Math (P)Review Part II:
Vector Calculus

Computer Graphics
CMU 15-462/662

Assignment 0.0 due / Assignment 0.5 out

- **Same story as last homework; second part on vector calculus.**
- **Autolab hand-in**

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1 Vector Calculus

1.1 Dot and Cross Product

In our study of linear algebra, we looked *inner products* in the abstract, *i.e.*, we said that an inner product $\langle \cdot, \cdot \rangle$ was *any* operation that is symmetric, bilinear, *etc.* In the context of vector calculus, we often work with one very special inner product called the **dot product**, which has a concrete geometric relationship to lengths and angles in $\mathbb{R}^n$. In particular, consider any two n-dimensional Euclidean vectors $\mathbf{u} = (u_1, \ldots, u_n)$ $\mathbf{v} = (u_n, \ldots, v_n)$ where the components u_i, v_i are expressed with respect to some orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The **dot product** is defined as

$$\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^{n} u_i n_i,$$

and satisfies the geometric relationship

$$\mathbf{u} \cdot \mathbf{v} := |\mathbf{u}||\mathbf{v}| \cos(\theta),$$

where $|\mathbf{u}|$ and $|\mathbf{v}|$ are the lengths of $\mathbf{u}$ and $\mathbf{v}$, respectively, and $\theta \geq 0$ is the (unsigned) angle between them.

Exercise 1. *Suppose we are working in $\mathbb{R}^2$ with the standard orthonormal basis $\mathbf{e}_1 := (1,0)$, $\mathbf{e}_2 := (0,1)$.*

*(a) Compute the Cartesian coordinates of a vector $\mathbf{u}$ with length $\ell_1 := 6$ and counter-clockwise angle $\theta_1 :=$ 0.100 relative to the positive $\mathbf{e}_1$-axis. [**Hint:** You may want to revisit our earlier discussion of polar coordinates.]*

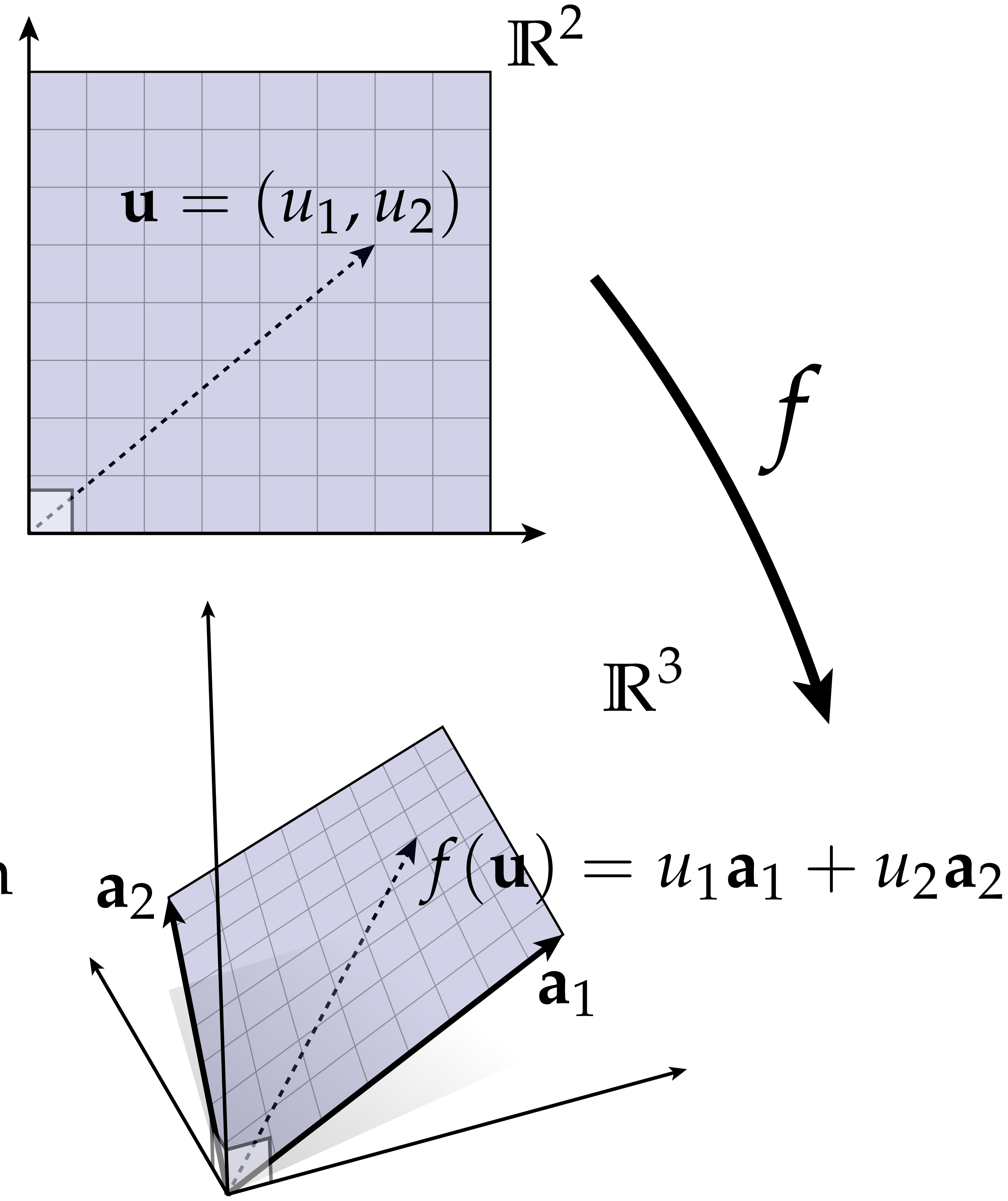
(b) Compute the Cartesian coordinates of a vector $\mathbf{v}$ with length $\ell_2 := 3$ and counter-clockwise angle $\theta_2 :=$

Last Time: Linear Algebra

- **Touched on a variety of topics:**

vectors & vector spaces
norm
L^2 norm/inner product
span
Gram-Schmidt
linear systems
quadratic forms
...

vectors as functions
inner product
linear maps
basis
frequency decomposition
bilinear forms
matrices
...

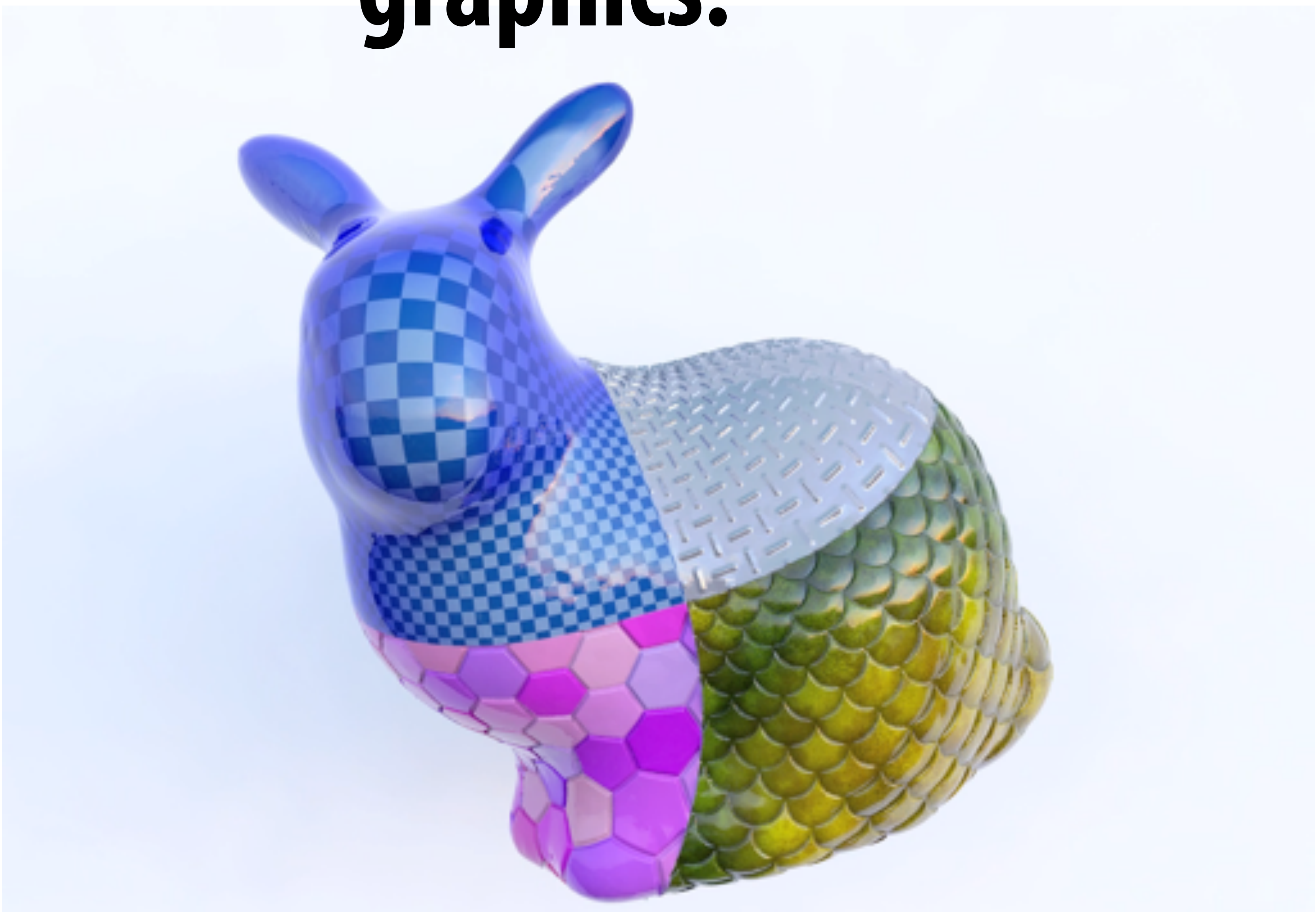


- **Don't have time to cover everything!**
- **But there are some fantastic lectures online:**

3Blue1Brown — Essence of Linear Algebra
Robert Ghrist — Calculus Blue
...
(Let us know about others online!)

Vector Calculus in Computer Graphics

- Today's topic: vector calculus.
- Why is vector calculus important for computer graphics?
  - Basic language for talking about spatial relationships, transformations, etc.
  - Much of modern graphics (physically-based animation, geometry processing, etc.) formulated in terms of partial differential equations (PDEs) that use div, curl, Laplacian...
  - As we saw last time, vector-valued data is everywhere in graphics!

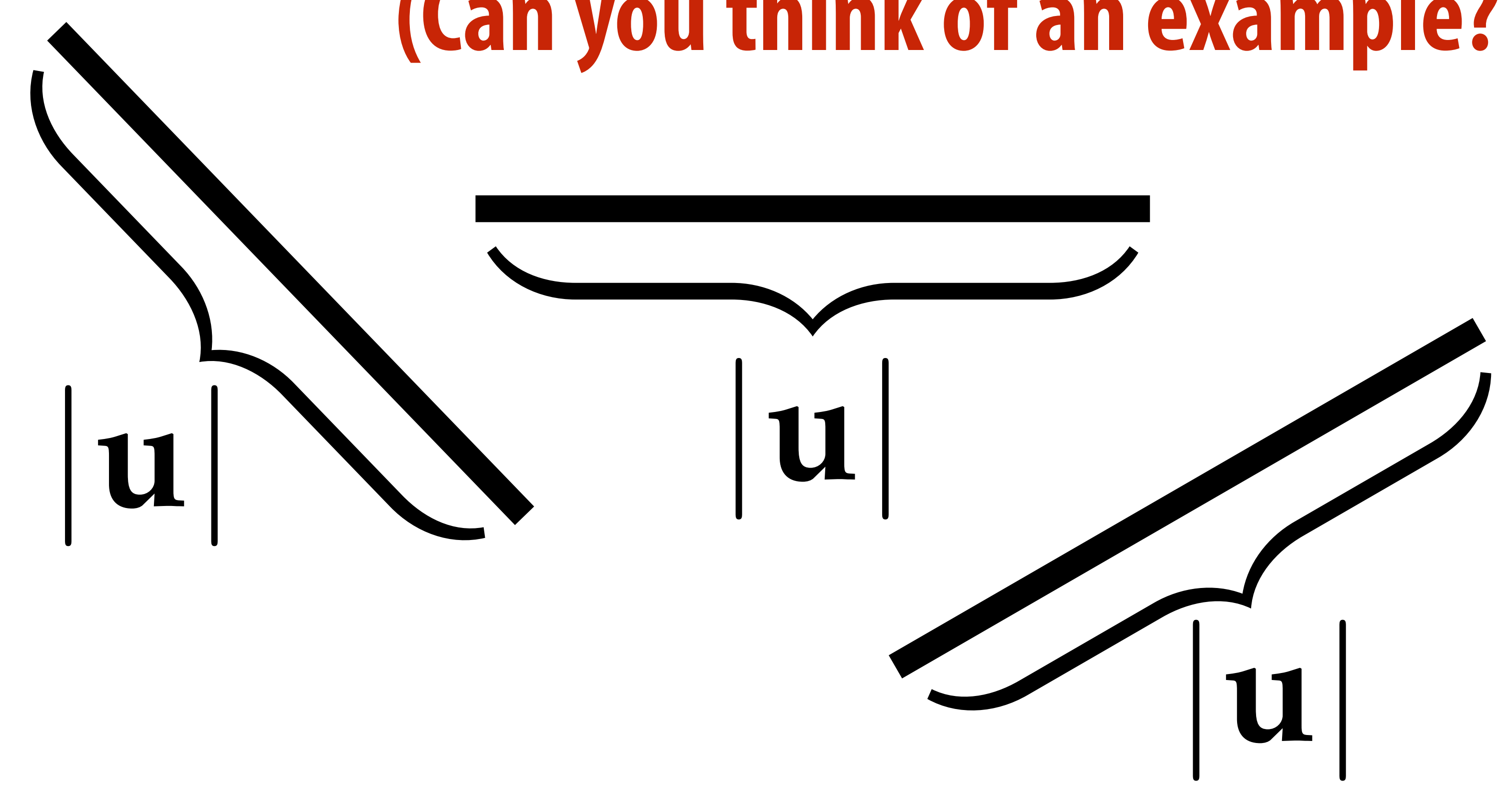


Euclidean Norm

- Last time, developed idea of norm, which measures total size, length, volume, intensity, etc.
- For geometric calculations, the norm we most often care about is the Euclidean norm
- Euclidean norm is any notion of length preserved by rotations/translations/reflections of space.
- In orthonormal coordinates:

Not true for all norms!
(Can you think of an example?)

$$|\mathbf{u}| := \sqrt{u_1^2 + \cdots + u_n^2}$$

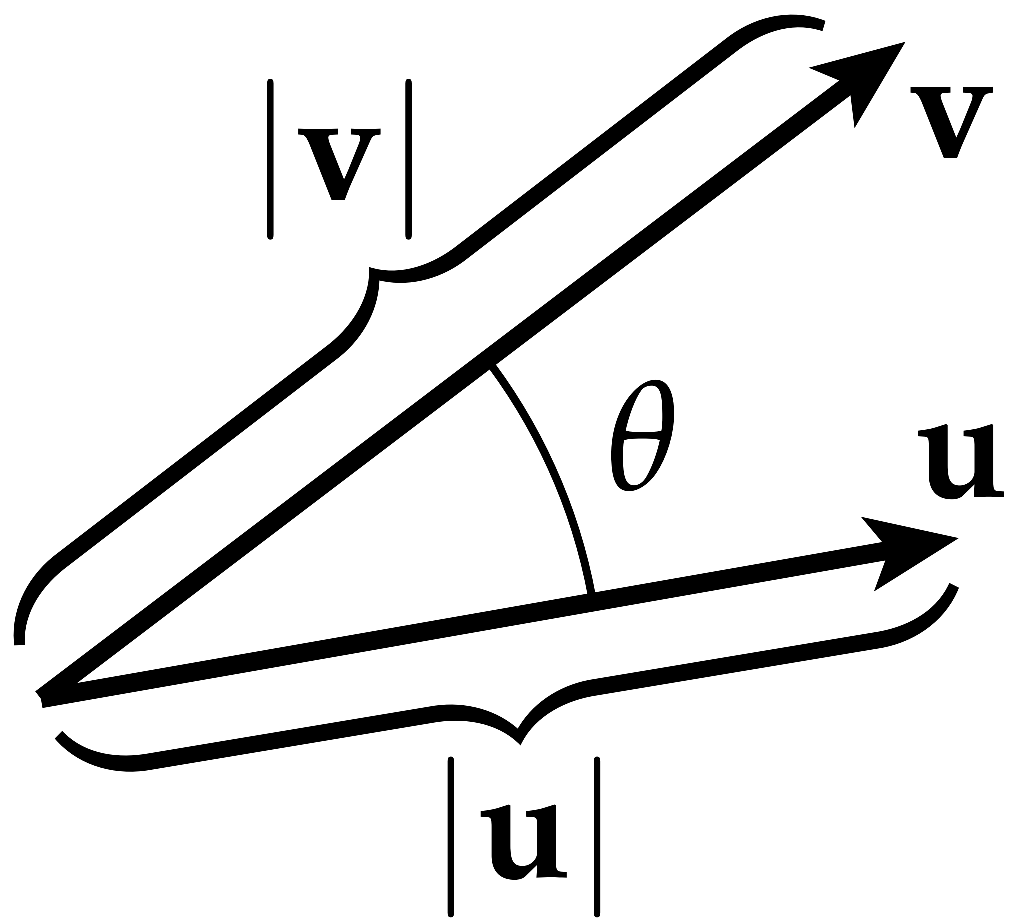


WARNING: This quantity does not encode geometric length unless vectors are encoded in an orthonormal basis. (Common source of bugs!)

Euclidean Inner Product / Dot Product

- Likewise, lots of possible inner products—intuitively, measure some notion of "alignment."
- For geometric calculations, want to use inner product that captures something about geometry!
- For n-dimensional vectors, Euclidean inner product defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle := |\mathbf{u}||\mathbf{v}| \cos(\theta)$$



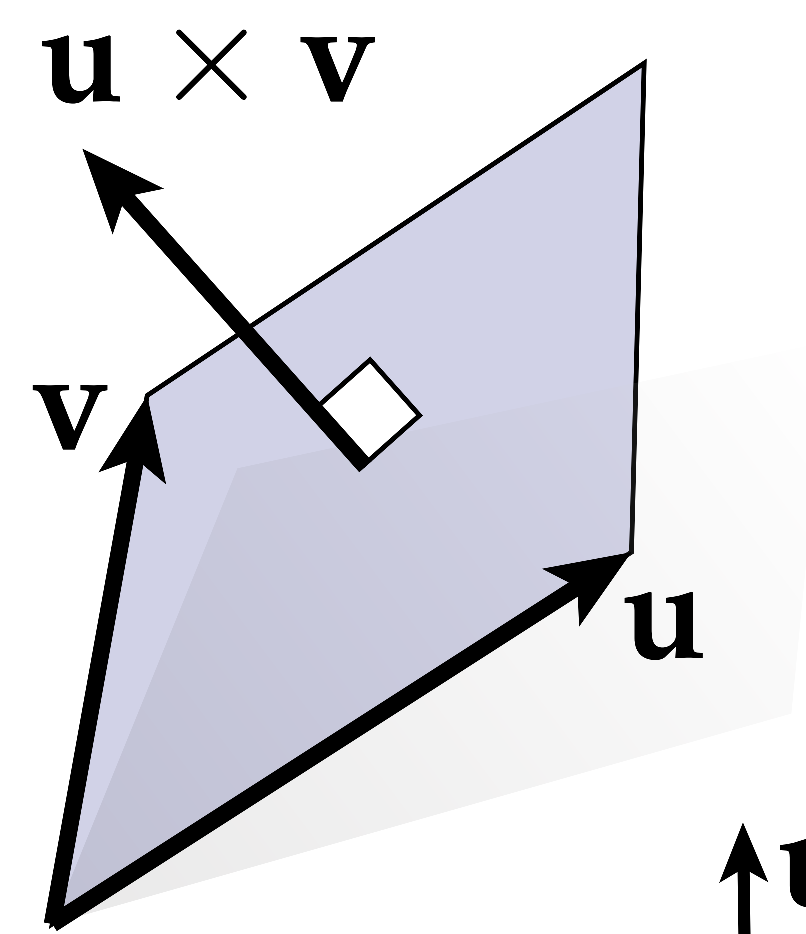
- In orthonormal Cartesian coordinates, can be represented via the dot product

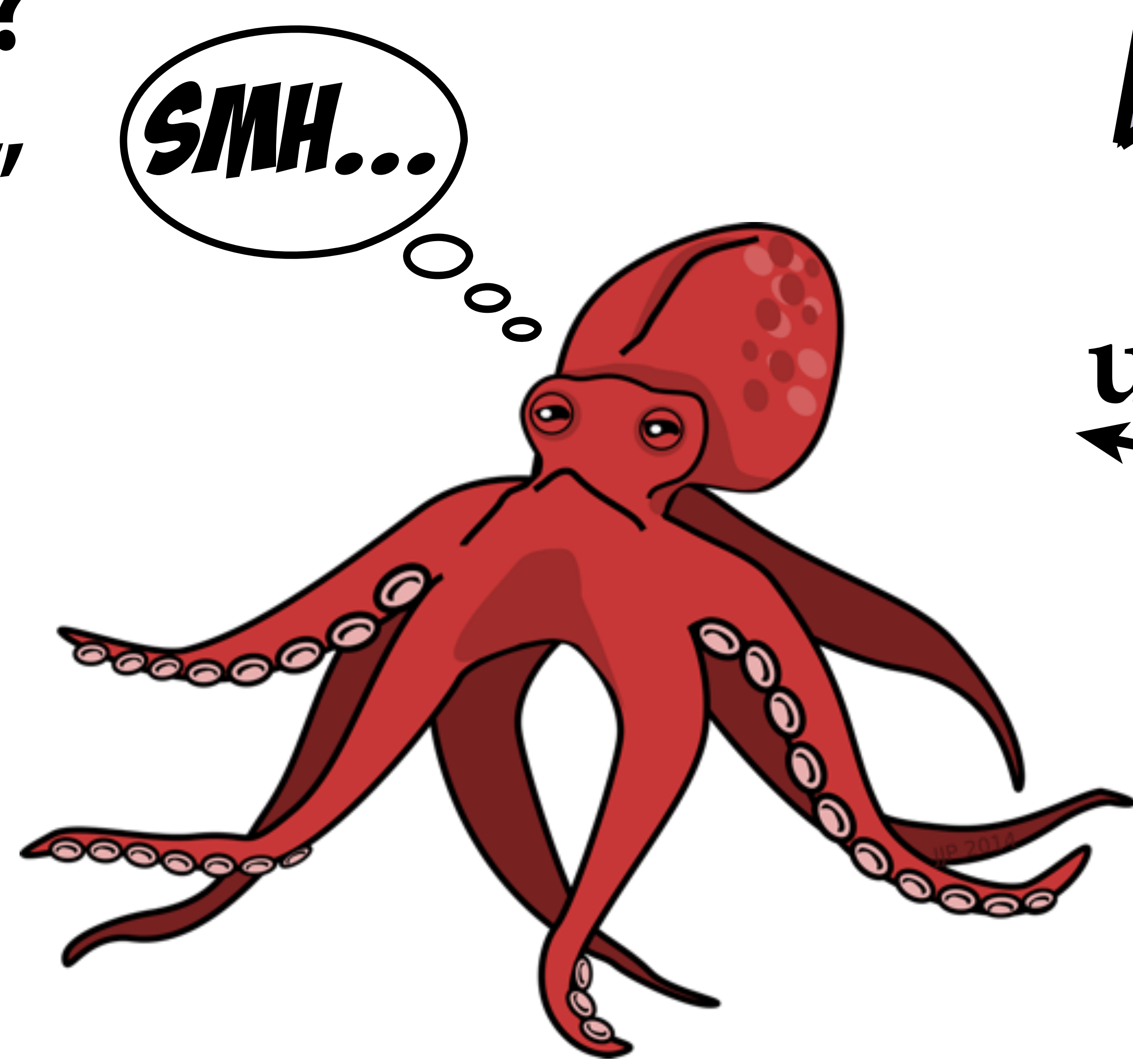
$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n$$

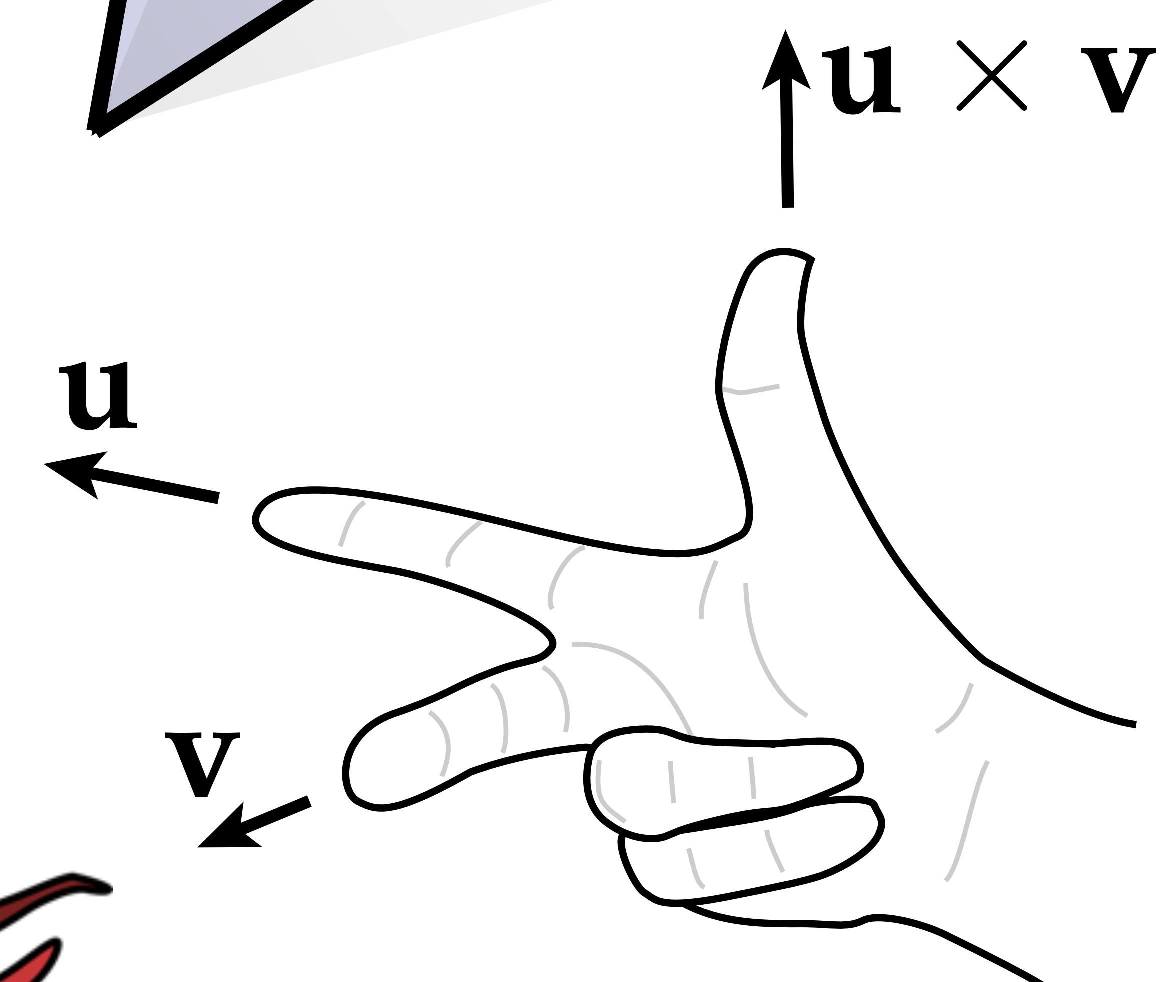
- WARNING: As with Euclidean norm, no geometric meaning unless coordinates come from an orthonormal basis.

