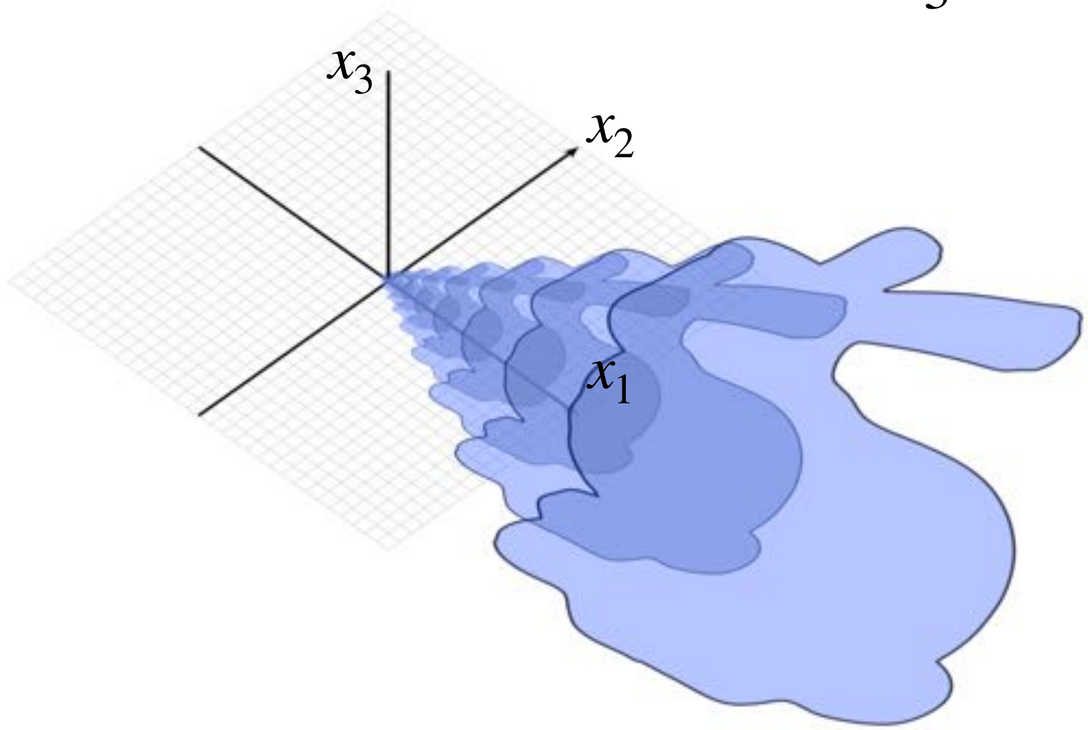



**2D rotation ↔ rotate around $x_3$**




**2D scale ↔ scale $x_1$ and $x_2$; preserve $x_3$**

**(Q: what happens to 2D shape if you scale $x_1$, $x_2$, and $x_3$ uniformly?)**




**2D translate ↔ shear**

**Now easy to compose all these transformations**

# 3D Transformations in Homogeneous Coordinates

- Not much changes in three (or more) dimensions: just append one "homogeneous coordinate" to the first three
- Matrix representations of 3D linear transformations just get an additional identity row/column; translation is again a shear

point in 3D

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

rotate $(x, y, z)$ around $y$ by $\theta$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

shear $(x, y)$ by $z$ in $(s, t)$ direction

$$\begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

scale $x, y, z$ by $a, b, c$

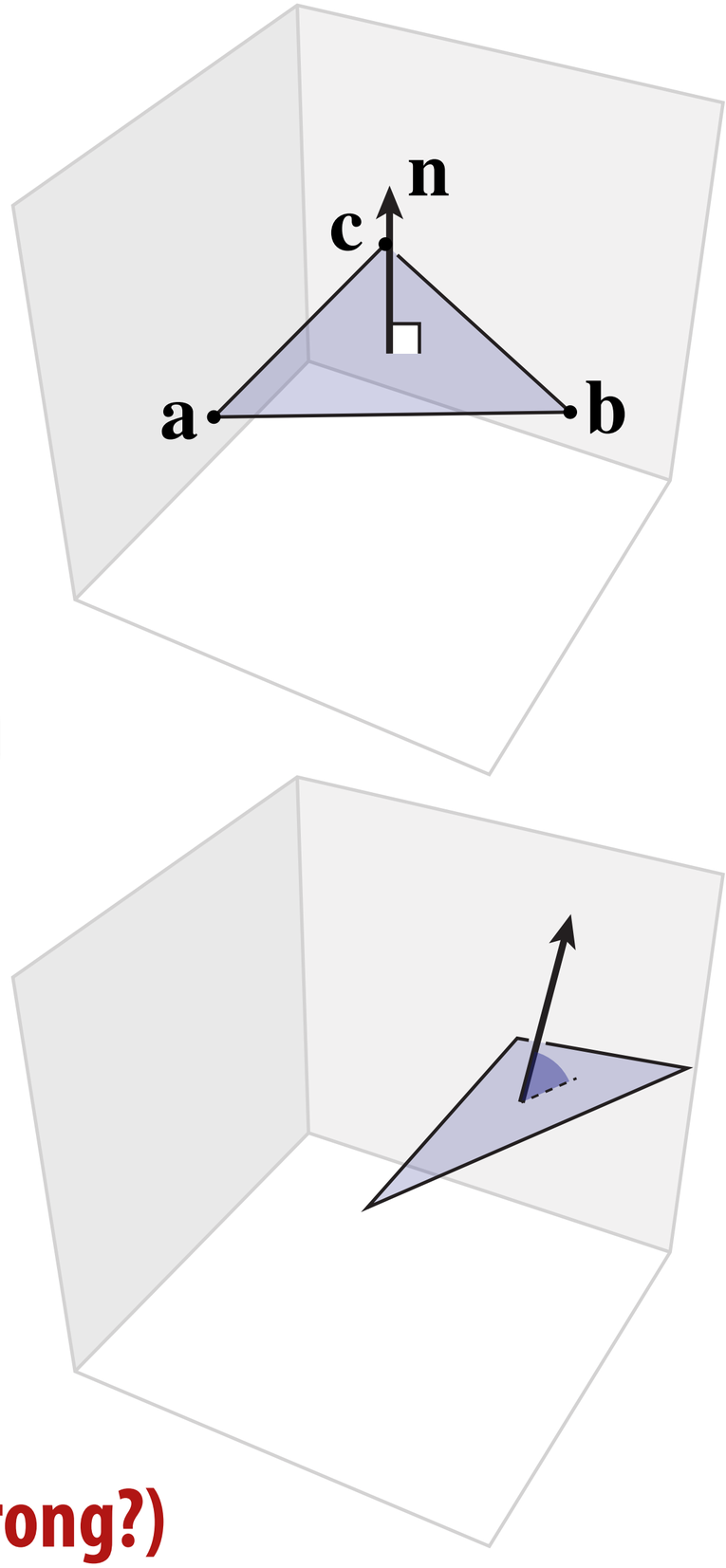
$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

translate $(x, y, z)$ by $(u, v, w)$

$$\begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Points vs. Vectors

- Homogeneous coordinates have another useful feature: distinguish between <u>points</u> and <u>vectors</u>
- Consider for instance a triangle with:
  - vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$
  - normal vector $\mathbf{n} \in \mathbb{R}^3$




- Suppose we transform the triangle by appending "1" to $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{n}$ and multiplying by this matrix:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



**Normal is not orthogonal to triangle! (What went wrong?)**

# Points vs. Vectors (continued)

- Let's think about what happens when we multiply the normal vector $\mathbf{n}$ by our matrix:

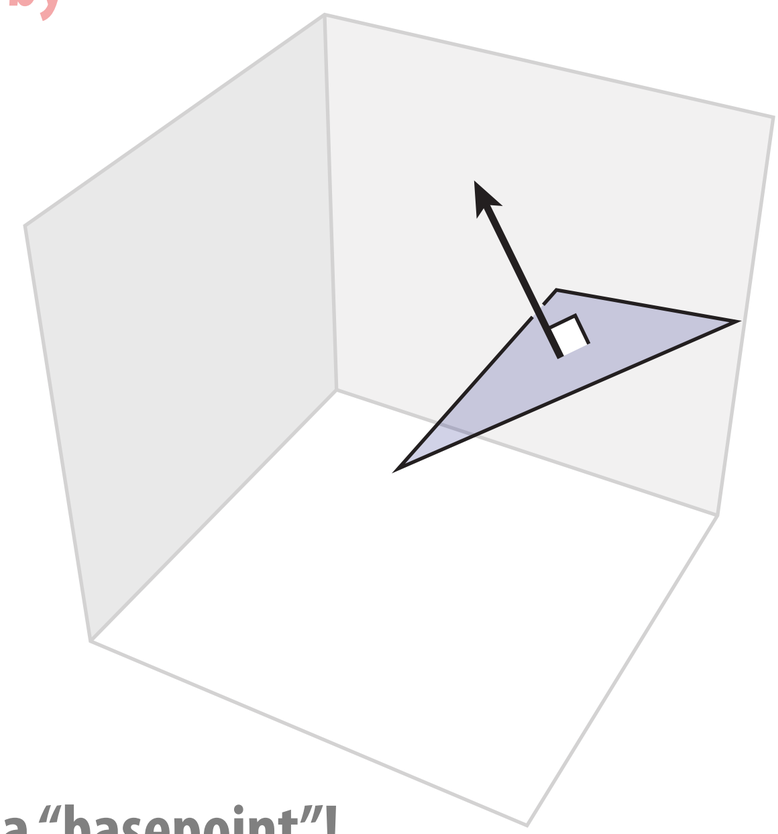
rotate normal around $y$ by $\theta$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{bmatrix}$$

- But when we rotate/translate a triangle, its normal should <u>just rotate!*</u>
- Solution? Just set homogeneous coordinate to zero!
- Translation now gets ignored; normal is orthogonal to triangle

translate normal by $(u, v, w)$

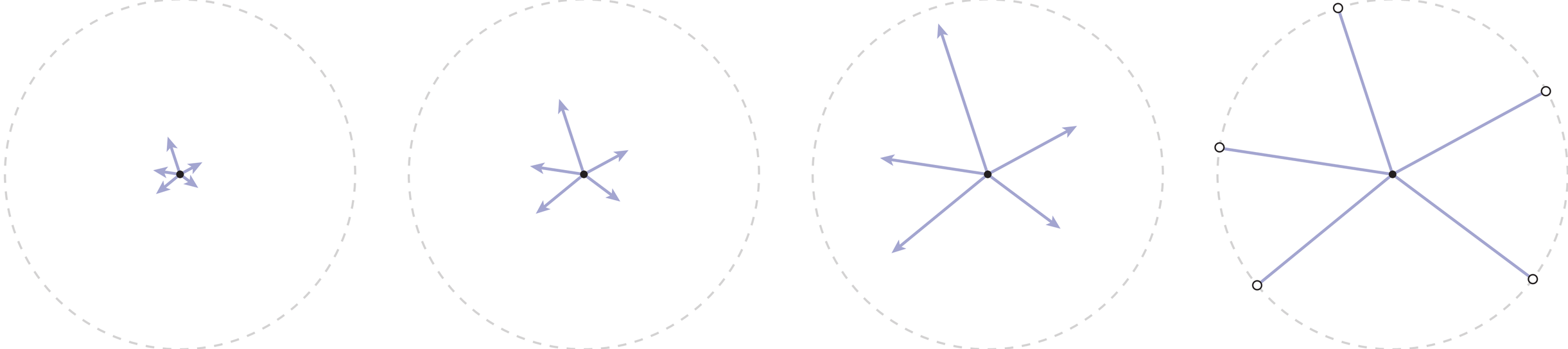
$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}$$



*Recall that <u>vectors</u> just have direction and magnitude—they don't have a "basepoint"!

