Introduction to
Optimization

Computer Graphics
CMU 15-462/15-662
Last time: physically-based animation

- Use dynamics to drive motion
- Complexity from simple models
- Technique: numerical integration
  - formulate equations of motion
  - take little steps forward in time
  - general, powerful tool
- Today: numerical optimization
  - another general, powerful tool
  - basic idea: “ski downhill” to get a better solution
  - used everywhere in graphics (not just animation)
  - image processing, geometry, rendering, ...
What is an optimization problem?

- Natural human desire: find the best solution among all possibilities (subject to certain constraints)
- E.g., cheapest flight, shortest route, tastiest restaurant ...
- Has been studied since antiquity, e.g., isoperimetric problem:

“The first optimization problem known in history was practically solved by Dido, a clever Phoenician princess, who left her Tyrian home and emigrated to North Africa, with all her property and a large retinue, because her brother Pygmalion murdered her rich uncle and husband Acerbas, and plotted to defraud her of the money which he left. On landing in a bay about the middle of the north coast of Africa she obtained a grant from Hiarbas, the native chief of the district, of as much land as she could enclose with an ox-hide. She cut the ox-hide into an exceedingly long strip, and succeeded in enclosing between it and the sea a very valuable territory on which she build Carthage.”

—Lord Kelvin, 1893

“Obvious” solution is a circle... ...but wait, what about the coastline?
Optimization in Graphics

Sumit Jain, Yuting Ye, and C. Karen Liu, “Optimization-based Interactive Motion Synthesis”
Optimization in Graphics

Niloy J. Mitra, Leonidas Guibas, Mark Pauly, “Symmetrization”
Optimization in Graphics

Moritz Bächer, Emily Whiting, Bernd Bickel, Olga Sorkine-Hornung,
“Spin-It: Optimizing Moment of Inertia for Spinnable Objects”
Optimization in Graphics

Nobuyuki Umetani, Yuki Koyama, Ryan Schmidt & Takeo Igarashi, “Pteromys: Interactive Design and Optimization of Free-formed Free-flight Model Airplanes”
Continuous vs. Discrete Optimization

**DISCRETE:**
- Domain is a discrete set (e.g., finite or integers)
- Example: best vegetable to put in a stew
  - Basic strategy? Try them all! (exponential)
  - Sometimes clever strategy (e.g., MST)
  - More often, NP-hard (e.g., TSP)

**CONTINUOUS:**
- Domain is not discrete (e.g., real numbers)
- Example: best temperature to cook an egg
- Still many (NP-)hard problems, but also large classes of “easy” problems (e.g., convex)
Optimization Problem in Standard Form

- Can formulate most continuous optimization problems this way:
  
  \[
  \min_{x \in \mathbb{R}^n} f_0(x) \quad \text{subject to} \quad f_i(x) \leq b_i, \ i = 1, \ldots, m
  \]

  "objective": how much does solution x cost?

  \[(f_i : \mathbb{R}^n \rightarrow \mathbb{R}, \ i = 0, \ldots, m)\]
  
  often (but not always) continuous, differentiable, ...

  "constraints": what must be true about x? ("x is feasible")

- Optimal solution \(x^*\) has smallest value of \(f_0\) among all feasible x

- Q: What if we want to maximize something instead?
- A: Just flip the sign of the objective!

- Q: What if we want equality constraints, rather than inequalities?
- A: Include two constraints: \(g(x) \leq c\) and \(g(x) \leq -c\)
Local vs. Global Minima

- Global minimum is absolute best among all possibilities
- Local minimum is best “among immediate neighbors”

Philosophical question: does a local minimum “solve” the problem? Depends on the problem! (E.g., real protein folding is local minimum) Other times, local minima can be really bad (e.g., path planning)
Optimization Problem, Visualized

Q: Is this an optimization problem in standard form? A: Yes.

Q: Where is the optimal solution? A: There are two, (0,1), (0,-1).