Cross Product

- Inner product takes two vectors and produces a scalar
- In 3D, cross product is a natural way to take two vectors and get a vector, written as "u x v"
- Geometrically:
  - magnitude equal to parallelogram area
  - direction orthogonal to both vectors
  - …but which way?
- Use "right hand rule"





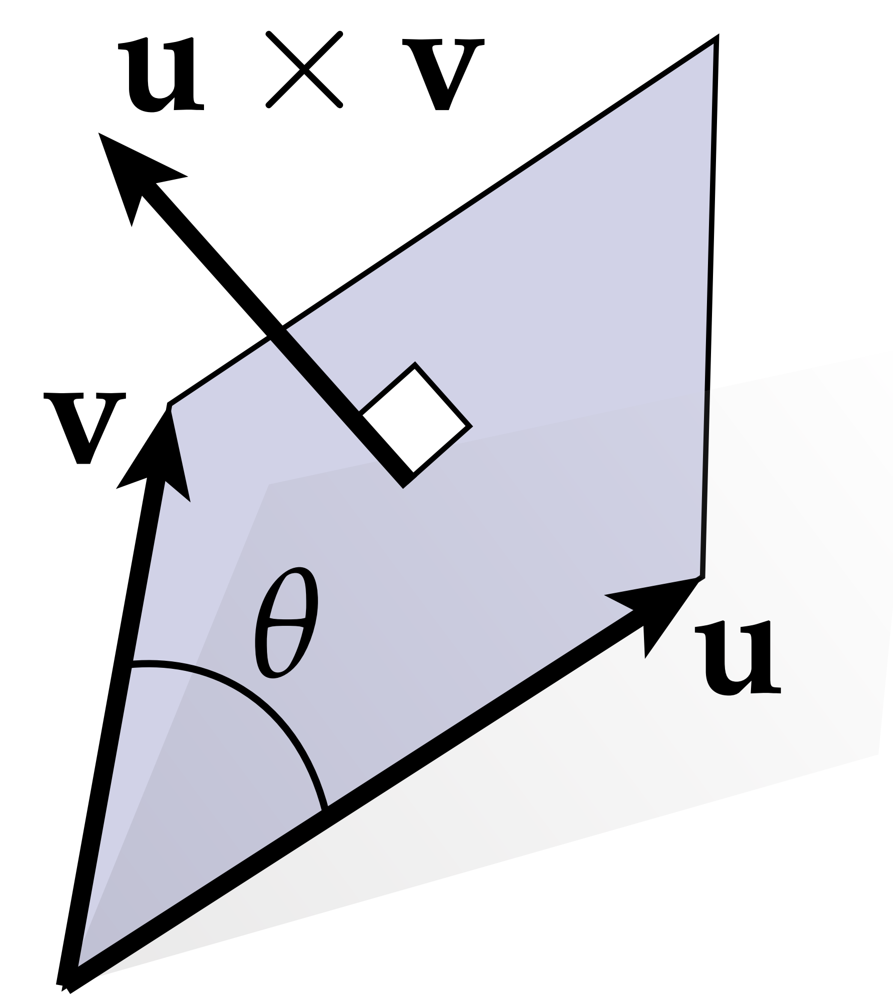


(Q: Why only 3D?)

Cross Product, Determinant, and Angle

- **More precise definition (that does not require hands):**

$$\sqrt{\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})} = |\mathbf{u}||\mathbf{v}|\sin(\theta)$$



- **θ is angle between u and v**
- **"det" is determinant of three column vectors**
- **Uniquely determines coordinate formula:**

$$\mathbf{u} \times \mathbf{v} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

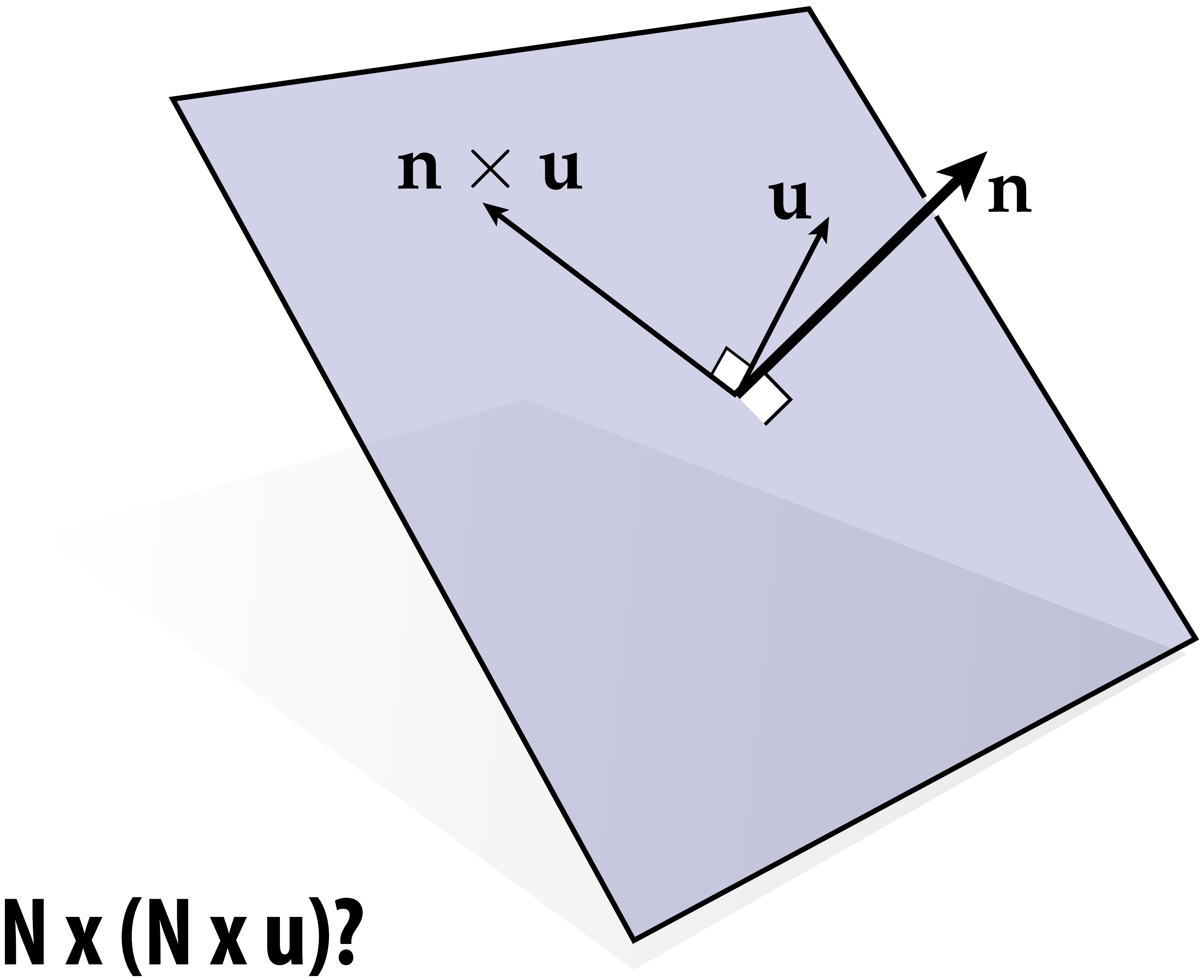
$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(mnemonic)

- **Useful abuse of notation in 2D:** $\mathbf{u} \times \mathbf{v} := u_1 v_2 - u_2 v_1$

Cross Product as Quarter Rotation

- **Simple but useful observation for manipulating vectors in 3D: cross product with a unit vector N is equivalent to a quarter-rotation in the plane with normal N:**



- **Q: What is N x (N x u)?**
- **Q: If you have u and N x u, how do you get a rotation by some arbitrary angle θ?**

Matrix Representation of Dot Product

- **Often convenient to express dot product via matrix product:**

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^{\mathsf{T}}\mathbf{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i$$

- **By the way, what about some other inner product?**
- **E.g., <u,v> := 2 u1 v1 + u1 v2 + u2 v1 + 3 u2 v2**

$$\underbrace{\begin{bmatrix} u_1 & u_2 \end{bmatrix}}_{\mathbf{u}^{\mathsf{T}}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2v_1 + v_2 \\ v_1 + 3v_2 \end{bmatrix}$$

$$= (2u_1v_1 + u_1v_2) + (u_2v_1 + 3u_2v_2). \quad \checkmark$$

Q: Why is matrix representing inner product always symmetric ($A^{\mathsf{T}}=A$)?

Matrix Representation of Cross Product

- **Can also represent cross product via matrix multiplication:**

$$\mathbf{u} := (u_1, u_2, u_3) \qquad \Rightarrow \widehat{\mathbf{u}} := \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

$$\mathbf{u} \times \mathbf{v} = \widehat{\mathbf{u}}\mathbf{v} \;\; = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

(Did we get it right?)

- **Q: Without building a new matrix, how can we express v x u?**
- **A: Useful to notice that v x u = -u x v (why?). Hence,**

$$\mathbf{v} \times \mathbf{u} = -\widehat{\mathbf{u}}\mathbf{v} \; = \widehat{\mathbf{u}}^{\mathsf{T}}\mathbf{v}$$

Determinant

- **Q: How do you compute the determinant of a matrix?**

$$\mathbf{A} := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- **A: Apply some algorithm somebody told me once upon a time:**

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

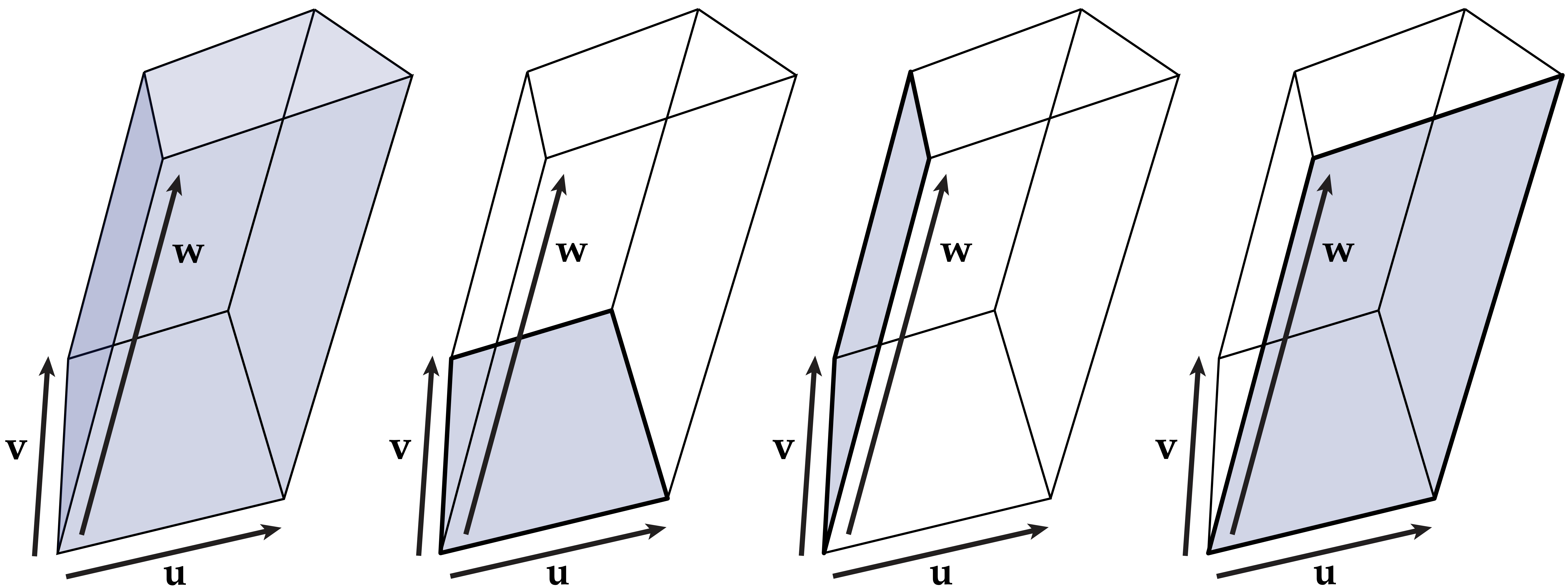
$$\det(\mathbf{A}) = a(ei - fh) + b(fg - di) + c(dh - eg)$$

Totally obvious… right?

- **Q: No! What the heck does this number mean?!**

Determinant, Volume and Triple Product

- **Better answer: det(u,v,w) encodes (signed) volume of parallelepiped with edge vectors u, v, w.**



$$\det(\mathbf{u},\mathbf{v},\mathbf{w}) \quad = \quad (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} \quad = \quad (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} \quad = \quad (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$$

- **Relationship known as a “triple product formula”**
- **(Q: What happens if we reverse order of cross product?)**

Determinant of a Linear Map

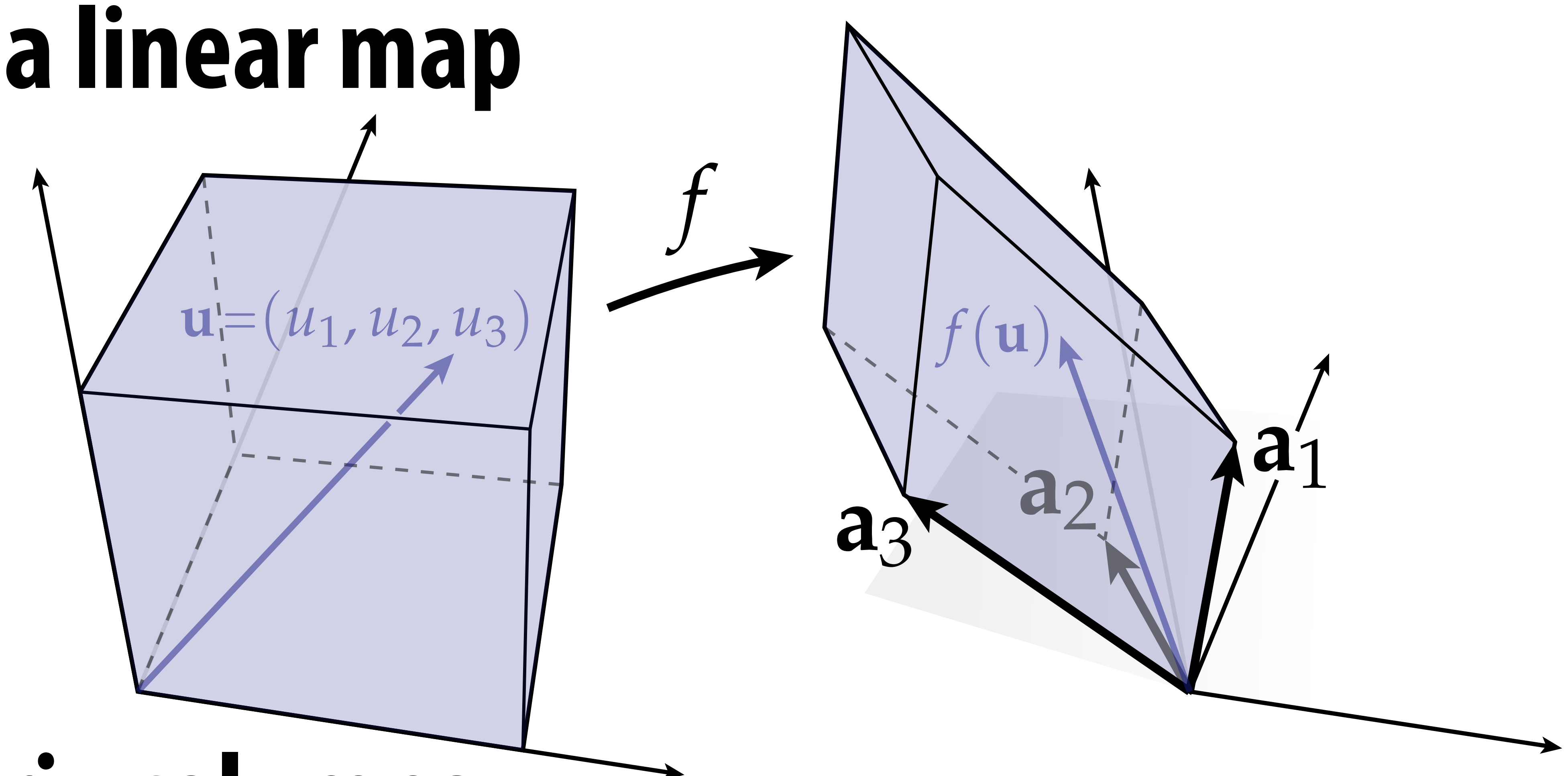
- **Q: If a matrix A encodes a linear map f, what does det(A) mean?**

(First: need to recall how a matrix encodes a linear map!)

Representing Linear Maps via Matrices

- **Key example: suppose I have a linear map**

$$f(\mathbf{u}) = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3$$



- **How do I encode as a matrix?**
- **Easy: "a" vectors become matrix columns:**

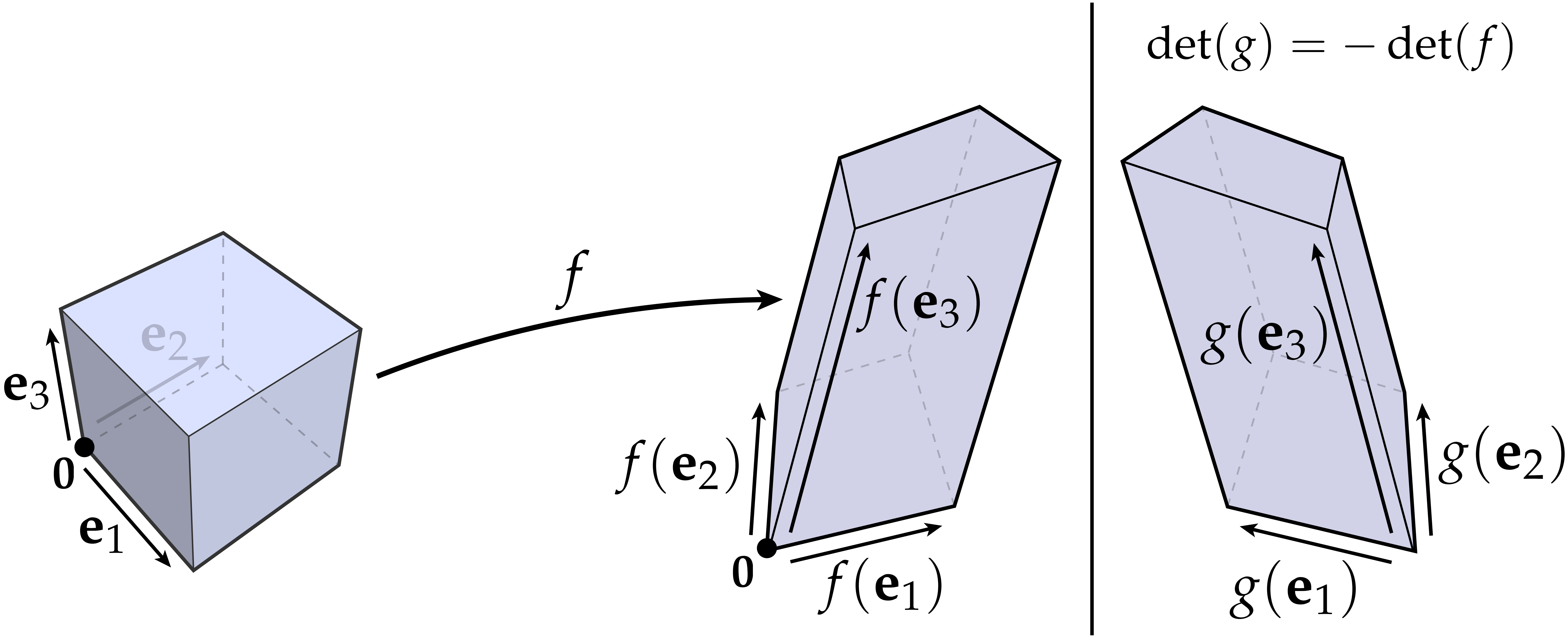
$$A := \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} a_{1,x} & a_{2,x} & a_{3,x} \\ a_{1,y} & a_{2,y} & a_{3,y} \\ a_{1,z} & a_{2,z} & a_{3,z} \end{bmatrix}$$

- **Now, matrix-vector multiply recovers original map:**

$$A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a_{1,x}u_1 + a_{2,x}u_2 + a_{3,x}u_3 \\ a_{1,y}u_1 + a_{2,y}u_2 + a_{3,y}u_3 \\ a_{1,z}u_1 + a_{2,z}u_2 + a_{3,z}u_3 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3$$

Determinant of a Linear Map

- Q: If a matrix A encodes a linear map f, what does det(A) mean?