# Points vs. Vectors in Homogeneous Coordinates

- In general:
  - A point has a <u>nonzero</u> homogeneous coordinate $(c = 1)$
  - A vector has a <u>zero</u> homogeneous coordinate $(c = 0)$
- But wait… what division by $c$ mean when it's equal to zero?
- Well consider what happens as $c \to 0$…



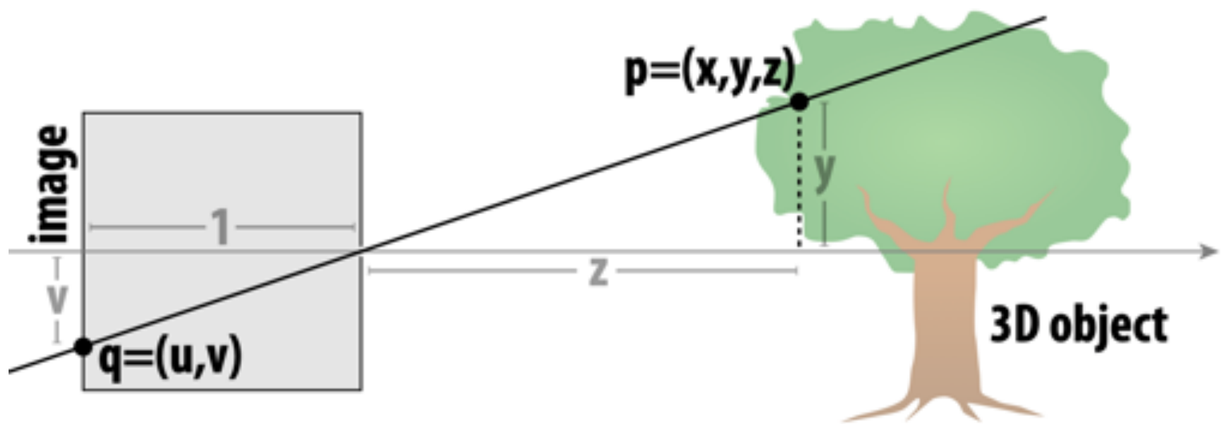
$(x, y)/1$ $(x, y)/0.5$ $(x, y)/0.25$ $(x, y)/0.001$

**Can think of vectors as "points at infinity" (sometimes called "ideal points")**

**(In practice: still need to check for divide by zero!)**

# Perspective Projection in Homogeneous Coordinates

- **Q: How can we perform perspective projection* using homogeneous coordinates?**




- **Remember from our pinhole camera model that the basic idea was to "divide by $z$"**

$$(x, y, z) \mapsto (x/z, y/z)$$

- **So, we can build a matrix that "copies" the $z$ coordinate into the homogeneous coordinate**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z \end{bmatrix}$$

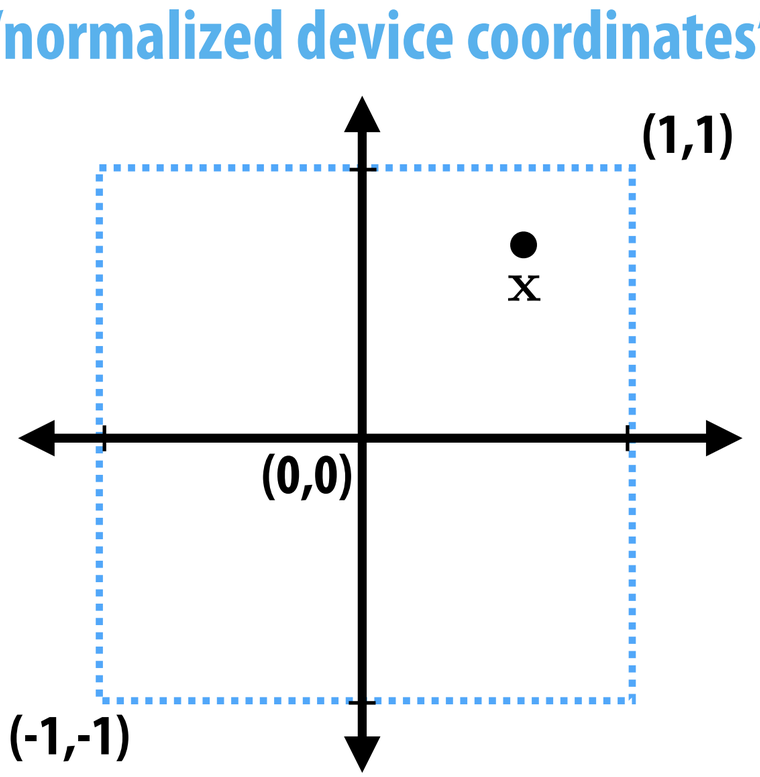
- **Division by the homogeneous coordinate now gives us perspective projection onto the plane $z = 1$**

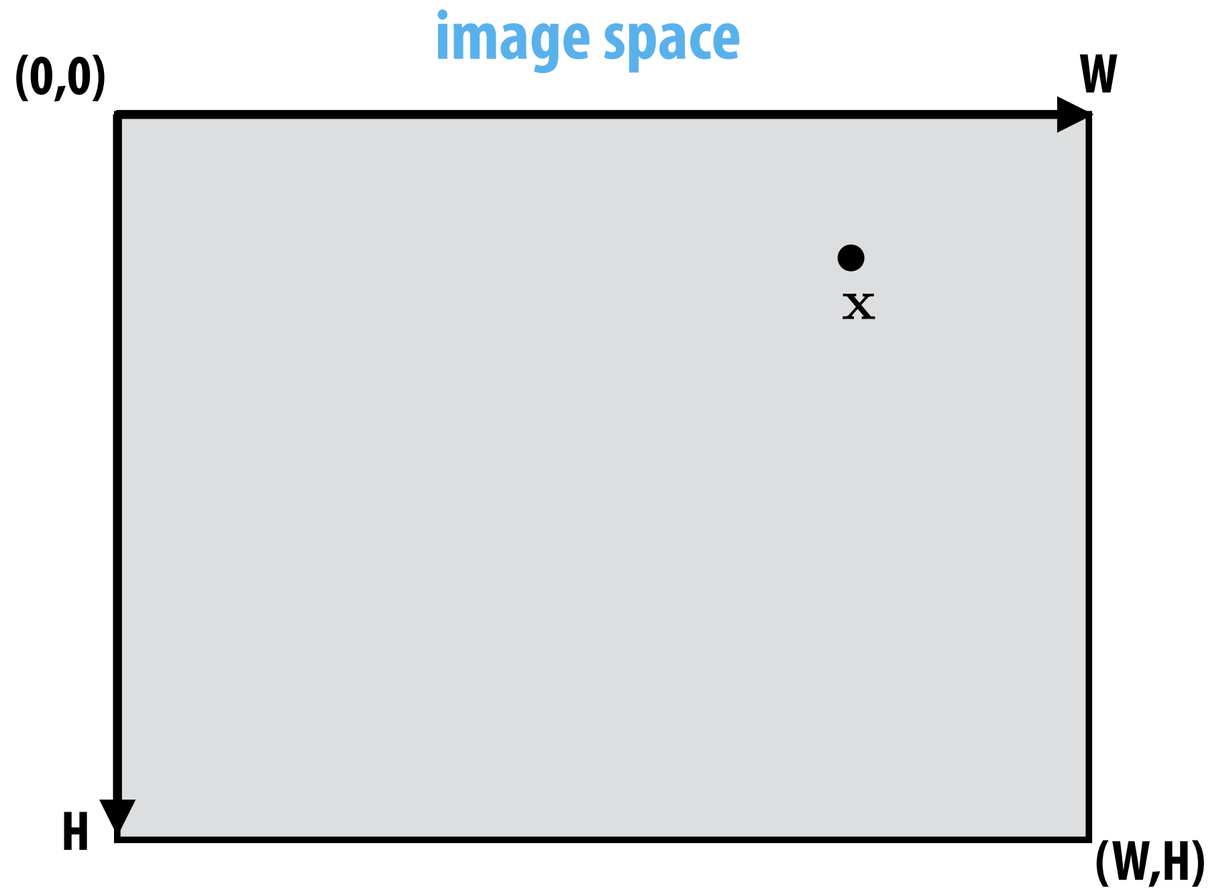
$$\Longrightarrow \begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$$

*Assuming a pinhole camera at (0,0,0) looking down the z-axis

# Screen Transformation

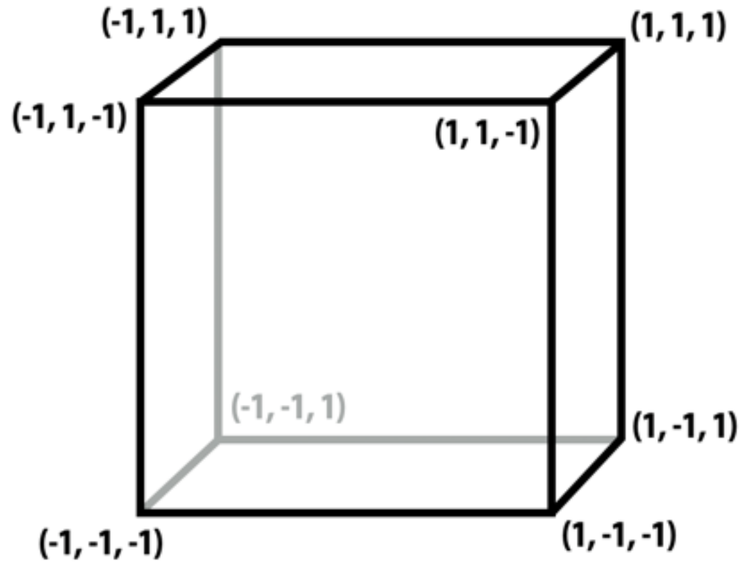
- One last transformation is needed in the rasterization pipeline: transform from viewing plane to pixel coordinates
- E.g., suppose we want to draw all points that fall inside the square [-1,1] x [-1,1] on the z = 1 plane, into a W x H pixel image



