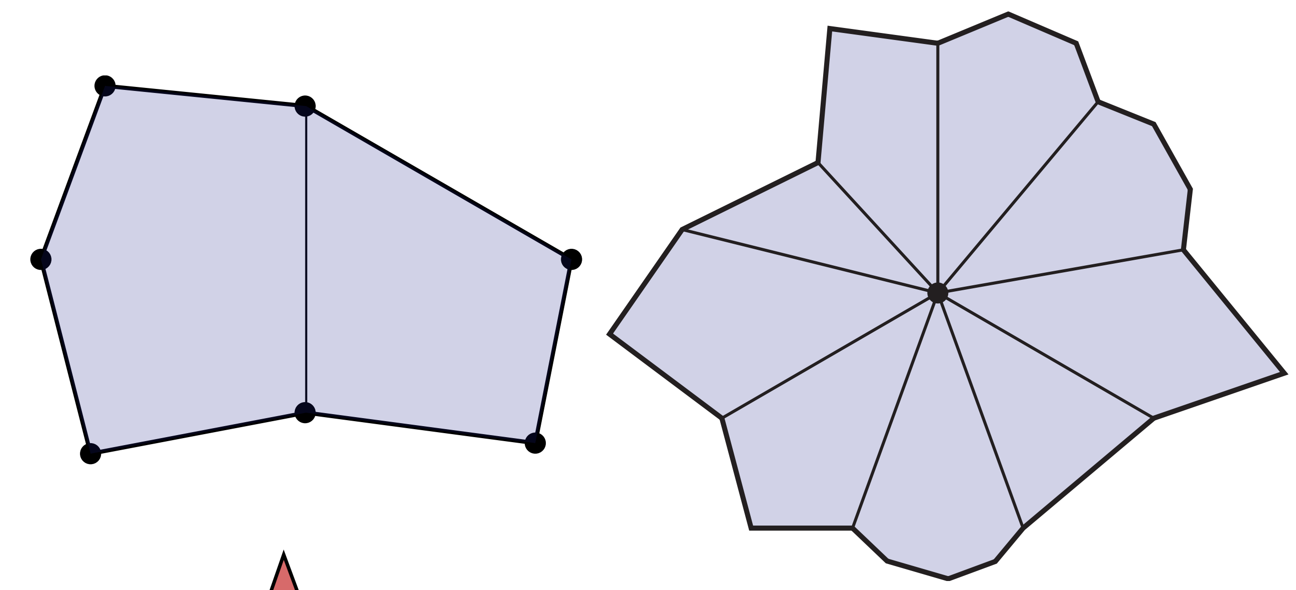
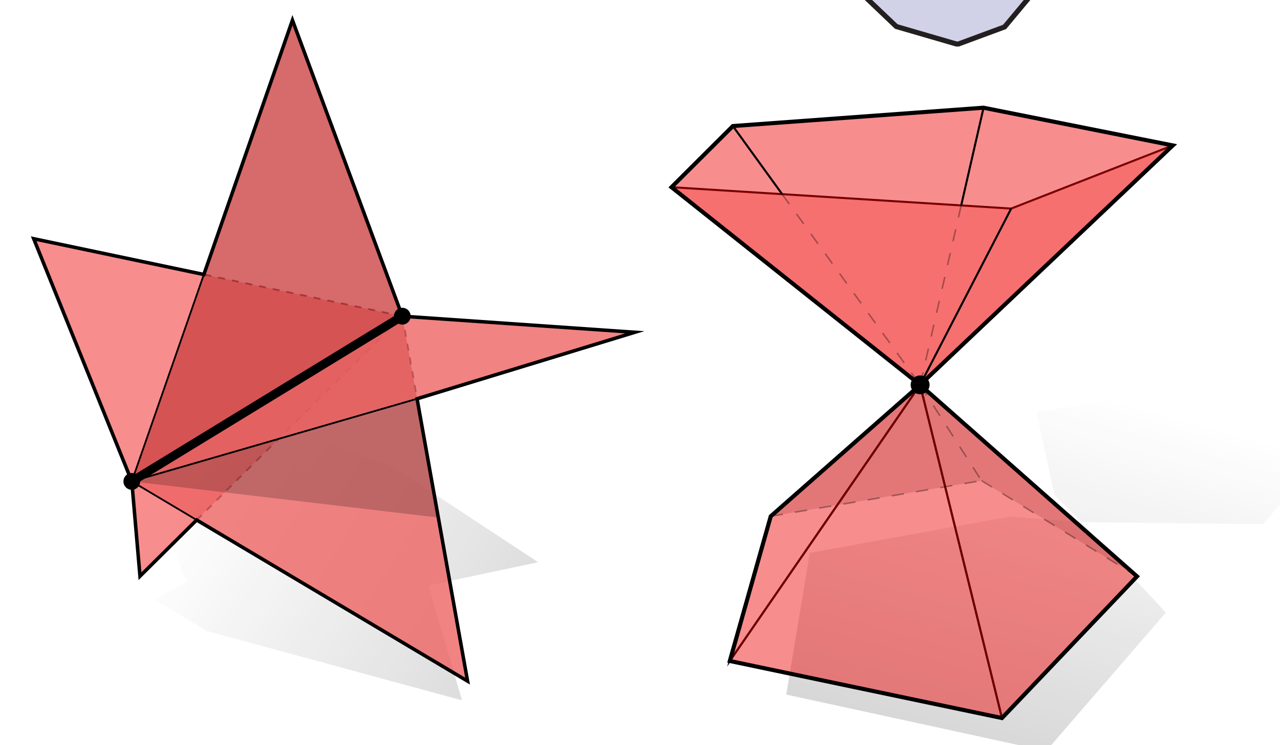
Digital Geometry Processing

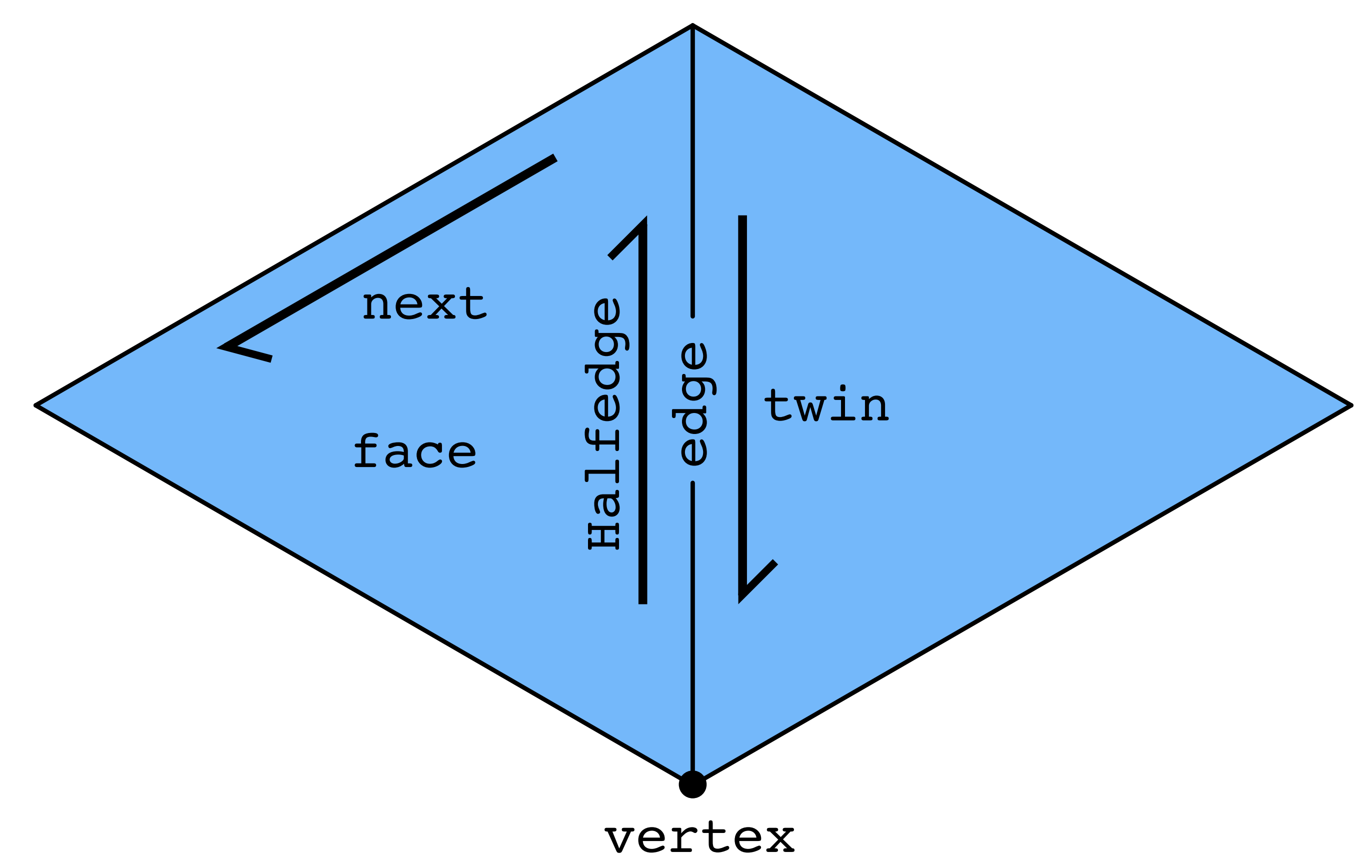
Computer Graphics
CMU 15-462/15-662

Last time: Meshes & Manifolds

- **Mathematical description of geometry**
  - simplifying assumption: manifold
  - for polygon meshes: "fans, not fins"
- **Data structures for surfaces**
  - polygon soup
  - halfedge mesh
  - storage cost vs. access time, etc.
- **Today:**
  - how do we manipulate geometry?
  - geometry processing / resampling

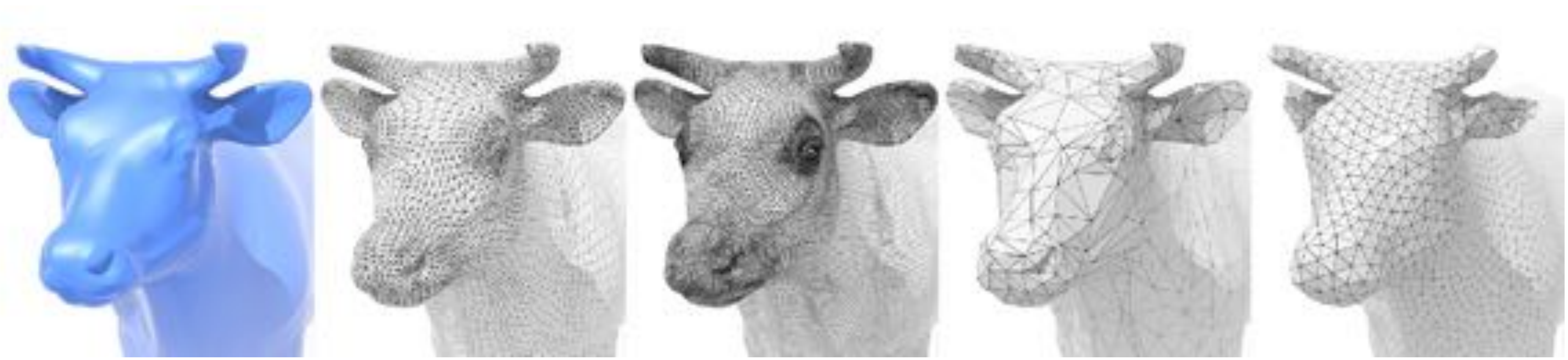






Today: Geometry Processing & Queries

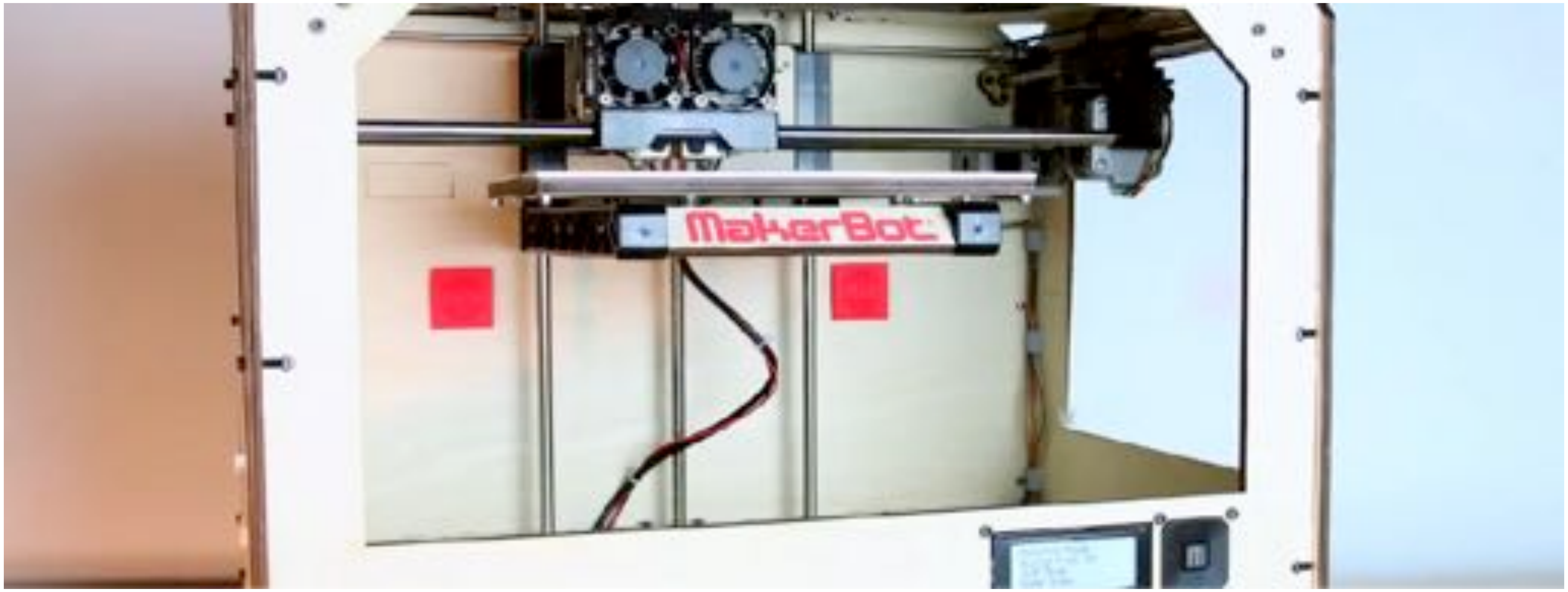
- Extend traditional digital signal processing (audio, video, etc.) to deal with geometric signals:
  - upsampling / downsampling / resampling / filtering ...
  - aliasing (reconstructed surface gives "false impression")
- Also ask some basic questions about geometry:
  - What's the closest point? Do two triangles intersect? Etc.
- Beyond pure geometry, these are basic building blocks for many algorithms in graphics (rendering, animation...)