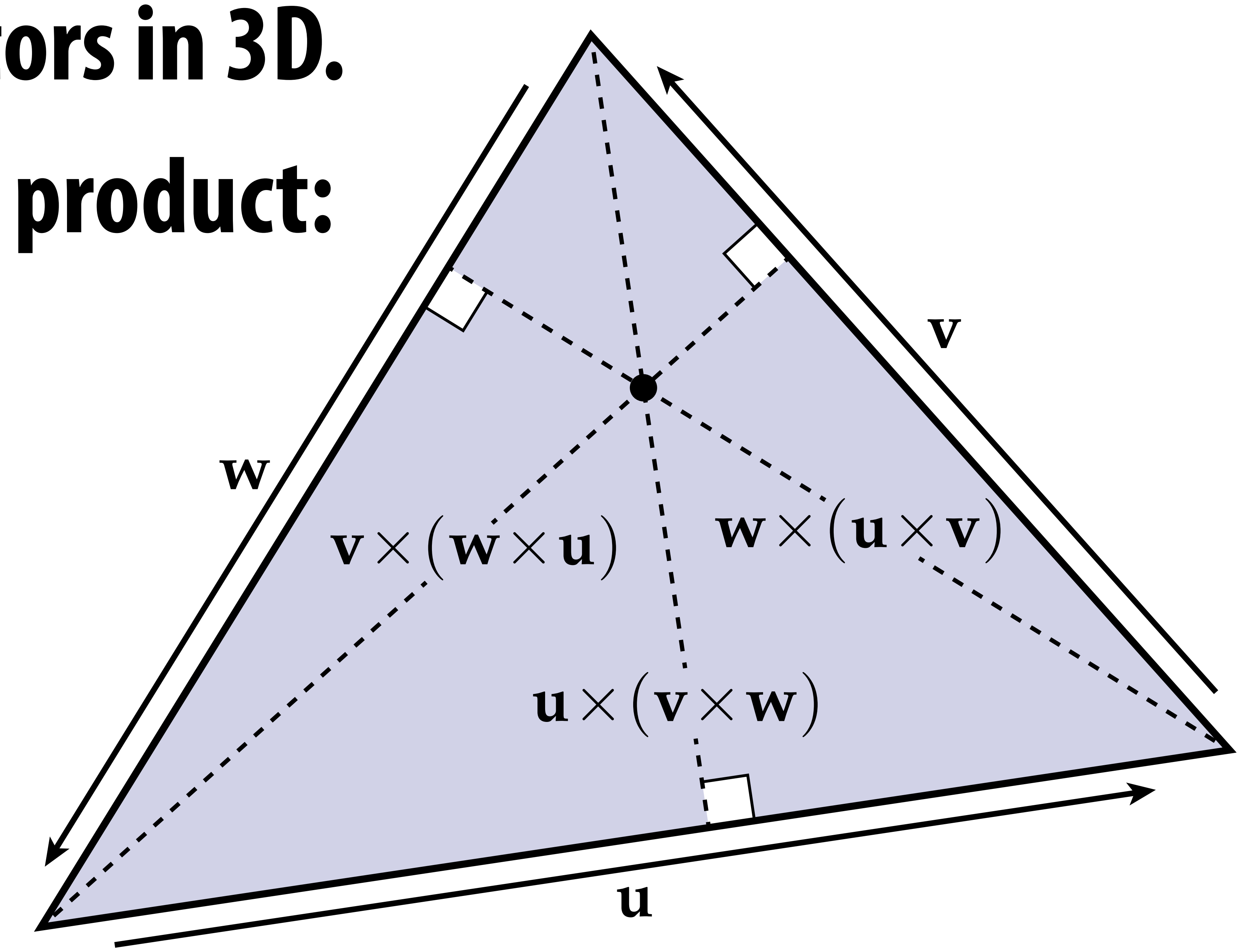
- A: It measures the change in volume.
- Q: What does the sign of the determinant tell us, in this case?
- A: It tells us whether orientation was reversed (det(A) < 0)

(Do we really need a matrix in order to talk about the determinant of a linear map?)

Other Triple Products

- Super useful for working w/ vectors in 3D.
- E.g., Jacobi identity for the cross product:

$$\begin{aligned} \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) &\quad + \\ \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) &\quad + \\ \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) &\quad = \quad 0 \end{aligned}$$



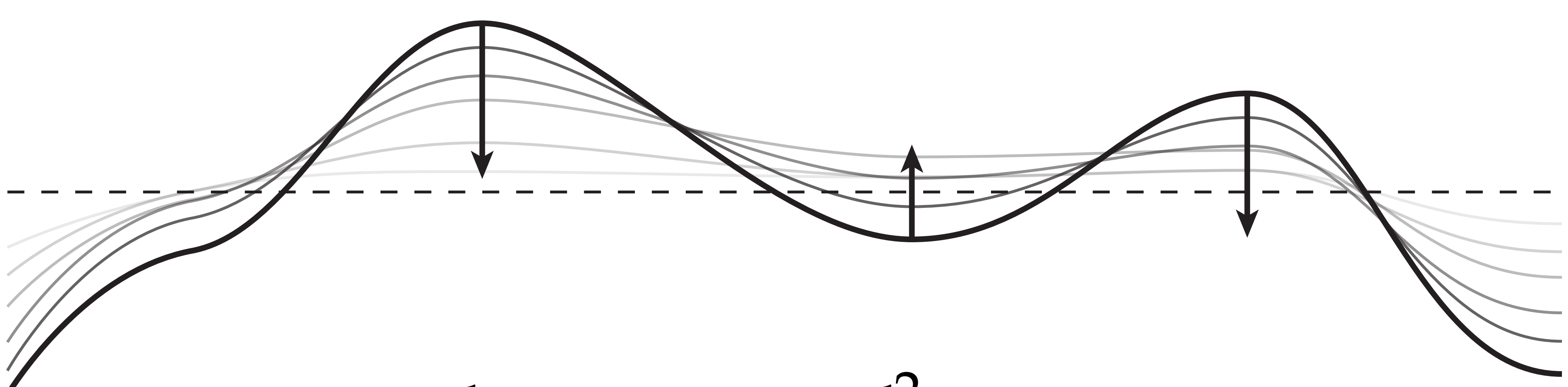
- Why is it true, geometrically?
- There is a geometric reason, but not nearly as obvious as det: has to do w/ fact that triangle's altitudes meet at a point.
- Yet another triple product: Lagrange's identity

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{u}\cdot\mathbf{v})$$

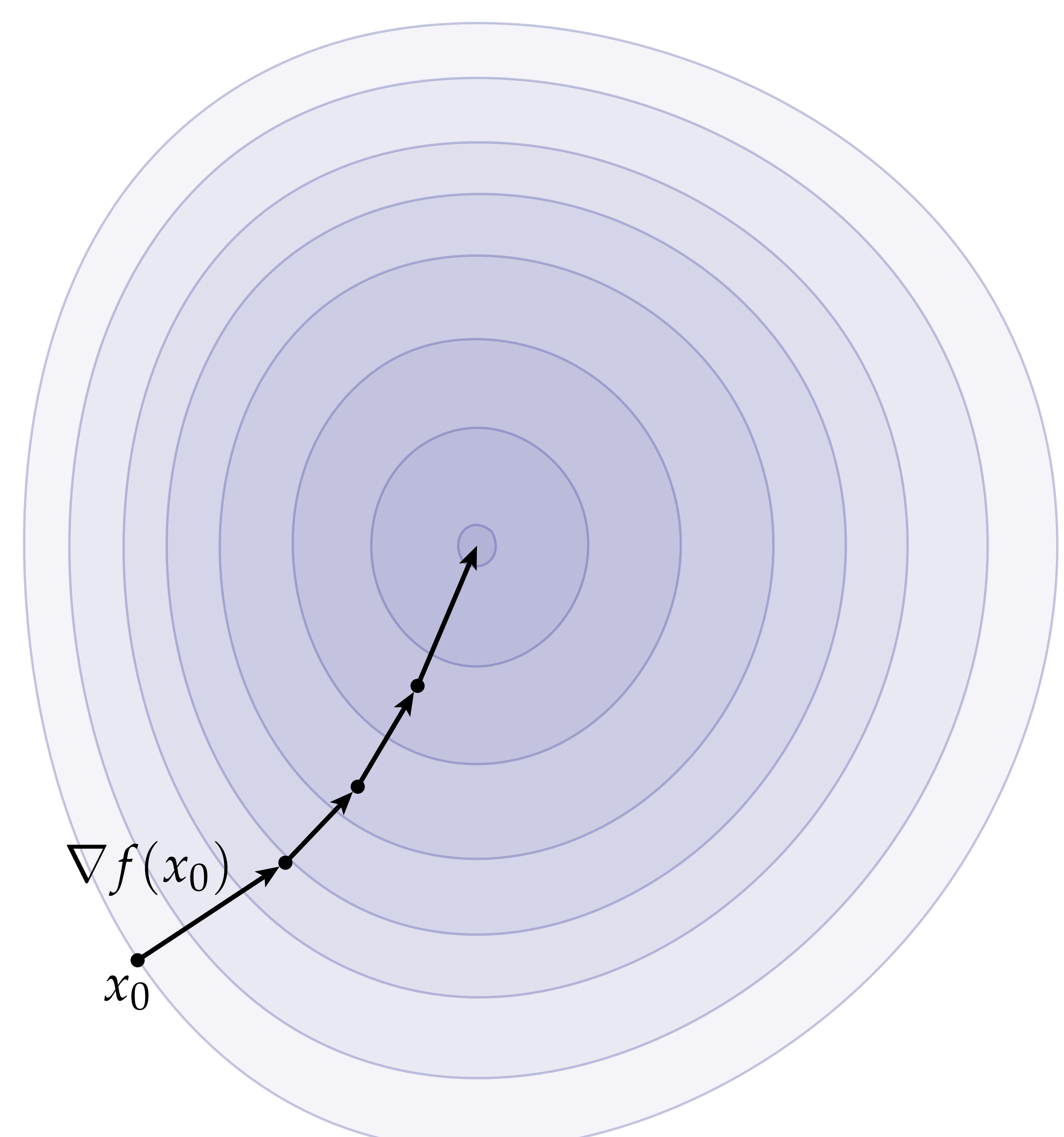
(Can you come up with a geometric interpretation?)

Differential Operators - Overview

- **Next up: differential operators and vector fields.**
- **Why is this useful for computer graphics?**
  - Many physical/geometric problems expressed in terms of relative rates of change (ODEs, PDEs).
  - These tools also provide foundation for numerical optimization—e.g., minimize cost by following the gradient of some objective.



$$\frac{d}{dt}\phi(x) = \frac{d^2}{dx^2}\phi(x)$$



Derivative as Slope

- Consider a function f(x): R → R
- What does its derivative f′ mean?
- One interpretation "rise over run"
- Corresponds to standard definition:

$$f'(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon}$$

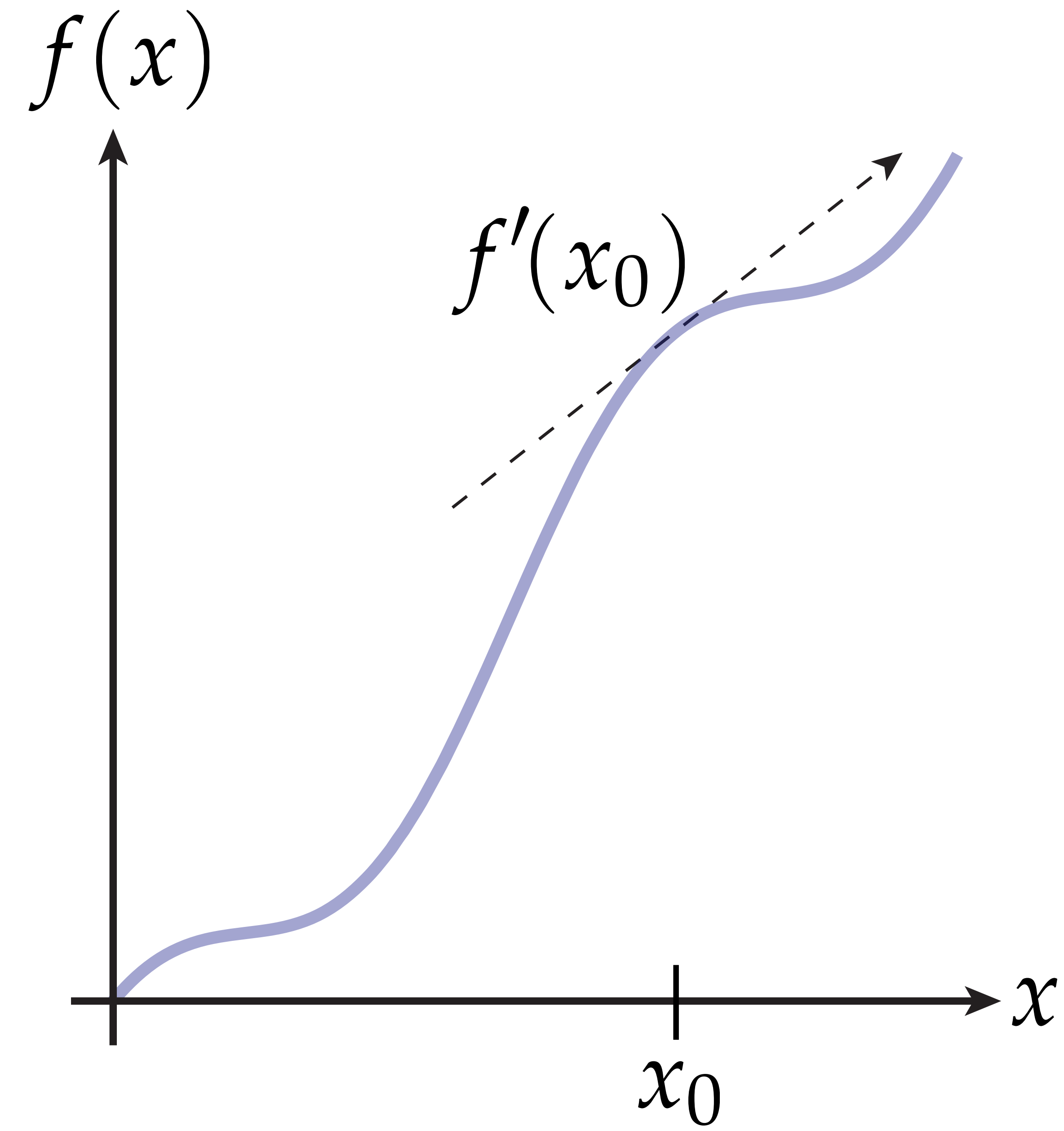
- Careful! What if slope is different when we walk in opposite direction?

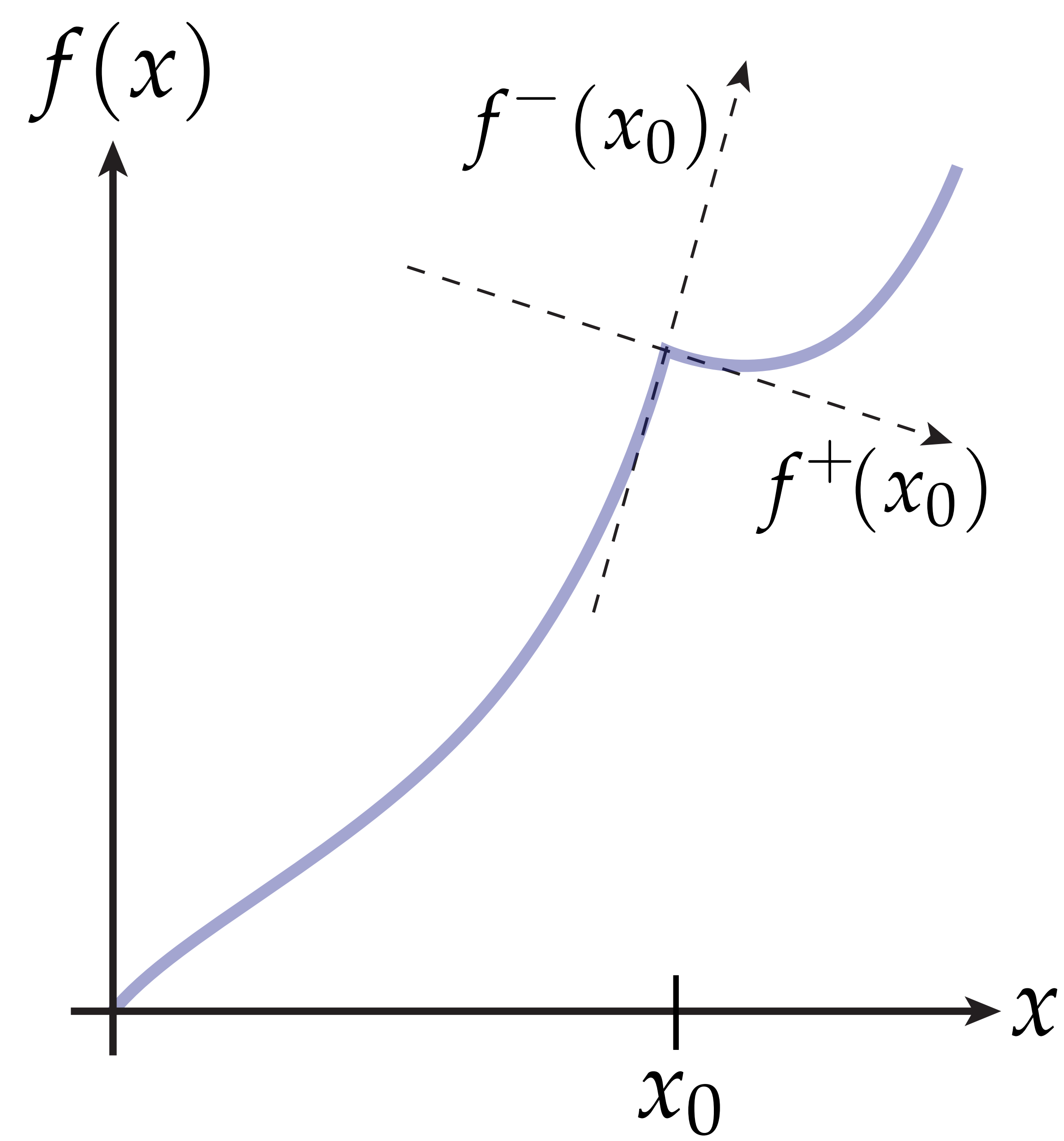
$$f^+(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon}$$

$$f^-(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0) - f(x_0-\varepsilon)}{\varepsilon}$$

- Differentiable at x0 if $f^+ = f^-$.

Many functions in graphics are NOT differentiable!



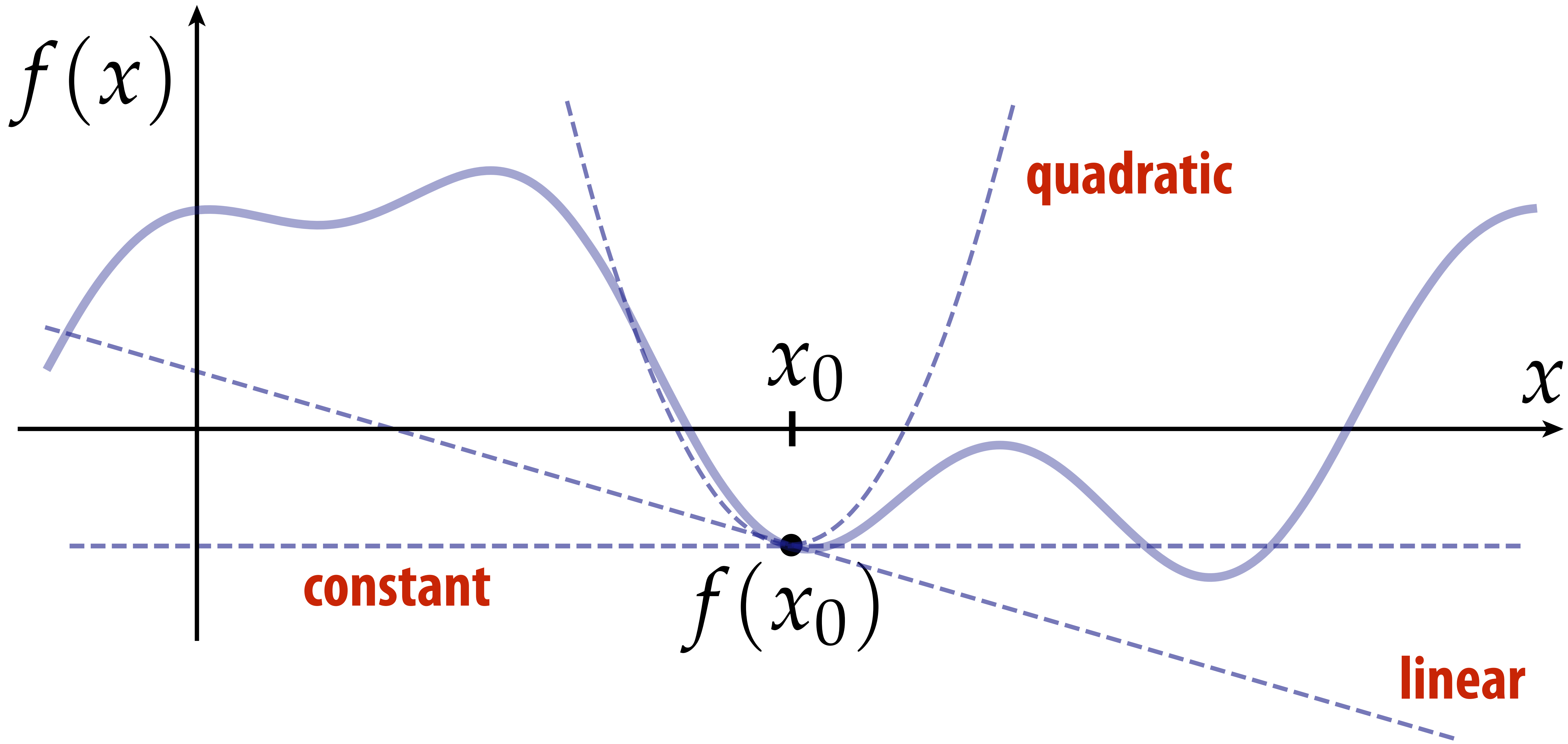


Derivative as Best Linear Approximation

- **Any smooth function f(x) can be expressed as a Taylor series:**

constant — linear — quadratic

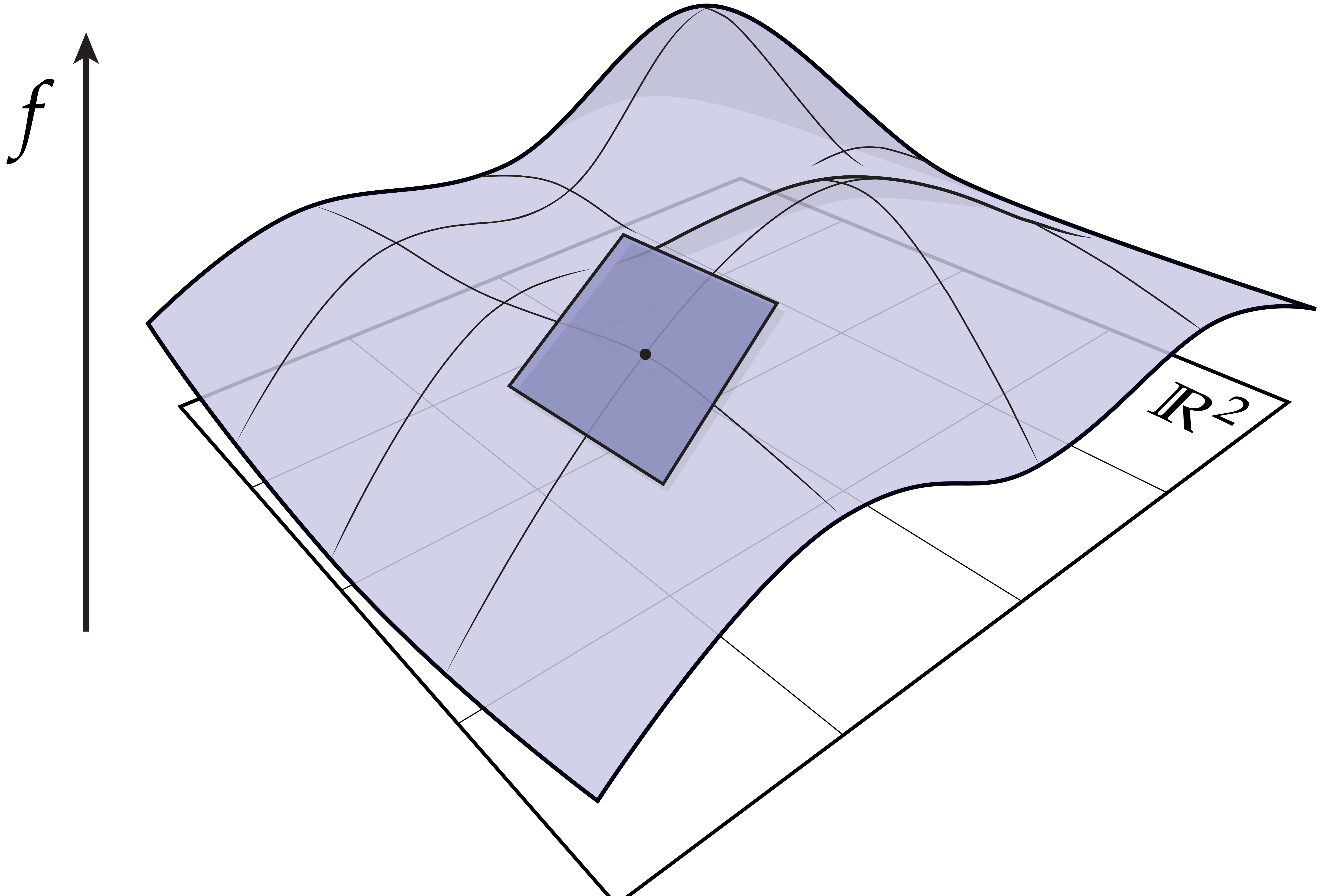
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$



- **Replacing complicated functions with a linear (and sometimes quadratic) approximation is a powerful trick in graphics algorithms—we'll see many examples.**

Derivative as Best Linear Approximation

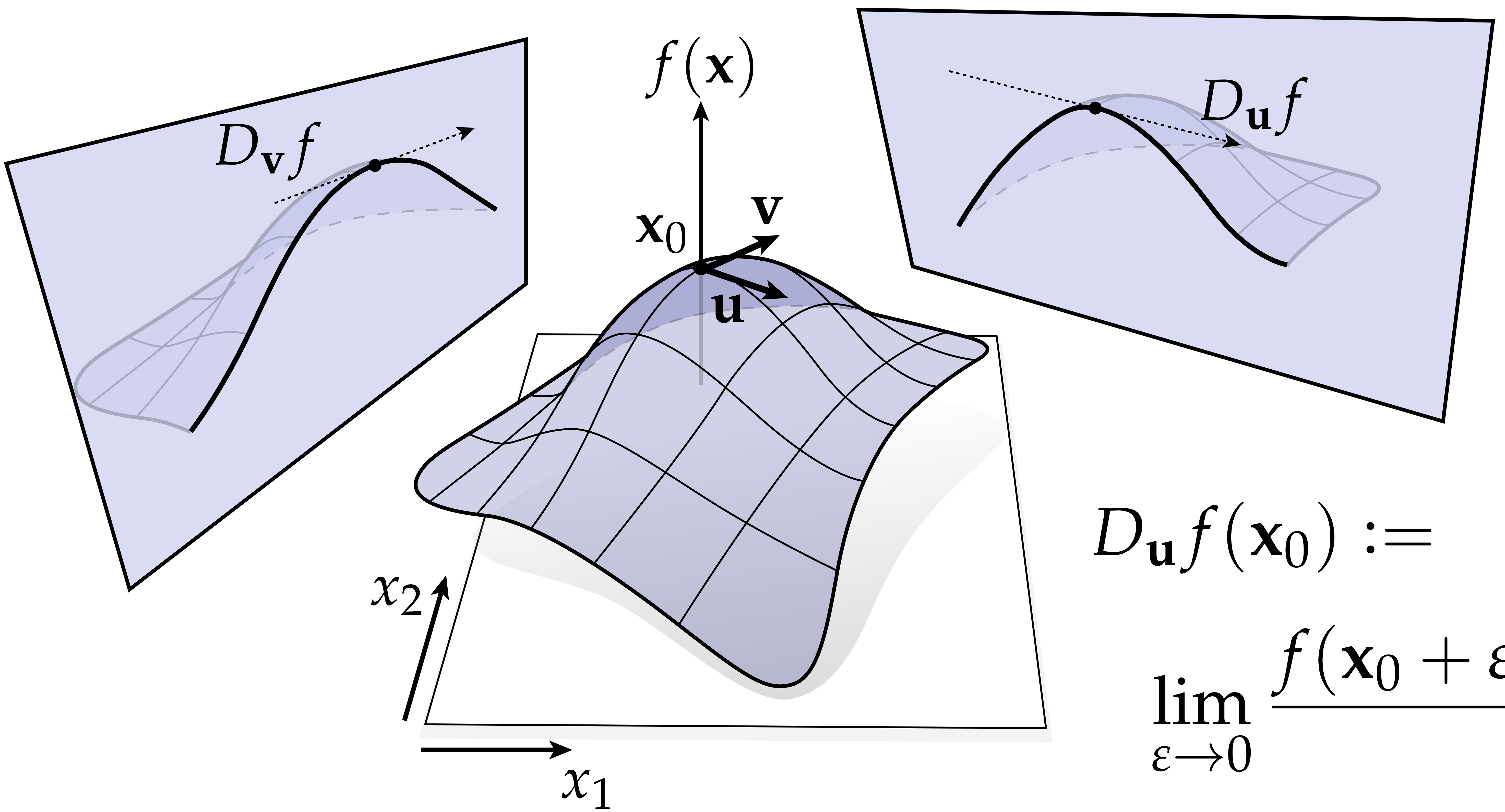
- **Intuitively, same idea applies for functions of multiple variables:**



How do we think about derivatives for a function that has multiple variables?