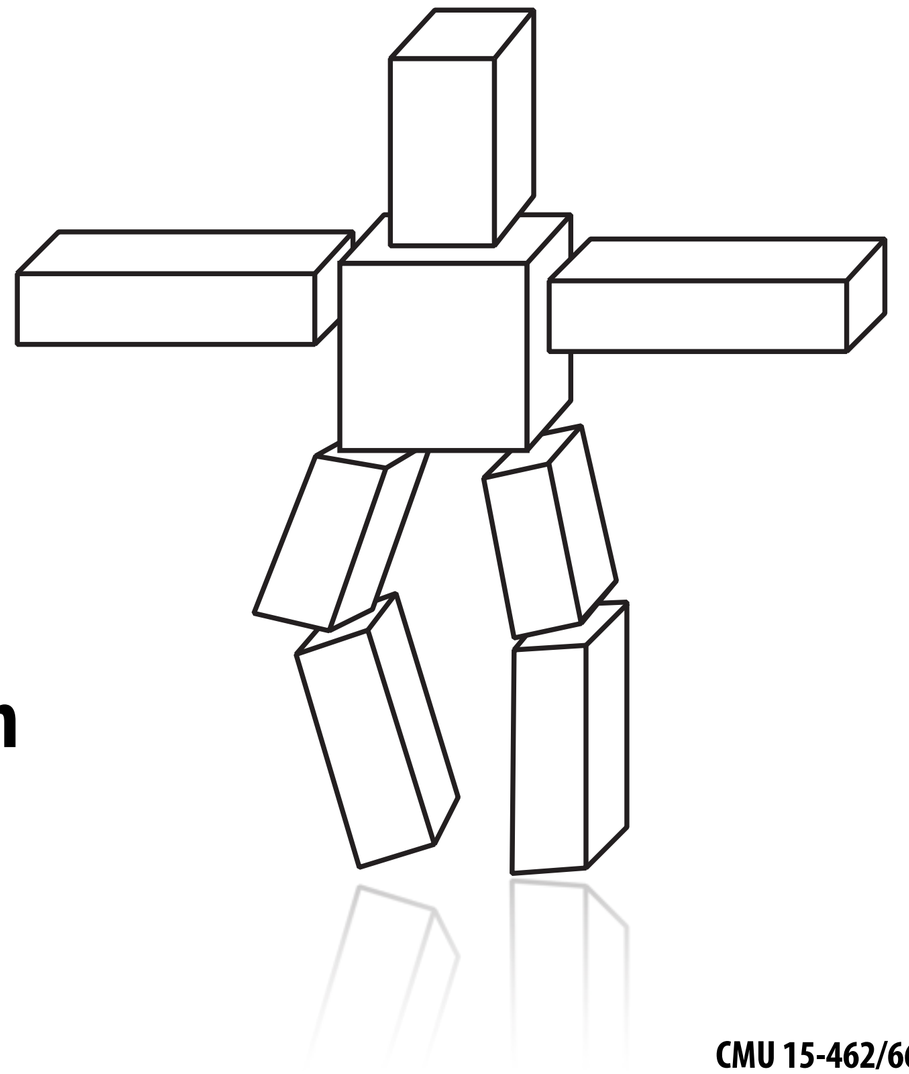




**Q: What transformation(s) would you apply? (Careful: $y$ is now down!)**

# Scene Graph

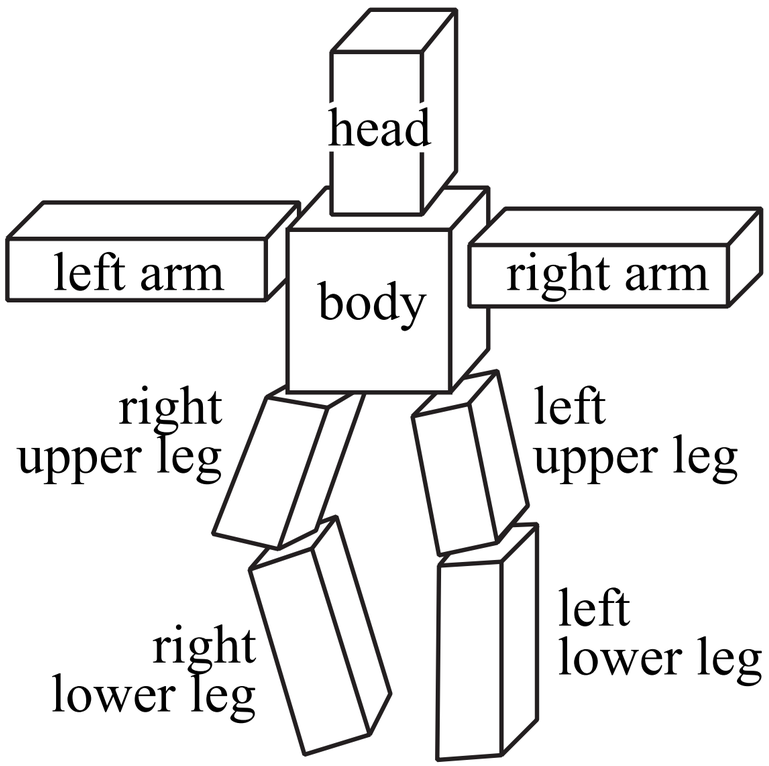


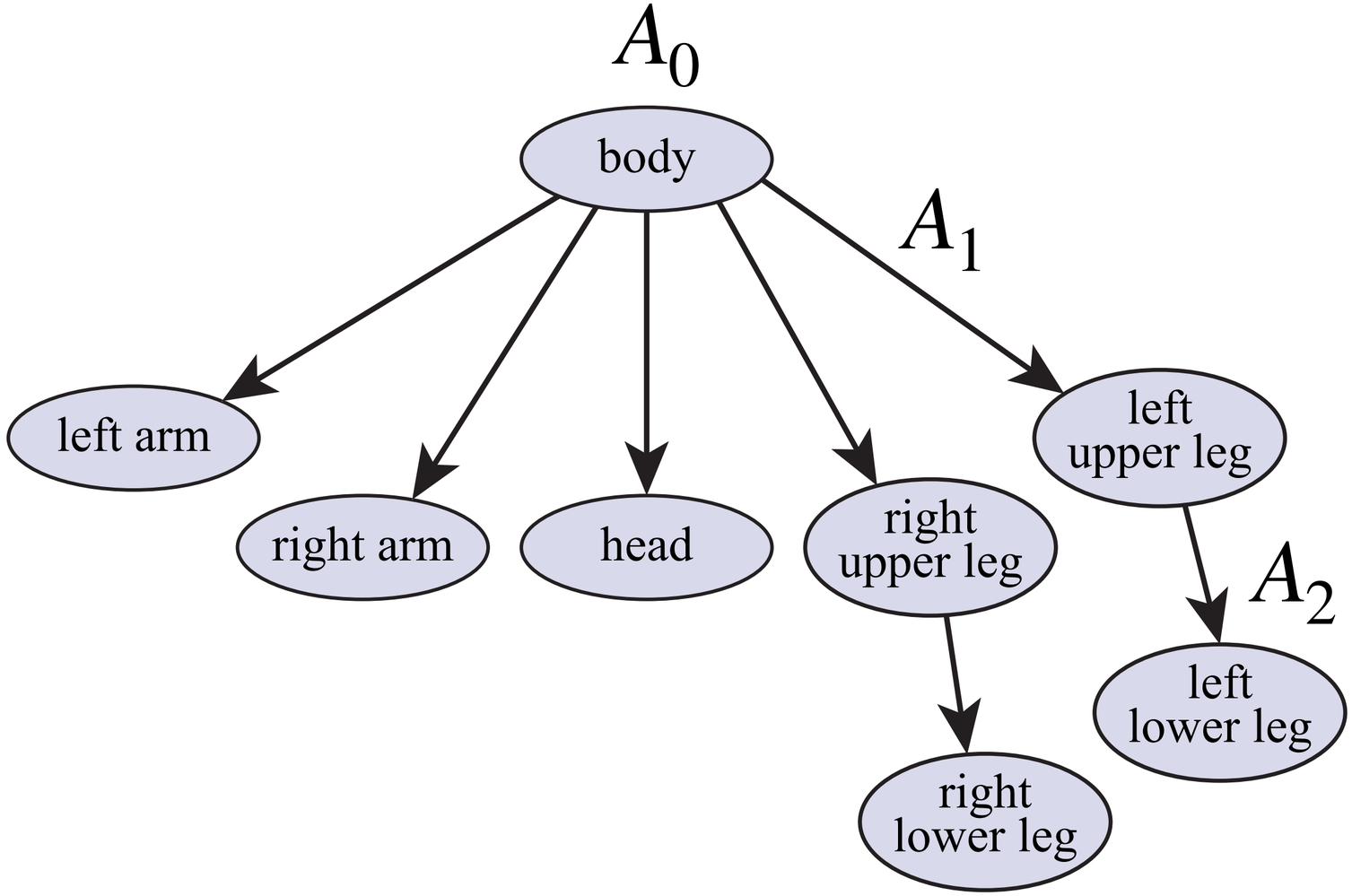

- For complex scenes (e.g., more than just a cube!) scene graph can help organize transformations
- Motivation: suppose we want to build a "cube creature" by transforming copies of the unit cube
- Difficult to specify each transformation directly
- Instead, build up transformations of "lower" parts from transformations of "upper" parts
  - E.g., first position the body
  - Then transform upper arm relative to the body
  - Then transform lower arm relative to upper arm
  - …



# Scene Graph (continued)

- Scene graph stores relative transformations in directed graph
- Each edge (+root) stores a linear transformation (e.g., a 4x4 matrix)
- Composition of transformations gets applied to nodes

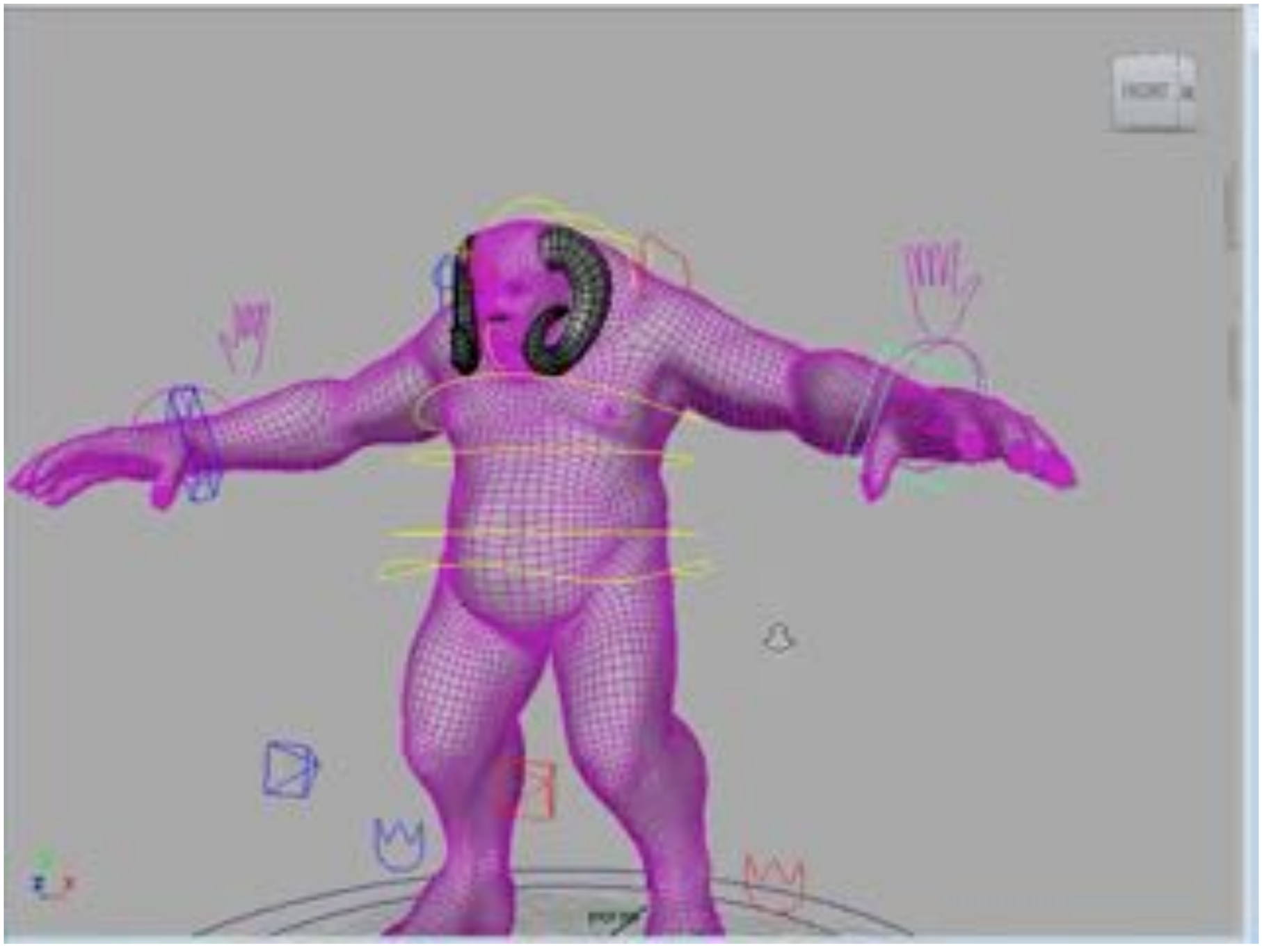



- E.g., $A_1A_0$ gets applied to left upper leg; $A_2A_1A_0$ to left lower leg
- Keep transformations on a stack to reduce redundant multiplication