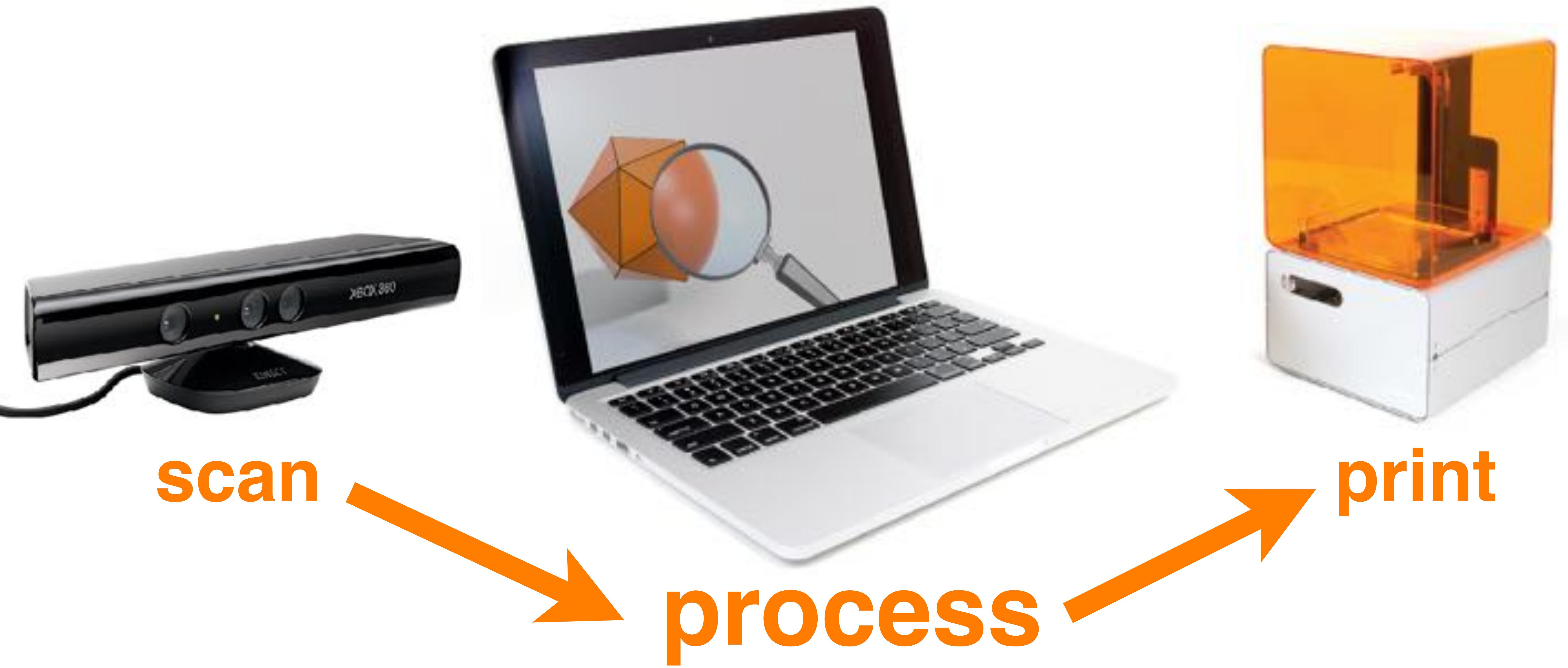
Digital Geometry Processing: Motivation

3D Scanning

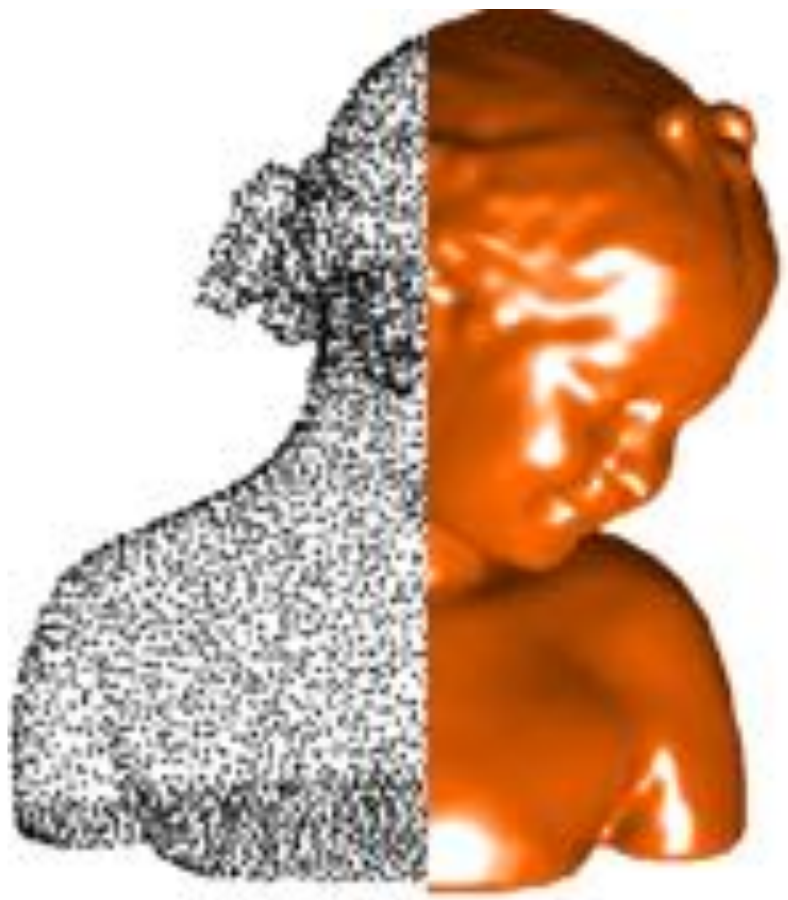
3D Printing



Geometry Processing Pipeline

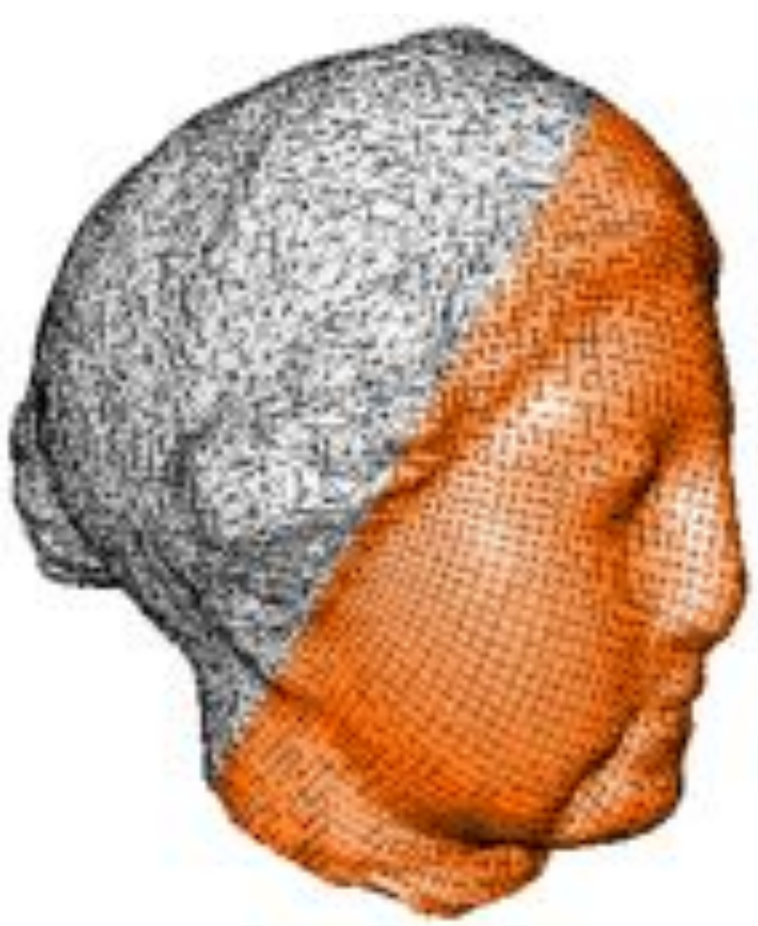


Geometry Processing Tasks



reconstruction

filtering



remeshing

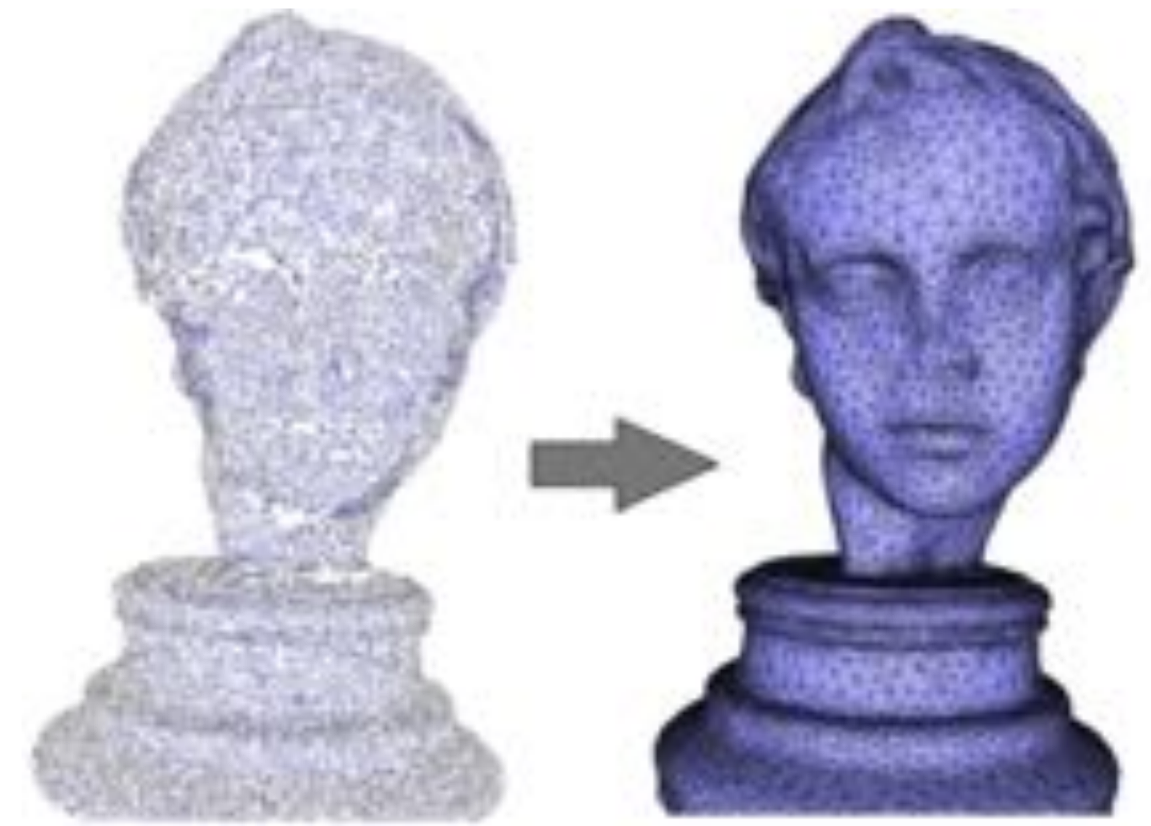
shape analysis

parameterization

compression

Geometry Processing: Reconstruction

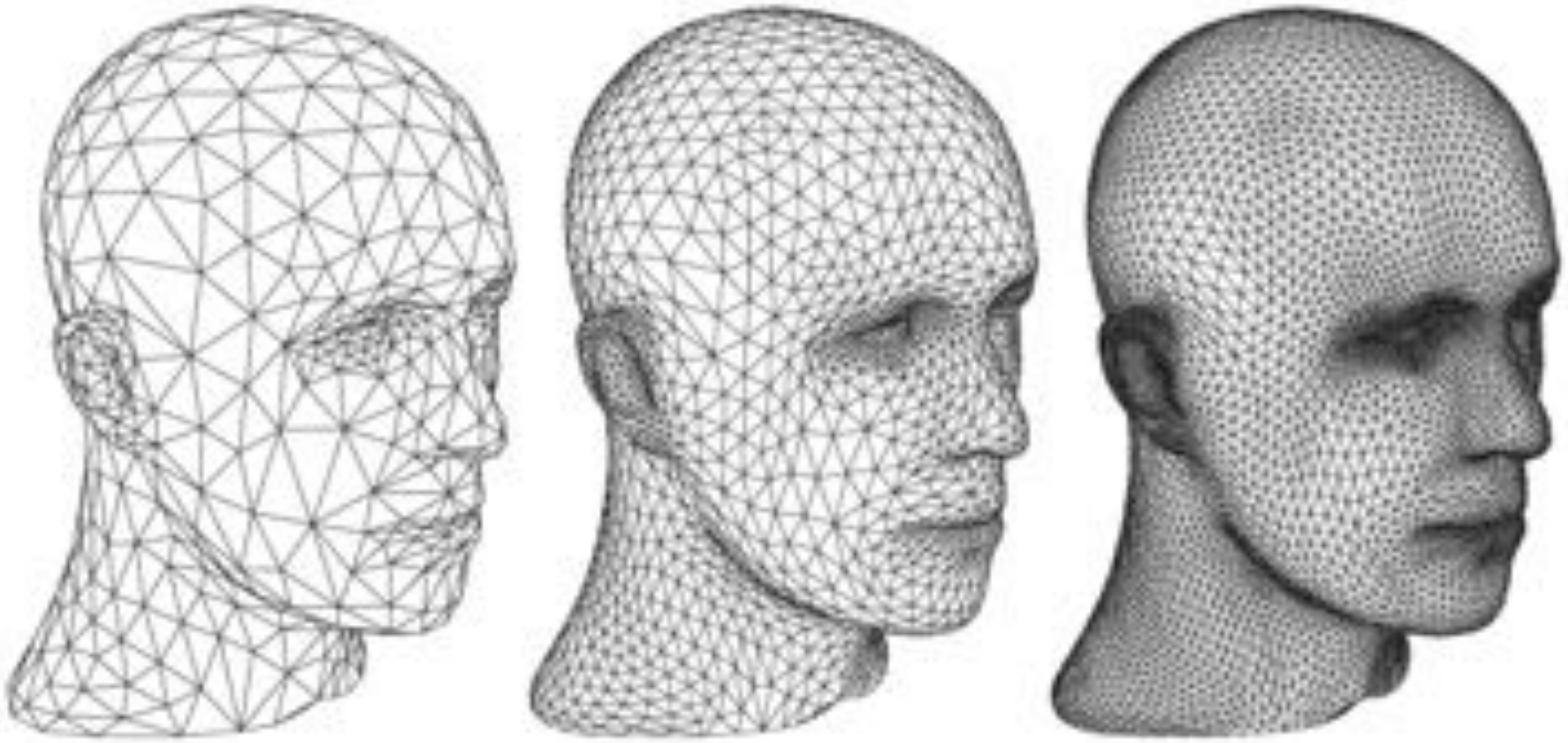
- Given samples of geometry, reconstruct surface
- What are "samples"? Many possibilities:
  - points, points & normals, ...
  - image pairs / sets (multi-view stereo)
  - line density integrals (MRI/CT scans)



- How do you get a surface? Many techniques:
  - silhouette-based (visual hull)
  - Voronoi-based (e.g., power crust)
  - PDE-based (e.g., Poisson reconstruction)
  - Radon transform / isosurfacing (marching cubes)

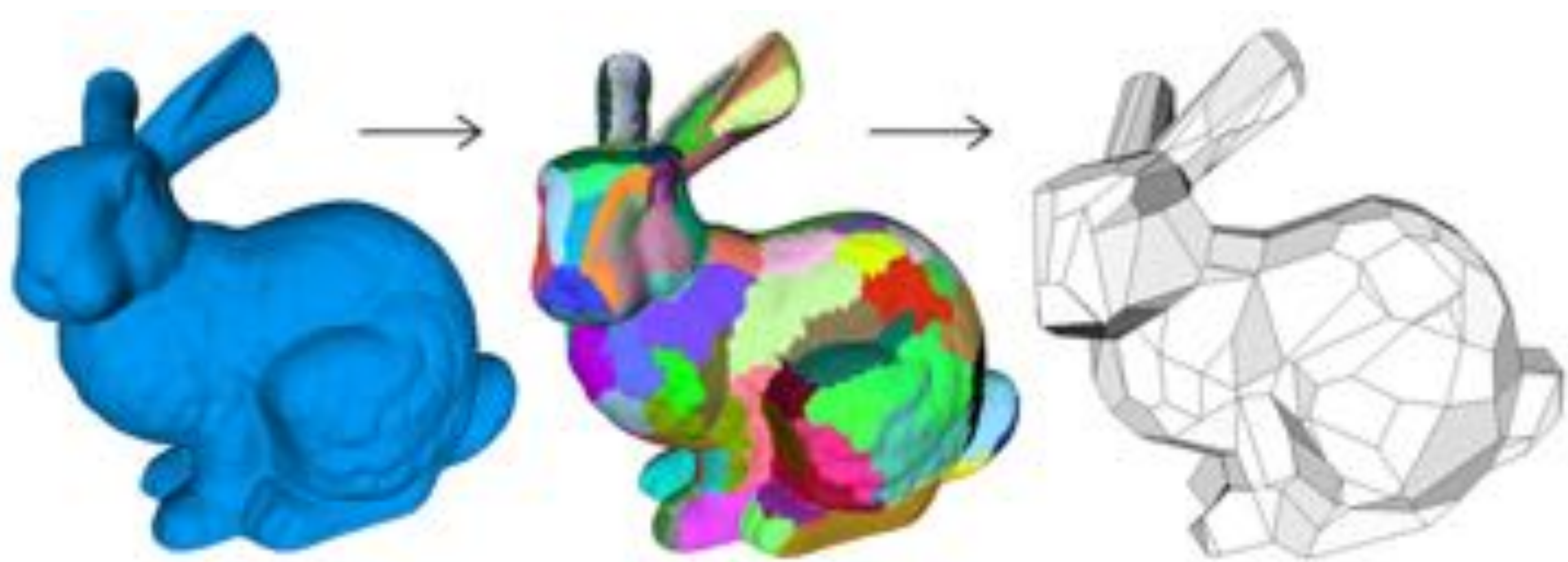
Geometry Processing: Upsampling

- Increase resolution via interpolation
- Images: e.g., bilinear, bicubic interpolation
- Polygon meshes:
  - subdivision
  - bilateral upsampling
  - ...



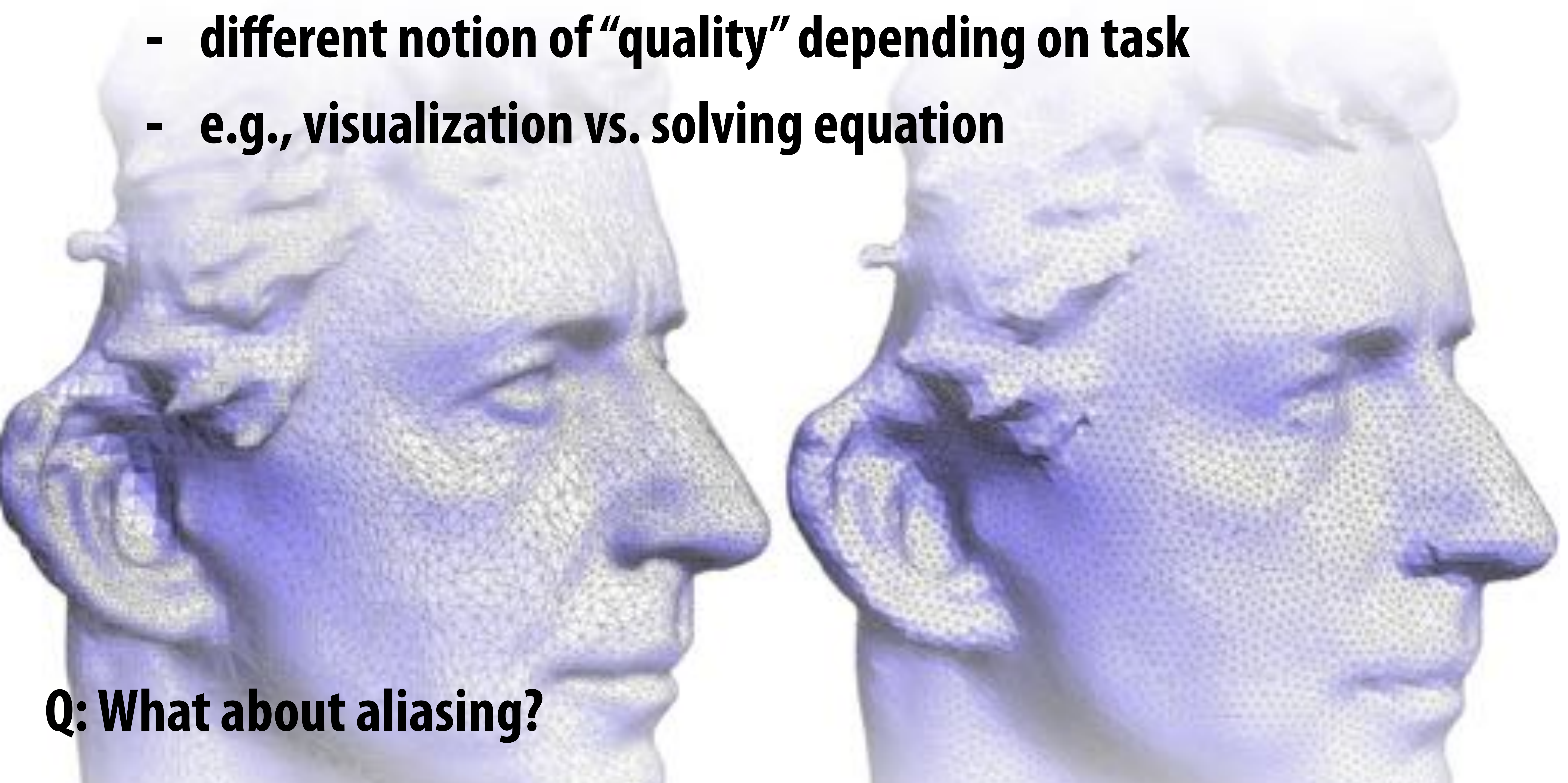
Geometry Processing: Downsampling

- Decrease resolution; try to preserve shape/appearance
- Images: nearest-neighbor, bilinear, bicubic interpolation
- Point clouds: subsampling (just take fewer points!)
- Polygon meshes:
  - iterative decimation, variational shape approximation, ...



Geometry Processing: Resampling

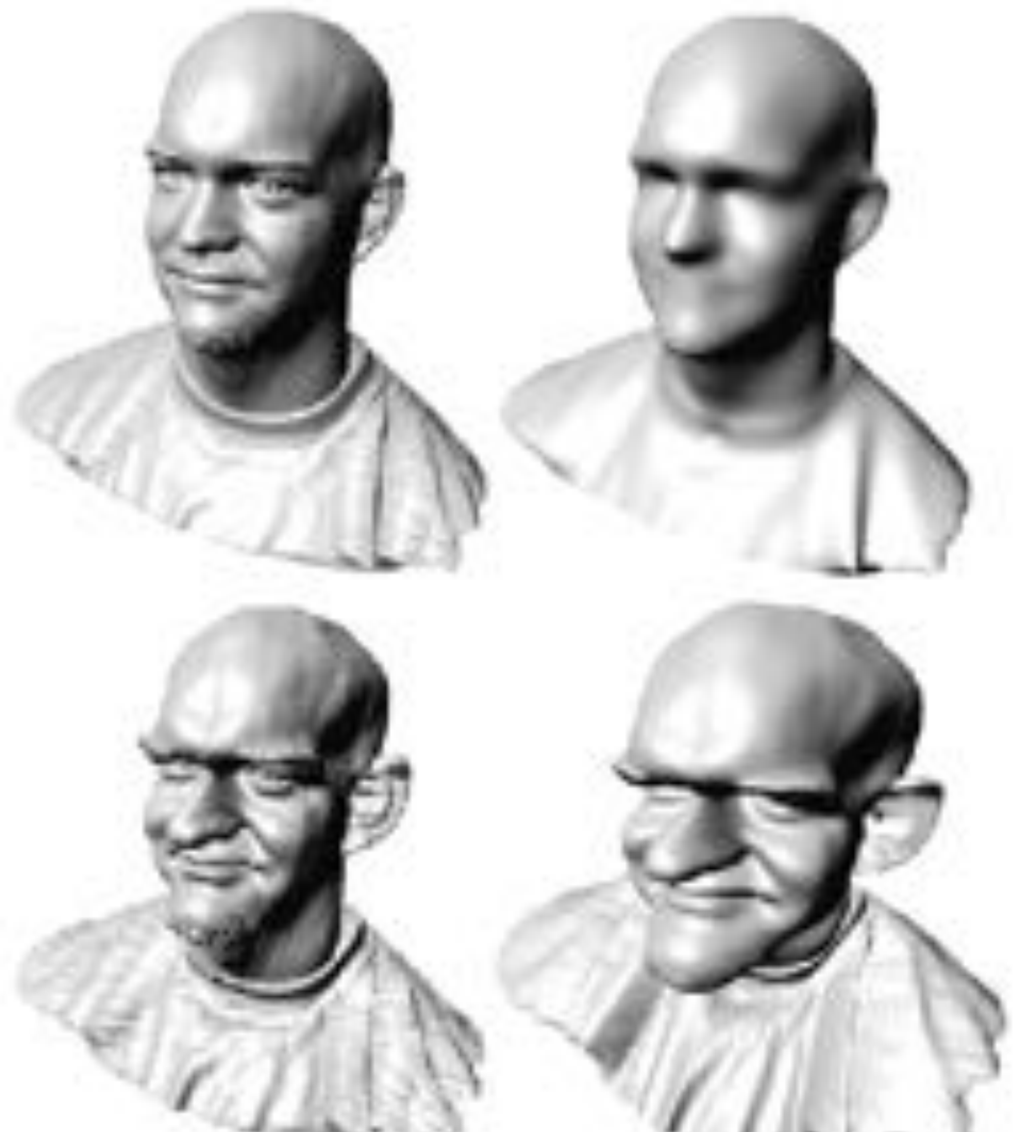
- Modify sample distribution to improve quality
- Images: not an issue! (Pixels always stored on a regular grid)
- Meshes: shape of polygons is extremely important!
  - different notion of "quality" depending on task
  - e.g., visualization vs. solving equation