Directional Derivative

- One way: suppose we have a function f(x1,x2)
  - Take a "slice" through the function along some line
  - Then just apply the usual derivative!
  - Called the directional derivative



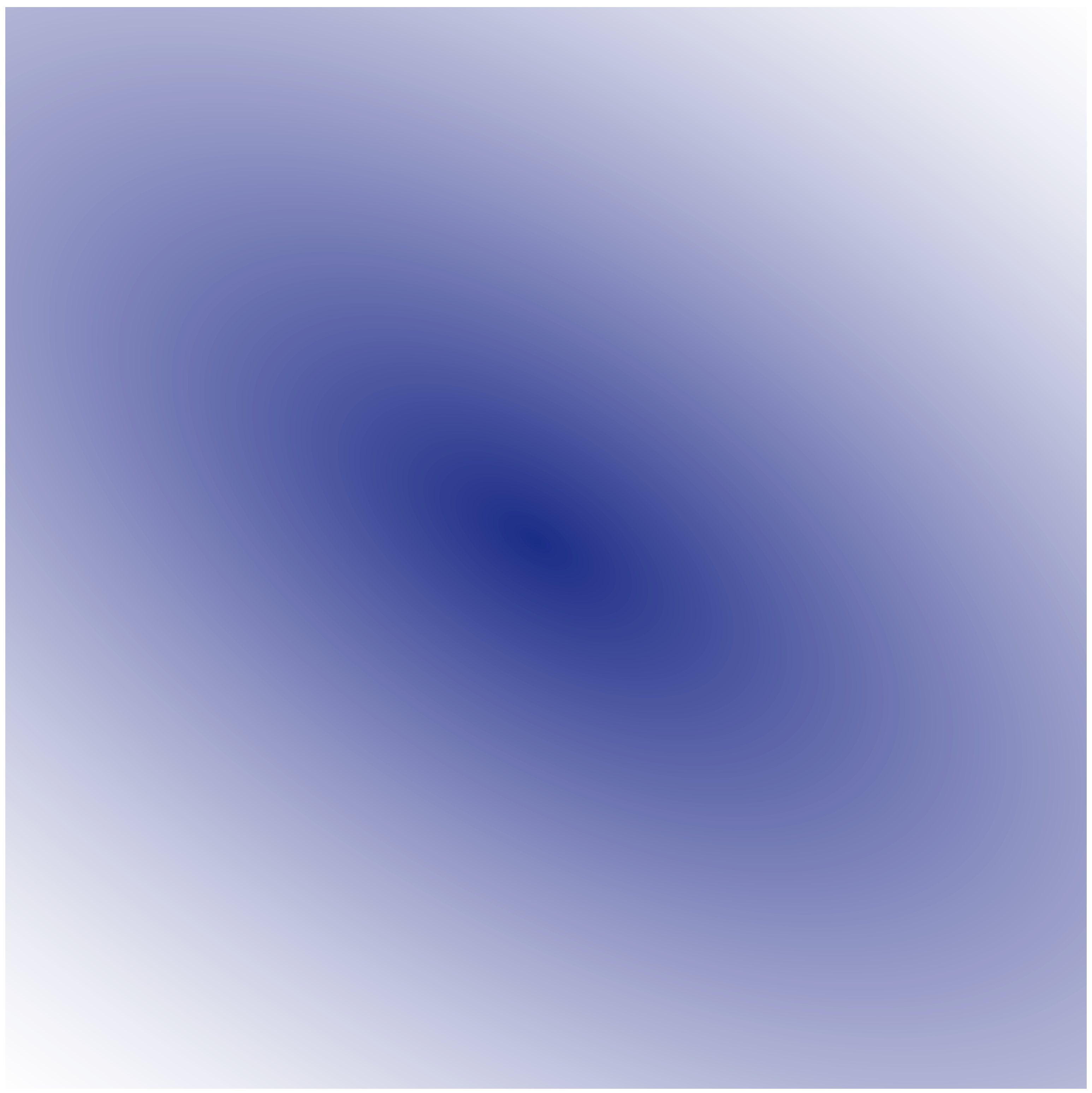
take a small step along u

$$D_{\mathbf{u}}f(\mathbf{x}_0) := \lim_{\varepsilon \to 0} \frac{f(\mathbf{x}_0 + \varepsilon\mathbf{u}) - f(\mathbf{x}_0)}{\varepsilon}$$

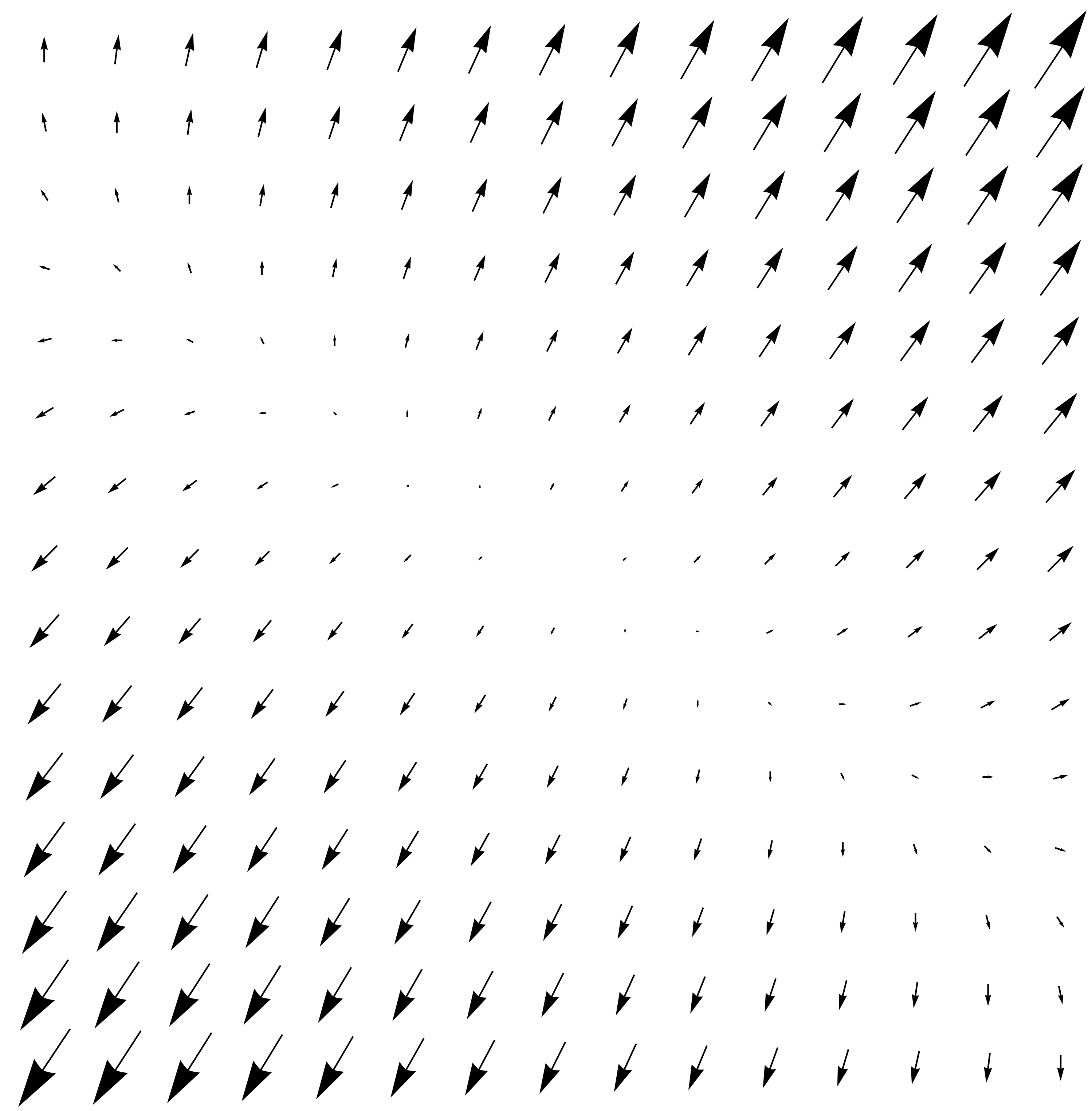
Gradient

"nabla"

- **Given a multivariable function $f(\mathbf{x})$, gradient $\nabla f(\mathbf{x})$ assigns a vector at each point:**



$f(\mathbf{x})$



$\nabla f(\mathbf{x})$

- **(Ok, but which vectors, exactly?)**

Gradient in Coordinates

- **Most familiar definition: list of partial derivatives**
- **I.e., imagine that all but one of the coordinates are just constant values, and take the usual derivative**

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

- **Two potential problems:**
  - **Role of inner product is not clear (more later!)**
  - **No way to differentiate functions of functions F(f) since we don't have a finite list of coordinates $x_1, \ldots, x_n$**
- **Still, extremely common way to calculate the gradient…**

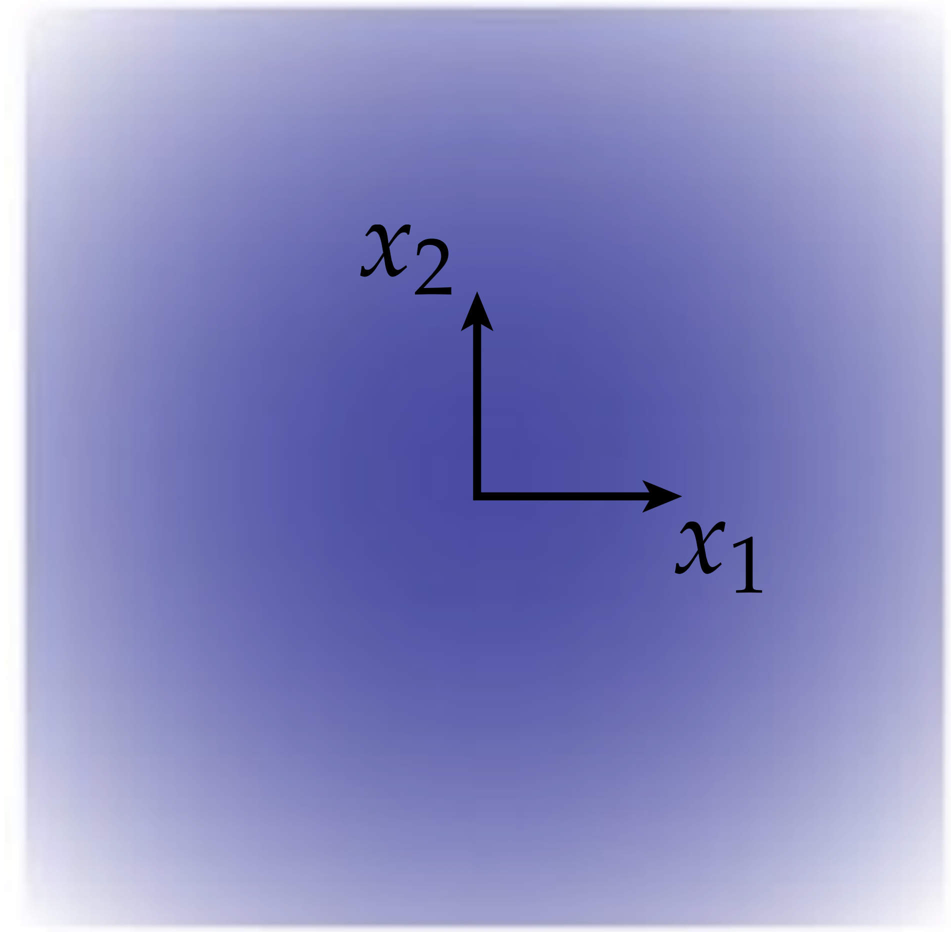
Example: Gradient in Coordinates

$$f(\mathbf{x}) := x_1^2 + x_2^2$$

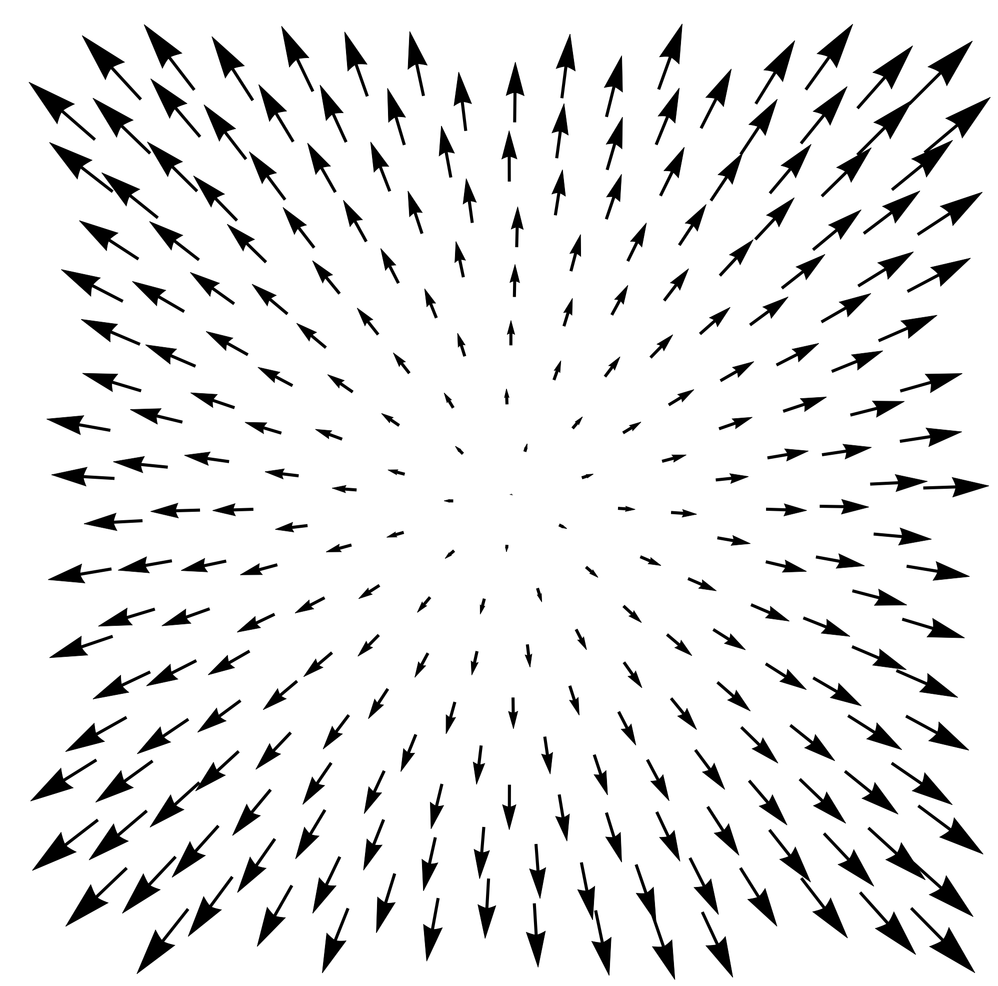
$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1} x_1^2 + \frac{\partial}{\partial x_1} x_2^2 = 2x_1 + 0$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2} x_1^2 + \frac{\partial}{\partial x_2} x_2^2 = 0 + 2x_2$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 2\mathbf{x}$$



$f(\mathbf{x})$

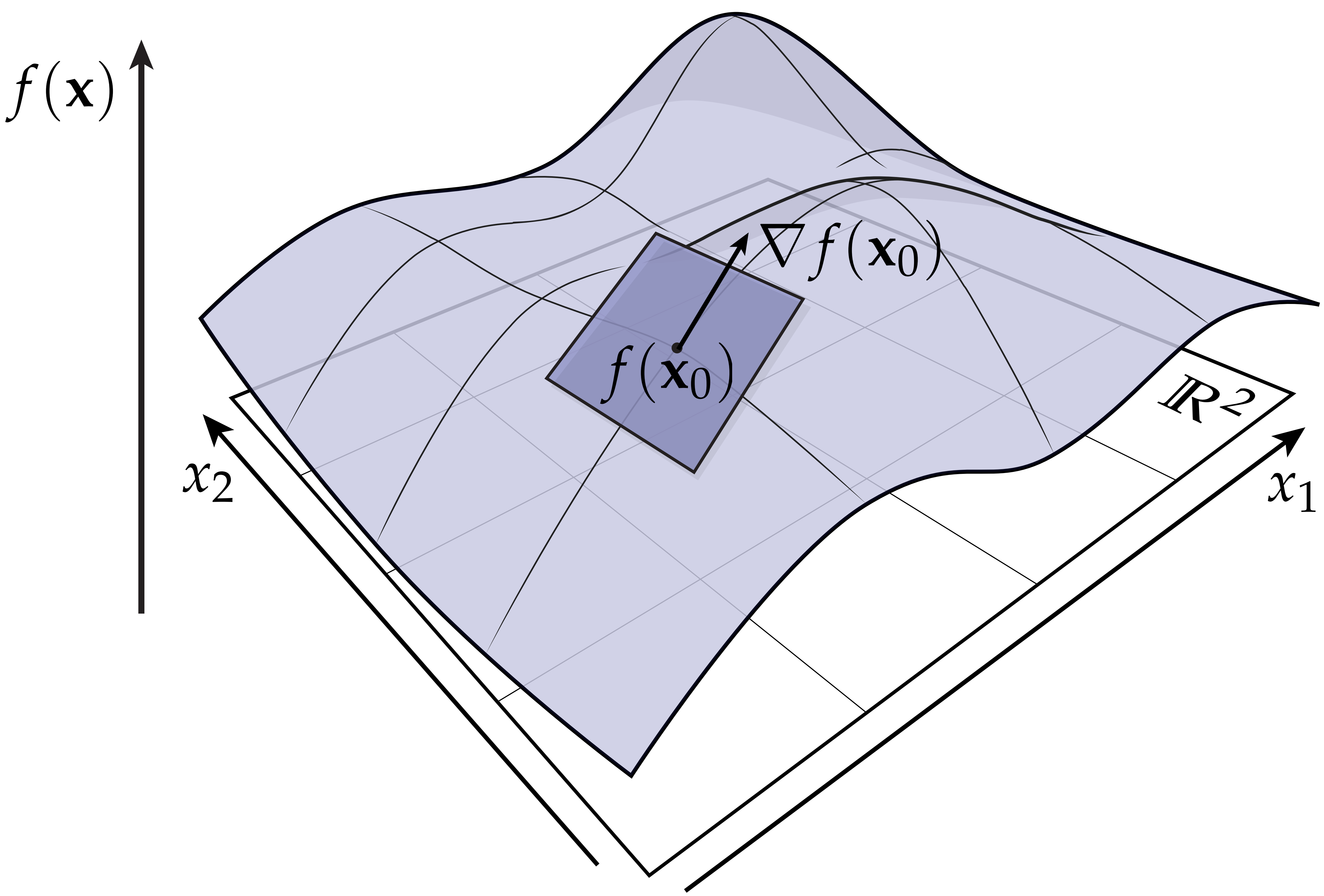


$\nabla f(\mathbf{x})$

Gradient as Best Linear Approximation

Another way to think about it: at each point x0, gradient is the vector $\nabla f(\mathbf{x}_0)$ that leads to the best possible approximation

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle$$

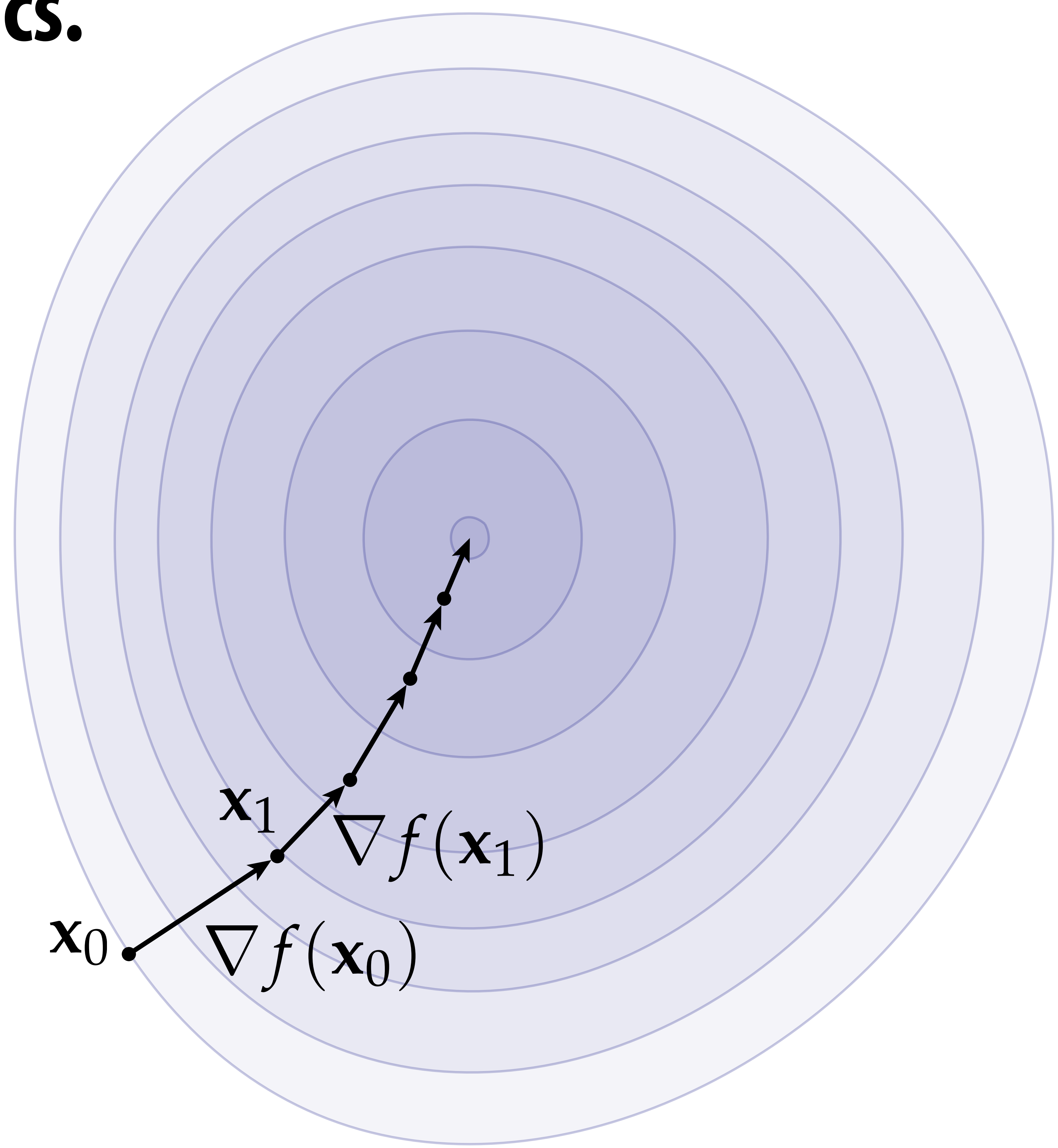


Starting at x_0, this term gets:

- **bigger if we move in the direction of the gradient,**
- **smaller if we move in the opposite direction, and**
- **doesn't change if we move orthogonal to gradient.**

The gradient takes you uphill…

- Another way to think about it: direction of "steepest ascent"
- I.e., what direction should we travel to increase value of function as quickly as possible?
- This viewpoint leads to algorithms for optimization, commonly used in graphics.