# Scene Graph—Example

## Often used to build up complex “rig”:

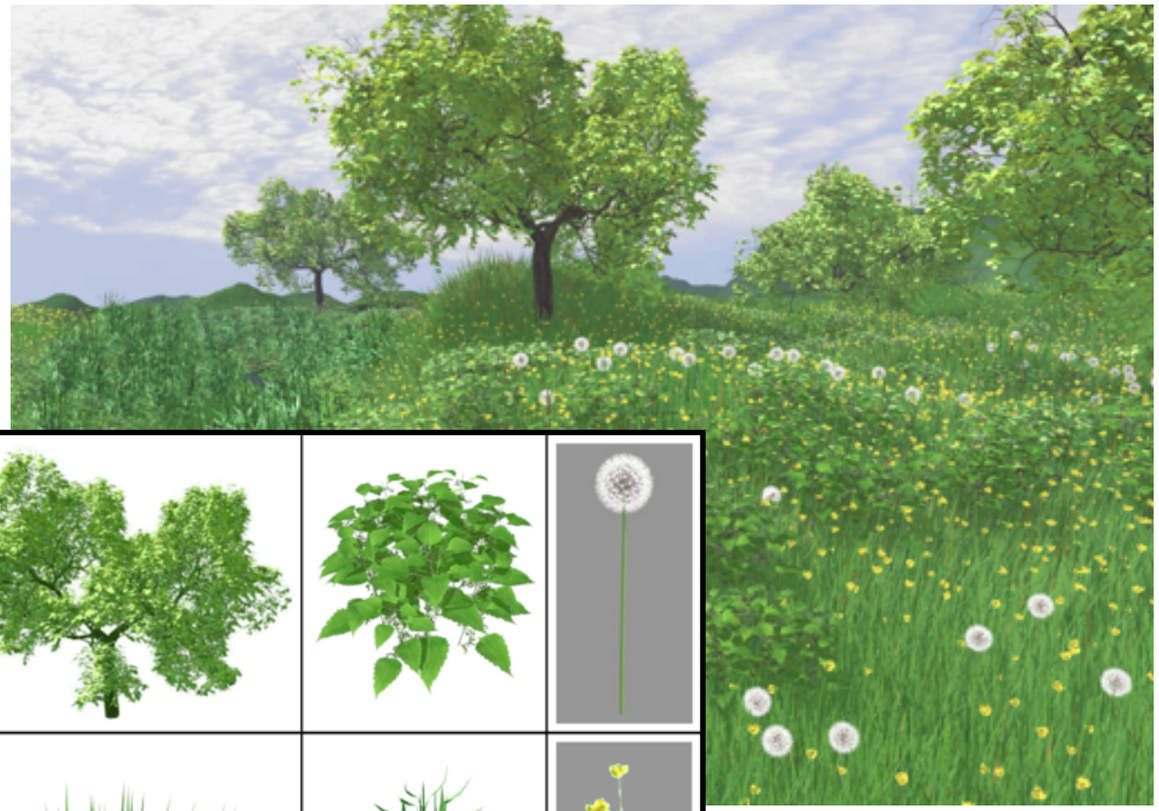


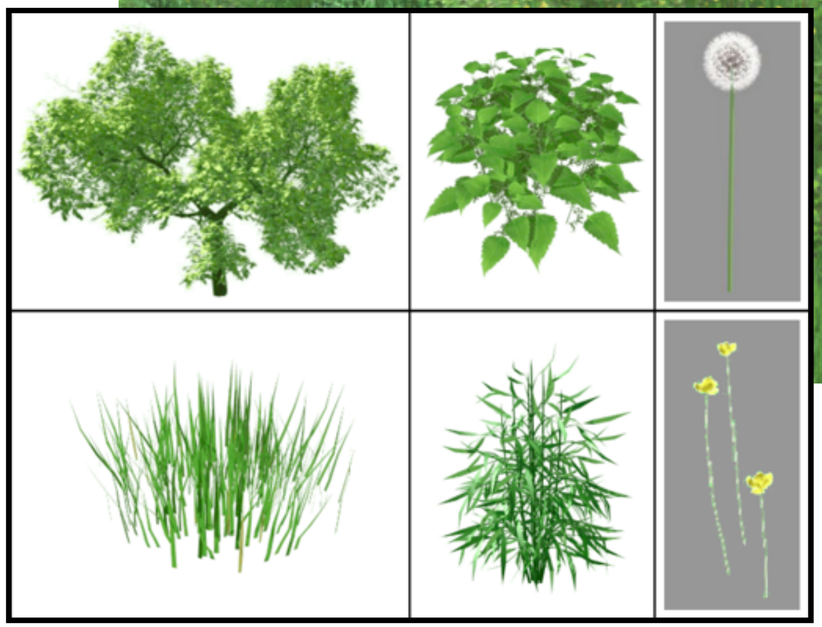
## In general, scene graph also includes other models, lights, cameras, …

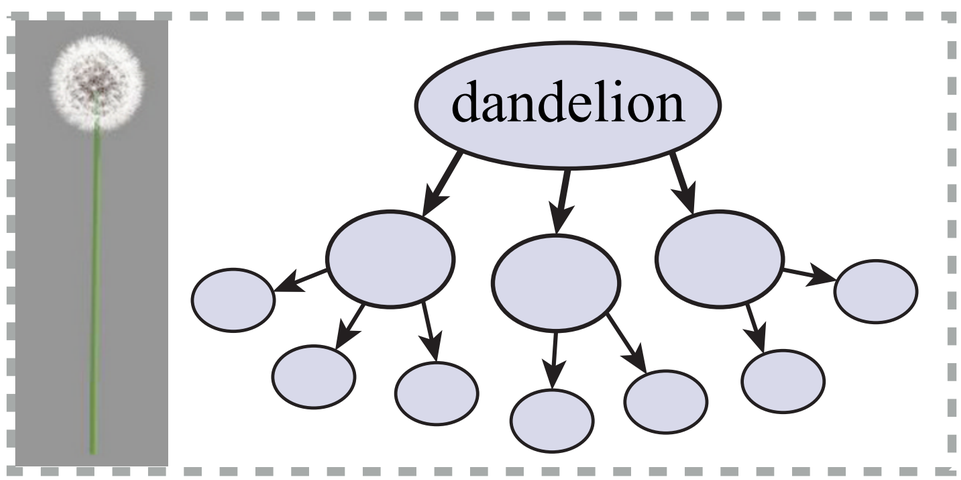
# Instancing

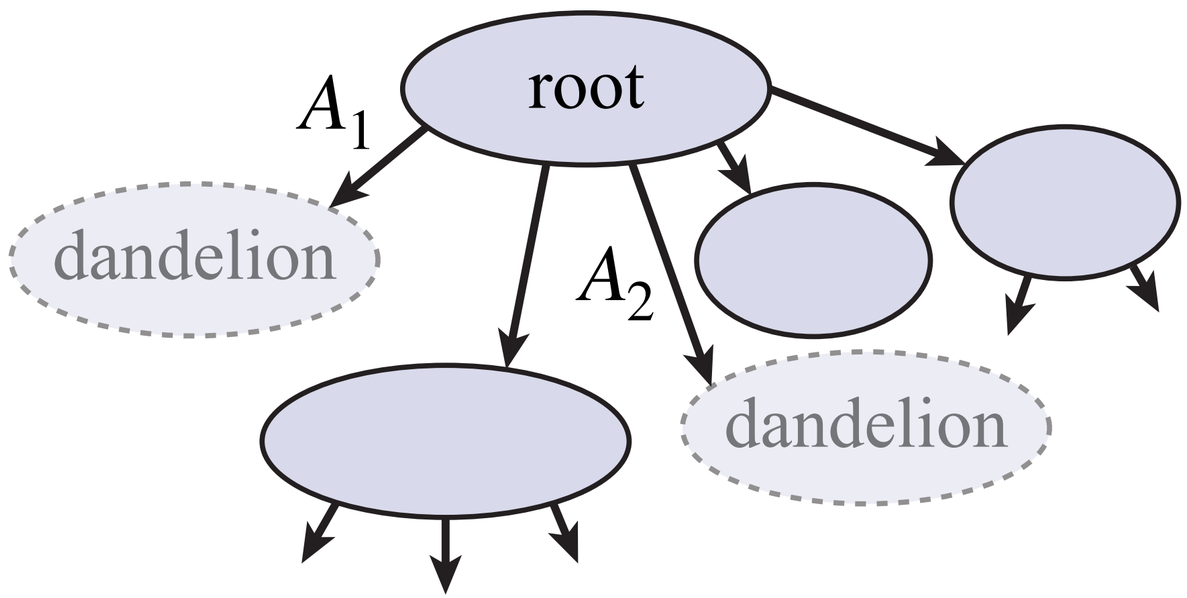
- What if we want many copies of the same object in a scene?
- Rather than have many copies of the geometry, scene graph, etc., can just put a "pointer" node in our scene graph
- Like any other node, can specify a different transformation on each incoming edge

Deussen et al, "Realistic modeling and rendering of plant ecosystems"