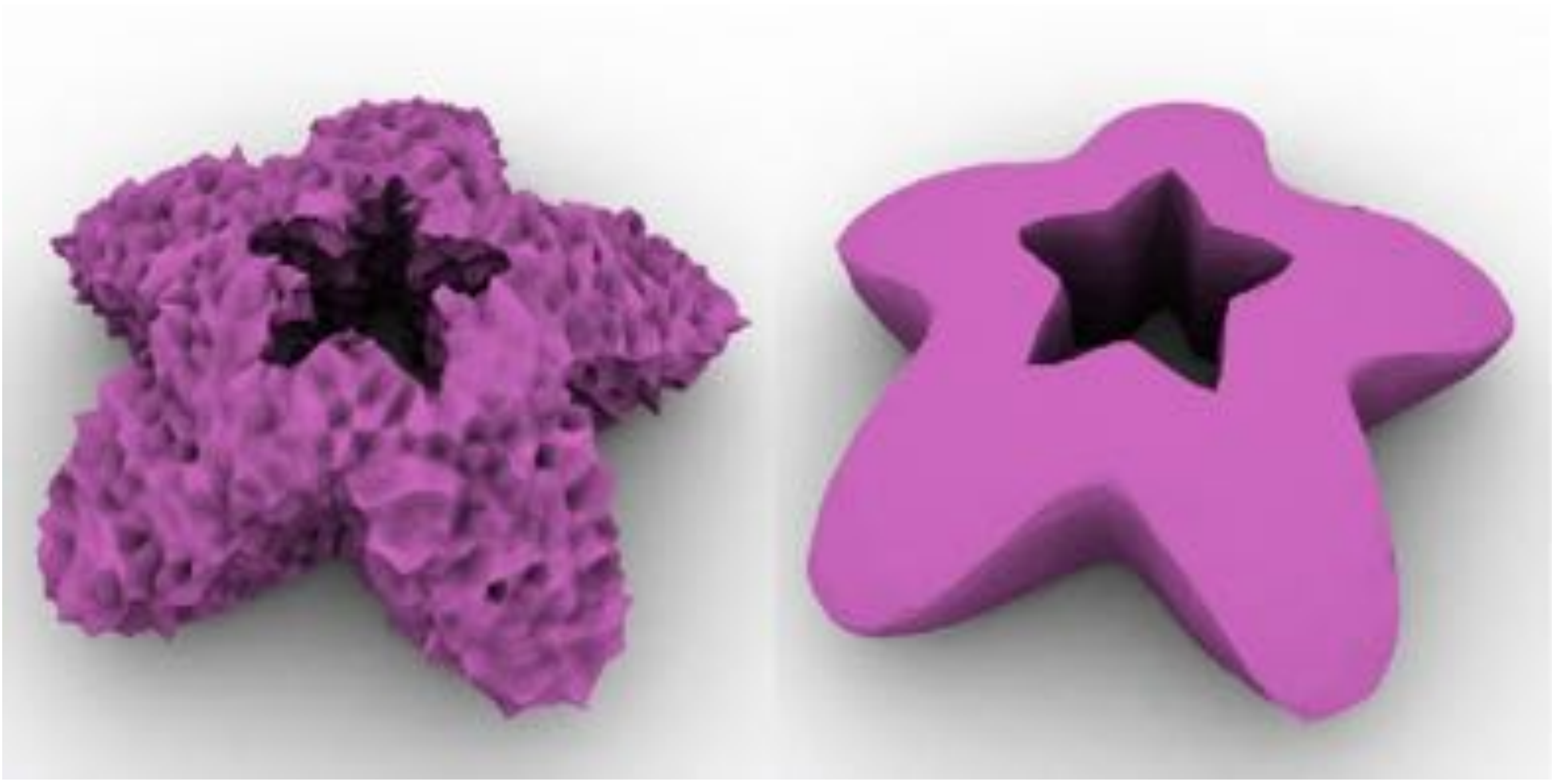


Q: What about aliasing?

Geometry Processing: Filtering

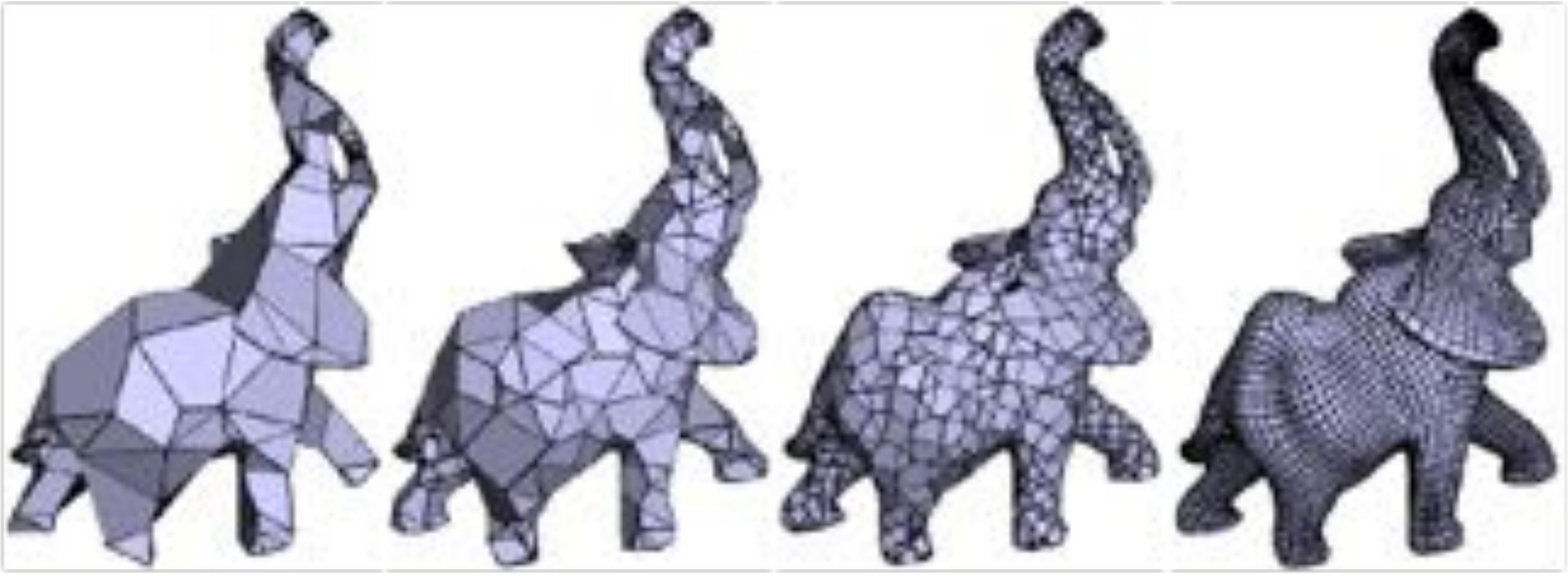
- Remove noise, or emphasize important features (e.g., edges)
- Images: blurring, bilateral filter, edge detection, ...
- Polygon meshes:
  - curvature flow
  - bilateral filter
  - spectral filter





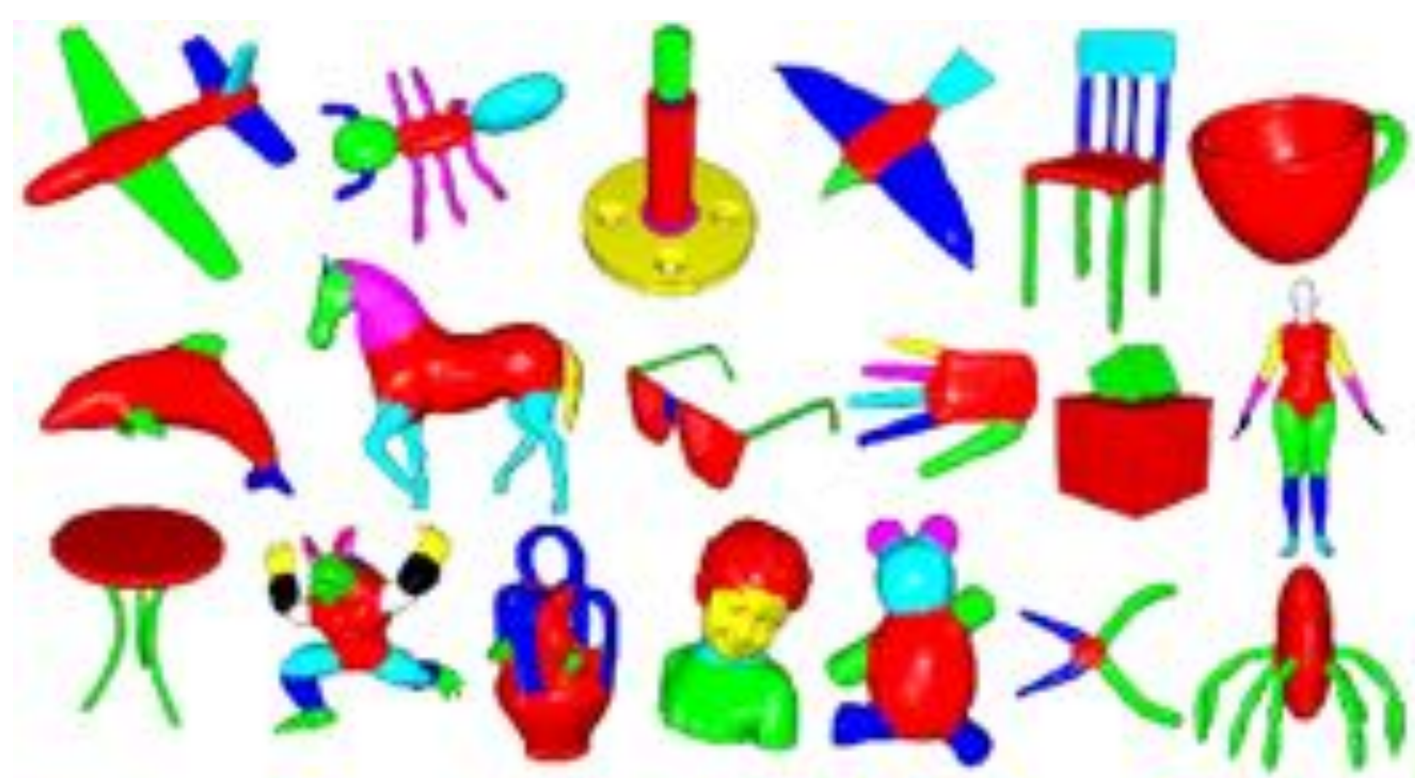
Geometry Processing: Compression

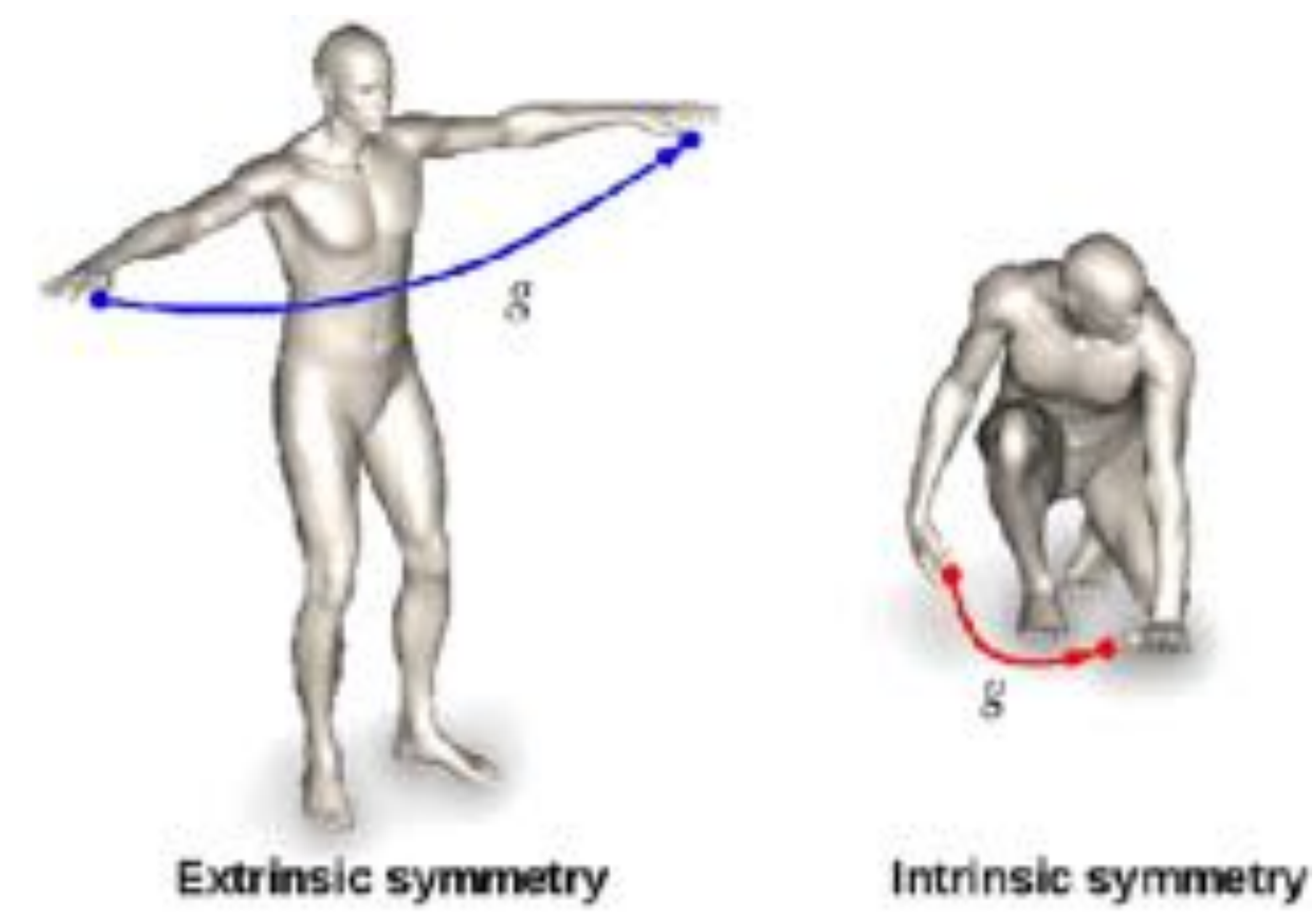
- Reduce storage size by eliminating redundant data/ approximating unimportant data
- Images:
  - run-length, Huffman coding - lossless
  - cosine/wavelet (JPEG/MPEG) - lossy
- Polygon meshes:
  - compress geometry and connectivity
  - many techniques (lossy & lossless)

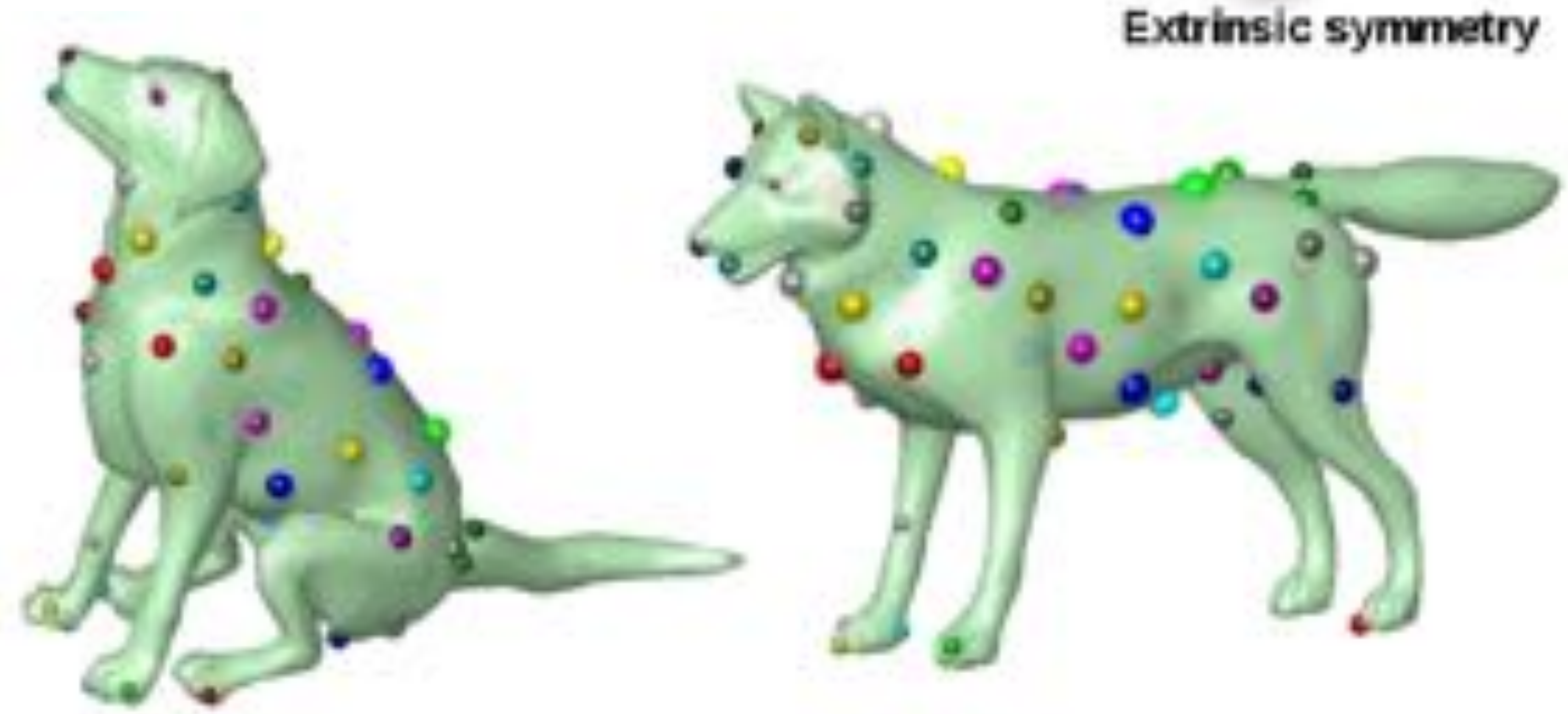


Geometry Processing: Shape Analysis

- Identify/understand important semantic features
- Images: computer vision, segmentation, face detection, ...
- Polygon meshes:
  - segmentation, correspondence, symmetry detection, ...



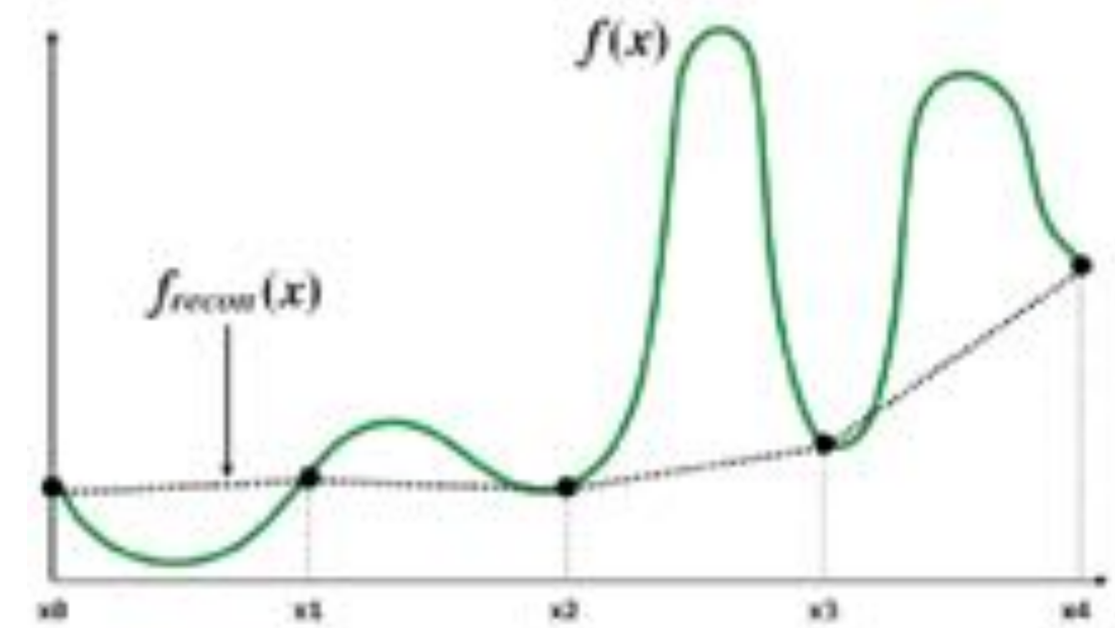




Enough overview—
Let's process some geometry!