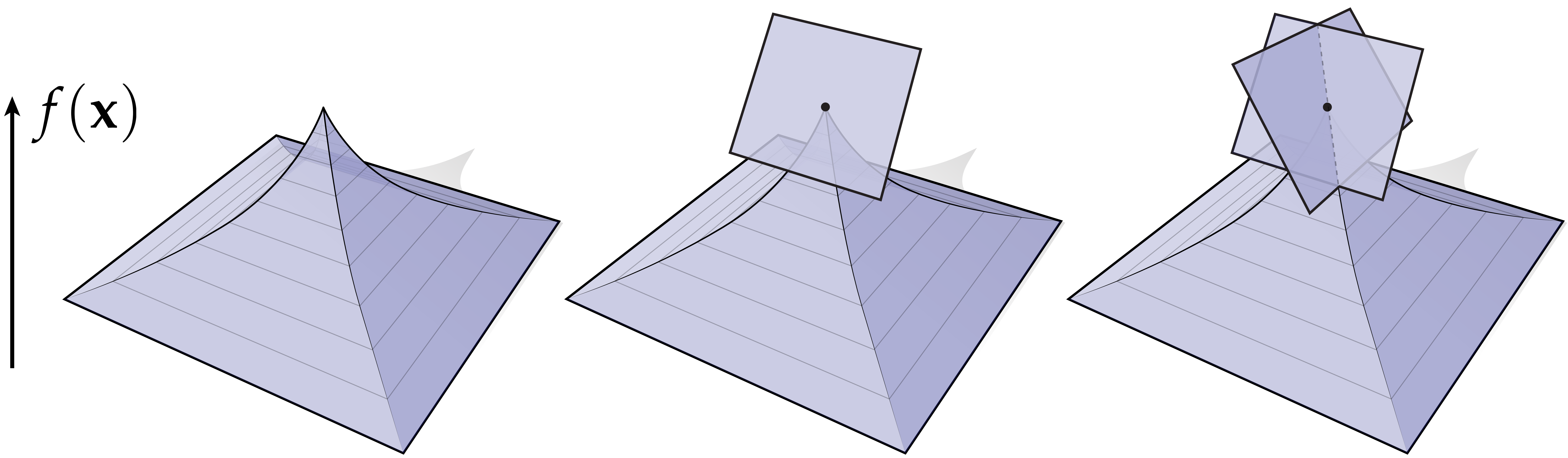
Gradient and Directional Derivative

At each point x, gradient is unique vector $\nabla f(\mathbf{x})$ such that

$$\langle \nabla f(x), \mathbf{u} \rangle = D_{\mathbf{u}} f(\mathbf{x})$$

for all u. In other words, such that taking the inner product w/ this vector gives you the directional derivative in any direction u.

Can't happen if function is not differentiable!



(Notice: gradient also depends on choice of inner product...)

Example: Gradient of Dot Product

- **Consider the dot product expressed in terms of matrices:**

$$f := \mathbf{u}^\mathsf{T}\mathbf{v}$$

- **What is gradient of f with respect to u?**
- **One way: write it out in coordinates:**

$$\mathbf{u}^\mathsf{T}\mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

(equals zero unless i = k)

$$\frac{\partial}{\partial u_k}\sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} \frac{\partial}{\partial u_k}(u_i v_i) = v_k$$

$$\Rightarrow \nabla_{\mathbf{u}} f = \begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix}$$

In other words:

$$\boxed{\nabla_{\mathbf{u}}(\mathbf{u}^\mathsf{T}\mathbf{v}) = \mathbf{v}}$$

Not so different from $\frac{d}{dx}(xy) = y$!

Gradients of Matrix-Valued Expressions

- **EXTREMELY useful in graphics to be able to differentiate expressions involving matrices**
- **Ultimately, expressions look much like ordinary derivatives**

For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and **symmetric** matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$:

MATRIX DERIVATIVE	LOOKS LIKE
$\nabla_{\mathbf{x}}(\mathbf{x}^T\mathbf{y}) = \mathbf{y}$	$\frac{d}{dx}xy = y$
$\nabla_{\mathbf{x}}(\mathbf{x}^T\mathbf{x}) = 2\mathbf{x}$	$\frac{d}{dx}x^2 = 2x$
$\nabla_{\mathbf{x}}(\mathbf{x}^T A\mathbf{y}) = A\mathbf{y}$	$\frac{d}{dx}axy = ay$
$\nabla_{\mathbf{x}}(\mathbf{x}^T A\mathbf{x}) = 2A\mathbf{x}$	$\frac{d}{dx}ax^2 = 2ax$
$\ldots$	$\ldots$

Excellent resource: Petersen & Pedersen, "The Matrix Cookbook"

- **At least once in your life, work these out meticulously in coordinates (to convince yourself they're true).**
- **Then… forget about coordinates altogether!**

Advanced*: L² Gradient

- Consider a function of a function $F(f)$
- What is the gradient of F with respect to f?
- Can't take partial derivatives anymore!
- Instead, look for function ∇F such that for all functions u,

$$\langle\langle \nabla F, u \rangle\rangle = D_u F$$

- What is directional derivative of a function of a function??
- Don't freak out—just return to good old-fashioned limit:

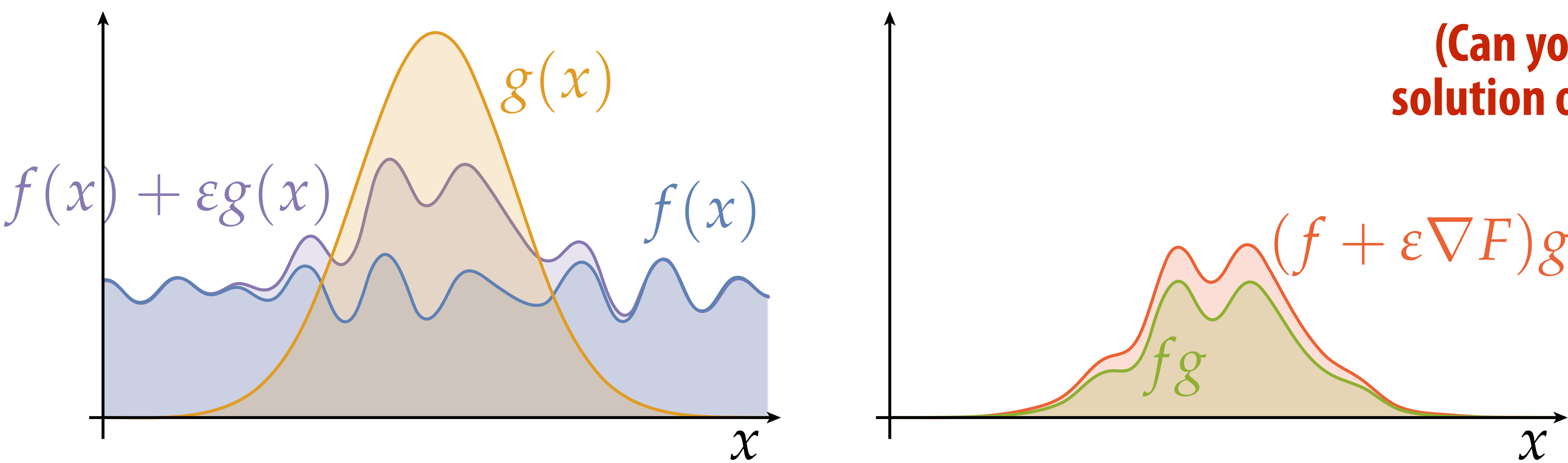
$$D_u F(f) = \lim_{\varepsilon \to 0} \frac{F(f + \varepsilon u) - F(f)}{\varepsilon}$$

- This strategy becomes much clearer w/ a concrete example...

*as in, NOT on the test! (But perhaps somewhere in the test of life…)

Advanced Visual Example: L^2 Gradient

- Consider function $F(f) := \langle\langle f, g \rangle\rangle$ for f,g: [0,1]→R
- I claim the gradient is: $\nabla F = g$
- Does this make sense intuitively? How can we increase inner product with g as quickly as possible?
  - inner product measures how well functions are "aligned"
  - g is definitely function best-aligned with g!
  - so to increase inner product, add a little bit of g to f



(Can you work this solution out formally?)

Advanced Example: L² Gradient

- **Consider function $F(f) := ||f||^2$ for arguments f: [0,1]→R**
- **At each "point" f0, we want function ∇F such that for all functions u**

$$\langle\langle \nabla F(f_0), u \rangle\rangle = \lim_{\varepsilon \to 0} \frac{F(f_0 + \varepsilon u) - F(f_0)}{\varepsilon}$$

- **Expanding 1st term in numerator, we get**

$$||f_0 + \varepsilon u||^2 = ||f_0||^2 + \varepsilon^2 ||u||^2 + 2\varepsilon \langle\langle f_0, u \rangle\rangle$$

- **Hence, limit becomes**

$$\lim_{\varepsilon \to 0} (\varepsilon ||u||^2 + 2\langle\langle f_0, u \rangle\rangle) = 2\langle\langle f_0, u \rangle\rangle$$

- **The only solution to $\langle\langle \nabla F(f_0), u \rangle\rangle = 2\langle\langle f_0, u \rangle\rangle$ for all u is**

$$\boxed{\nabla F(f_0) = 2f_0}$$

not much different from $\frac{d}{dx} x^2 = 2x$!

Key idea:

Once you get the hang of taking the gradient of ordinary functions, it's (superficially) not much harder for more exotic objects like matrices, functions of functions, …

Vector Fields

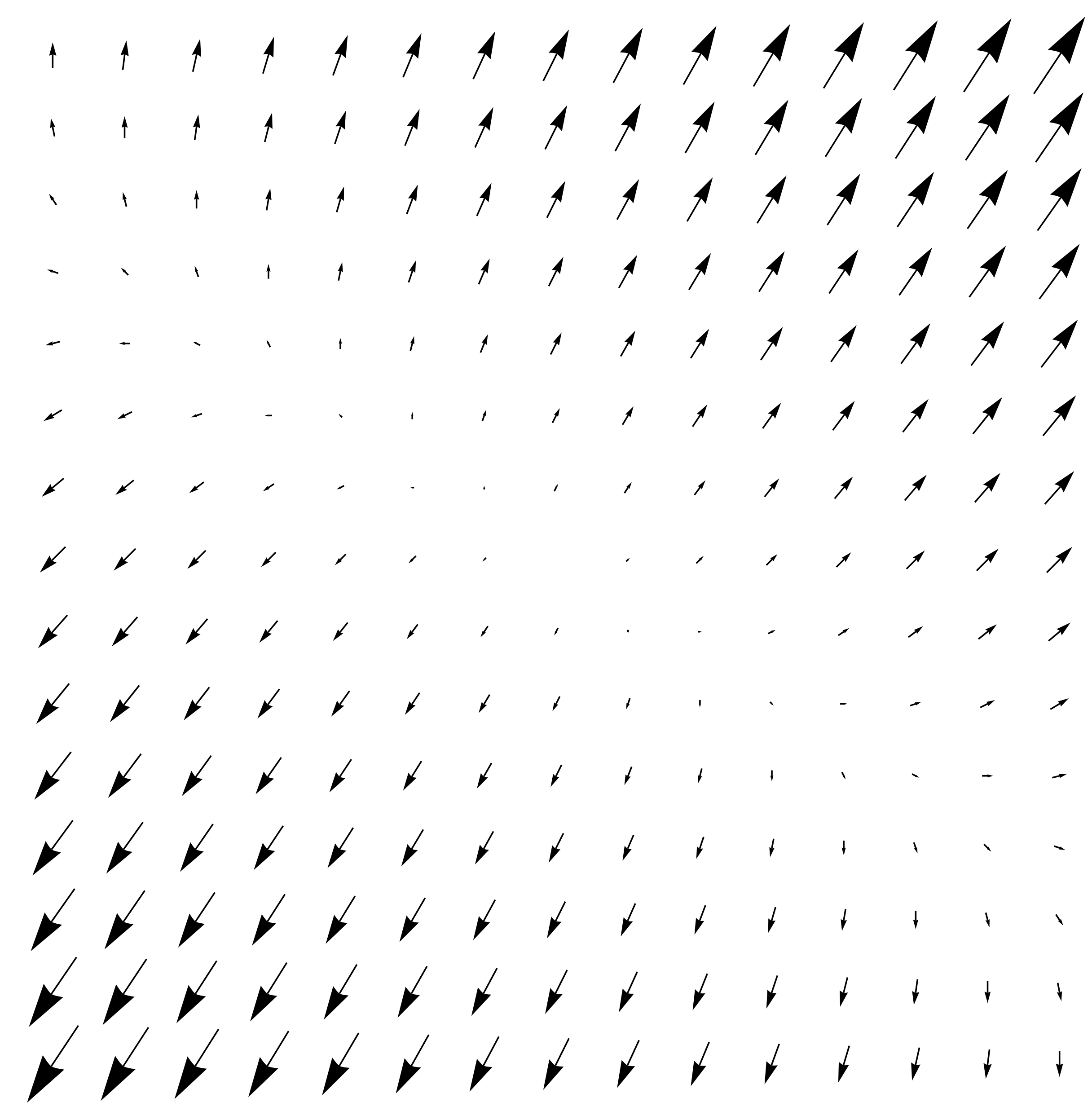
- Gradient was our first example of a vector field
- In general, a vector field assigns a vector to each point in space
- E.g., can think of a 2-vector field in the plane as a map

$$X : \mathbb{R}^2 \to \mathbb{R}^2$$

- For example, we saw a gradient field

$$\nabla f(x, y) = (2x, 2y)$$

(for the function $f(x,y) = x^2 + y^2$)



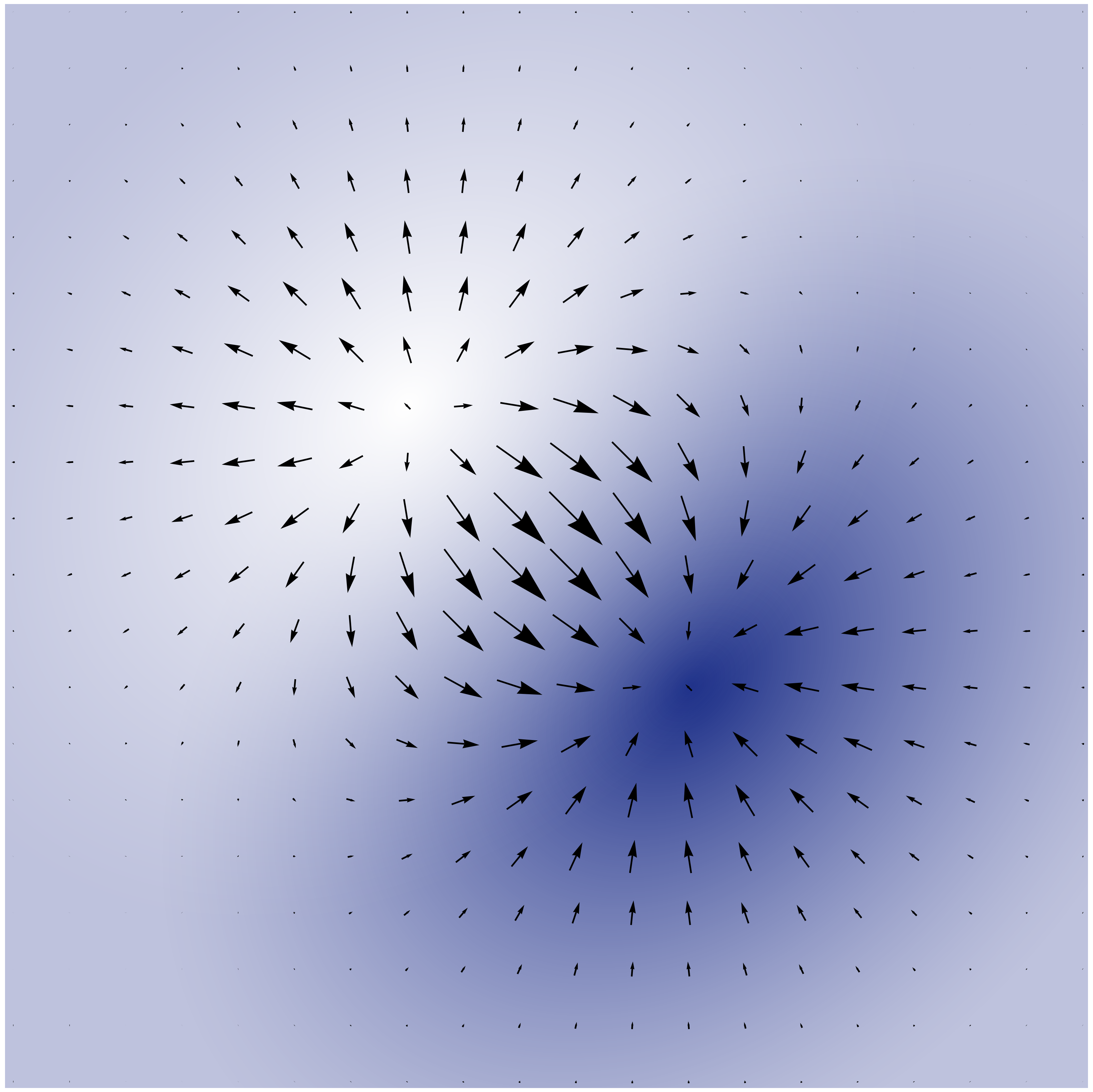
$\nabla f(\mathbf{x})$

Q: How do we measure the change in a vector field?

Divergence and Curl

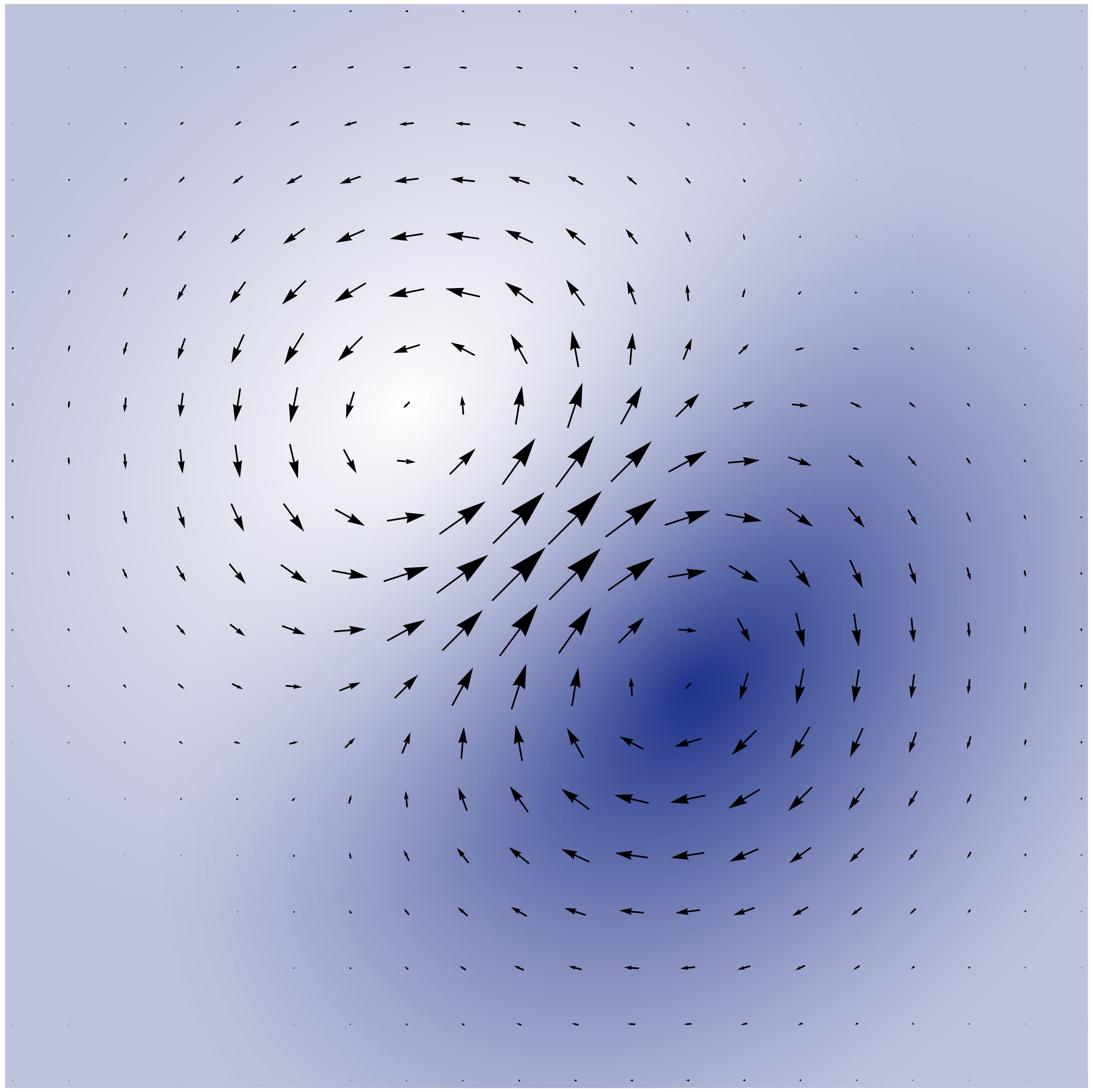
- **Two basic derivatives for vector fields:**

"How much is field shrinking/expanding?"



$\mathrm{div}\ X$

"How much is field spinning?"