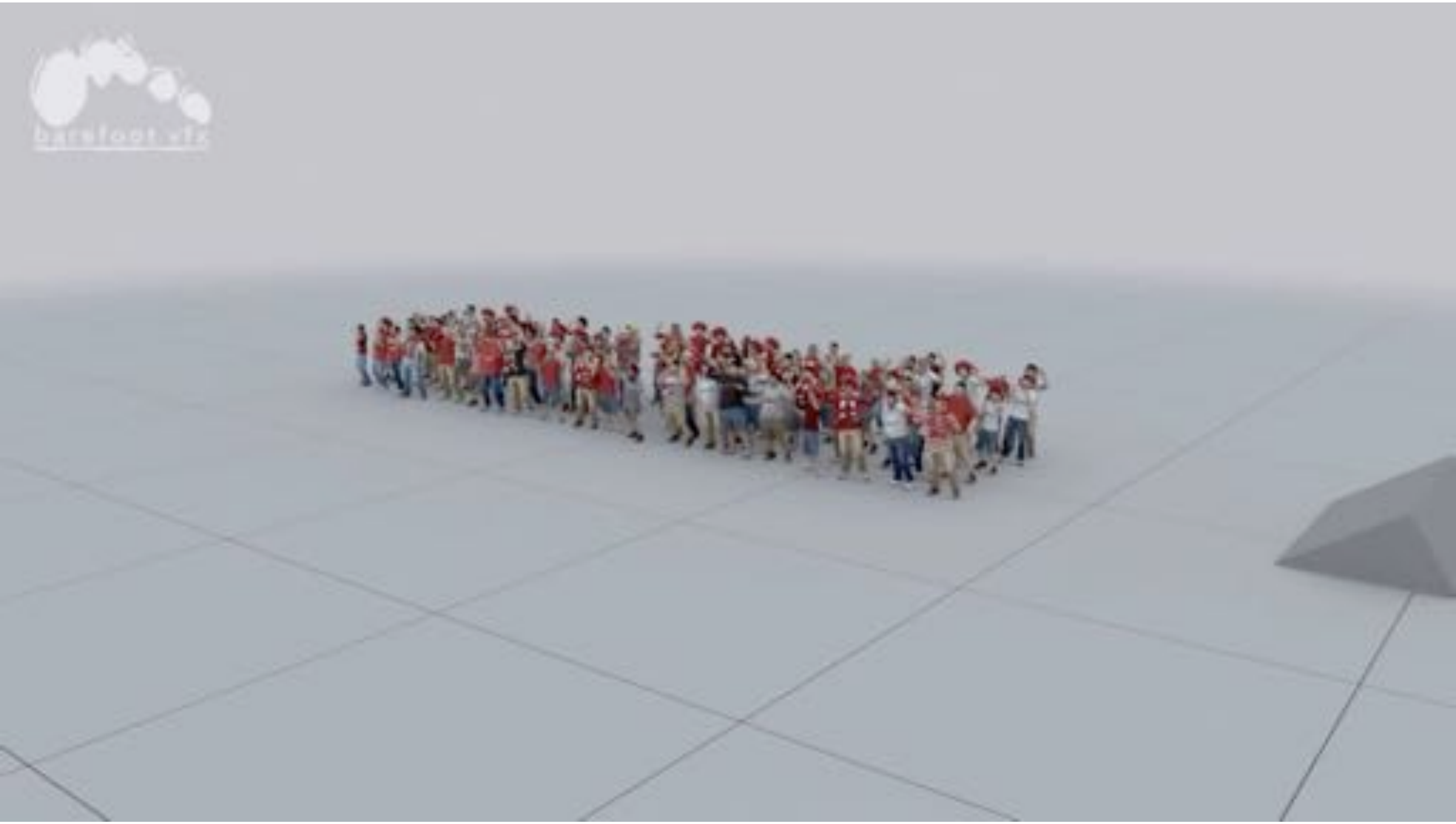




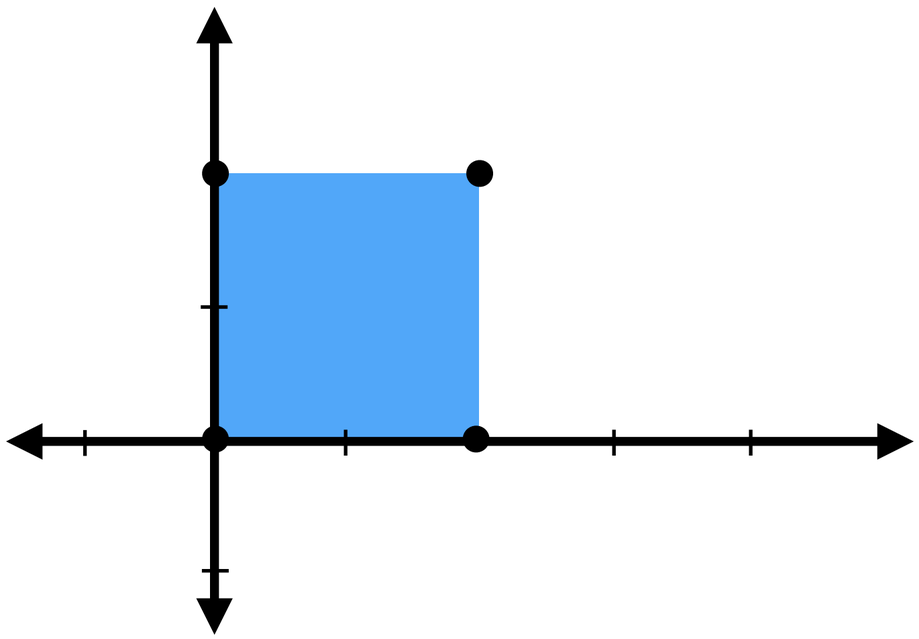


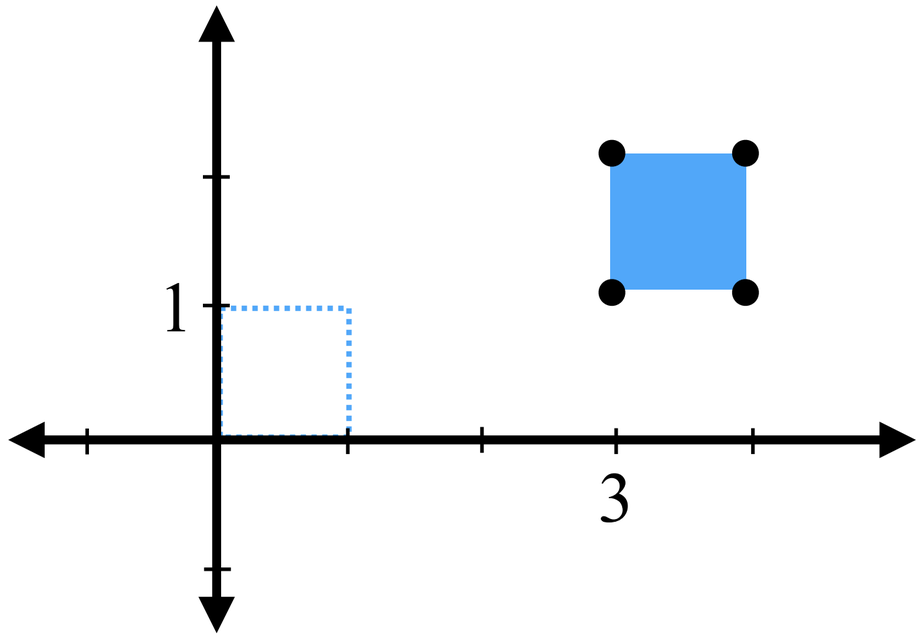
# Instancing—Example



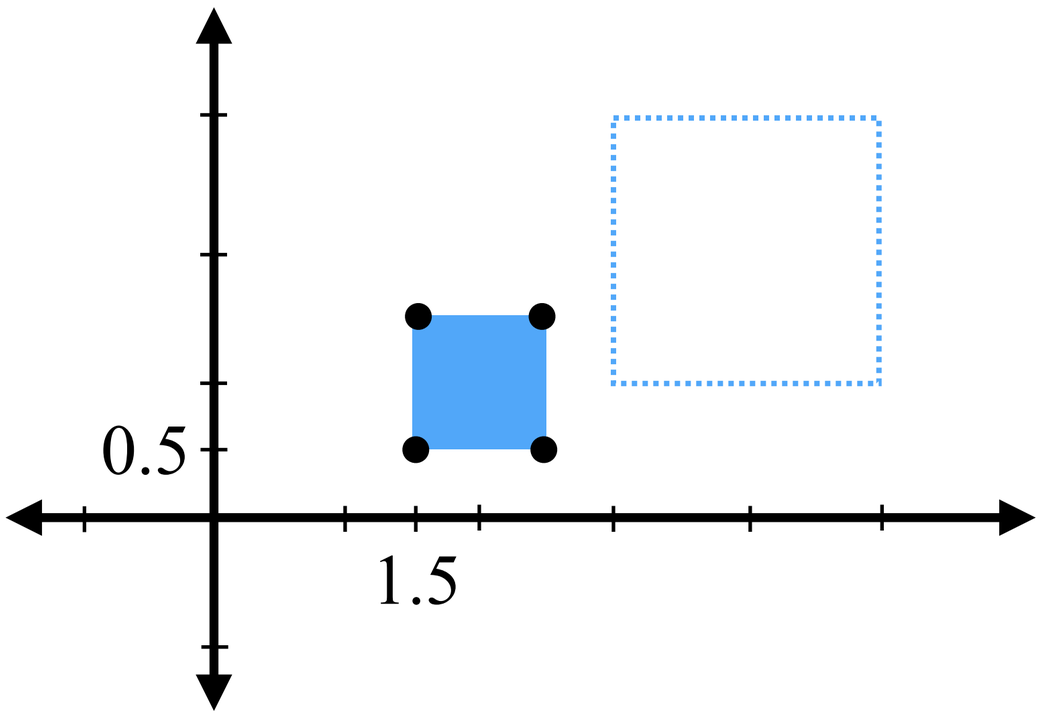
# Order matters when composing transformations!



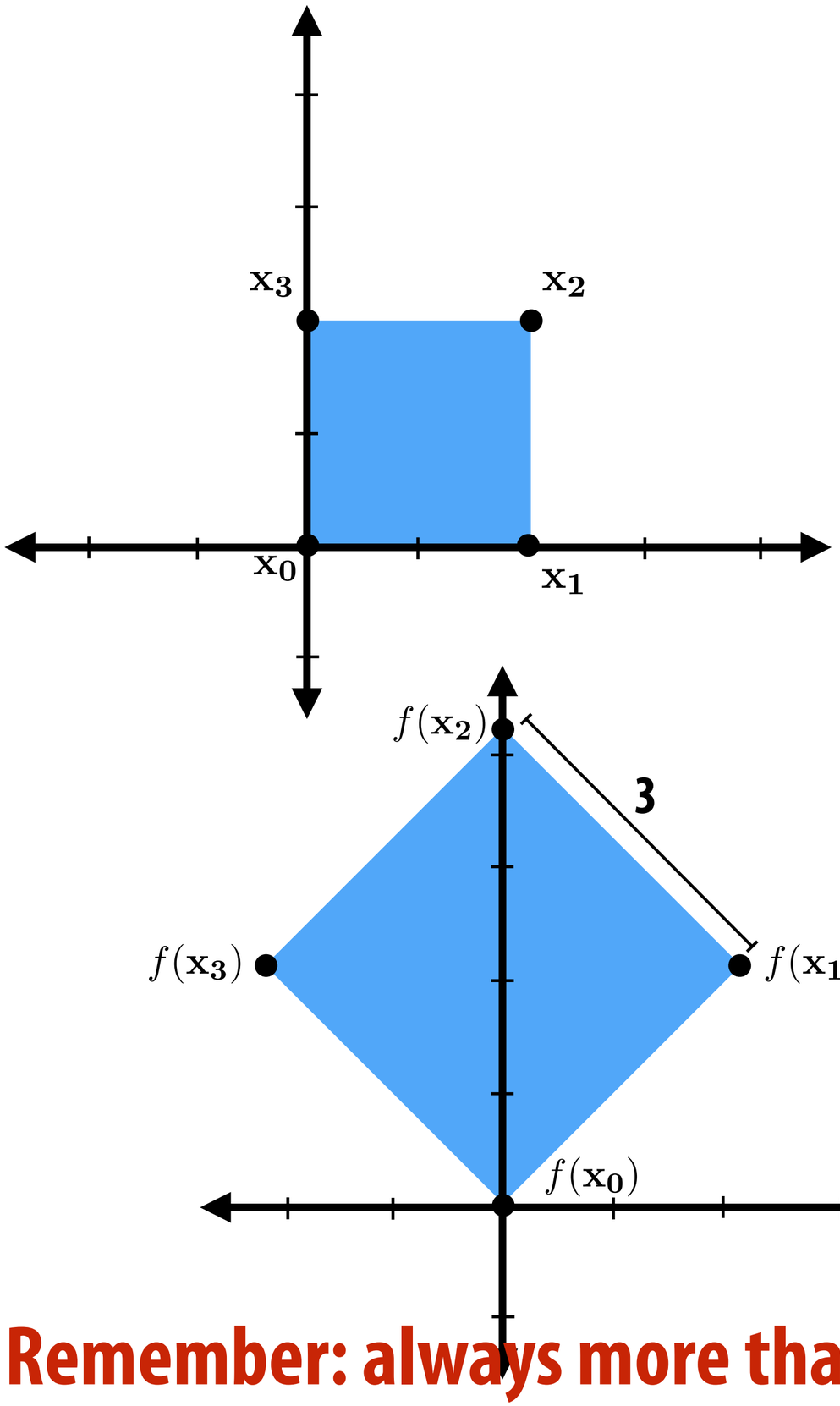
scale by 1/2, then translate by (3,1)