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad x_1^2 - x_2^2 \\
\text{s.t.} & \quad x_1^2 + x_2^2 - 1 \leq 0
\end{align*}
\]
Existence & Uniqueness of Minimizers

- Already saw that (global) minimizer is not unique.
- Does it always exist? Why?
- Just consider all possibilities and take the smallest one, right?

\[ f_0(x) \]

- WRONG! Not all objectives are bounded from below.
- It’s like that old adage: “no matter how good you are, there will always be someone better than you.”
Feasibility

- Ok, but suppose the objective is bounded from below.
- Then we can just take the best feasible solution, right?

\[
\min_{x \in \mathbb{R}^n} \quad 0 \\
\text{subject to} \quad f_i(x) \leq b_i, \quad i = 1, \ldots, m
\]

value of objective doesn’t depend on x; all feasible solutions are equally good

- Not if there aren’t any!
- Every system of equations is an optimization problem.
- But not all problems have solutions!
- (You’ll appreciate this more as you get older.)
Q: Is this problem feasible?

\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad \sin(x_1) + x_2^2 \\
\text{s.t.} & \quad (x_1 - 2)^2 + x_2^2 \leq 1, \\
& \quad x_1 \leq -1
\end{align*}
\]

A: No—the two sublevel sets (points where \( f_i(x) \leq 0 \)) have no common points, i.e., they do not overlap.
Existence & Uniqueness of Minimizers, cont.

- Even being bounded from below is not enough:

\[
\min_{x \in \mathbb{R}} e^{-x}
\]

- No matter how big \( x \) is, we never achieve the lower bound \((0)\)
- So when does a solution exist? Two sufficient conditions:
  - Extreme value theorem: continuous objective & compact domain
  - Coercivity: objective goes to \(+\infty\) as we travel (far) in any direction
Characterization of Minimizers

- Ok, so we have some sense of when a minimizer might exist.
- But how do we know a given point $x$ is a minimizer?

Checking if a point is a global minimizer is (generally) hard.
- But we can certainly test if a point is a local minimum (ideas?).
- (Note: a global minimum is also a local minimum!)
Characterization of Local Minima

- Consider an objective $f_0 : \mathbb{R} \rightarrow \mathbb{R}$. How do you find a minimum?
- (Hint: you may have memorized this formula in high school!)

$$f_0'(x^*) = 0$$

- Also need to check second derivative (how?)
- Make sure it’s positive
- Ok, but what does this all mean for more general functions $f_0$?

$$f_0''(x^*) \geq 0$$
Optimality Conditions (Unconstrained)

- In general, our objective is $f_0: \mathbb{R}^n \to \mathbb{R}$
- How do we test for a local minimum?
- 1st derivative becomes gradient; 2nd derivative becomes Hessian

\[ \nabla f := \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \nabla^2 f := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \]

**GRADIENT** (measures "slope")

**HESSIAN** (measures "curvature")

- Optimality conditions?
  \[ \nabla f_0(x^\star) = 0 \]
  1st order
  \[ \nabla^2 f_0(x^\star) \geq 0 \]
  2nd order

Positive semidefinite (PSD) \((u^\top A u \geq 0 \text{ for all } u)\)
Optimality Conditions (Constrained)

- What if we have constraints?
- Is gradient at minimizer still zero?
- Is Hessian at minimizer still PSD?
- Not necessarily! (See example above)

In general, any (local or global) minimizer must at least satisfy the Karush–Kuhn–Tucker (KKT) conditions:

\[ \exists \lambda_i \text{ s.t. } \nabla f_0(x^*) = -\sum_{i=1}^{n} \lambda_i \nabla f_i(x^*) \]

stationarity

\[ f_i(x^*) \leq 0, \ i = 1, \ldots, n \]

primal feasibility

\[ \lambda_i \geq 0, \ i = 1, \ldots, n \]

dual feasibility

\[ \lambda_i f_i(x^*) = 0, \ i = 1, \ldots, n \]

complementary slackness

...we won’t work with these in this class! (But good to know where to look.)
Convex Optimization

- Special class of problems that are almost always “easy” to solve (polynomial-time!)