$\mathrm{curl}\ Y$

Divergence

- Also commonly written as $\nabla \cdot X$
- Suggests a coordinate definition for divergence
- Think of ∇ as a "vector of derivatives"

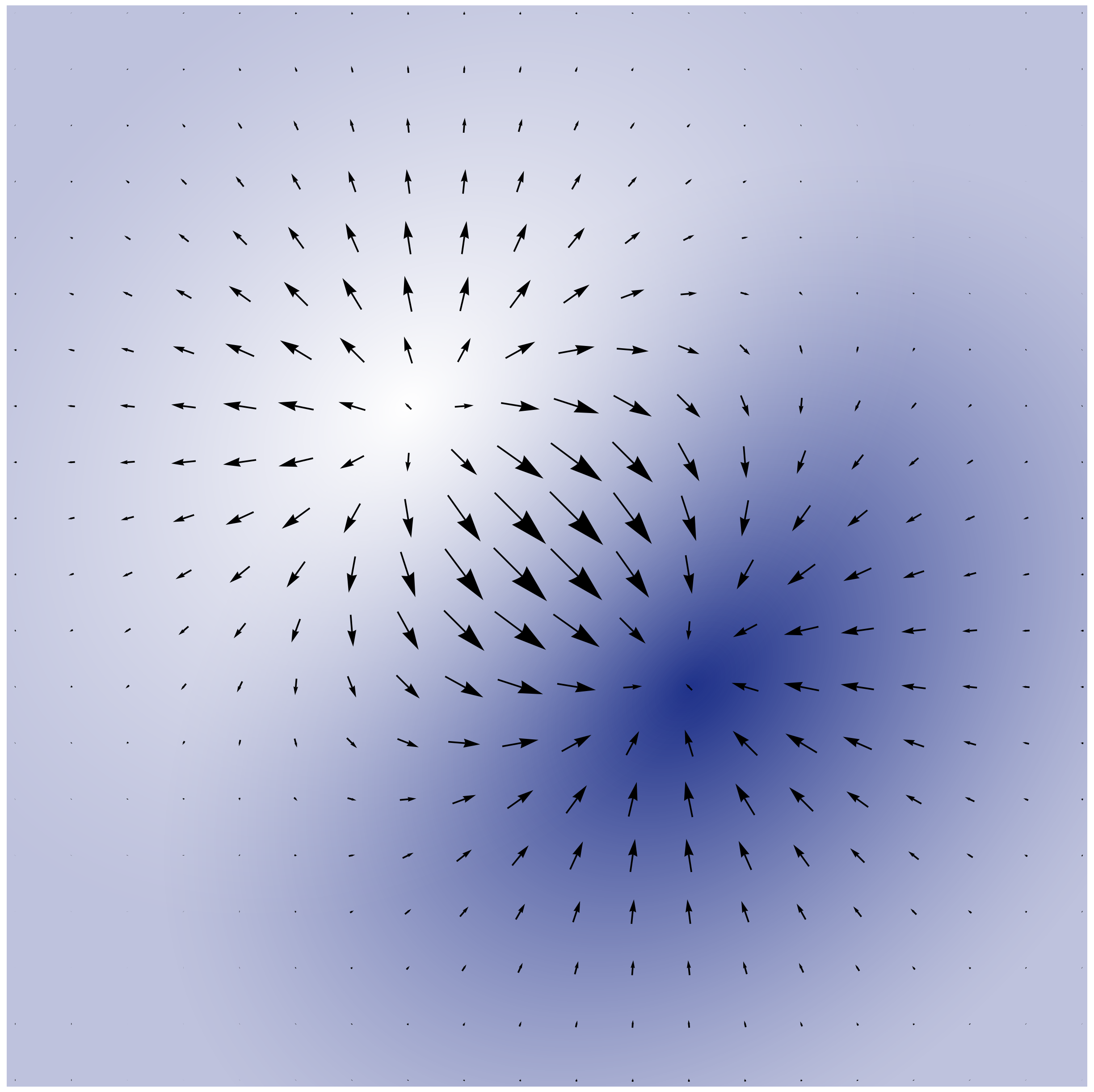
$$\nabla = \left(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n} \right)$$

- Think of X as a "vector of functions"

$$X(\mathbf{u}) = (X_1(\mathbf{u}), \ldots, X_n(\mathbf{u}))$$

- Then divergence is

$$\nabla \cdot X := \sum_{i=1}^{n} \partial X_i / \partial u_i$$

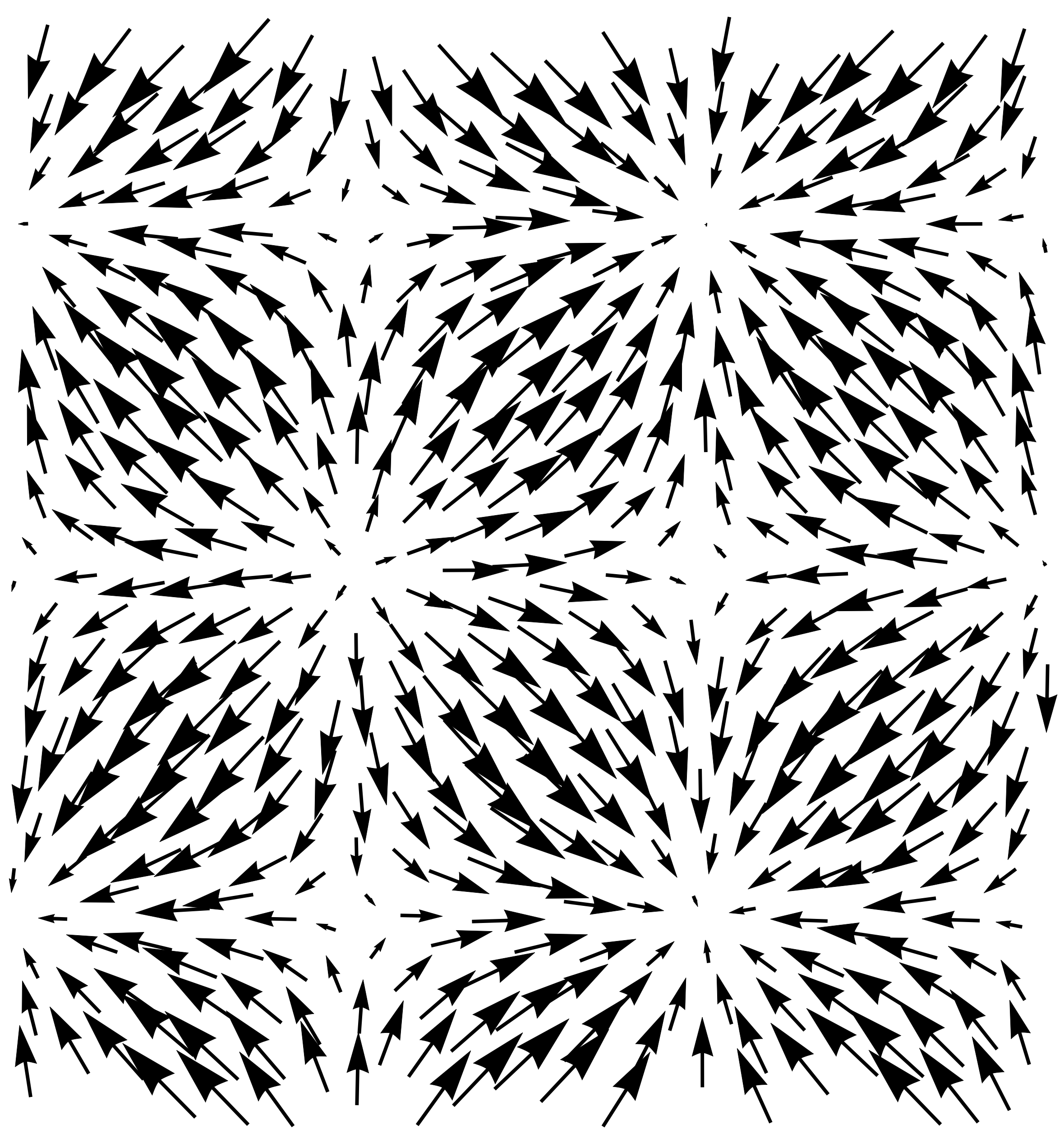


$\nabla \cdot X$

Divergence - Example

- **Consider the vector field** $X(u,v) := (\cos(u), \sin(v))$
- **Divergence is then**

$$\nabla \cdot X = \frac{\partial}{\partial u}\cos(u) + \frac{\partial}{\partial v}\sin(v) = -\sin(u) + \cos(v).$$



X

$\nabla \cdot X$

Curl

- **Also commonly written as** $\nabla \times X$
- **Suggests a coordinate definition for curl**
- **This time, think of ∇ as a vector of just three derivatives:**

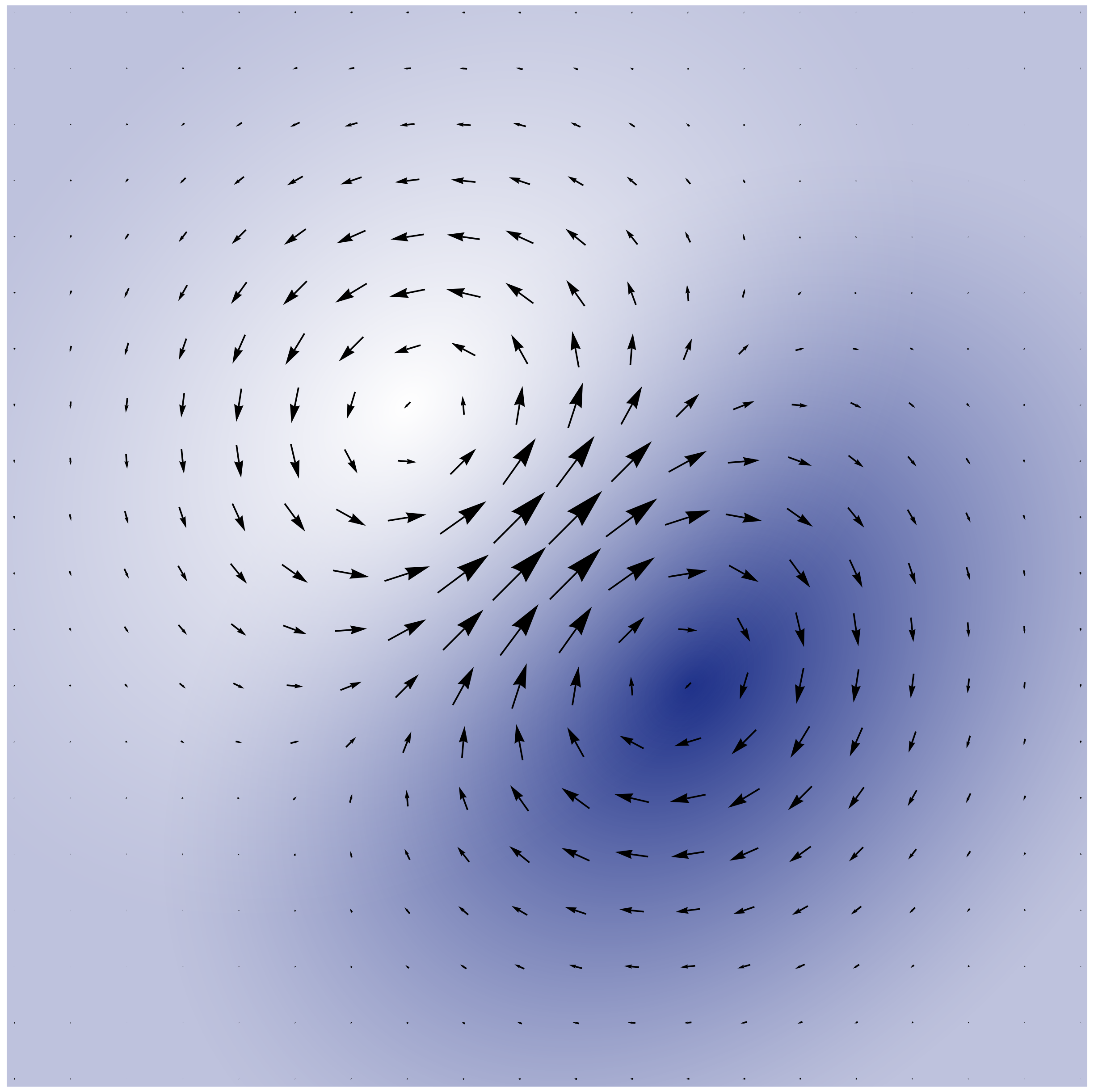
$$\nabla = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3} \right)$$

- **Think of X as vector of three functions:**

$$X(\mathbf{u}) = (X_1(\mathbf{u}), X_2(\mathbf{u}), X_3(\mathbf{u}))$$

- **Then curl is**

$$\nabla \times X := \left[\begin{array}{c} \partial X_3 / \partial u_2 - \partial X_2 / \partial u_3 \\ \partial X_1 / \partial u_3 - \partial X_3 / \partial u_1 \\ \partial X_2 / \partial u_1 - \partial X_1 / \partial u_2 \end{array} \right]$$



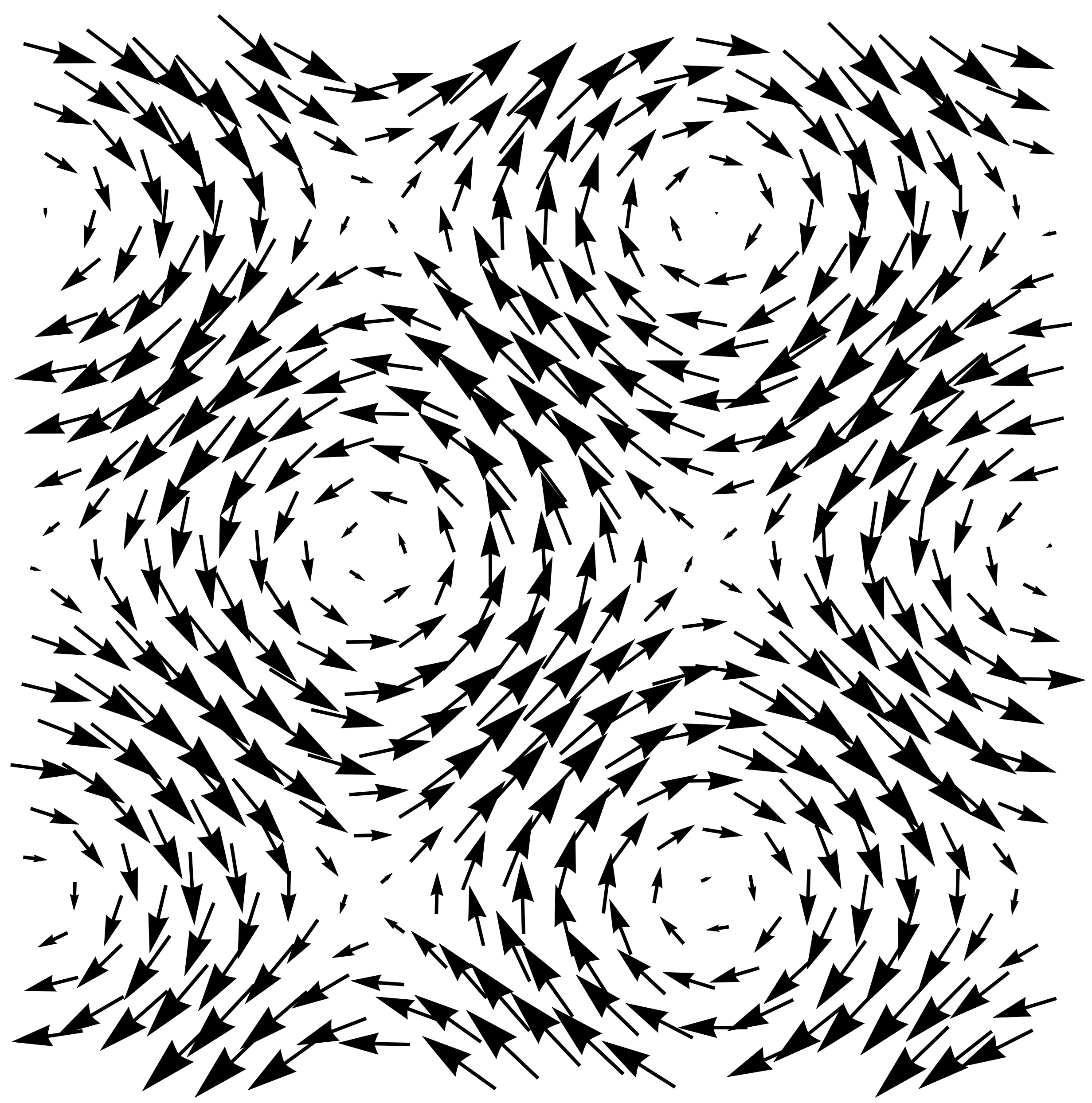
$\nabla \times X$

(2D "curl": $\nabla \times X := \partial X_2 / \partial u_1 - \partial X_1 / \partial u_2$**)**

Curl - Example

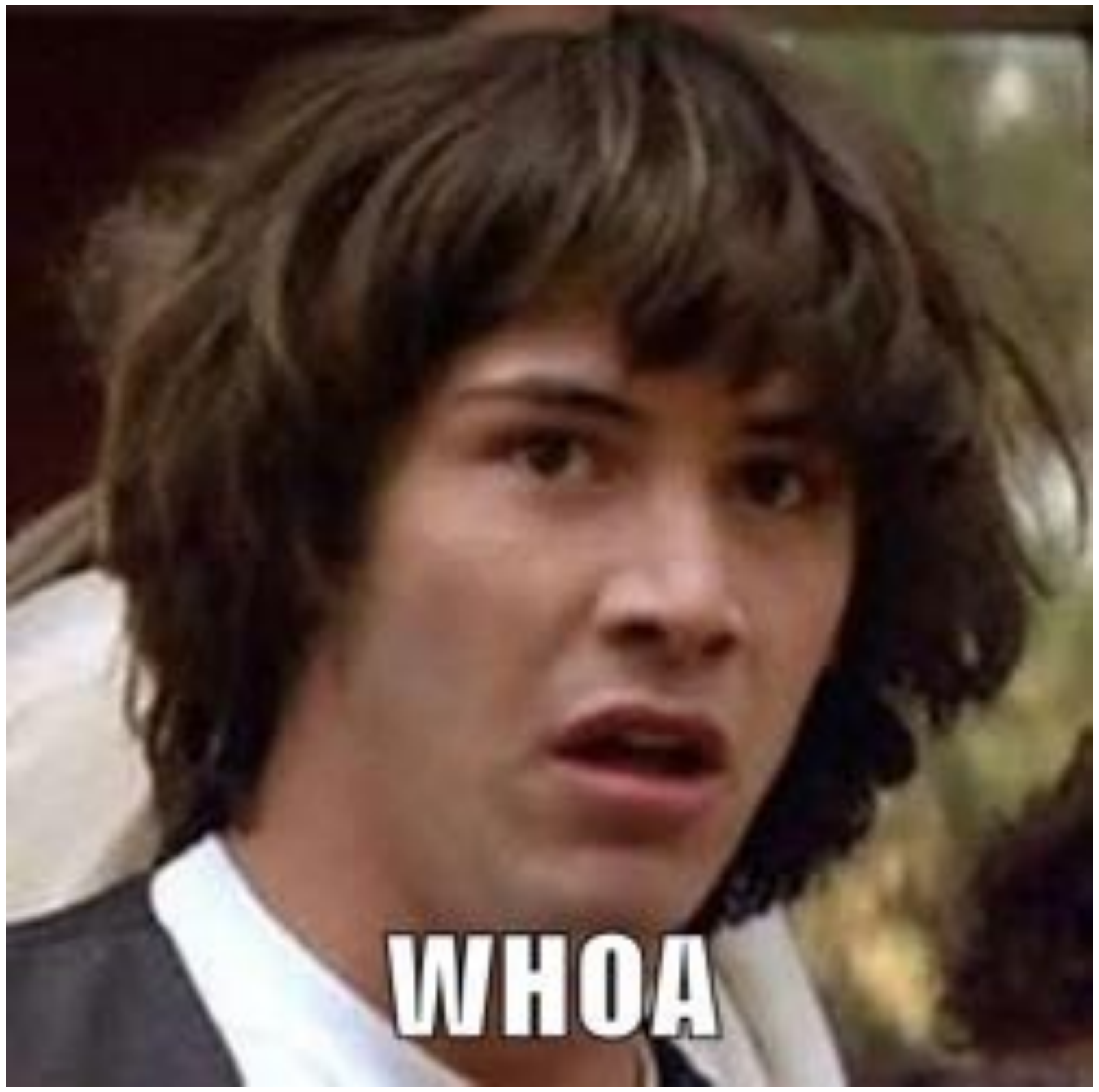
- **Consider the vector field** $X(u,v) := (-\sin(v), \cos(u))$
- **(2D) Curl is then**

$$\nabla \times X = \frac{\partial}{\partial u}\cos(u) - \frac{\partial}{\partial v}(-\sin(v)) = -\sin(u) + \cos(v).$$



X

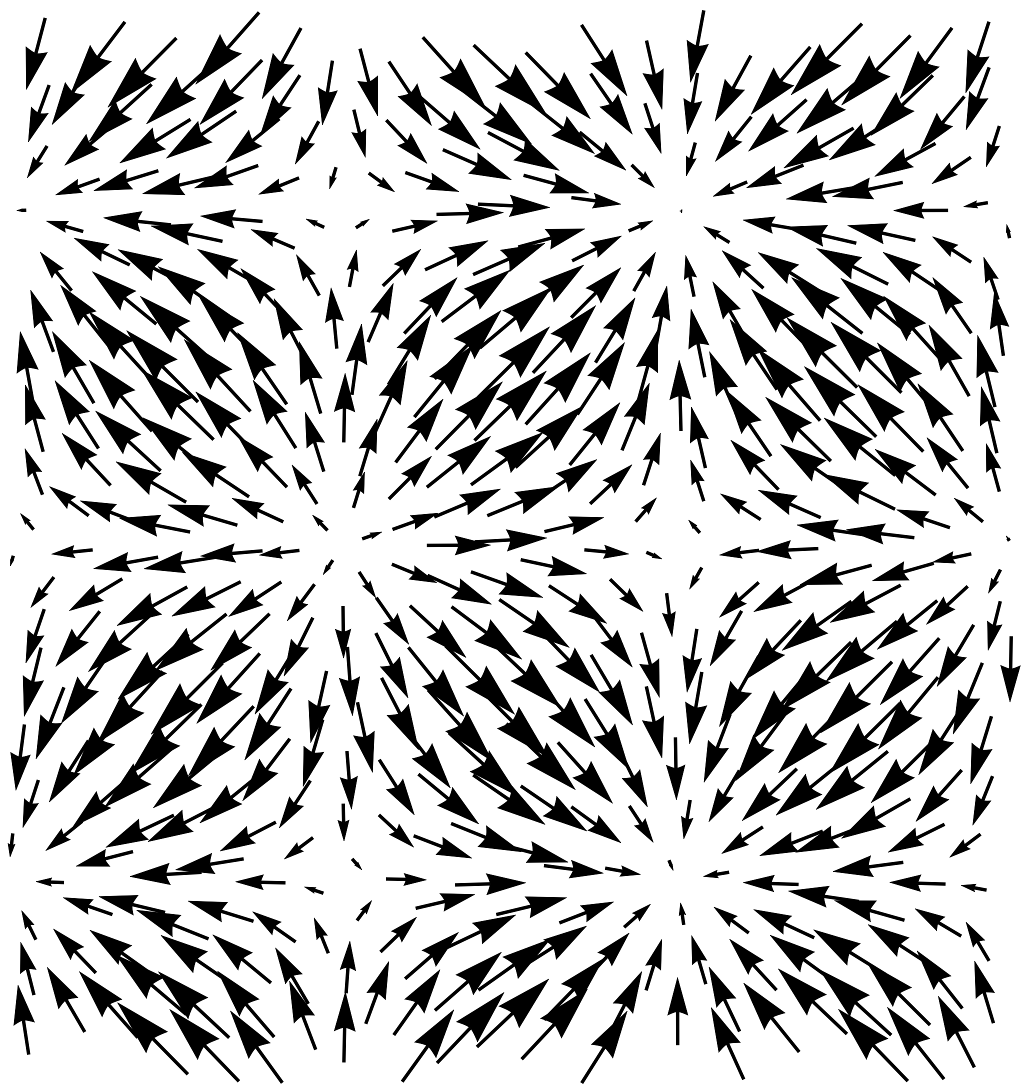
$\nabla \times X$



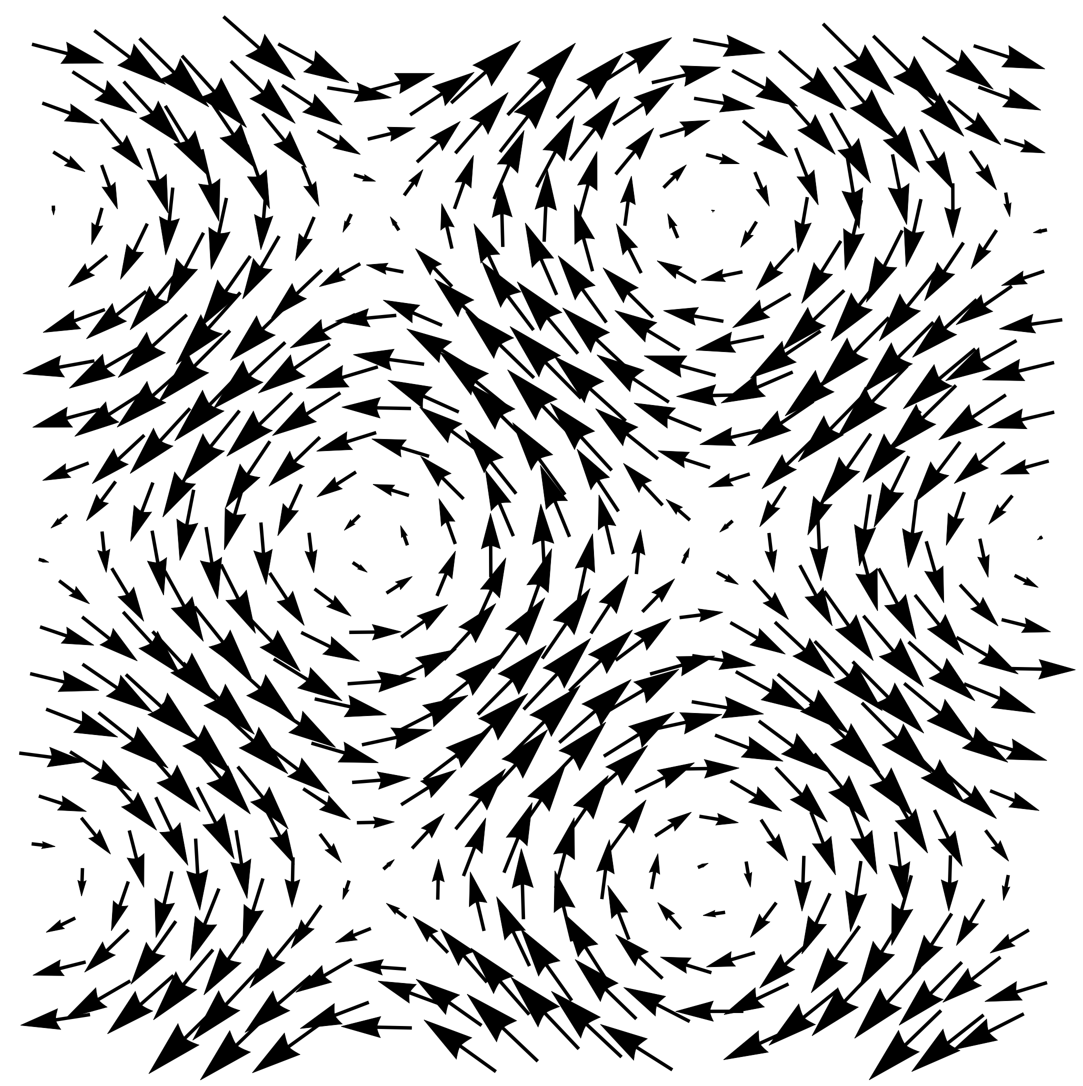
Notice anything about the relationship between curl and divergence?