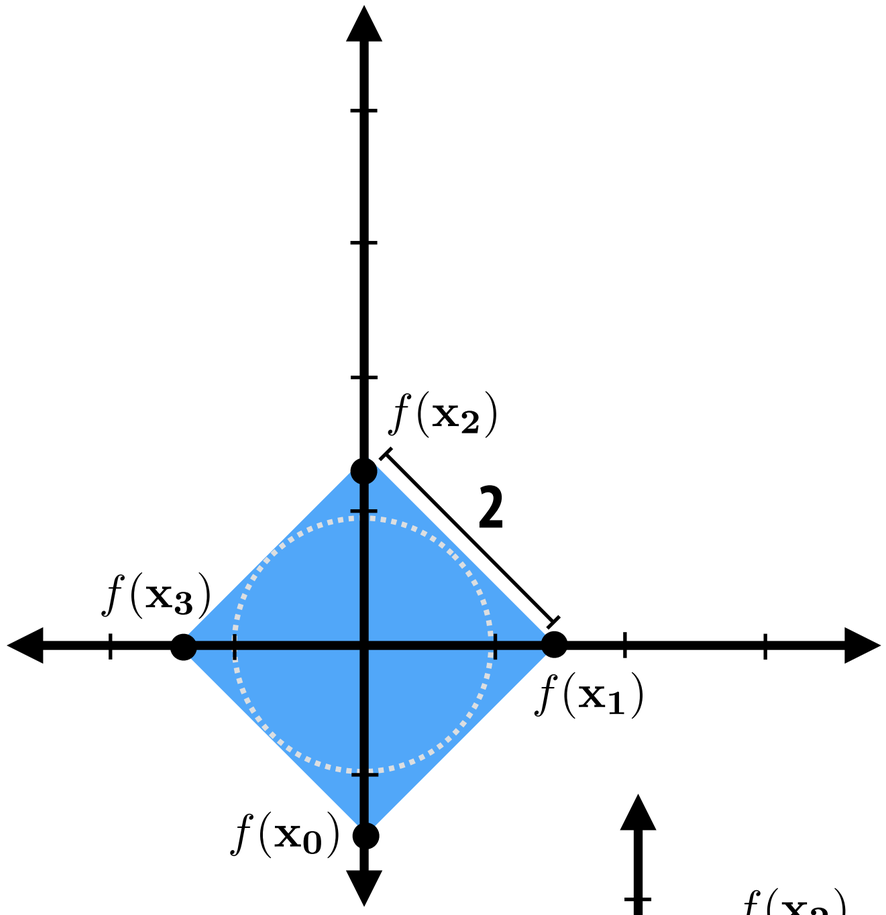


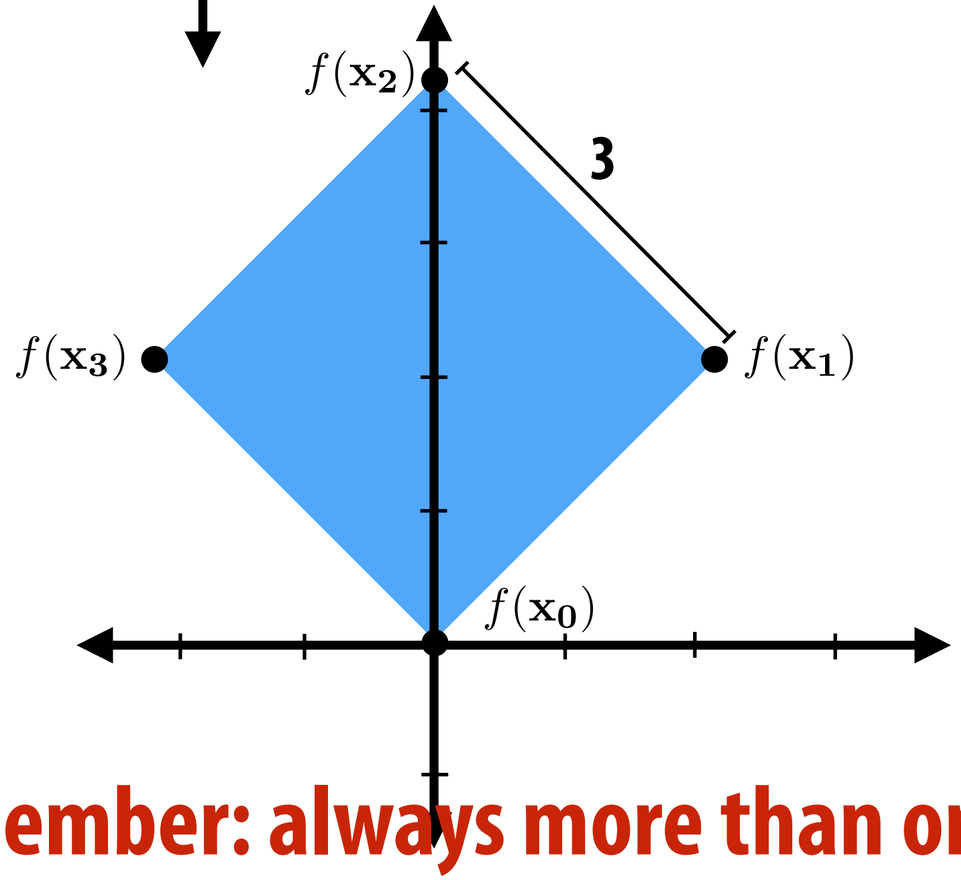

translate by (3,1), then scale by 1/2

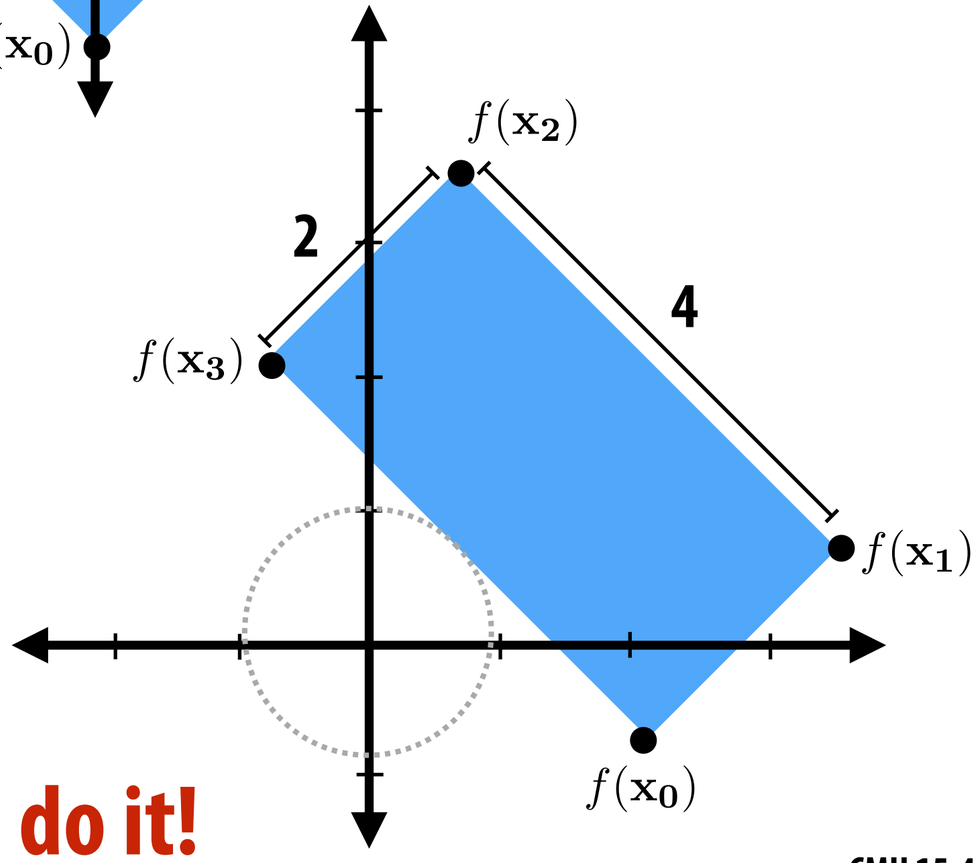


# How would you perform these transformations?



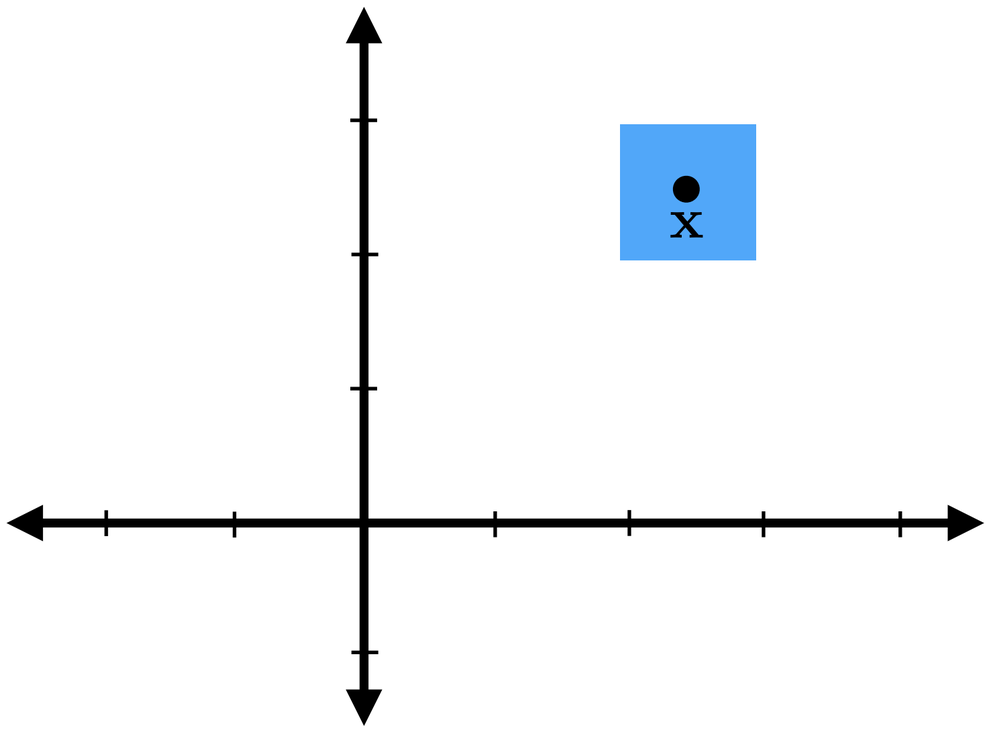



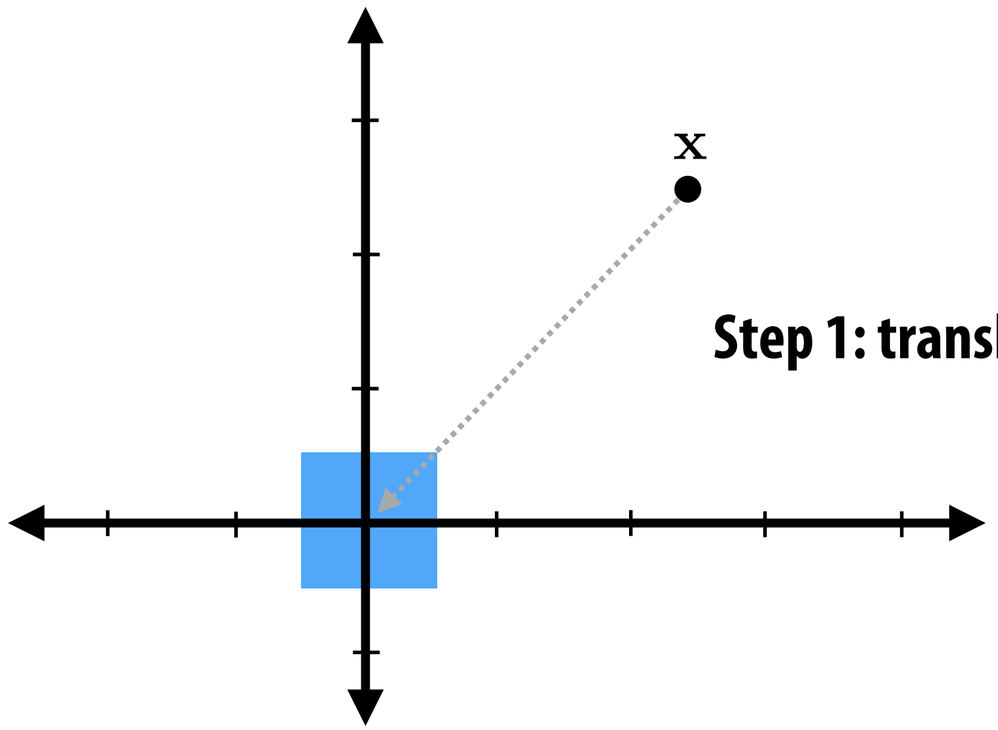



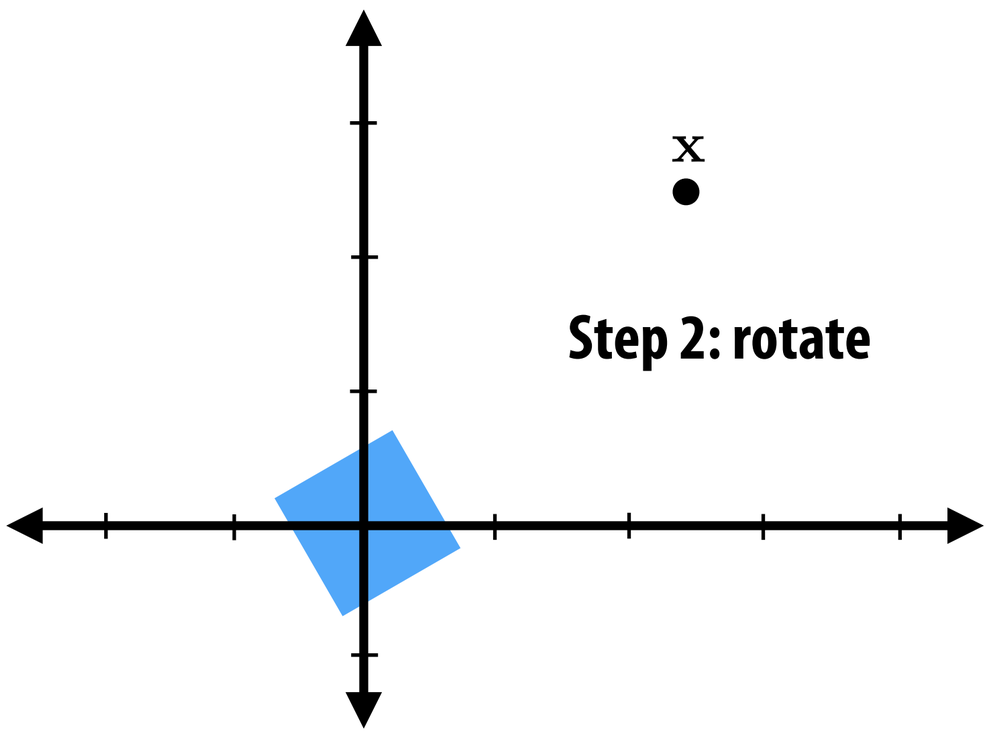



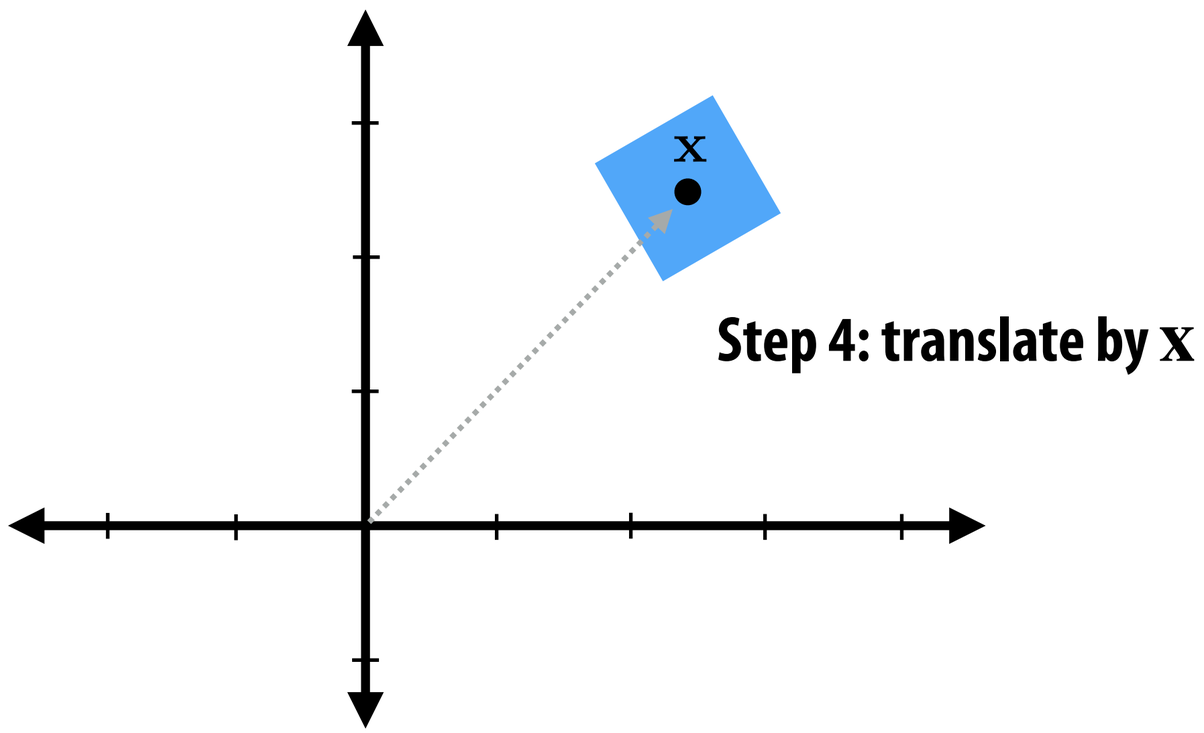

**Remember: always more than one way to do it!**

# Common task: rotate about a point x



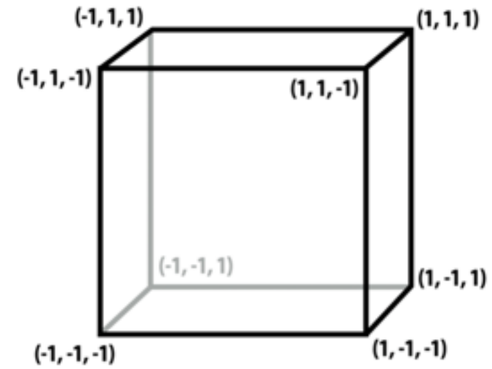