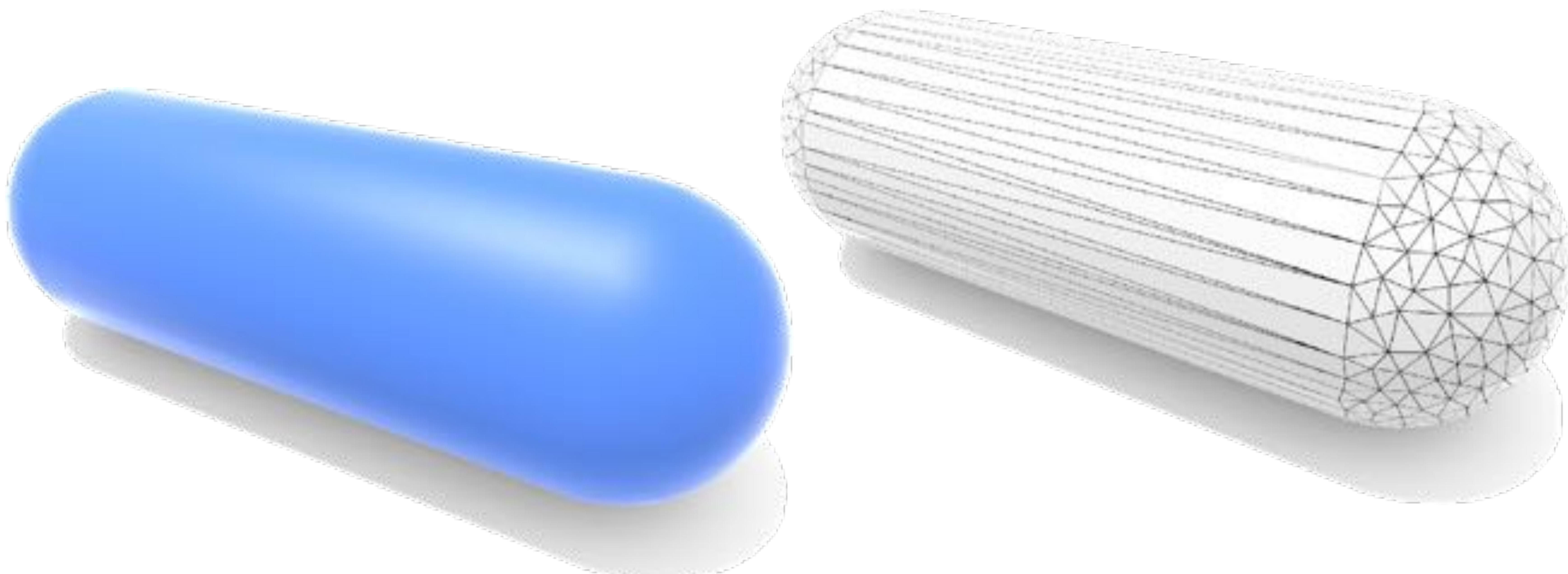
Remeshing as resampling

- Remember our discussion of aliasing
- Bad sampling makes signal appear different than it really is
- E.g., undersampled curve looks flat
- Geometry is no different!
    - undersampling destroys features
    - oversampling bad for performance



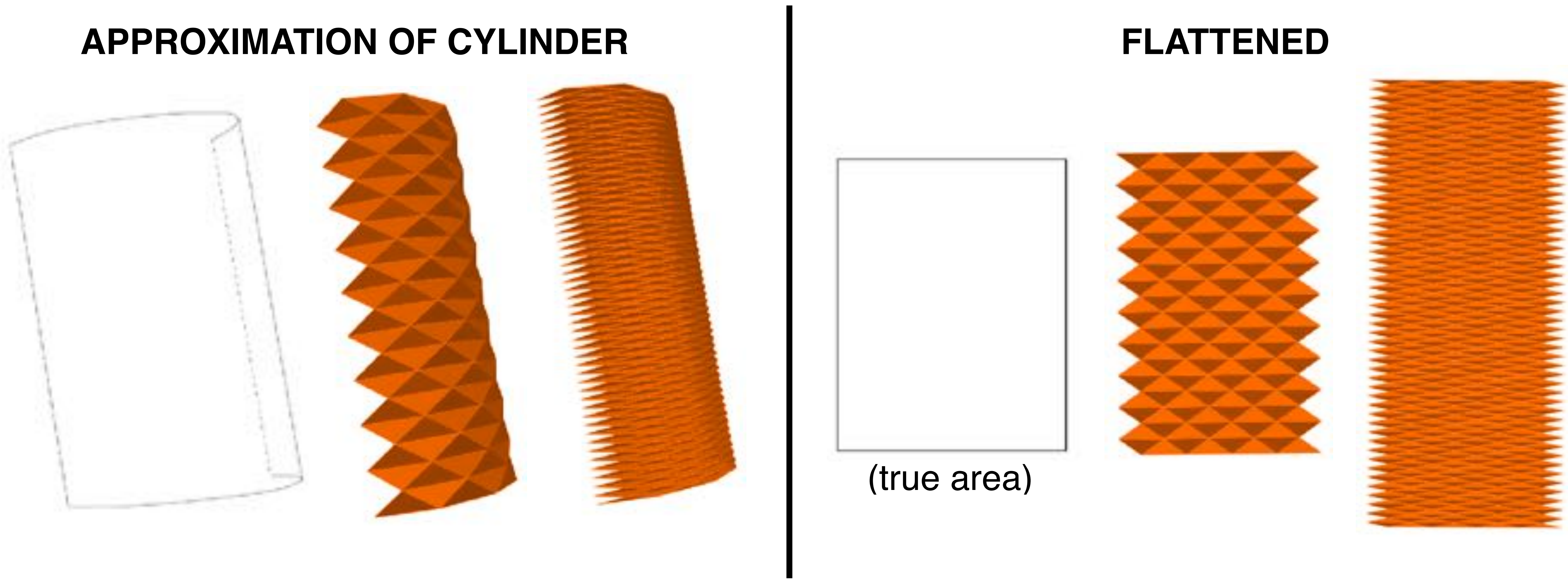
What makes a "good" mesh?

- One idea: good approximation of original shape!
- Keep only elements that contribute information about shape
- Add additional information where, e.g., curvature is large



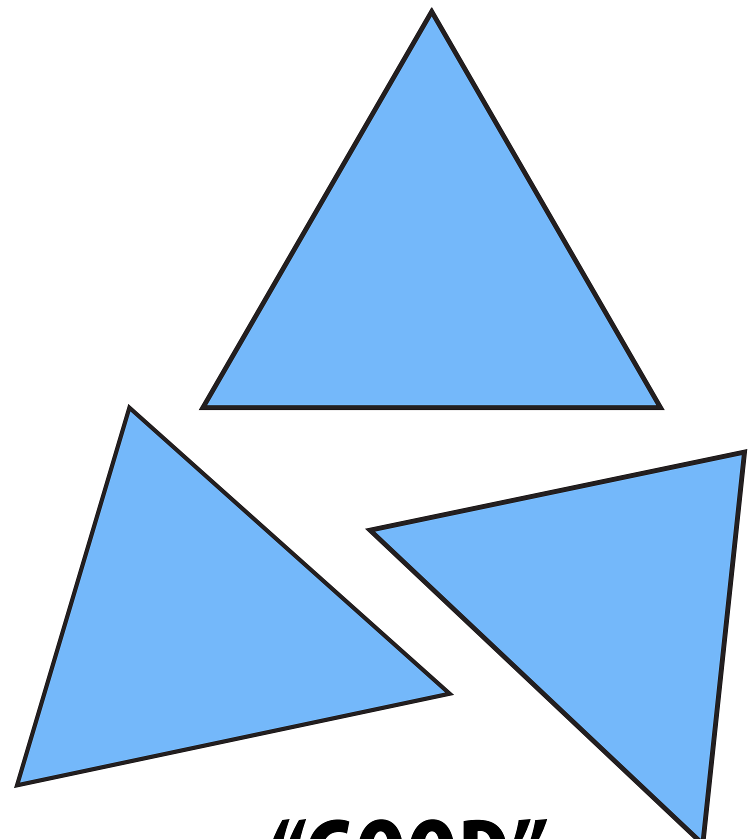
Approximation of position is not enough!

- Just because the vertices of a mesh are very close to the surface it approximates does not mean it's a good approximation!
- Need to consider other factors, e.g., close approximation of surface normals
- Otherwise, can have wrong appearance, wrong area, wrong…

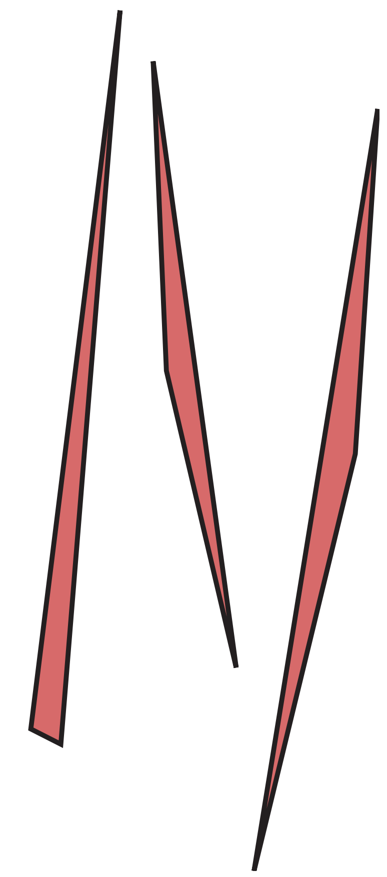


What else makes a "good" triangle mesh?

- Another rule of thumb: triangle



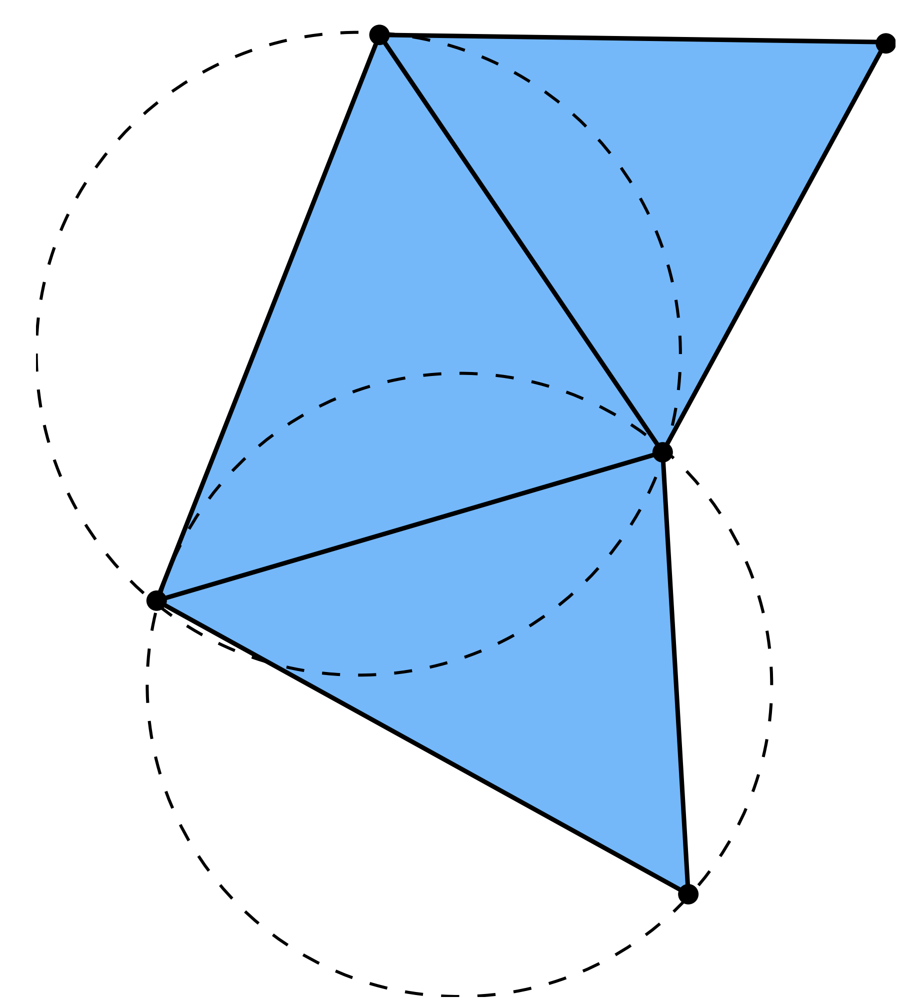
"GOOD"



"BAD"

(pronunciation)

DELAUNAY

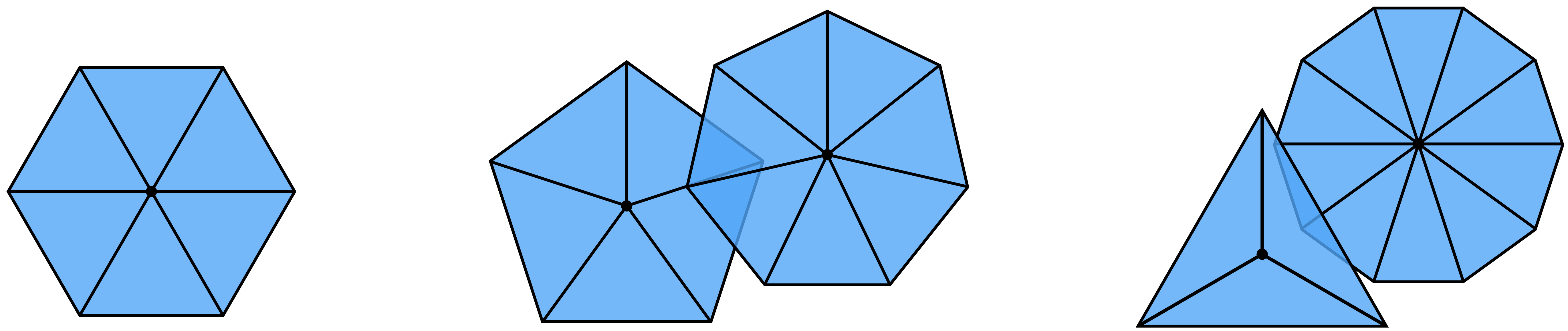


- E.g., all angles close to 60 degrees
- More sophisticated condition: Delaunay
- Can help w/ numerical accuracy/stability
- Tradeoffs w/ good geometric approximation*
  - e.g., long & skinny might be "more efficient"

*See Shewchuk, "What is a Good Linear Element"

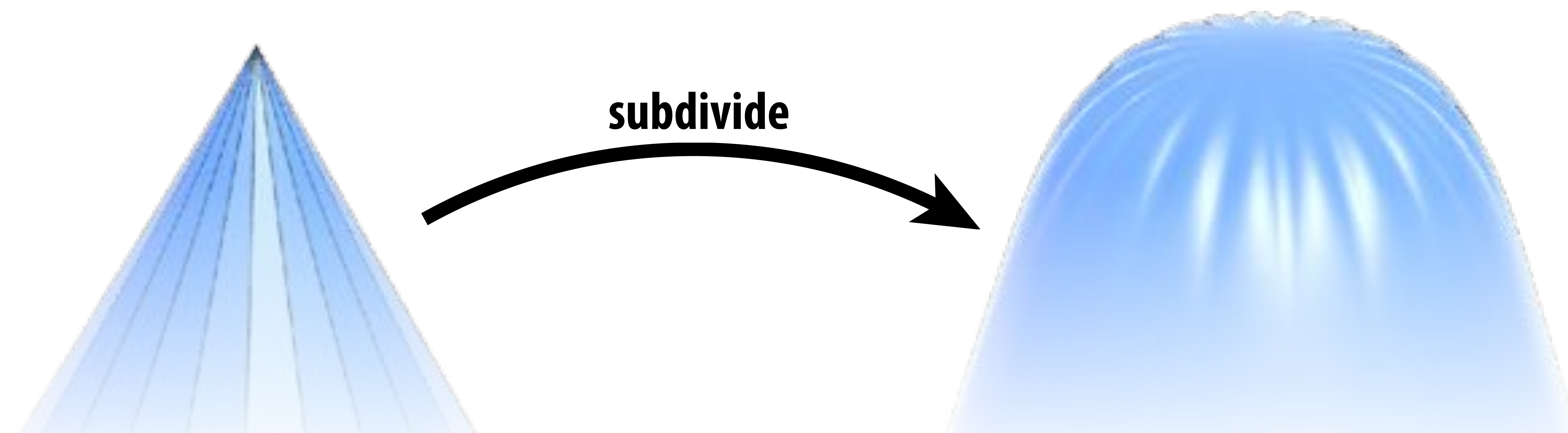
What else constitutes a good mesh?

- Another rule of thumb: regular vertex degree
- E.g., valence 6 for triangle meshes (equilateral)



"GOOD" "OK" "BAD"

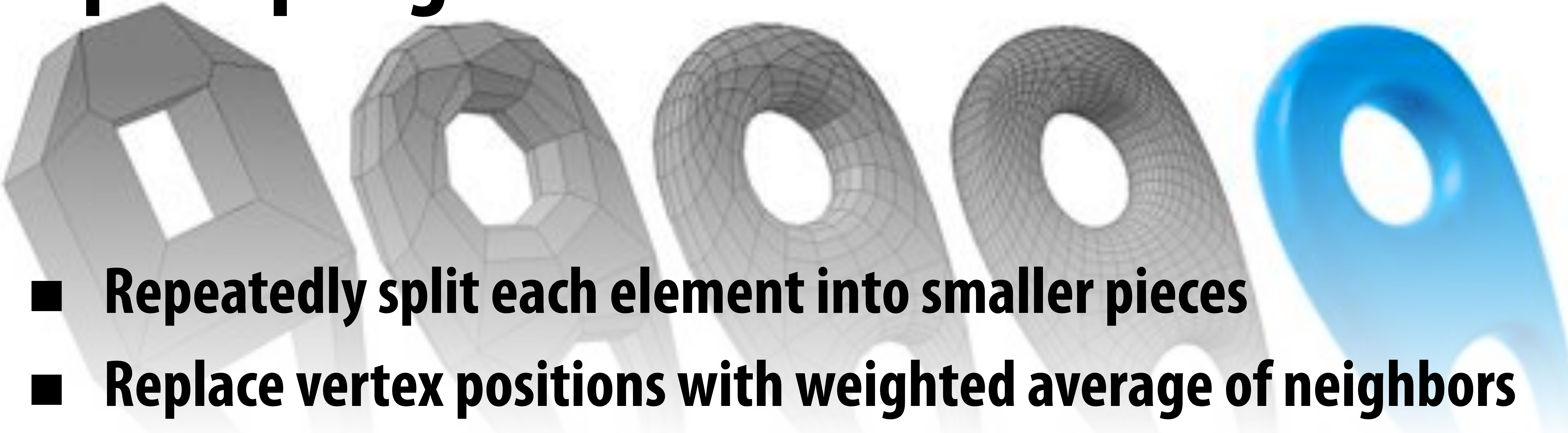
- Why? Better polygon shape, important for (e.g.) subdivision:



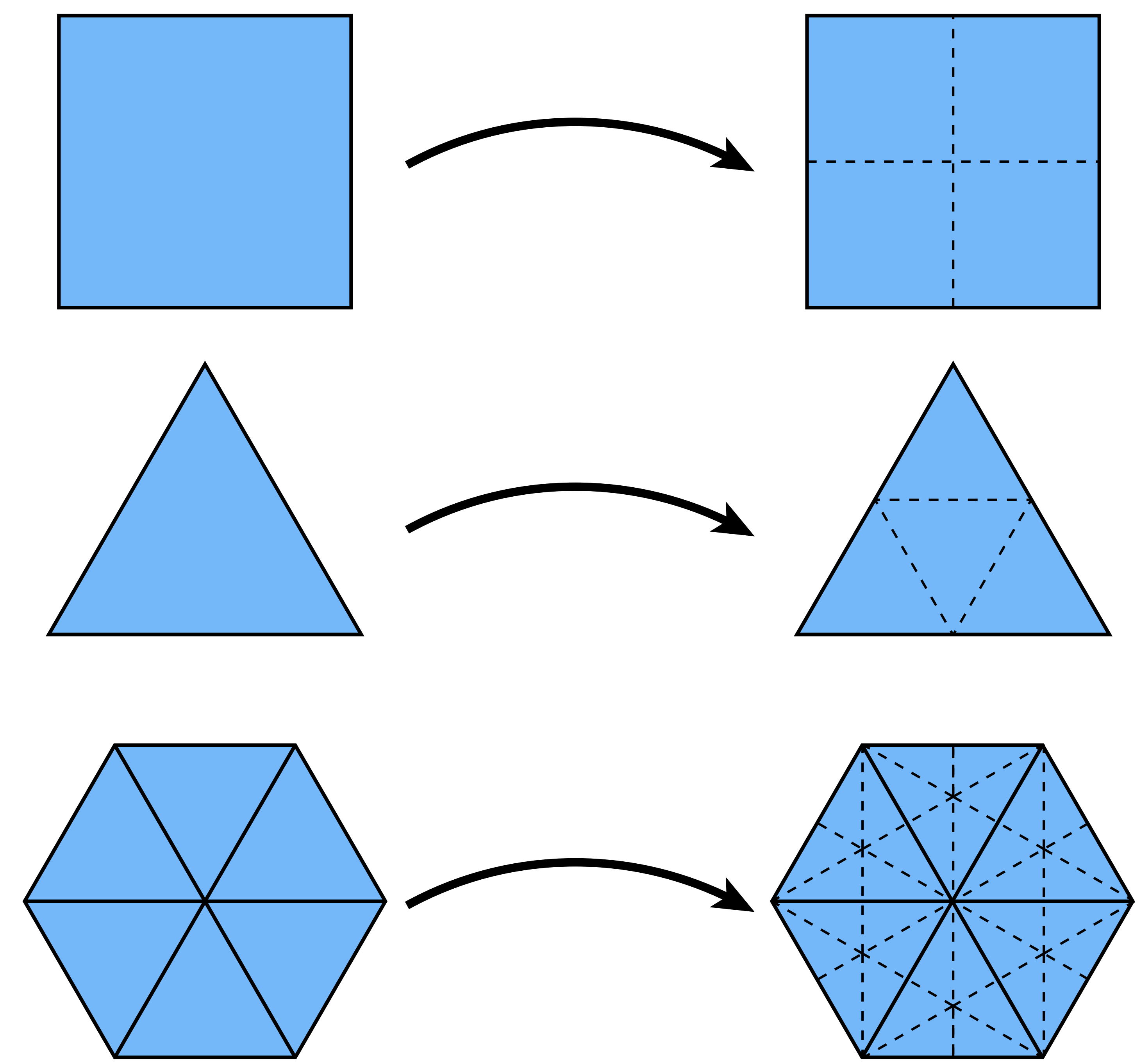
- FACT: Can't have perfect valence everywhere! (except on torus)

How do we upsample a mesh?

Upsampling via Subdivision

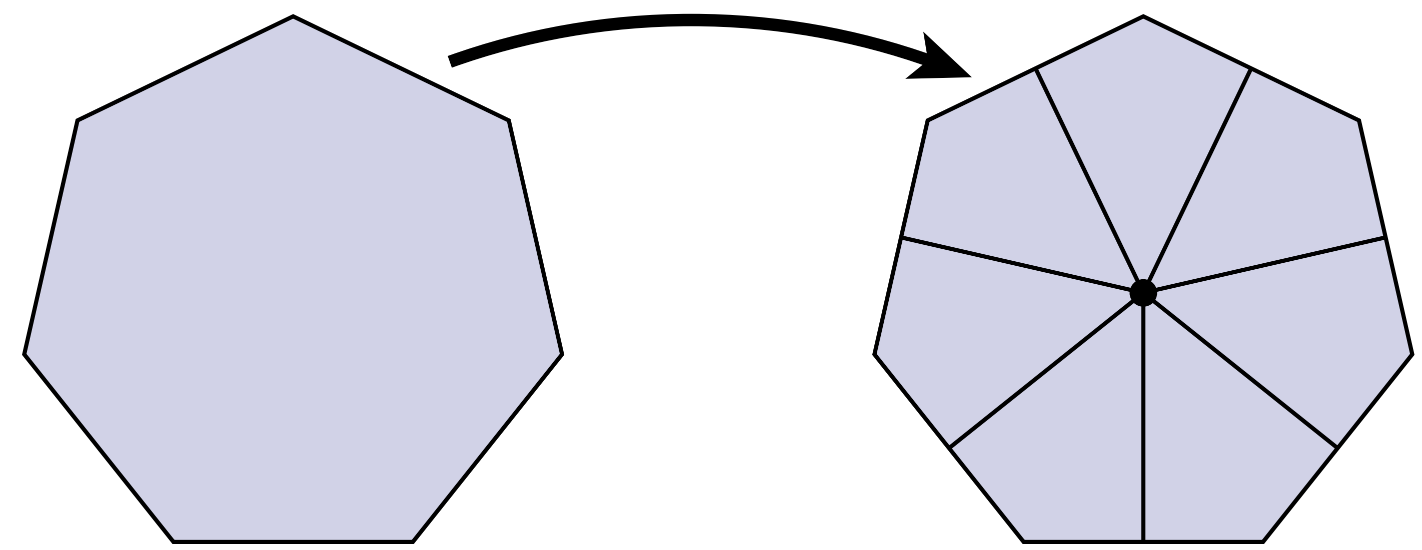


- Repeatedly split each element into smaller pieces
- Replace vertex positions with weighted average of neighbors
- Main considerations:
  - interpolating vs. approximating
  - limit surface continuity (C^1, C^2, ...)
  - behavior at irregular vertices
- Many options:
  - Quad: Catmull-Clark
  - Triangle: Loop, Butterfly, Sqrt(3)



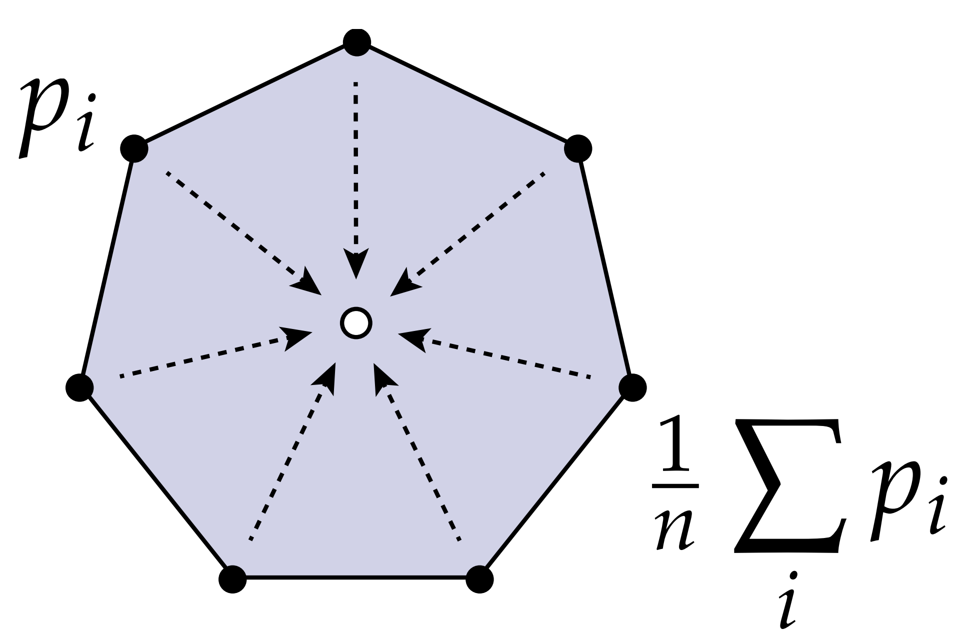
Catmull-Clark Subdivision

- **Step 0: split every polygon (any # of sides) into quadrilaterals:**

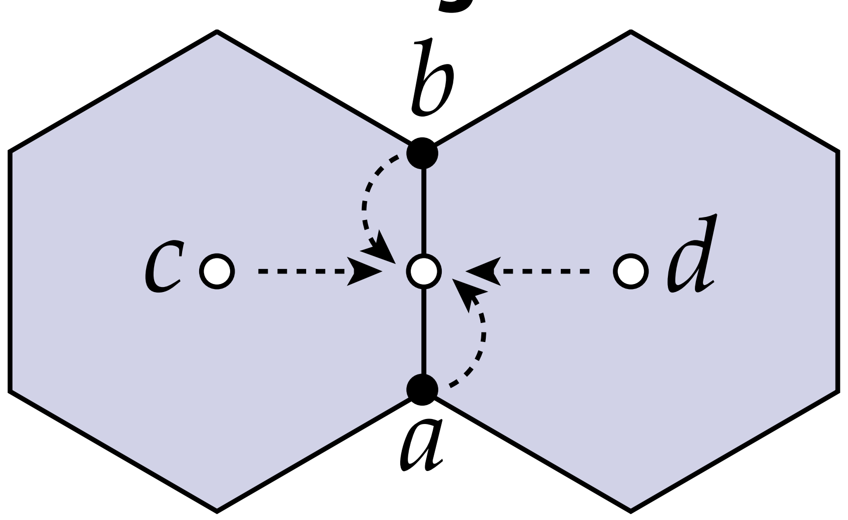


- **New vertex positions are weighted combination of old ones:**

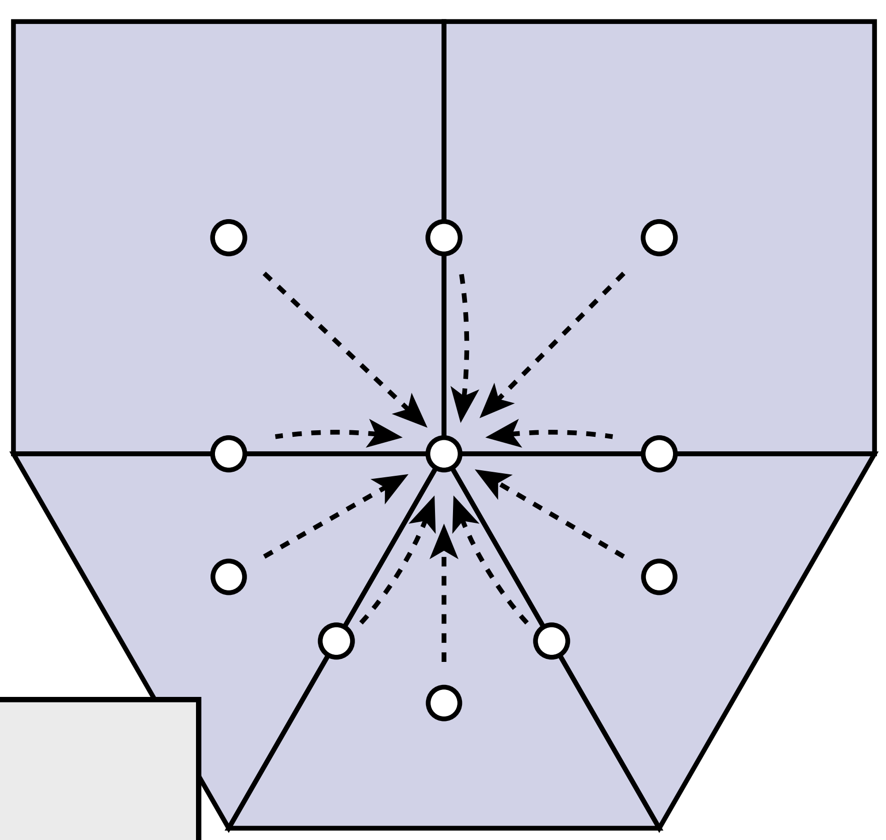
STEP 1: Face coords



STEP 2: Edge coords



STEP 3: Vertex coords

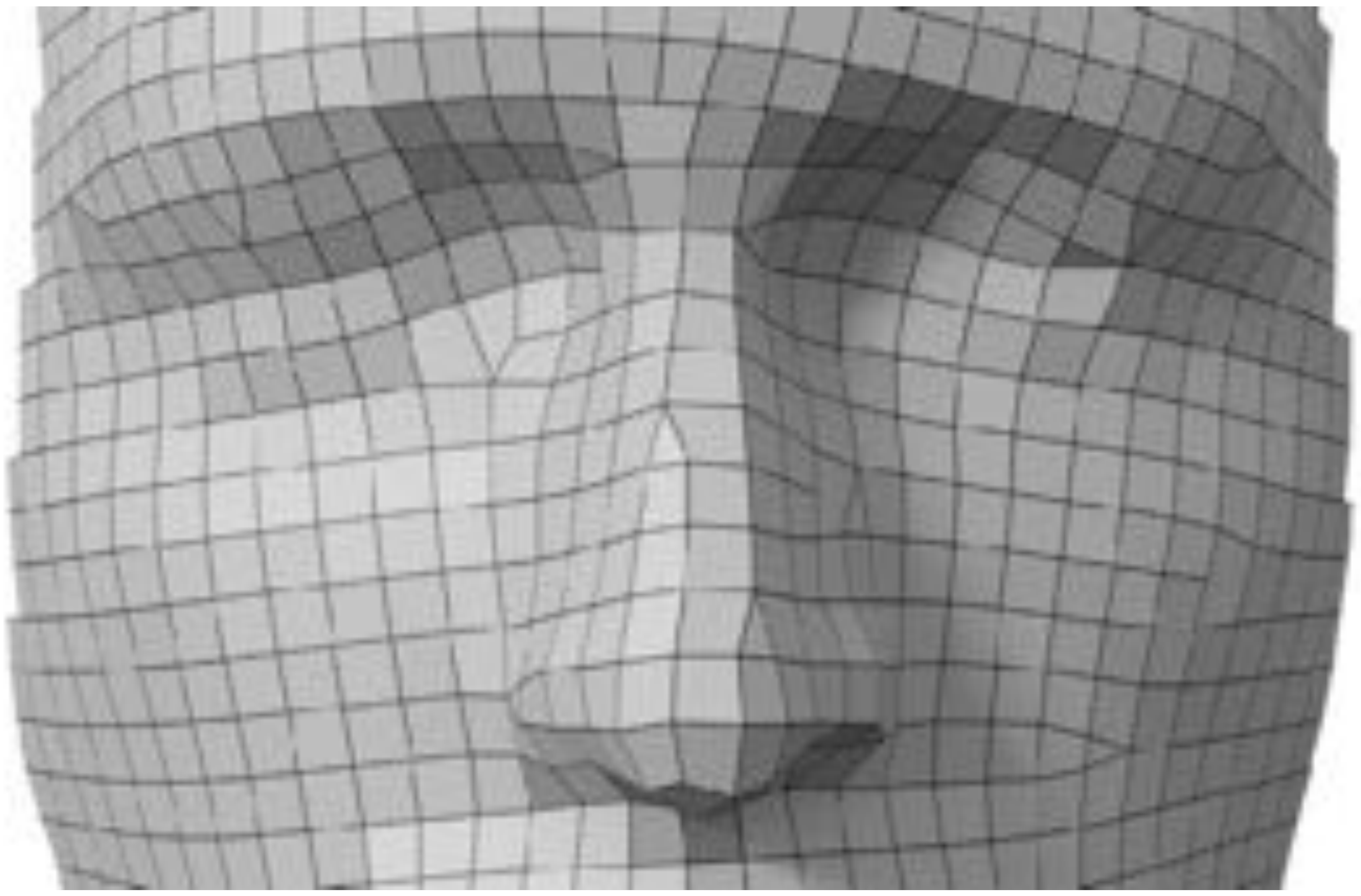


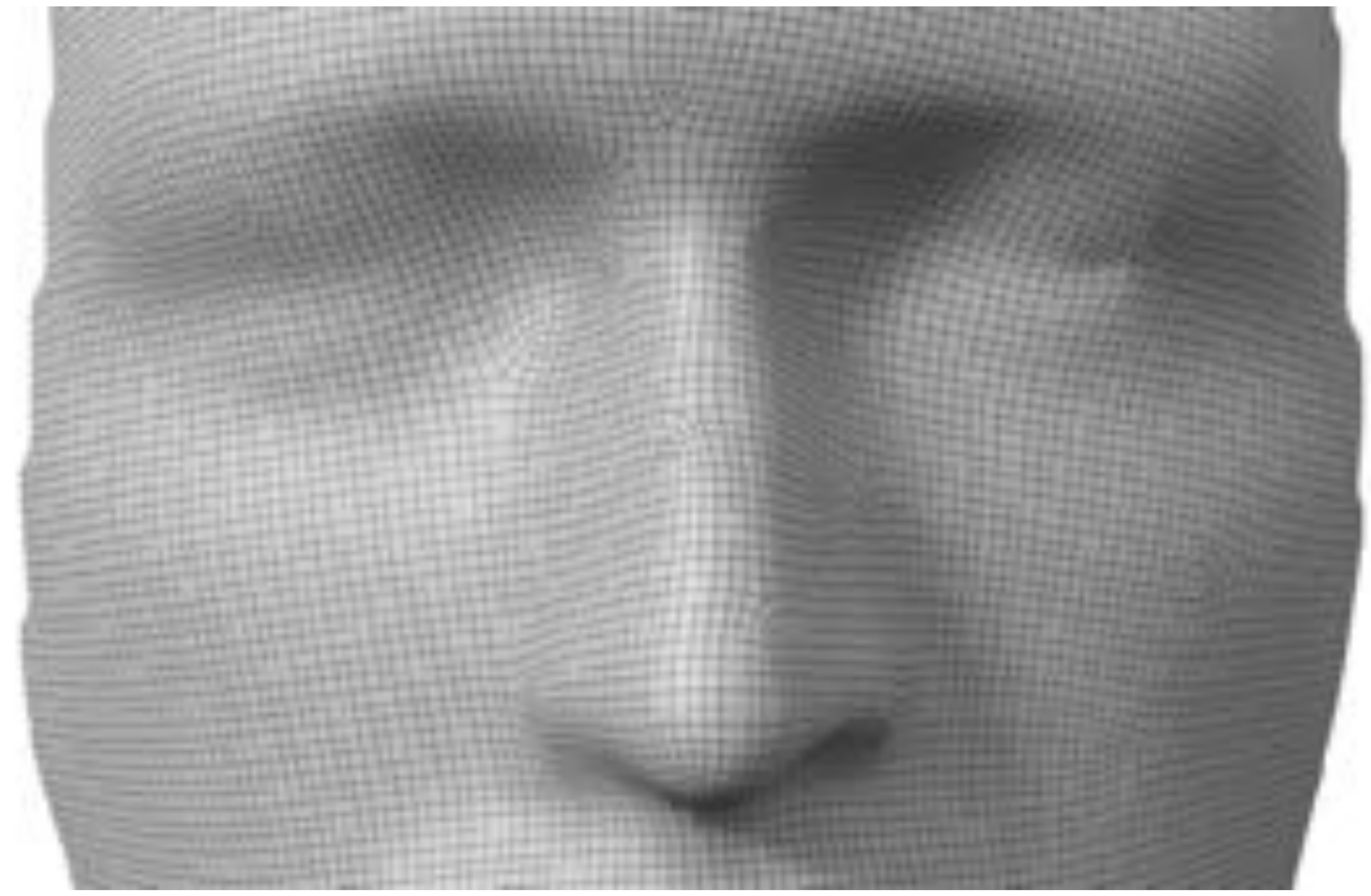
New vertex coords:

$$\frac{Q + 2R + (n-3)S}{n}$$

n – **vertex degree**
Q – **average of face coords around vertex**
R – **average of edge coords around vertex**
S – **original vertex position**