Divergence vs. Curl (2D)

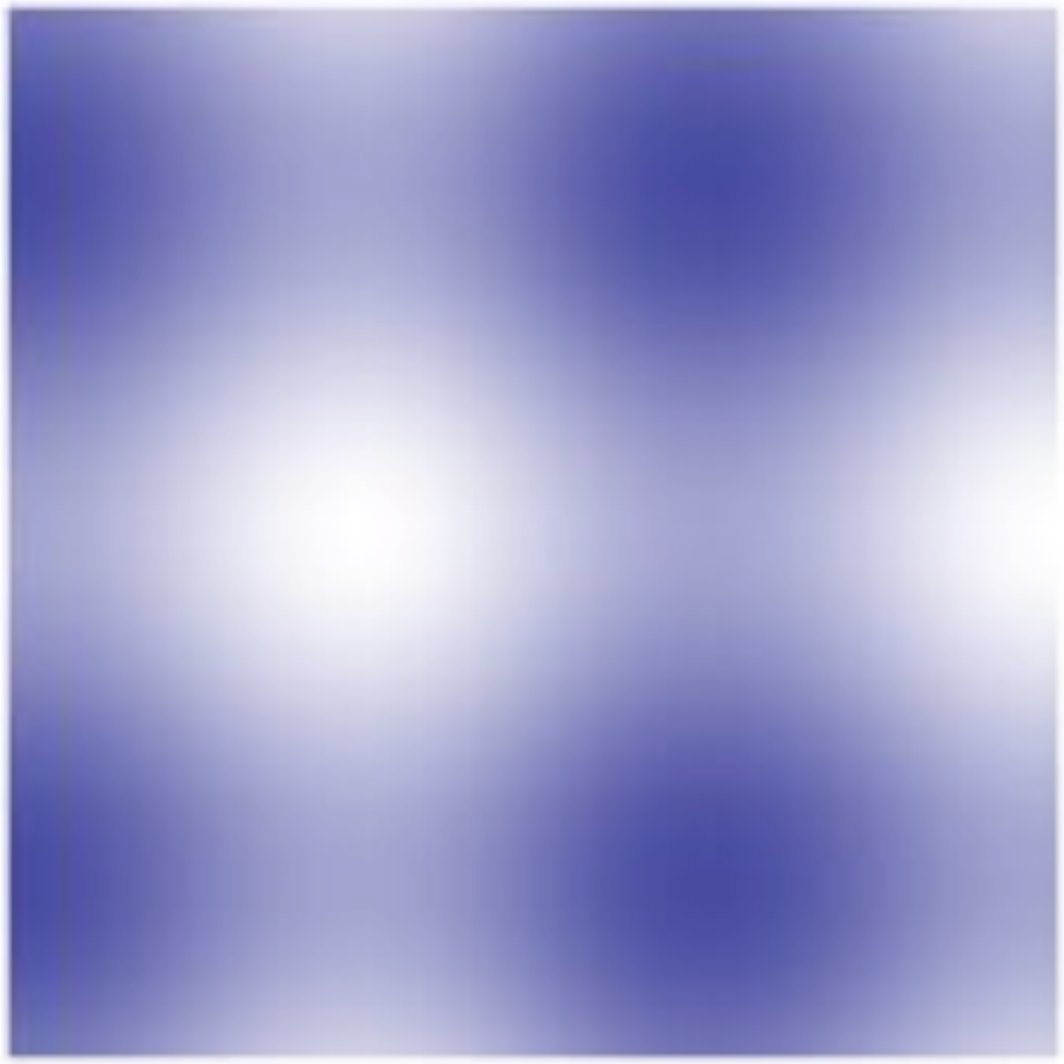
- **Divergence of X is the same as curl of 90-degree rotation of X:**



X



$X^{\perp}$

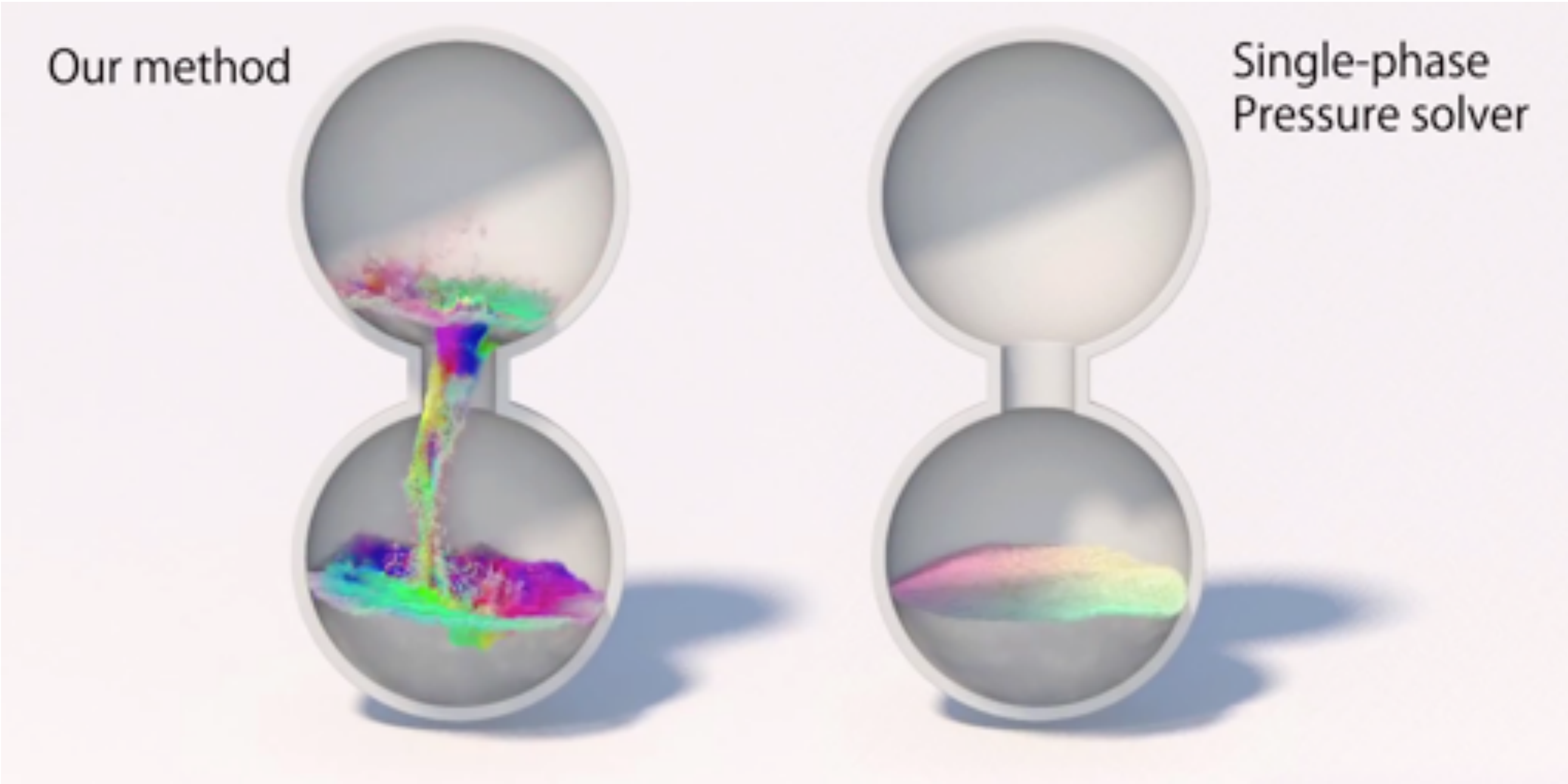


$\nabla \cdot X = \nabla \times X^{\perp}$

- **Playing these kinds of games w/ vector fields plays an important role in algorithms (e.g., fluid simulation)**

- **(Q: Can you come up with an analogous relationship in 3D?)**

Example: Fluids w/ Stream Function



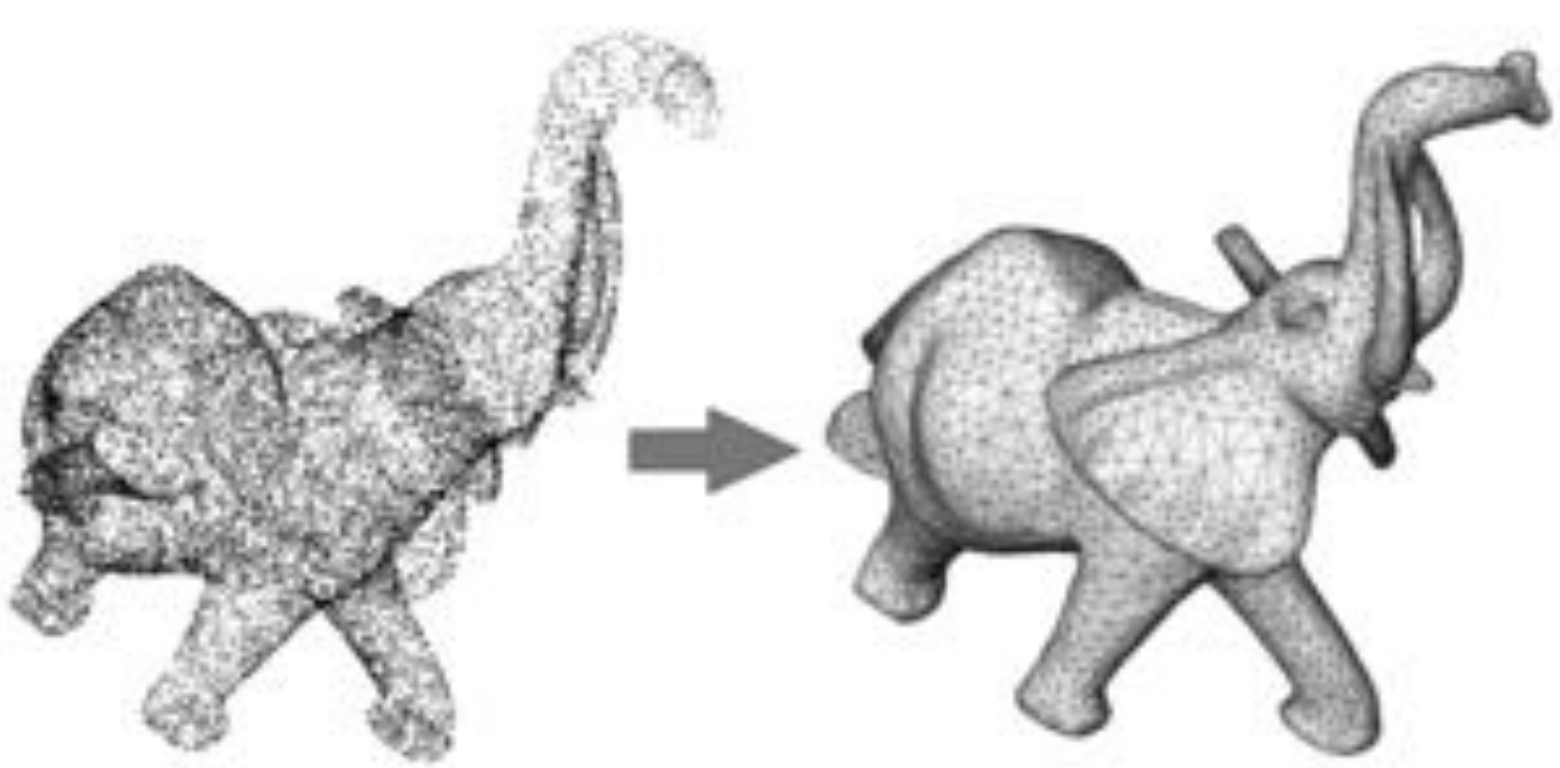
$$\min_{\Psi} ||u^* - \nabla \times \Psi||^2$$
$$u = \nabla \times \Psi$$

$$\Delta p = \nabla \cdot u^*$$
$$u = u^* - \nabla p$$

Ando et al, "A Stream Function Solver for Liquid Simulations" (SIGGRAPH 2015)

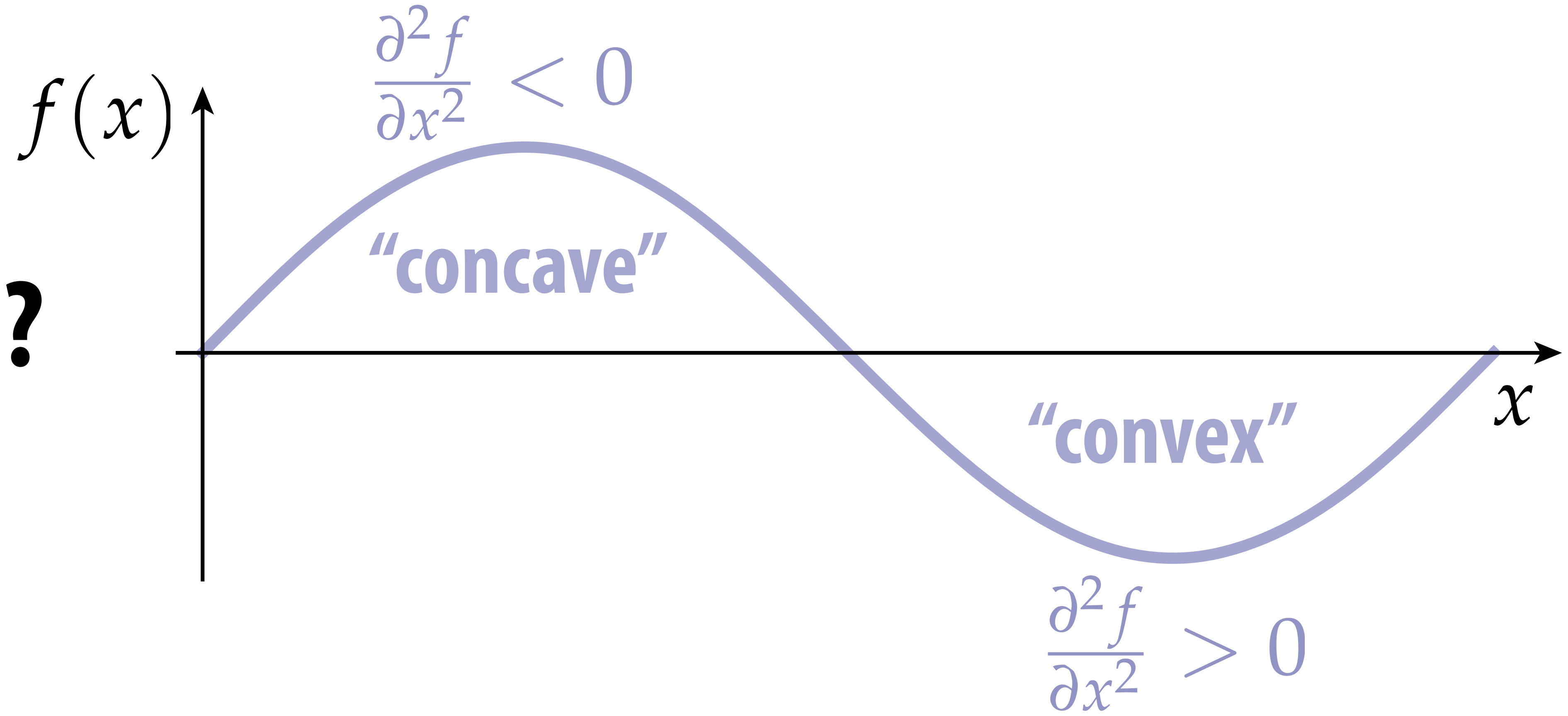
Laplacian

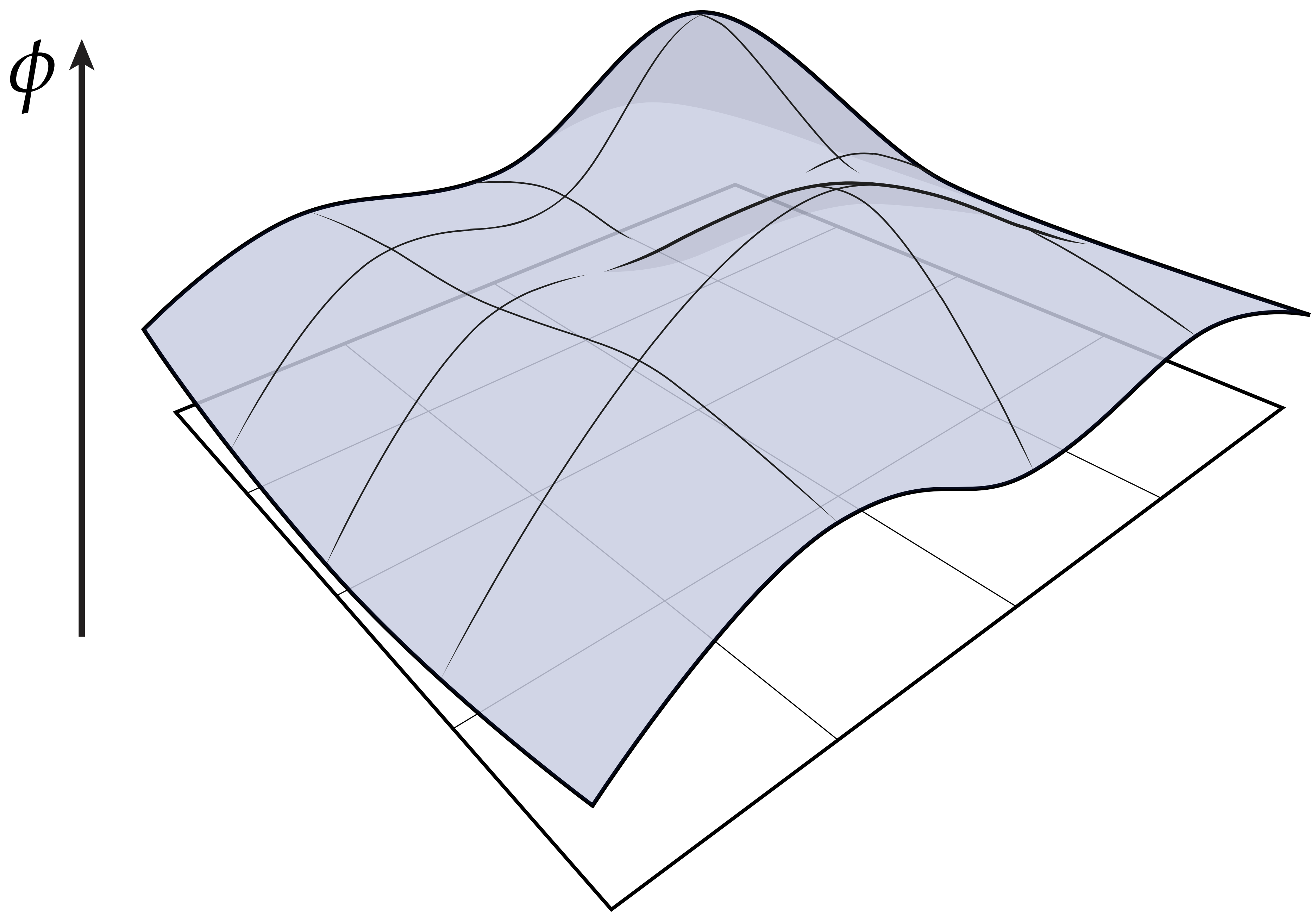
- One more operator we haven't seen yet: the Laplacian
- Unbelievably important object in graphics, showing up across geometry, rendering, simulation, imaging
  - basis for Fourier transform / frequency decomposition
  - used to define model PDEs (Laplace, heat, wave equations)
  - encodes rich information about geometry

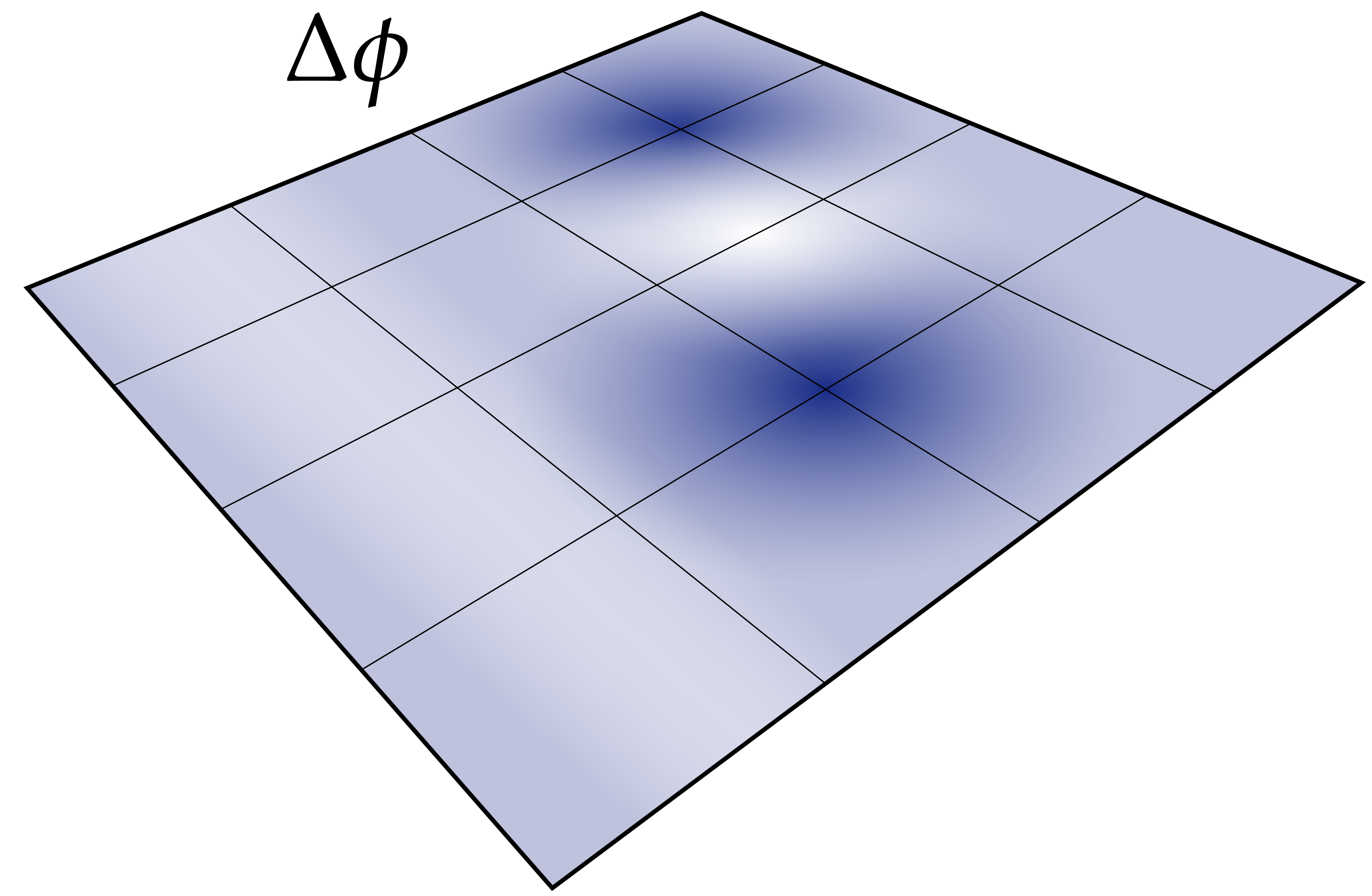


Laplacian—Visual Intuition

Q: For ordinary function f(x), what does 2nd derivative tell us?







Likewise, Laplacian measures "curvature" of a function.