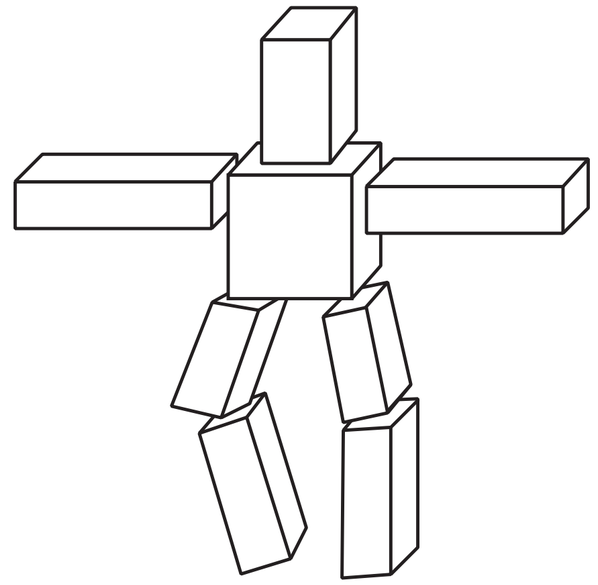




**Q: What happens if we just rotate without translating first?**

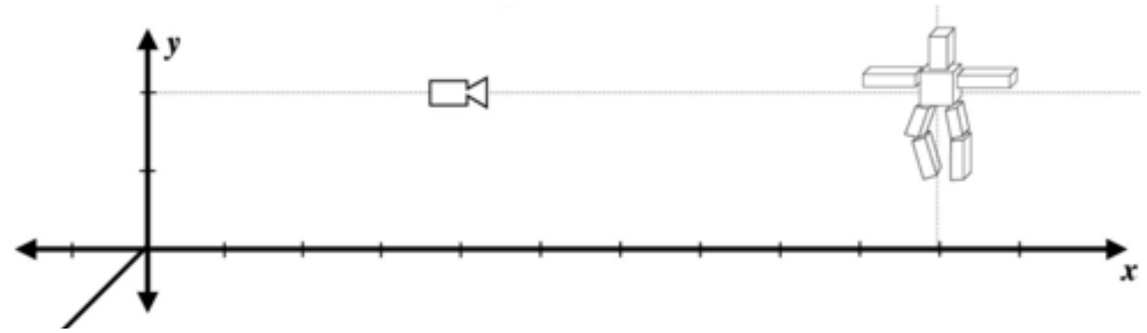
# Drawing a Cube Creature

- Let's put this all together: starting with our 3D cube, we want to make a 2D, perspective-correct image of a "cube creature"

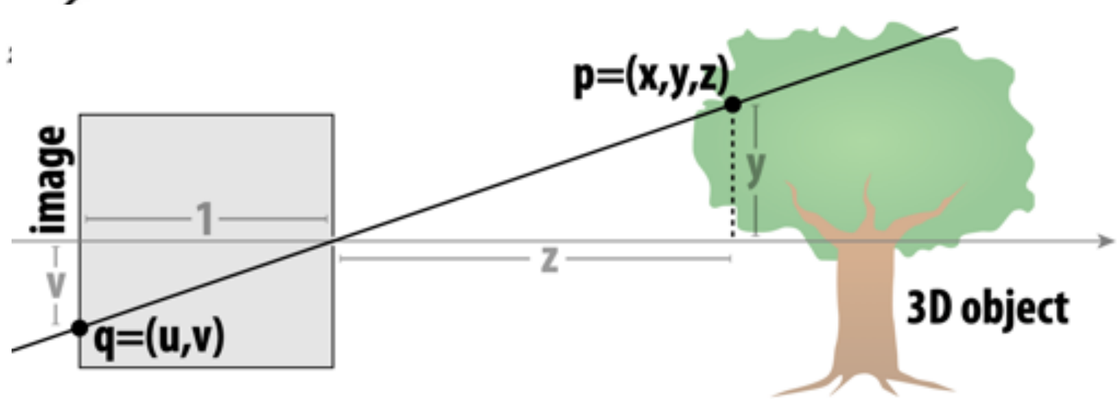


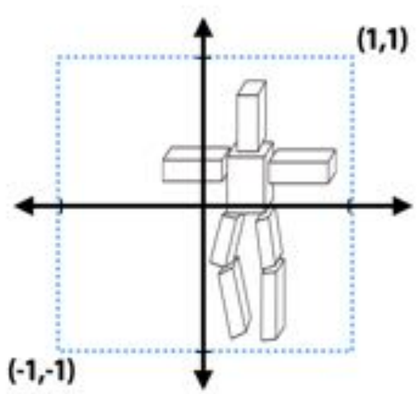



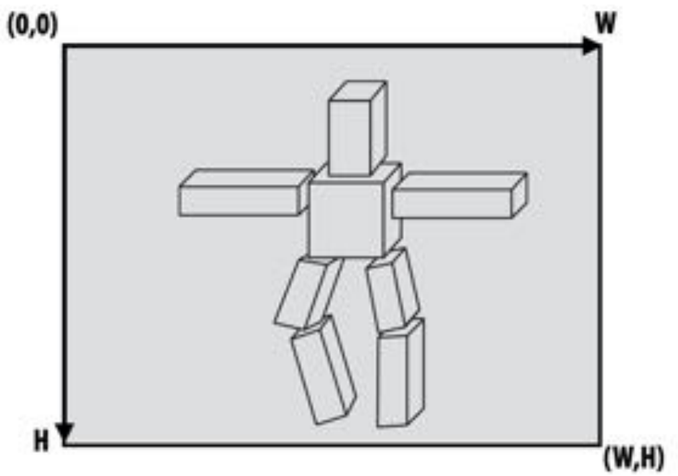
- First we use our scene graph to apply 3D transformations to several copies of our cube