Catmull-Clark on quad mesh

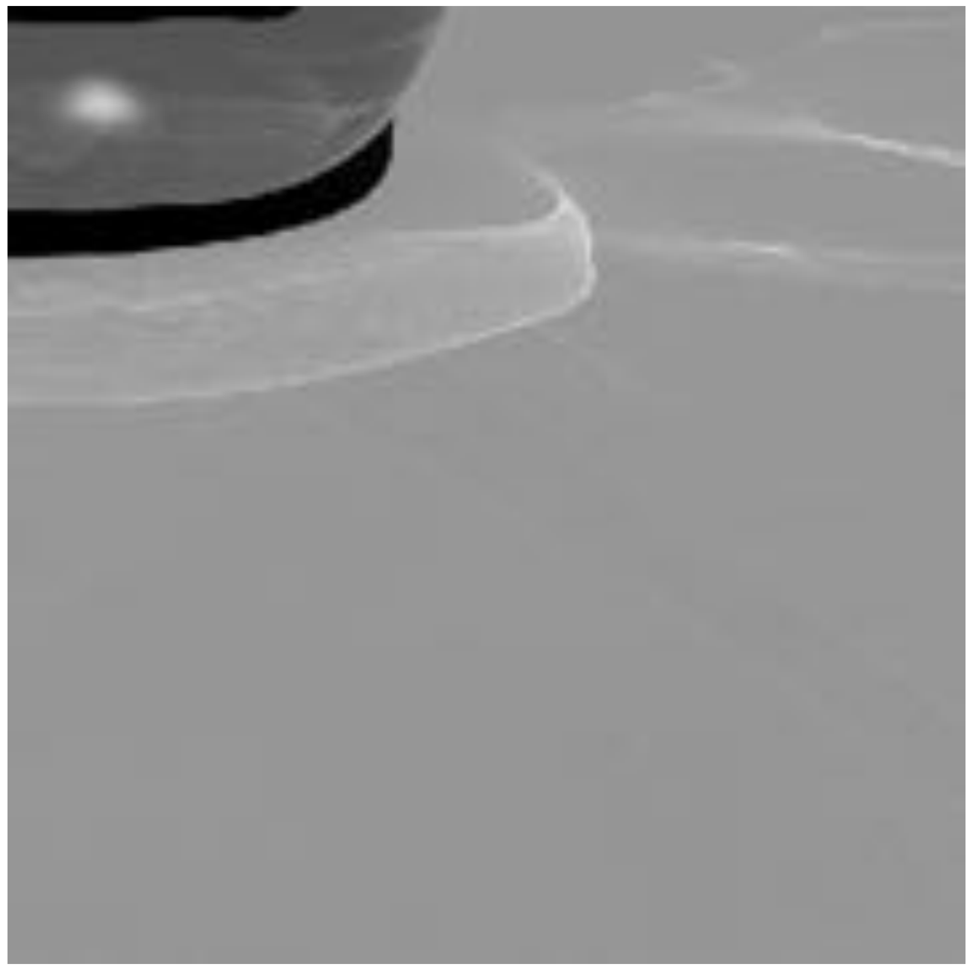




(very few irregular vertices)

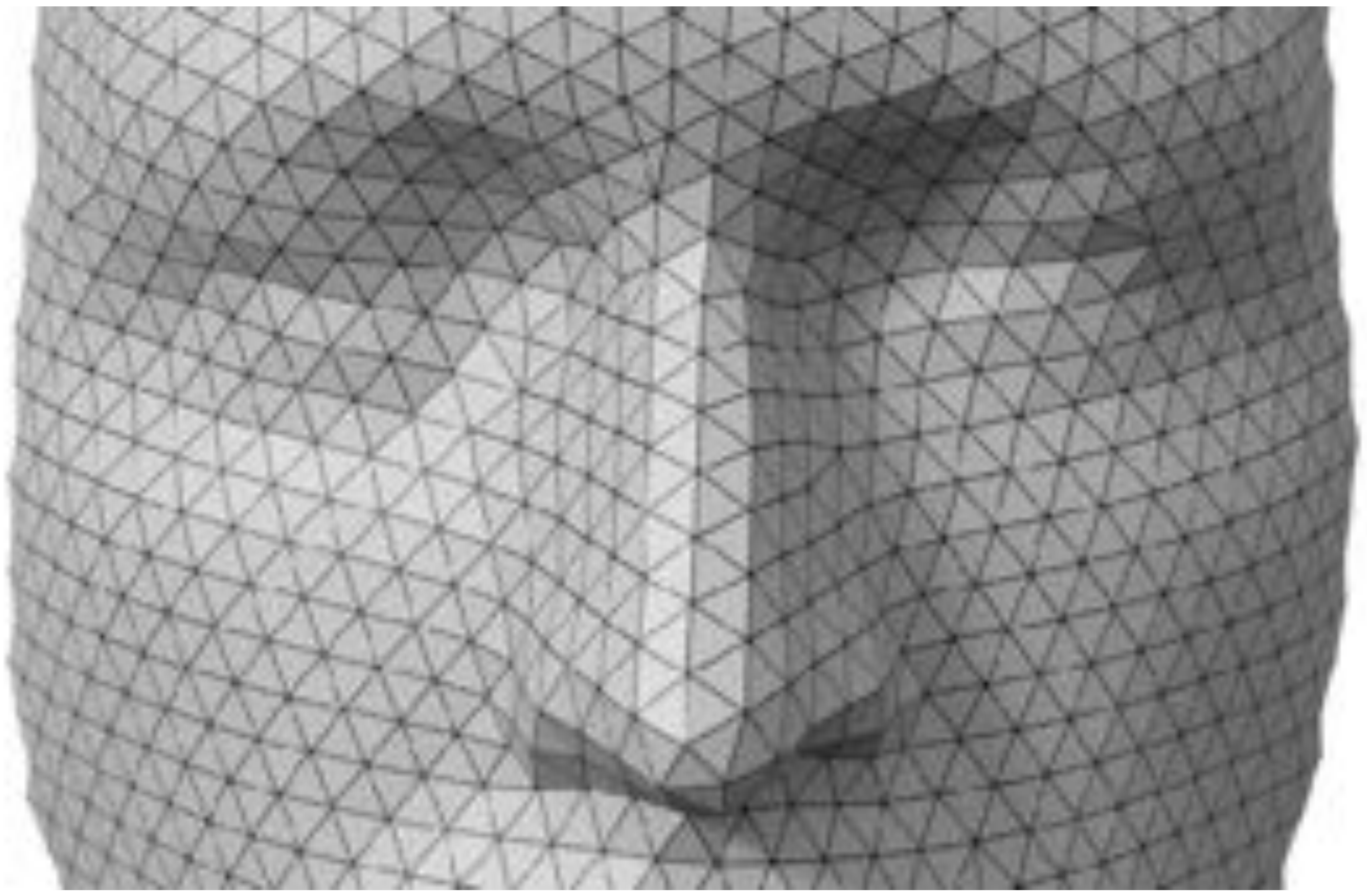
Good normal approximation almost everywhere:

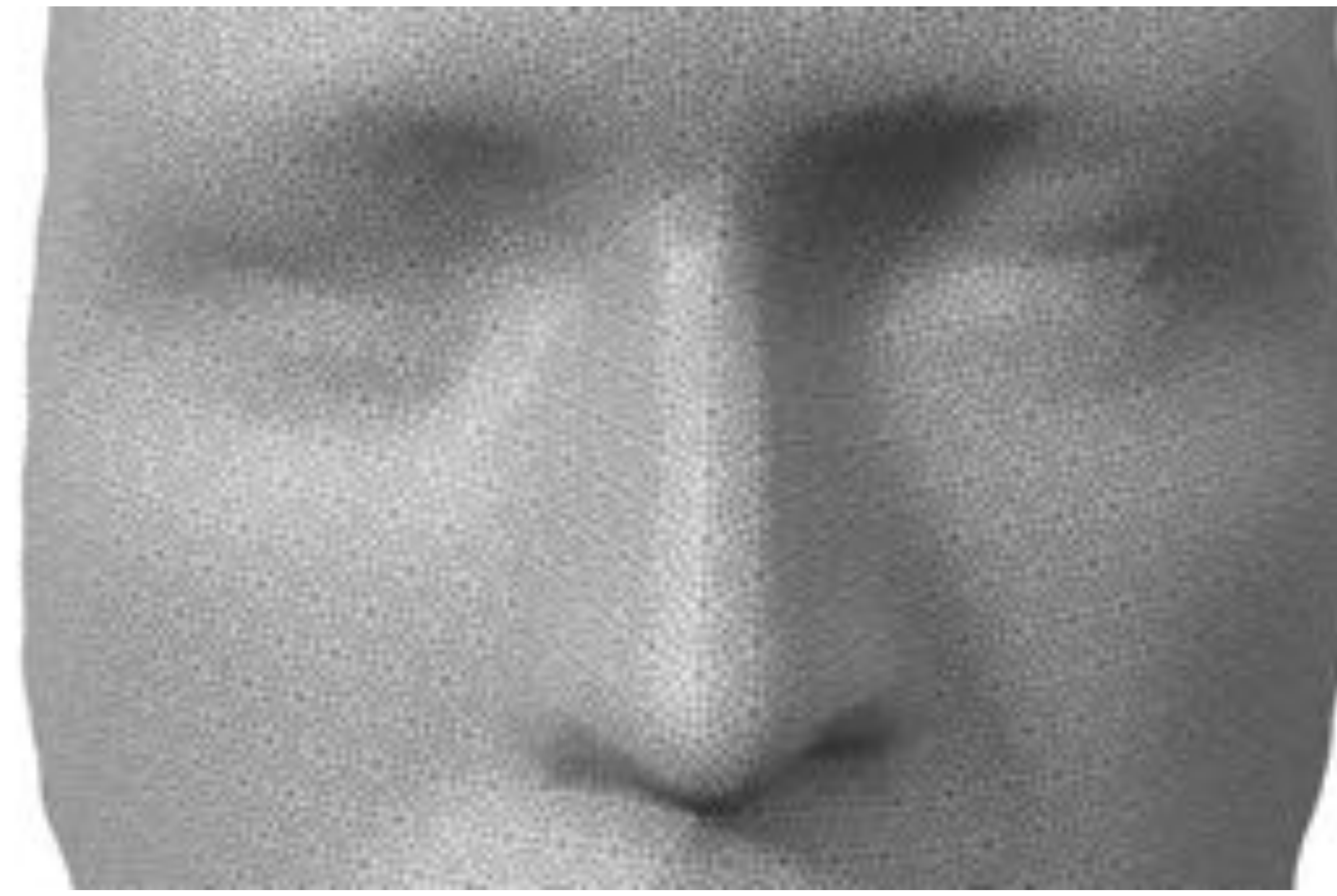
smooth
reflection lines



smooth
caustics

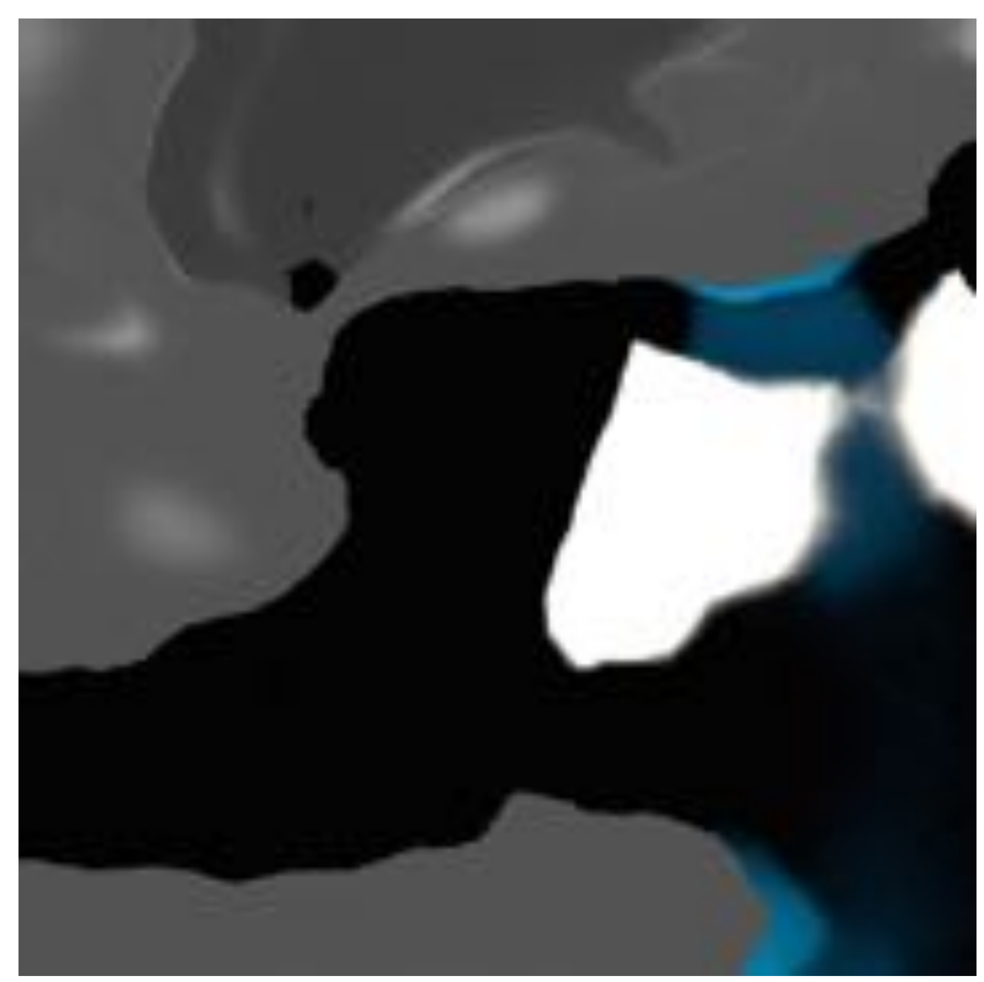
Catmull-Clark on triangle mesh



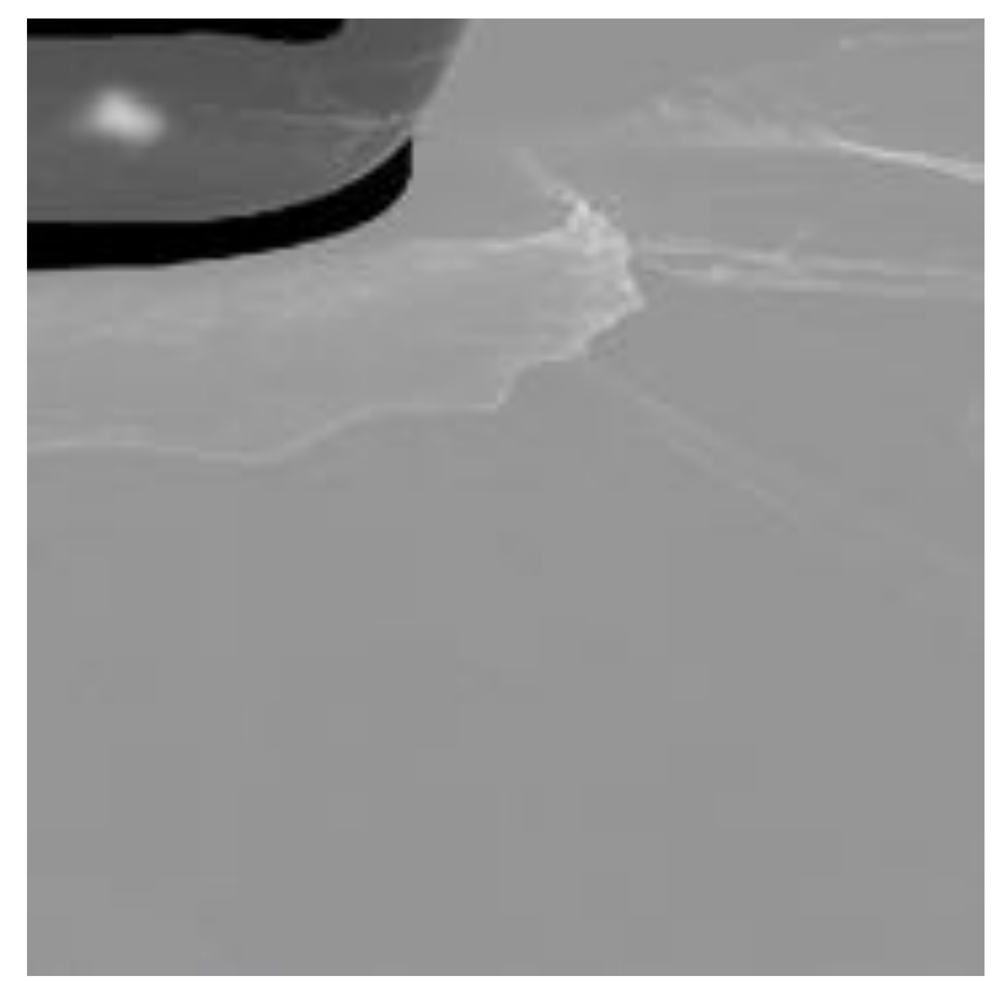


(huge number of irregular vertices!)

Poor normal approximation almost everywhere:



jagged reflection lines

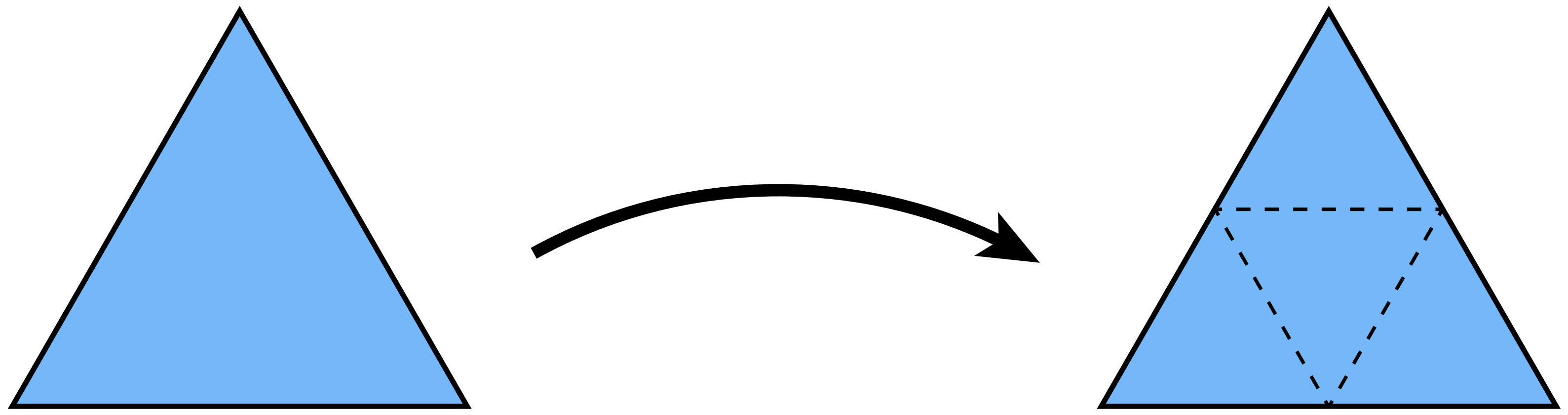


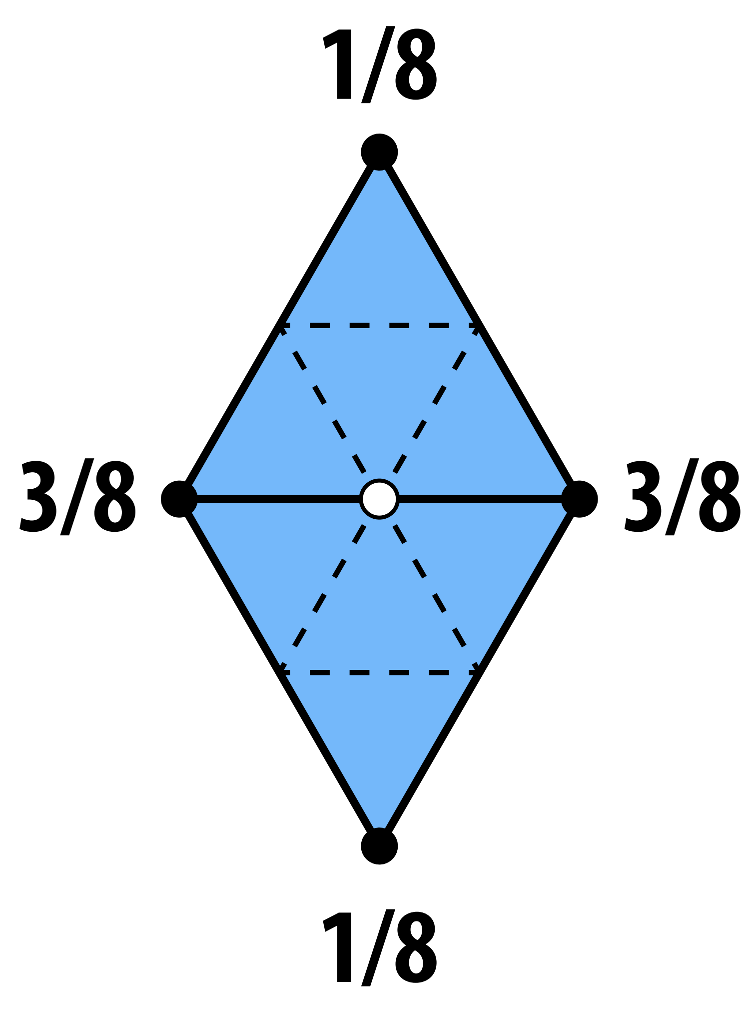
jagged caustics

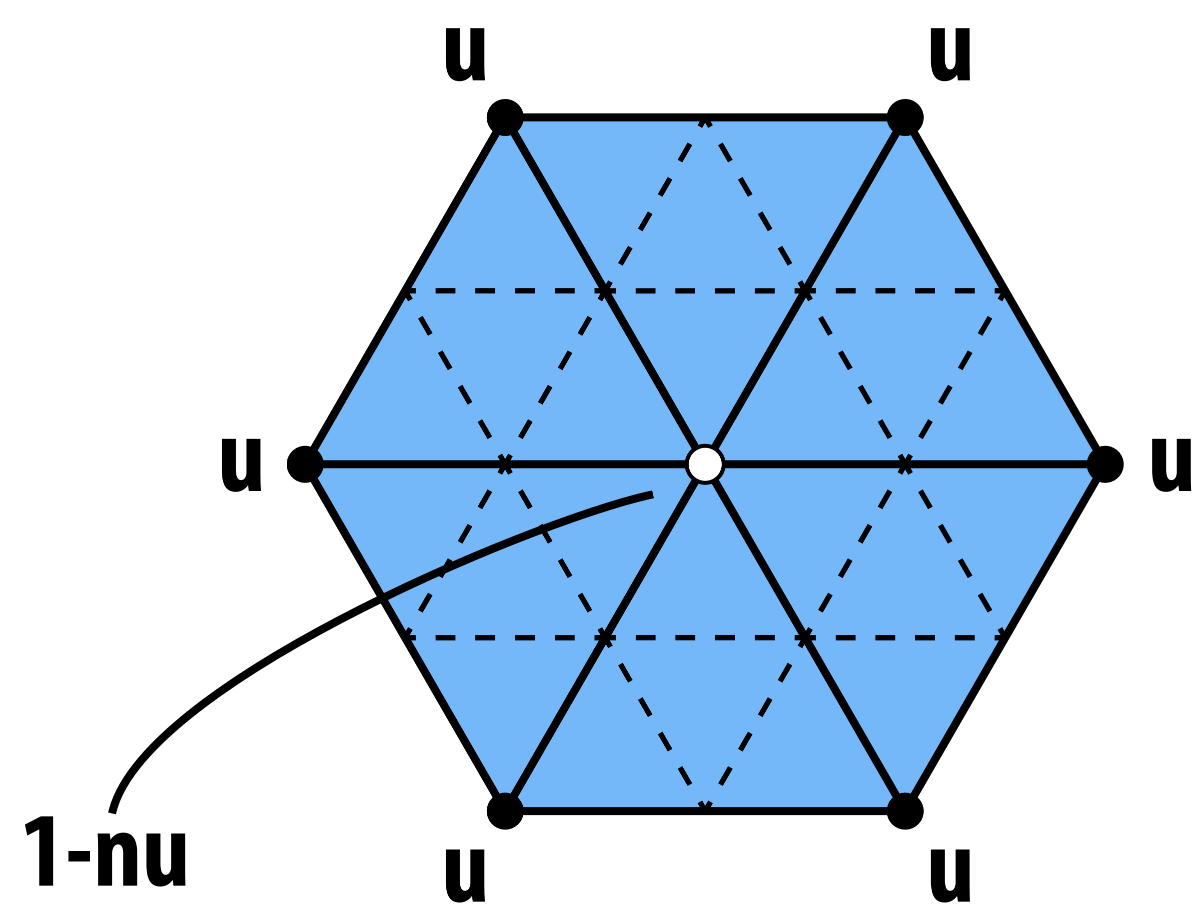
ALIASING!

Loop Subdivision

- Alternative subdivision scheme for triangle meshes
- Curvature is continuous away from irregular vertices ("C^2")
- Algorithm:
  - Split each triangle into four
  - Assign new vertex positions according to weights:





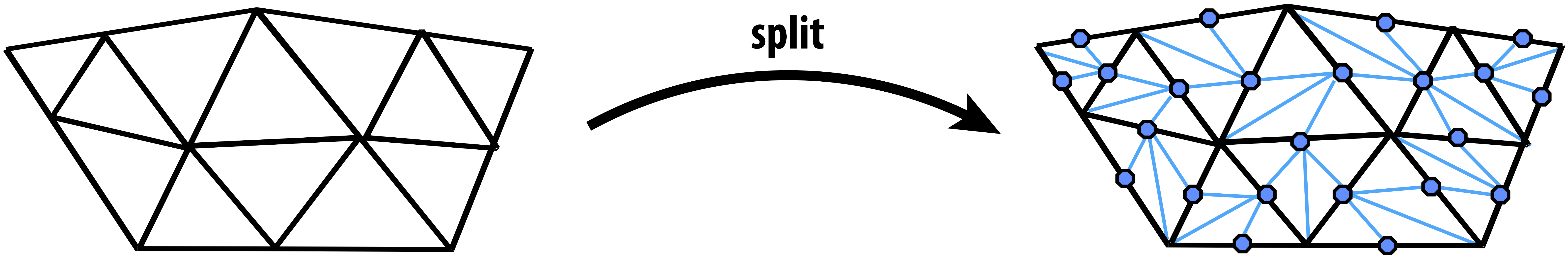


n: vertex degree

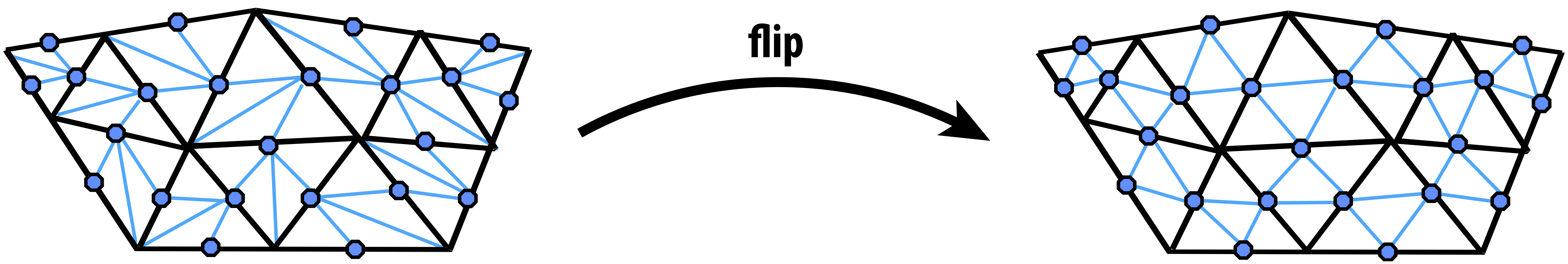
u: 3/16 if n=3, 3/(8n) otherwise

Loop Subdivision via Edge Operations

- **First, split edges of original mesh in any order:**



- **Next, flip new edges that touch a new & old vertex:**



(Don't forget to update vertex positions!)

Images cribbed from Denis Zorin.

What if we want fewer triangles?

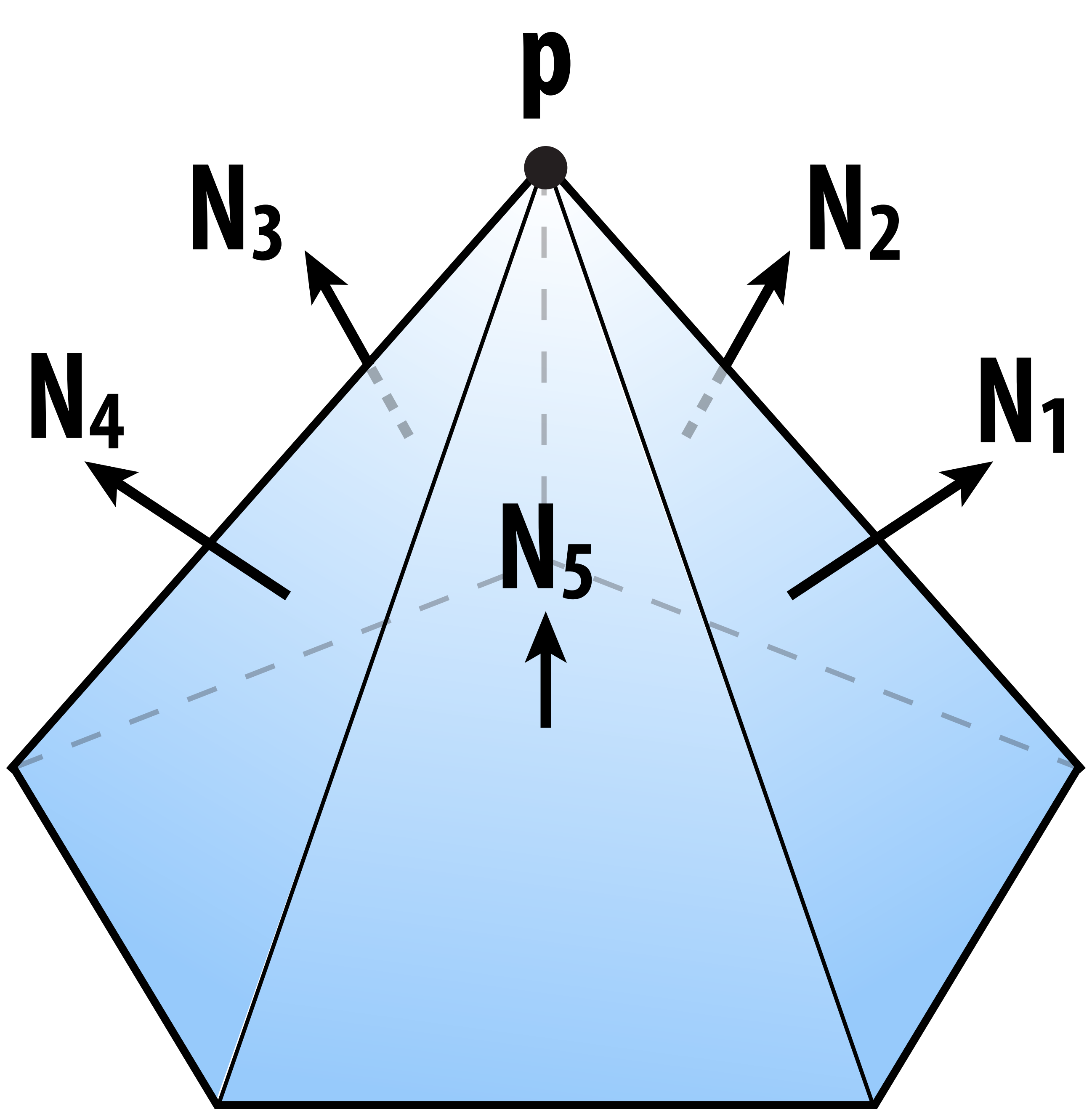
Simplification via Edge Collapse

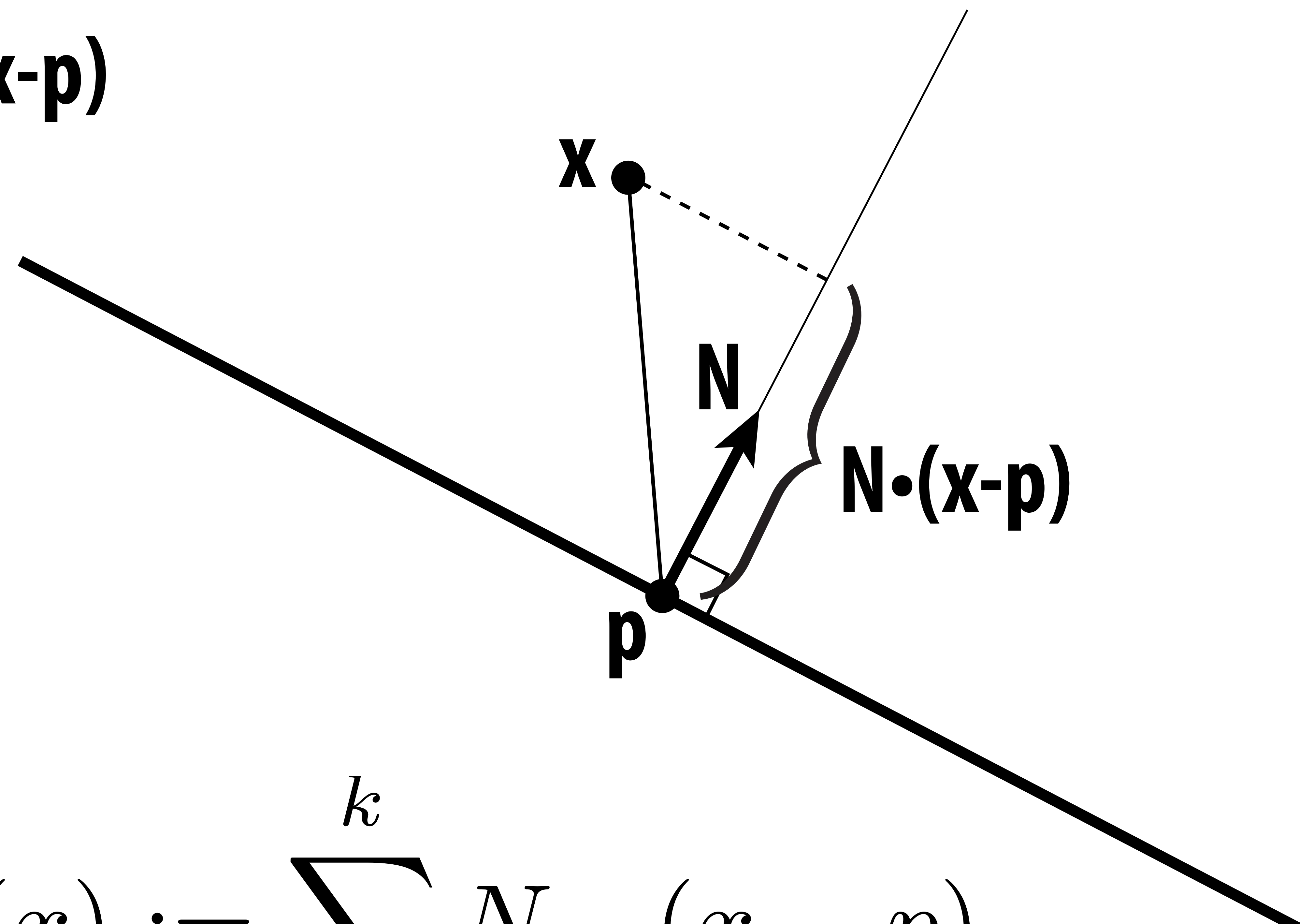
- One popular scheme: iteratively collapse edges
- Greedy algorithm:
  - assign each edge a cost
  - collapse edge with least cost
  - repeat until target number of elements is reached
- Particularly effective cost function: quadric error metric*

*invented here at CMU! (Garland & Heckbert 1997)

Quadric Error Metric

- Approximate distance to a collection of triangles
- Distance is sum of point-to-plane distances
  - Q: Distance to plane w/ normal N passing through point p?
  - A: d(x) = N•x - N•p = N•(x-p)
- Sum of distances:





$$d(x) := \sum_{i=1}^{k} N_i \cdot (x - p)$$

Quadric Error - Homogeneous Coordinates

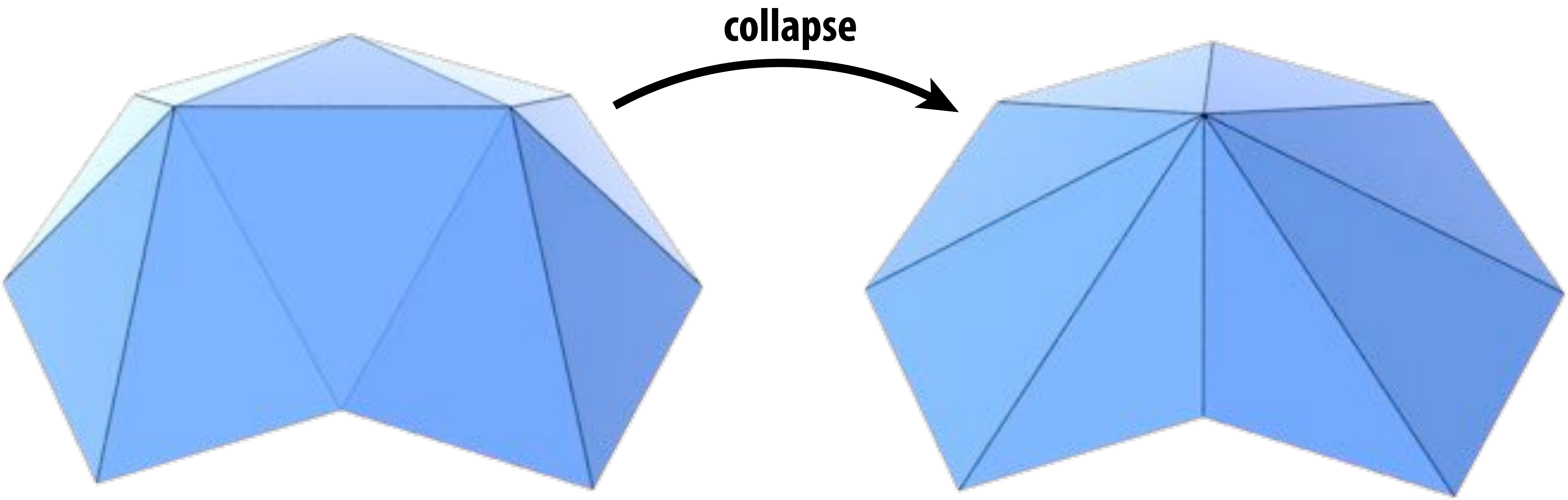
- **Suppose in coordinates we have**
  - **a query point (x,y,z)**
  - **a normal (a,b,c)**
  - **an offset d := -(p,q,r) • (a,b,c)**

$$K = \begin{bmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}$$

- **Then in homogeneous coordinates, let**
  - **u := (x,y,z,1)**
  - **v := (a,b,c,d)**
- **Signed distance to plane is then just u•v = ax+by+cz+d**
- **Squared distance is $(u^T v)^2 = u^T(vv^T)u =: u^T K u$**
- **Key idea: matrix K encodes distance to plane**
- **K is symmetric, contains 10 unique coefficients (small storage)**

Quadric Error of Edge Collapse

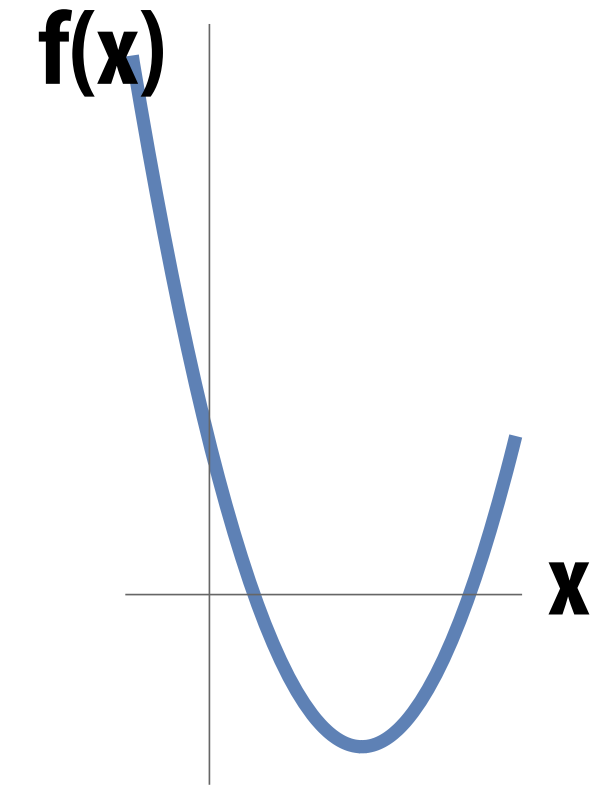
- How much does it cost to collapse an edge?
- Idea: compute edge midpoint, measure quadric error

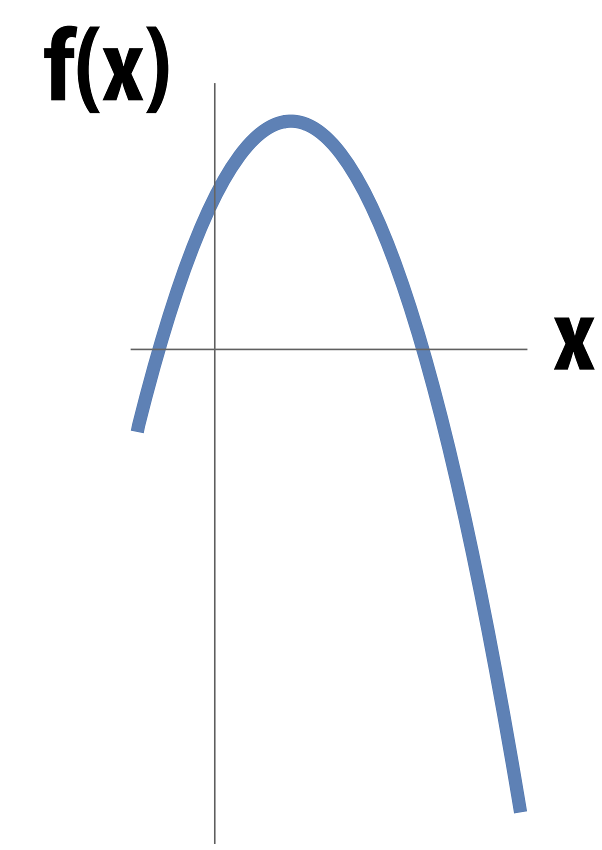


- Better idea: use point that minimizes quadric error as new point!
- Q: Ok, but how do we minimize quadric error?