- Problem convex if it has a convex domain and convex objective

Why care about convex problems in graphics?
- can make guarantees about solution (always the best)
- doesn’t depend on initialization (strong convexity)
- often quite efficient, but not always
Convex Quadratic Objectives & Linear Systems

- Very important example: convex quadratic objective
- Already saw this with, e.g., quadric error simplification
- Valuable “variational” way of looking at many common equations
- Can be expressed via positive-semidefinite (PSD) matrix:

\[ f_0(x) = \frac{1}{2} x^T A x - x^T b, \quad A \succeq 0 \]

- Q: 1st-order optimality condition? \[ Ax = b \]
- Q: 2nd-order optimality condition? \[ A \succeq 0 \]
Sadly, life is not usually that easy. How do we solve optimization problems in general?
Descent Methods

An idea as old as the hills:
Gradient Descent (1D)

- Basic idea: follow the gradient “downhill” until it’s zero
- (Zero gradient was our 1st-order optimality condition)

$$\frac{d}{dt} x(t) = -f'_0(x(t))$$

- Do we always end up at a (global) minimum?
- How do we compute gradient descent in practice?
Gradient Descent Algorithm (1D)

- Did you notice that gradient descent equation is an ODE?

- Q: How do we solve it numerically?
  \[
  \frac{d}{dt} x(t) = -f'(x(t))
  \]

- One way: forward Euler:
  \[
  x_{k+1} = x_k + \tau f_0'(x_k)
  \]

- Q: How do we pick the time step?

- If we’re not careful, we’ll go zipping all over the place; won’t make any progress.

- Basic idea: use “step control” to determine step size based on value of objective & derivatives.

- A careful strategy (e.g., Armijo-Wolfe) can guarantee convergence at least to a local minimum.

- For now we will do something simpler: make \( \tau \) really small!
Gradient Descent Algorithm (nD)

- Q: How do we write gradient descent equation in general?
  \[
  \frac{d}{dt} x(t) = -\nabla f_0(x(t))
  \]

- Q: What’s the corresponding discrete update?
  \[
  x_{k+1} = x_k - \tau \nabla f_0(x_k)
  \]

- Basic challenge in nD:
  - solution can “oscillate”
  - takes many, many small steps
  - very slow to converge
Higher Order Descent

- General idea: apply a coordinate transformation so that the local energy landscape looks more like a “round bowl”
- Gradient now points directly toward nearby minimizer
- Most basic strategy: Newton’s method:

\[ x_{k+1} = x_k - \tau (\nabla^2 f_0(x_k))^{-1} \nabla f_0(x_k) \]

- Great for convex problems (even proofs about # of steps!)
- For nonconvex problems, need to be more careful
- In general, nonconvex optimization is a BLACK ART
- Meta-strategy: try lots of solvers, see what works!
  - quasi-Newton, trust region, L-BFGS, ...
Example: Inverse Kinematics

Example 12: IK-driven robot claw
Forward Kinematics

- Many systems well-described by a kinematic chain
  - collection of rigid bodies, connected by joints
  - joints have various behaviors (ball, piston, ...)
  - also have constraints (e.g., range of angles)
  - hierarchical structure (body → leg → foot)
- In animation, often called a rig
- How do we specify the configuration of a “rig”? 
  - One way: artist sets each joint individually
  - Another way: ...optimization!
Simple Kinematic Chain

- Consider a simple path-like chain in 2D
- Q: How do we write $p_1$ in terms of the root position $p_0$, angles, & vectors $u := c_{i+1} - c_i$?

$$p_1 = p_0 + \begin{bmatrix} \cos \theta_0 & \sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{bmatrix} u_0$$

- (For brevity, can use complex numbers:)

$$p_1 = p_0 + e^{i\theta_0} u_0$$

- Q: How about $p_2$?

$$p_2 = p_0 + e^{i\theta_0} u_0 + e^{i\theta_0} e^{i\theta_1} u_1$$
Simple IK Algorithm

- Basic idea behind our IK algorithm:
  - write down distance between final point and “target”
  - compute gradient with respect to angles
  - apply gradient descent

- Objective?

  \[ f_0(\theta) = \frac{1}{2} |\tilde{p}_n - p_n|^2 \]

- Constraints?
  - None! The joint angle can take any value.
  - Though we could limit joint angles (for instance)
Coming up next: PDEs in Computer Graphics

Frank Losasso, Jerry O. Talton, Nipun Kwatra, and Ron Fedkiw, “Two-Way Coupled SPH and Particle Level Set Fluid Simulation”