- Then we apply a 3D transformation to position our camera
- Then a perspective projection
- Finally we convert to image coordinates (and rasterize)
- …Easy, right? :-)



# Spatial Transformations—Summary

## transformation defined by its invariants

### basic linear transformations

scaling
rotation
reflection
shear

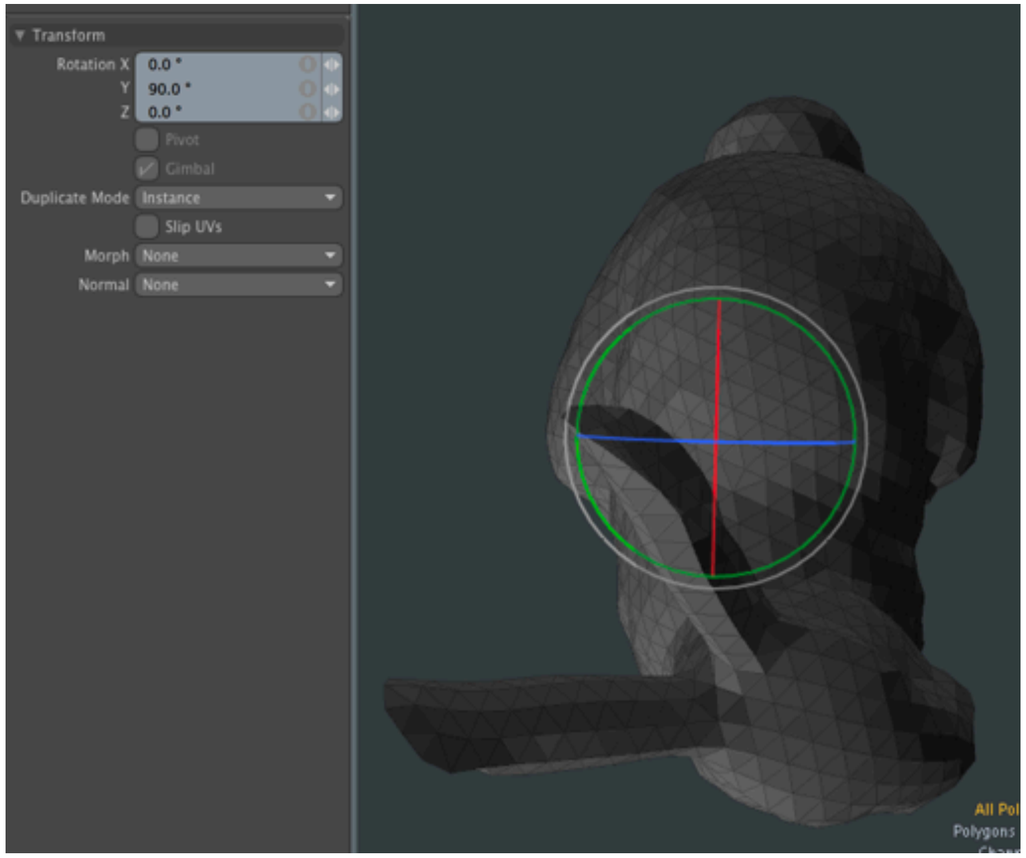
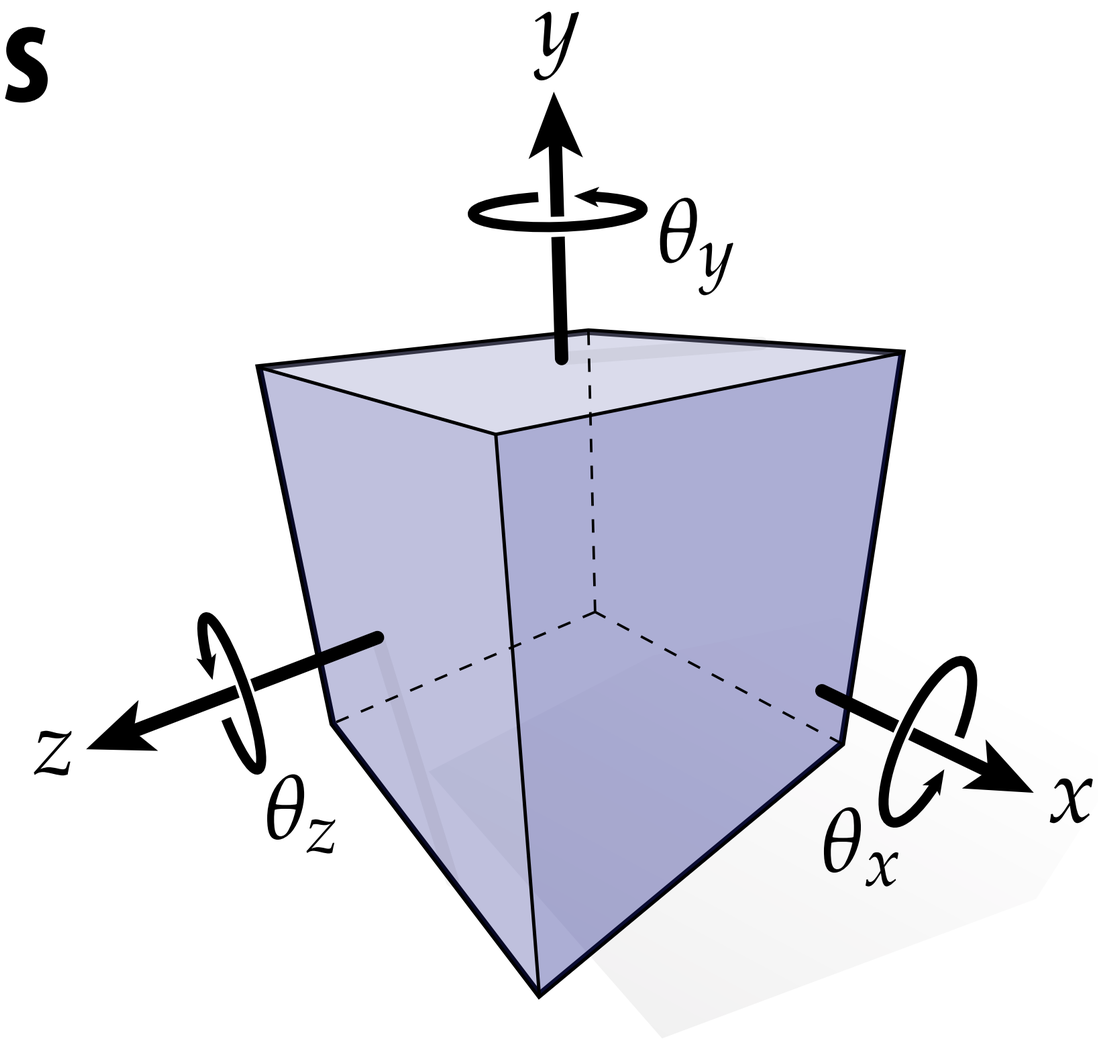
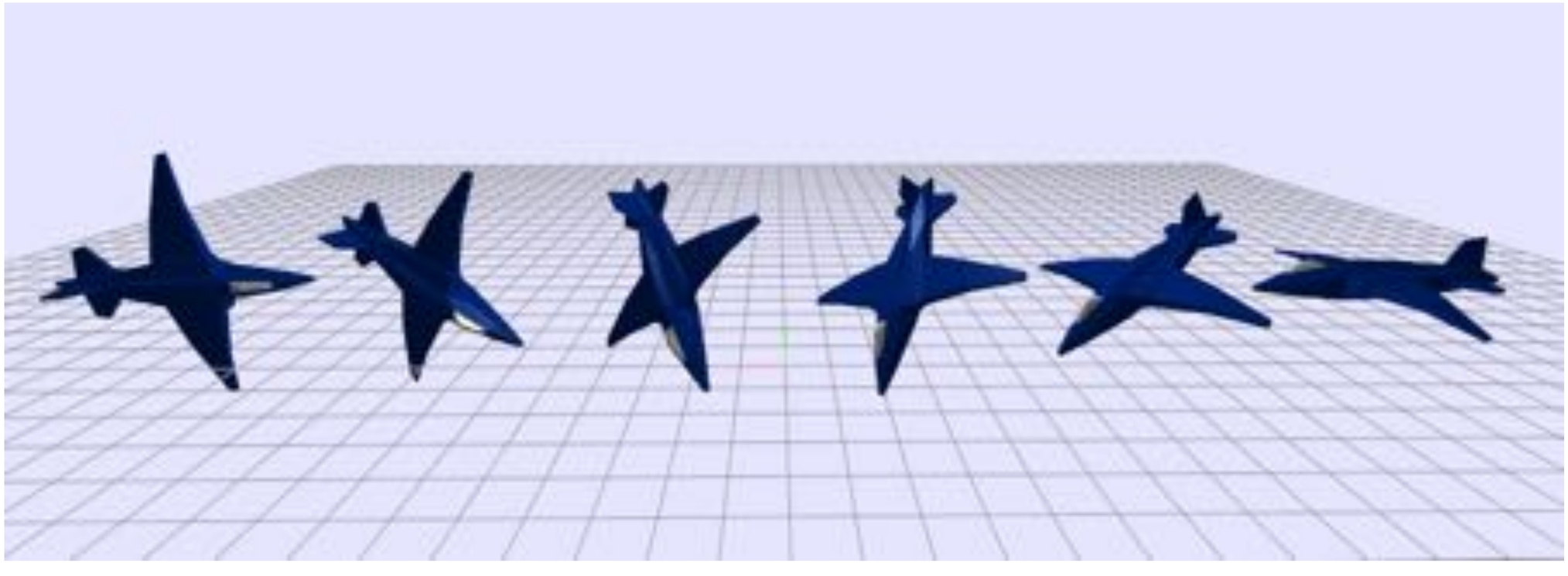
### basic nonlinear transformations

translation
perspective projection

linear when represented via homogeneous coords

homogeneous coords also distinguish points & vectors

### composite transformations

- compose basic transformations to get more interesting ones
- always reduces to a single 4x4 matrix (in homogeneous coordinates)
  - –simple, unified representation, efficient implementation
- order of composition matters!
- many ways to decompose a given transformation (polar, SVD, . . .)
- use scene graph to organize transformations
  - use instancing to eliminate redundancy

# Next time: 3D Rotations