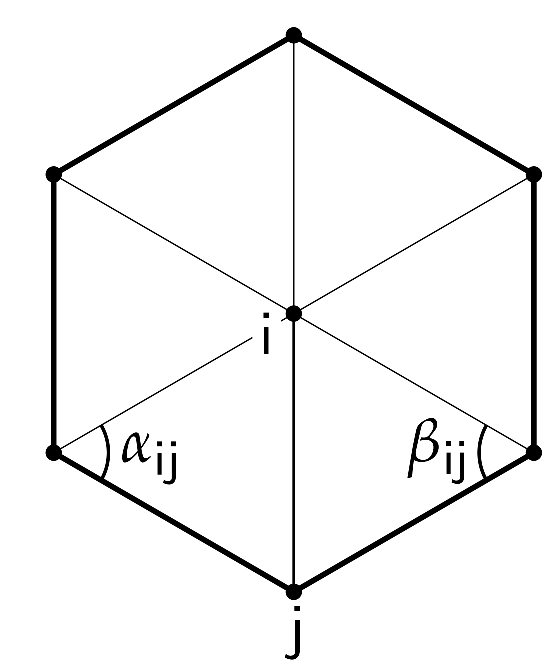
For further interpretations of the Laplacian, see https://youtu.be/oEq9ROI9Umk

Laplacian—Many Definitions

- **Maps a scalar function to another scalar function (linearly!)**
- **Usually* denoted by** Δ ← “Delta”
- **Many starting points for Laplacian:**
  - **divergence of gradient** $\Delta f := \nabla \cdot \nabla f = \mathrm{div}(\mathrm{grad} f)$
  - **sum of 2nd partial derivatives** $\Delta f := \sum_{i=1}^{n} \partial^2 f / \partial x_i^2$
  - **gradient of Dirichlet energy** $\Delta f := -\nabla_f \left(\frac{1}{2}||\nabla f||^2\right)$
  - **by analogy: graph Laplacian**
  - **variation of surface area**
  - **trace of Hessian …**

	1	
1	-4	1
	1	

$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2}$$



$$\frac{1}{2}\sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

*Or by ∇^2, but we’ll reserve this symbol for the Hessian

Laplacian—Example

- **Let's use coordinate definition:** $\Delta f := \sum_i \partial^2 f / \partial x_i^2$
- **Consider the function** $f(x_1, x_2) := \cos(3x_1) + \sin(3x_2)$
- **We have**

$$\frac{\partial^2}{\partial x_1^2} f = \frac{\partial^2}{\partial x_1^2}\cos(3x_1) + \cancelto{0}{\frac{\partial^2}{\partial x_1^2}\sin(3x_2)} =$$

$$-3\frac{\partial}{\partial x_1}\sin(3x_1) = -9\cos(3x_1).$$

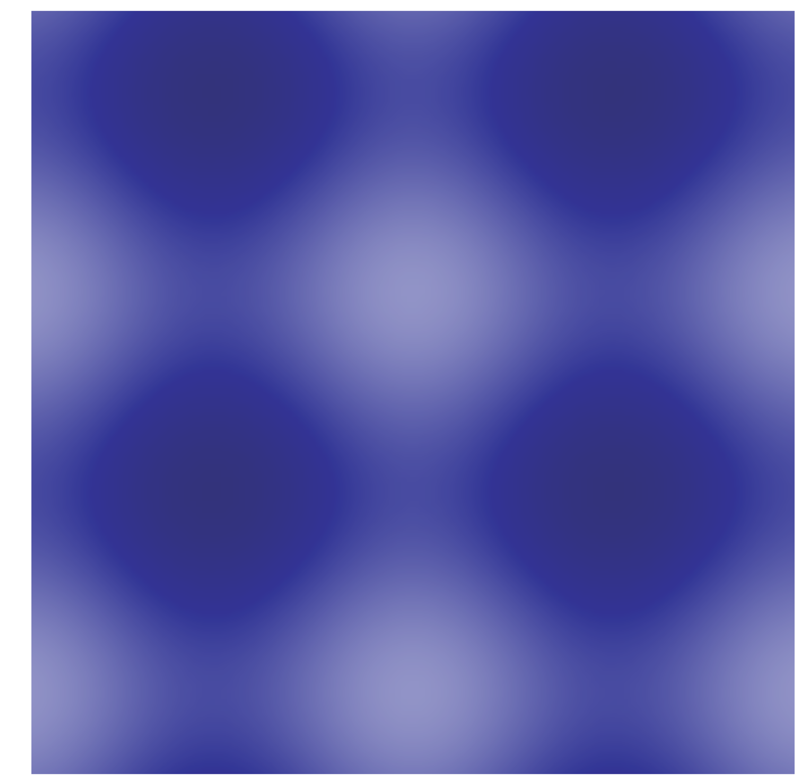
and

$$\frac{\partial^2}{\partial x_2^2} f = -9\sin(3x_2).$$

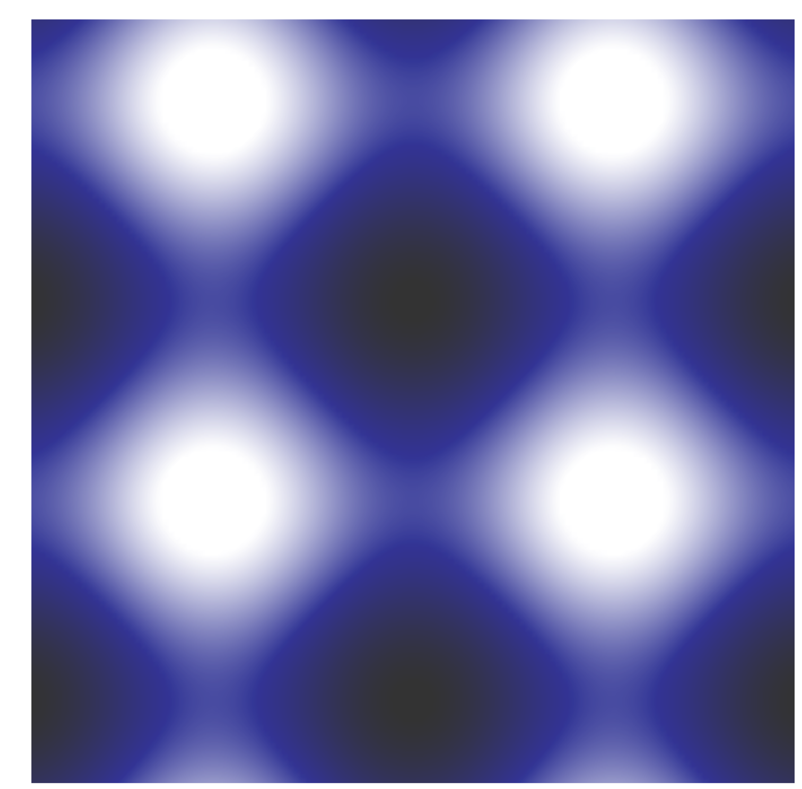
Hence,

$$\Delta f = -9(\cos(3x_1) + \sin(3x_2))$$
$$= -9f$$

Interesting! Does this always happen?



f

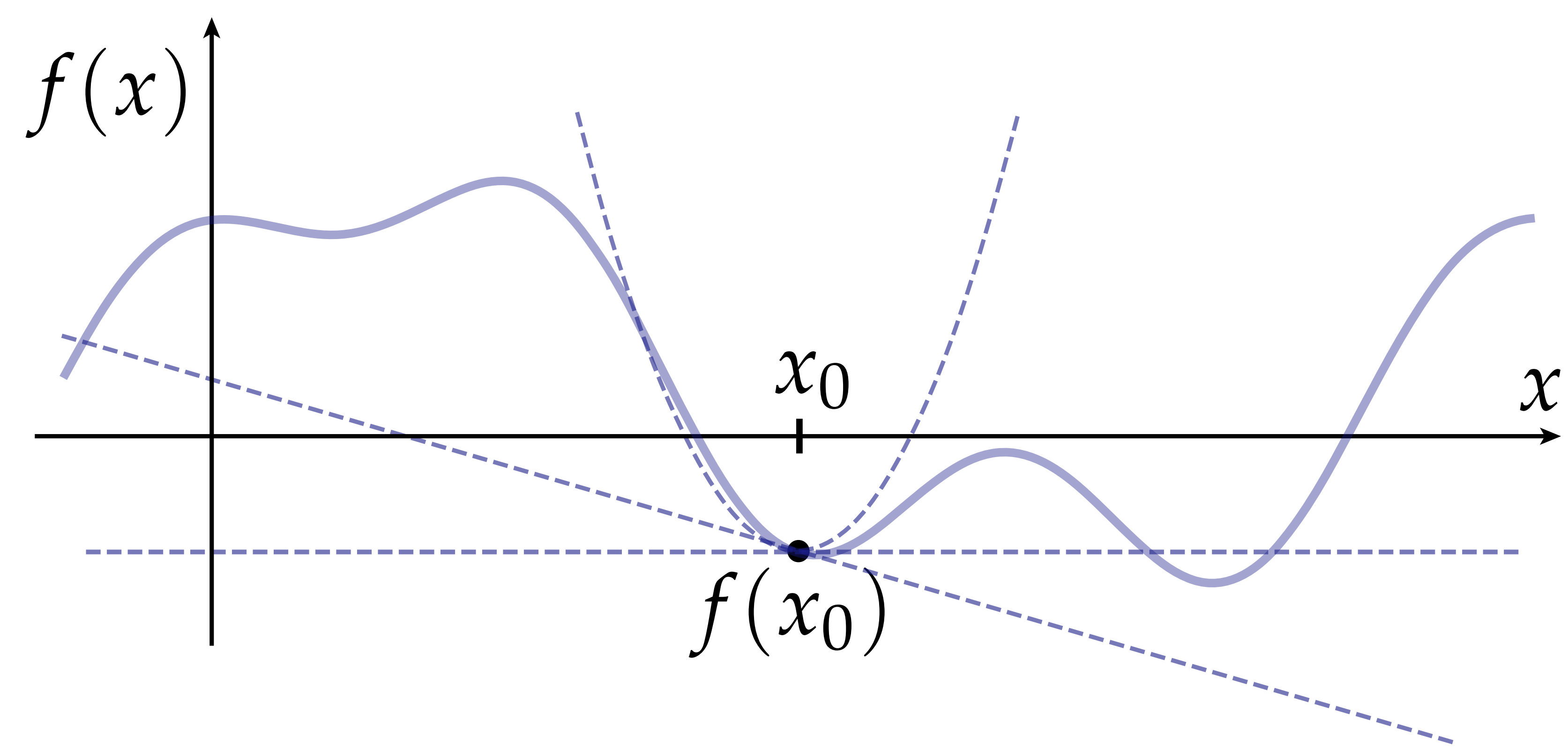


Δf

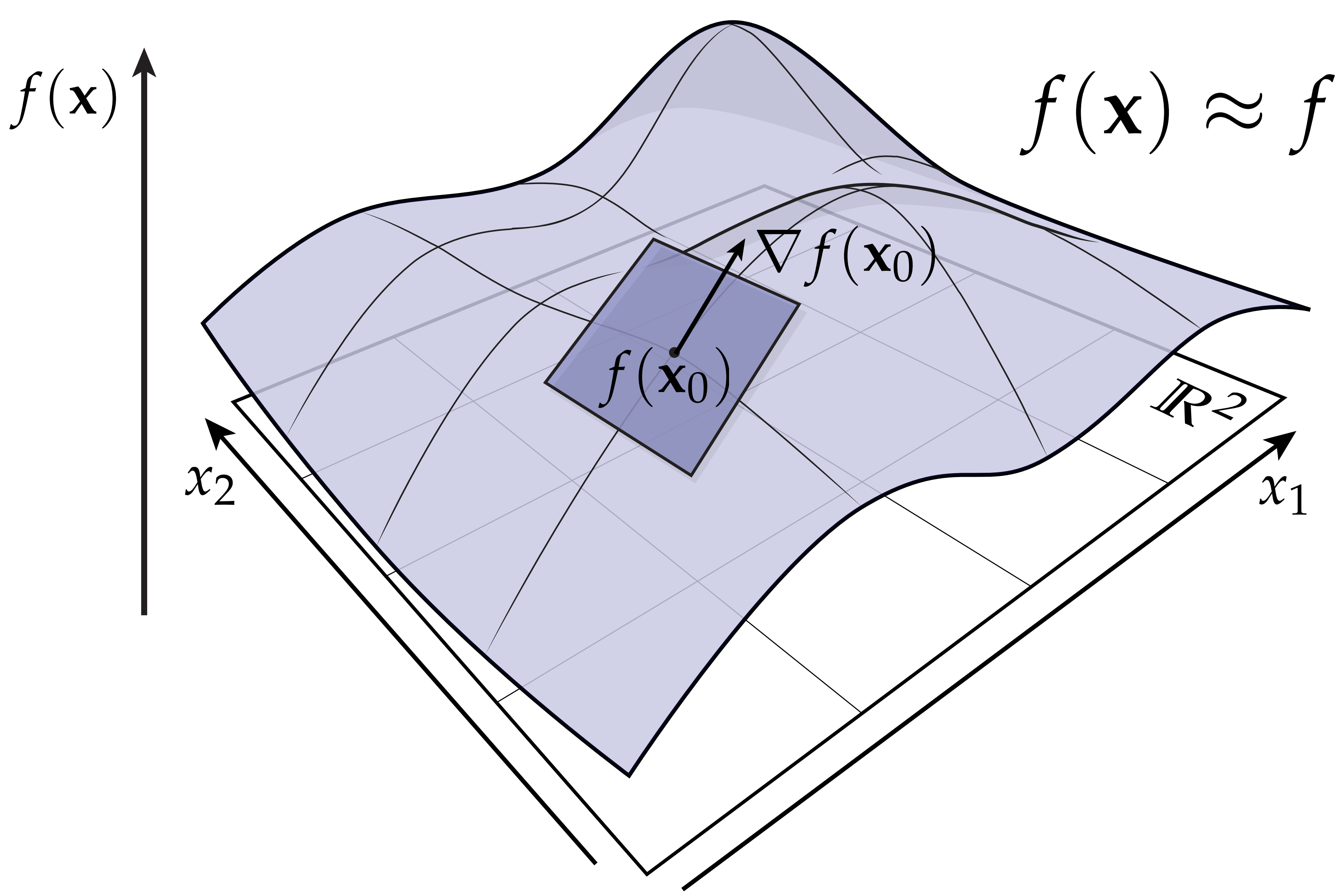
Hessian

- **Our final differential operator—Hessian will help us locally approximate complicated functions by a few simple terms**
- **Recall our Taylor series**

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$



- **How do we do this for multivariable functions?**
- **Already talked about best linear approximation, using gradient:**



$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle$$

Hessian gives us next, "quadratic" term.

Hessian in Coordinates

- **Typically denote Hessian by symbol** ∇^2
- **Just as gradient was "vector that gives us partial derivatives of the function," Hessian is "operator that gives us partial derivatives of the gradient":**

$$(\nabla^2 f)\mathbf{u} := D_{\mathbf{u}}(\nabla f)$$

- **For a function f(x): $R^n \to R$, can be more explicit:**

$$\nabla^2 f := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Q: Why is this matrix always symmetric?

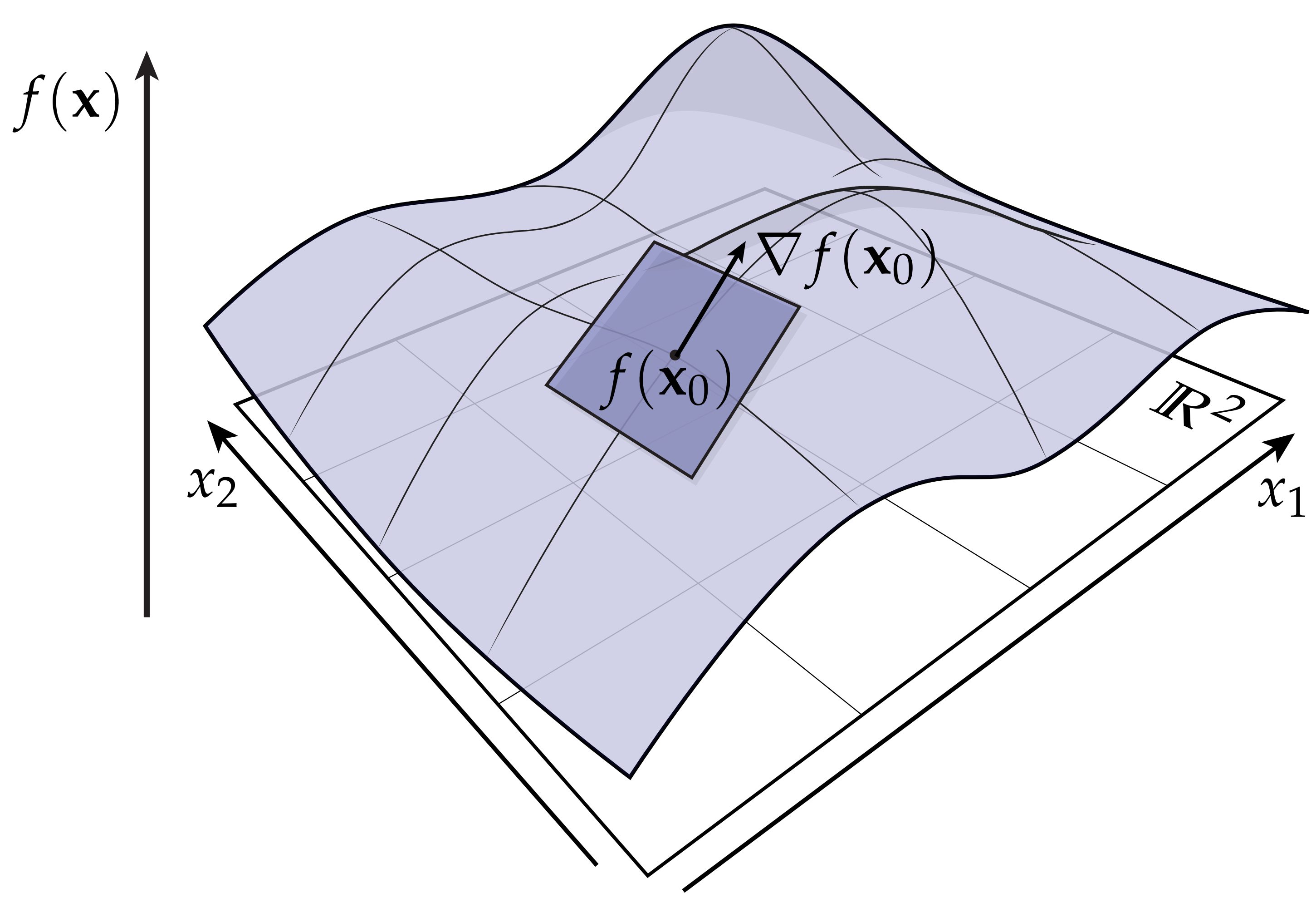
Taylor Series for Multivariable Functions

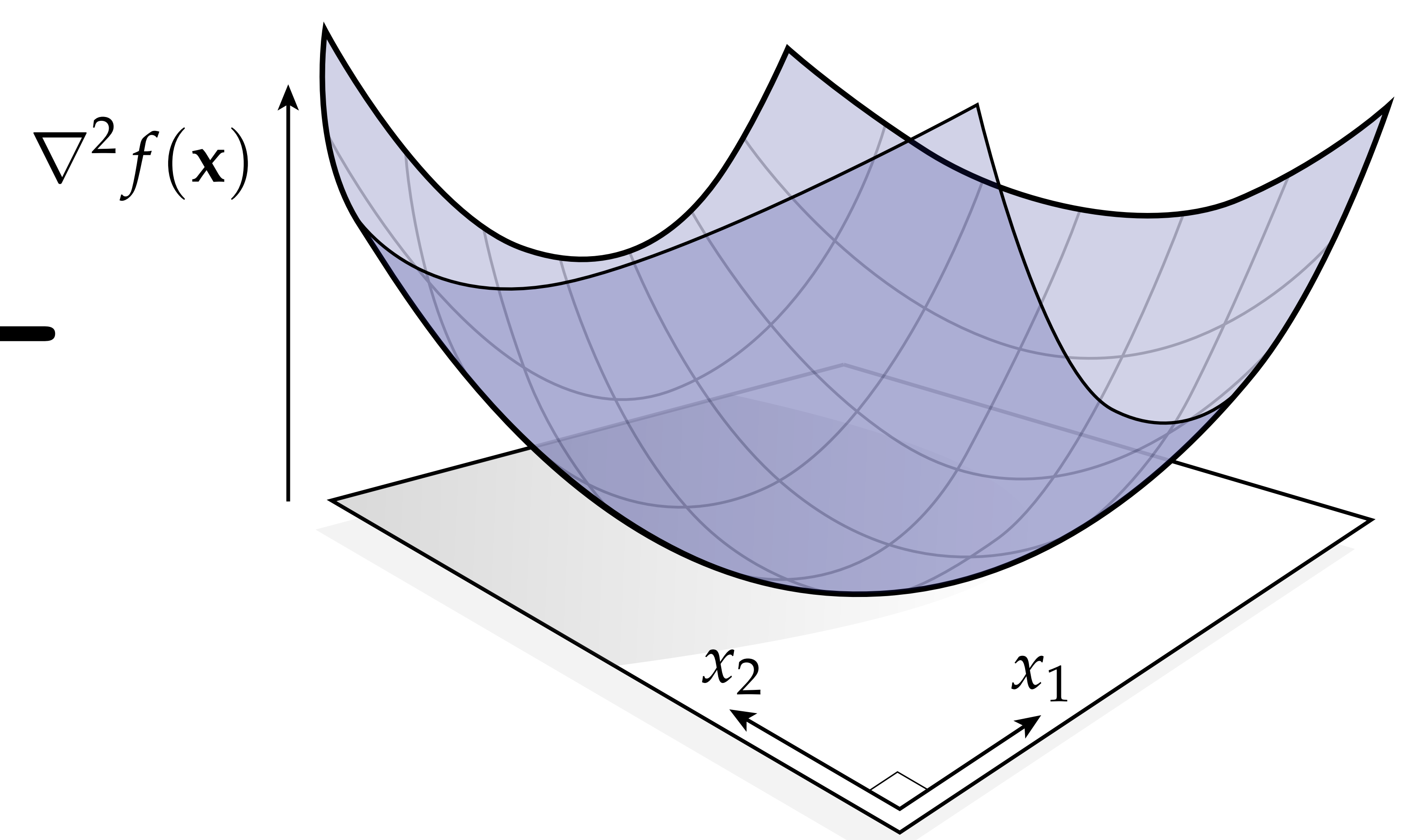
- Using Hessian, can now write 2nd-order approximation of any smooth, multivariable function f(x) around some point $\mathbf{x}_0$:

$$f(\mathbf{x}) \approx \overbrace{\underbrace{f(\mathbf{x}_0)}_{c\in\mathbb{R}}}^{\text{constant}} + \overbrace{\langle \underbrace{\nabla f(\mathbf{x}_0)}_{\mathbf{b}\in\mathbb{R}^n}, \mathbf{x}-\mathbf{x}_0\rangle}^{\text{linear}} + \overbrace{\langle \underbrace{\nabla^2 f(x_0)}_{\mathbf{A}\in\mathbb{R}^{n\times n}}(\mathbf{x}-\mathbf{x}_0), \mathbf{x}-\mathbf{x}_0\rangle/2}^{\text{quadratic}}$$

- Can write this in matrix form as

$$f(\mathbf{u}) \approx \tfrac{1}{2}\mathbf{u}^\top \mathbf{A}\mathbf{u} + \mathbf{b}^\top \mathbf{u} + c, \quad \mathbf{u} := \mathbf{x}-\mathbf{x}_0$$





Will see later on how this approximation is very useful for optimization!

MiniHW1 - Sampling and Aliasing

A major theme of Wednesday's lecture, and a major theme of our class, was how poor sampling and reconstruction can lead to aliasing.

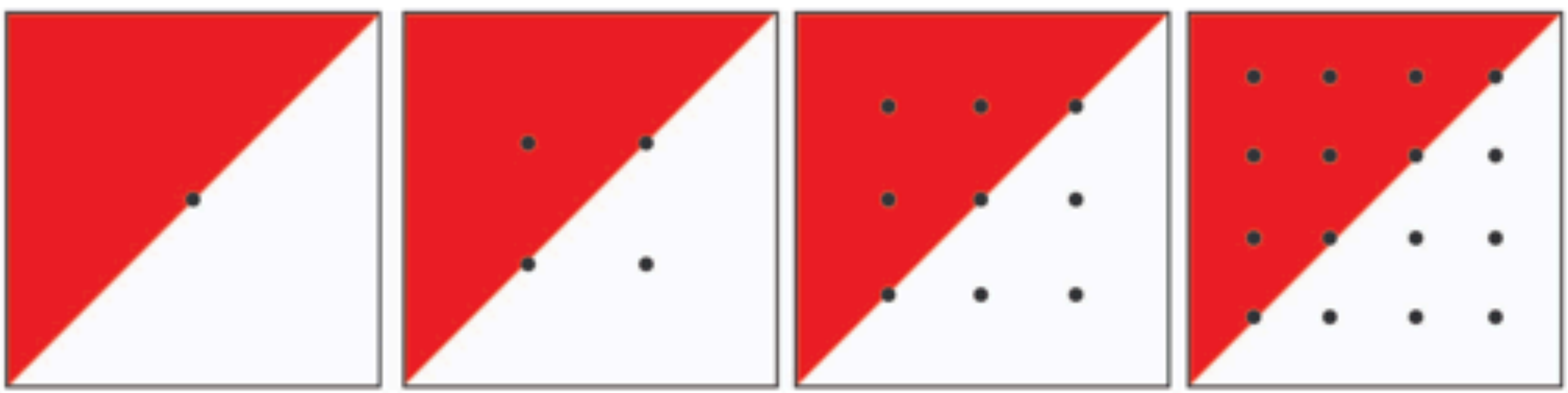
Recall that aliasing means, roughly speaking, when something appears to be what it is not. (In English, an "alias" essentially just means a false name or identity.) In computer graphics and signal processing, aliasing occurs because of a mismatch between sampling and reconstruction: the rate or manner in which a signal is sampled is insufficient to provide a faithful reconstruction of the original signal.

For this exercise we will be looking at how various sampling methods and resolutions can affect the reconstruction of the image. We will be using supersampling to compute the value of the same pixel. For each cell, the red triangle takes up exactly half of the pixel. **If the sample is being taken at the edge of the triangle, it is counted as being inside the triangle in this example.**

1. What is the percent red for each supersampled pixel? Please compute this for each of the 4 images below.

2. Plot a graph of the relative sampling error as we increase the supersample rate from 1 to 4. Recall that the relative error is abs(samplePercent - truePercent) / truePercent.

3. Based on your graph, what do you notice about the error? Does it increase or decrease in this case? What does that tell you about the pixel accuracy as we increase the supersample rate?



... (see the course webpages for the complete homework

Next time: Rasterization

- Next time, we'll talk about how to draw triangles
- A lot more interesting (and difficult!) than it might seem…
- Leads to a deep understanding of modern graphics hardware