Review: Minimizing a Quadratic Function

- Suppose I give you a function f(x) = ax²+bx+c
- Q: What does the graph of this function look like?
- Could also look like this!
- Q: How do we find the minimum?
- A: Look for the point where the function isn't changing (if we look "up close")
- I.e., find the point where the derivative vanishes





$$f'(x) = 0$$

$$2ax + b = 0$$

$$x = -b/2a$$

(What about our second example?)

Minimizing a Quadratic Form

- **A quadratic form is just a generalization of our quadratic polynomial from 1D to nD**
- **E.g., in 2D: $f(x,y) = ax^2 + bxy + cy^2 + dx + ey + g$**
- **Can always (always!) write quadratic polynomial using a symmetric matrix (and a vector, and a constant):**

$$f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + g$$

$$= \mathbf{x}^{\mathsf{T}} A \mathbf{x} + \mathbf{u}^{\mathsf{T}} x + g$$

(this expression works for any n!)

- **Q: How do we find a critical point (min/max/saddle)?**
- **A: Set derivative to zero!**

$$2A\mathbf{x} + \mathbf{u} = 0$$

$$\mathbf{x} = -\tfrac{1}{2} A^{-1} \mathbf{u}$$

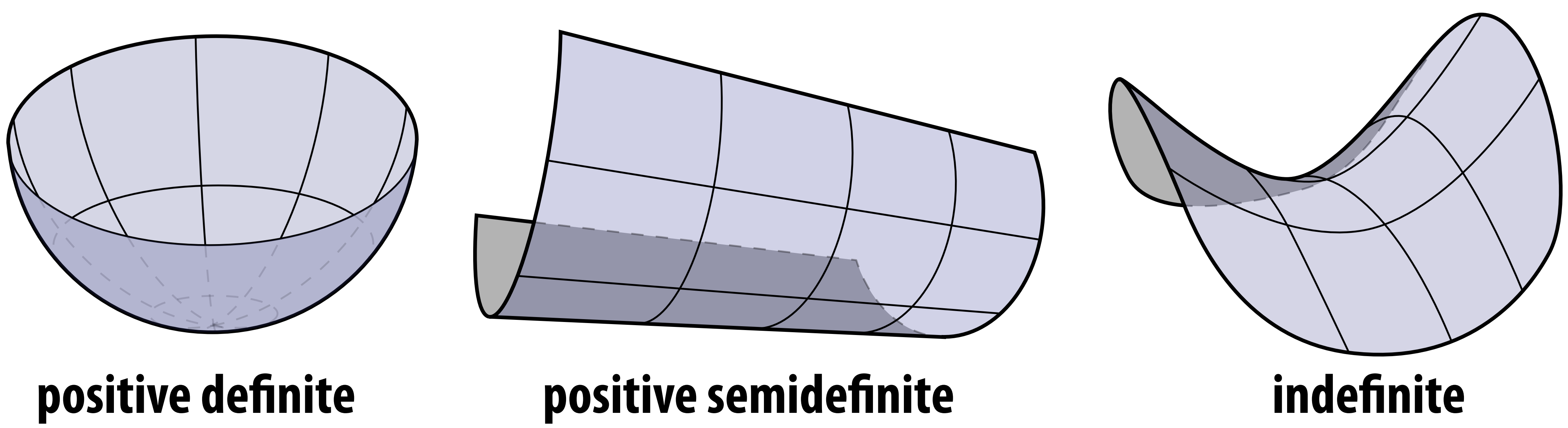
(Can you show this is true, at least in 2D?)

Positive Definite Quadratic Form

- Just like our 1D parabola, critcal point is not always a min!
- Q: In 2D, 3D, nD, when do we get a minimum?
- A: When matrix A is positive-definite:

$$\mathbf{x}^\mathsf{T} A\, \mathbf{x} > 0 \quad \forall\, \mathbf{x}$$

- 1D: Must have $xax = ax^2 > 0$. In other words: a is positive!
- 2D: Graph of function looks like a "bowl":



- Positive-definiteness is extremely important in computer graphics: it means we can find a minimum by solving linear equations. Basis of many, many modern algorithms (geometry processing, simulation, ...).

Minimizing Quadratic Error

- **Find "best" point for edge collapse by minimizing quad. form**

$$\min_u \mathbf{u}^\mathsf{T} K \mathbf{u}$$

- **Already know fourth (homogeneous) coordinate is 1!**
- **So, break up our quadratic function into two pieces:**

$$\begin{bmatrix} \mathbf{x}^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} B & \mathbf{w} \\ \mathbf{w} & d^2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

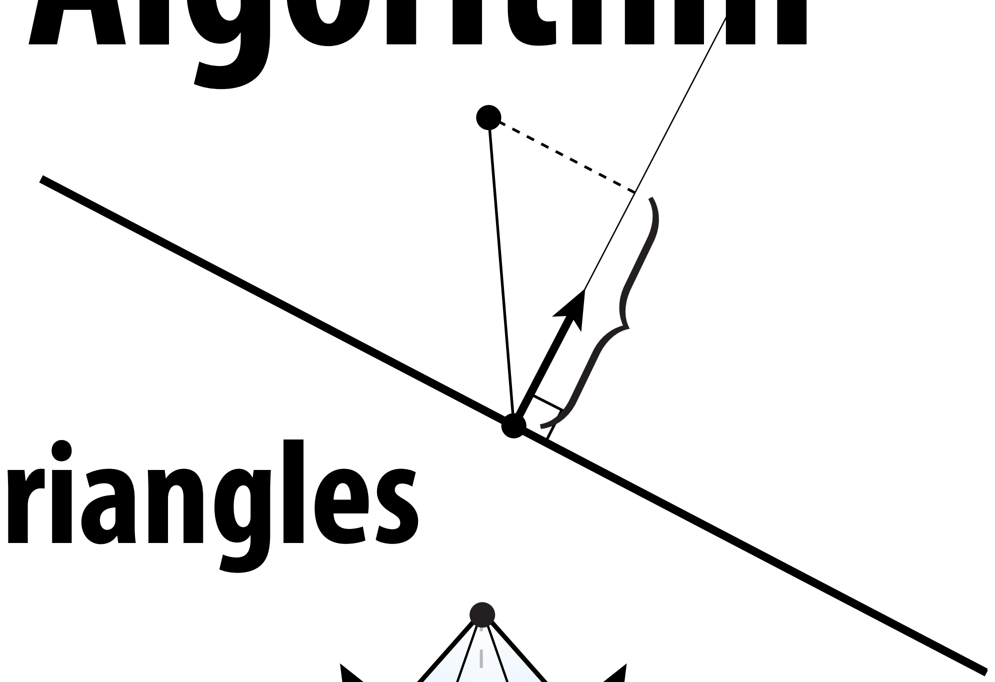
$$= \mathbf{x}^\mathsf{T} B \mathbf{x} + 2\mathbf{w}^\mathsf{T}\mathbf{x} + d^2$$

- **Now we have a quadratic form in the 3D position x.**
- **Can minimize as before:**

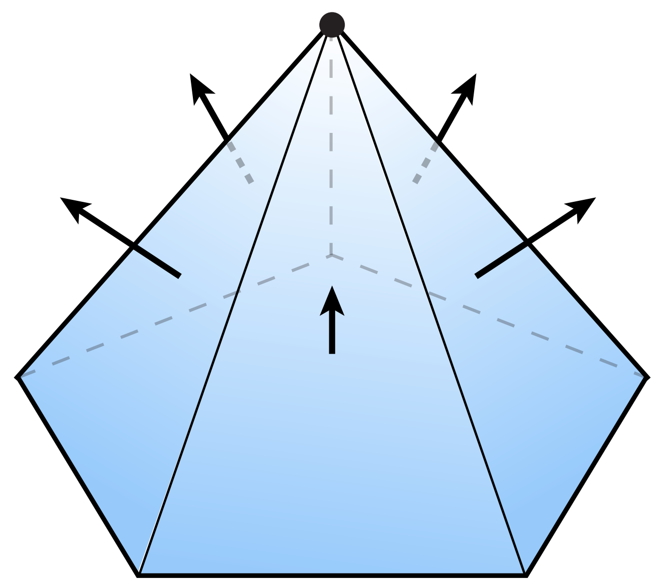
$$2B\mathbf{x} + 2\mathbf{w} = 0 \qquad \iff \qquad \mathbf{x} = -B^{-1}\mathbf{w}$$

(Q: Why should B be positive-definite?)

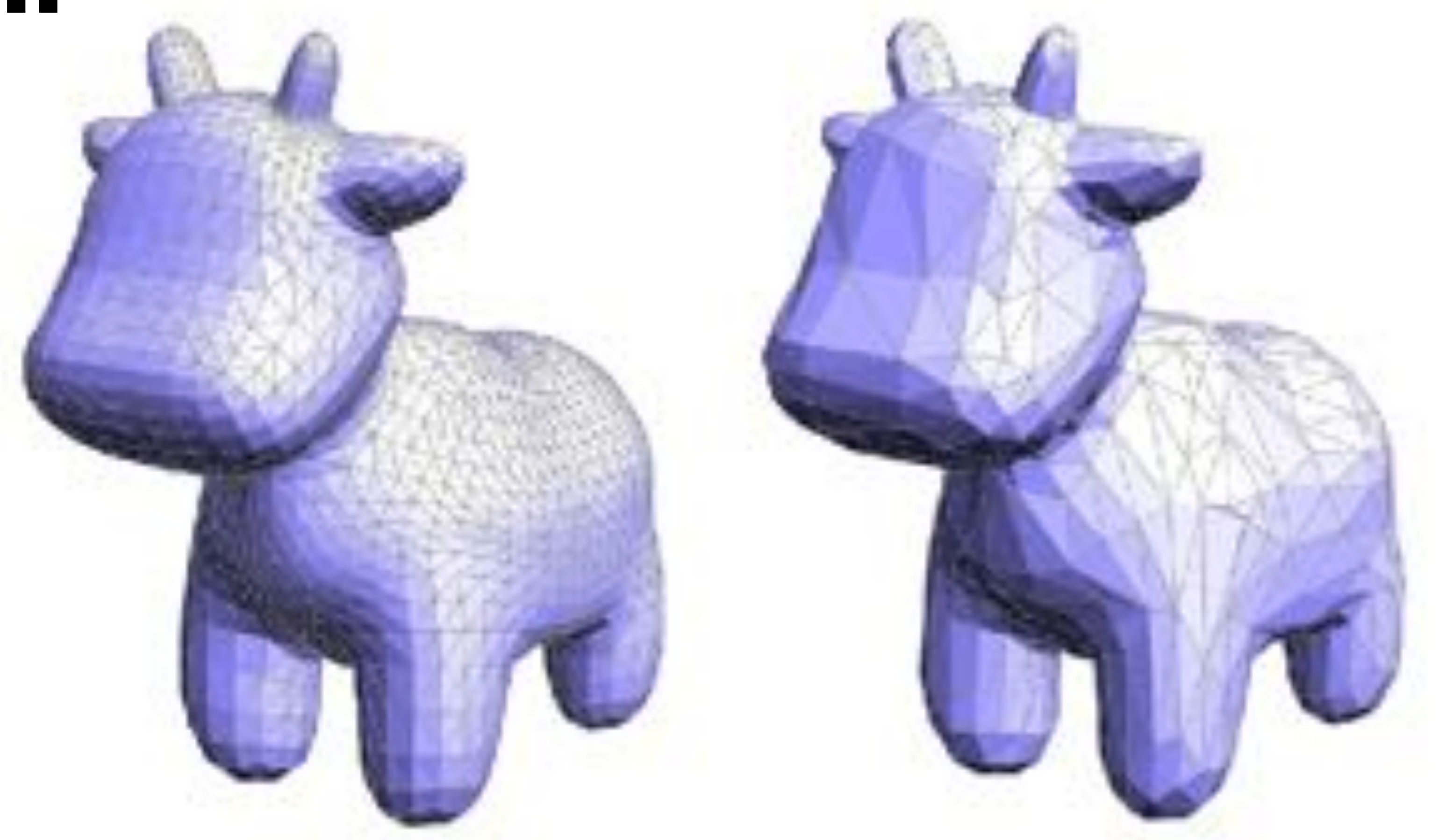
Quadric Error Simplification: Final Algorithm



- Compute K for each triangle (distance to plane)
- Set K at each vertex to sum of Ks from incident triangles
- Set K at each edge to sum of Ks at endpoints

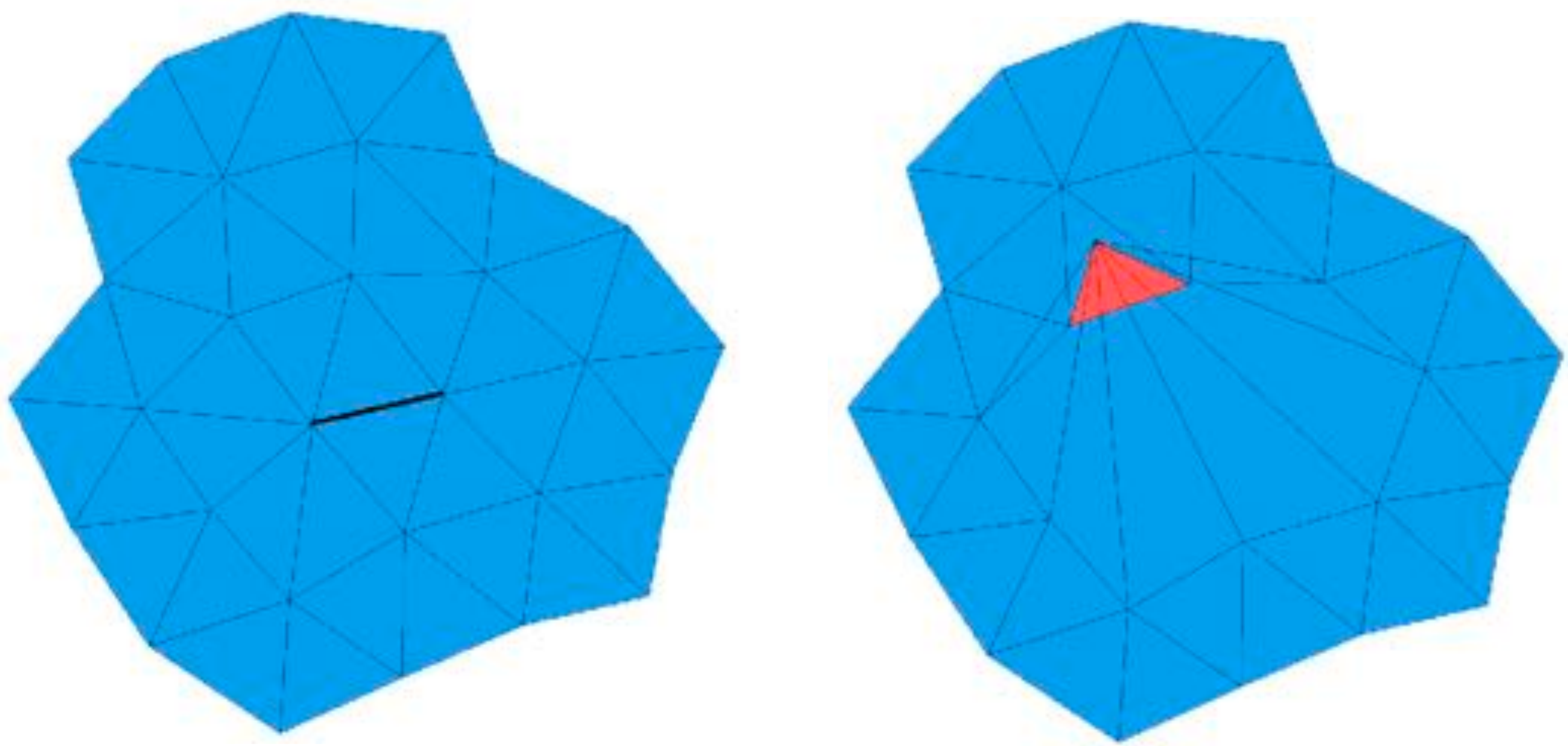


- Find point at each edge minimizing quadric error
- Until we reach target # of triangles:
  - collapse edge (i,j) with smallest cost to get new vertex m
  - add K_i and K_j to get quadric K_m at m
  - update cost of edges touching m
- More details in assignment writeup!



Quadric Simplification—Flipped Triangles

- Depending on where we put the new vertex, one of the new triangles might be "flipped" (normal points in instead of out):

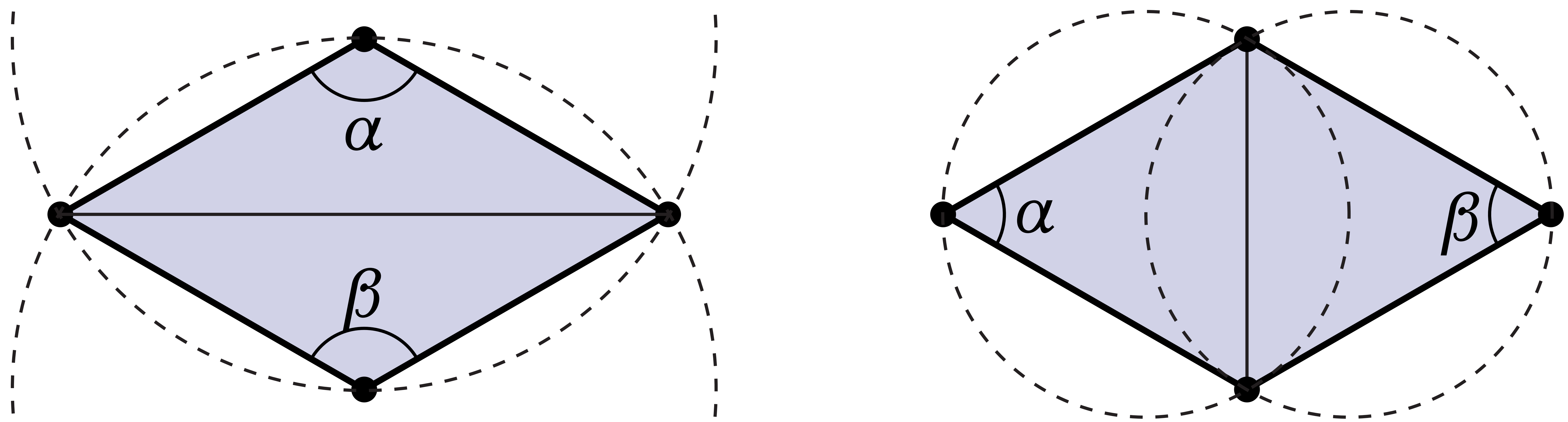


- Easy solution: check dot product between normals across edge
- If negative, don't collapse this edge!

What if we're happy with the number of triangles, but want to improve quality?

How do we make a mesh "more Delaunay"?

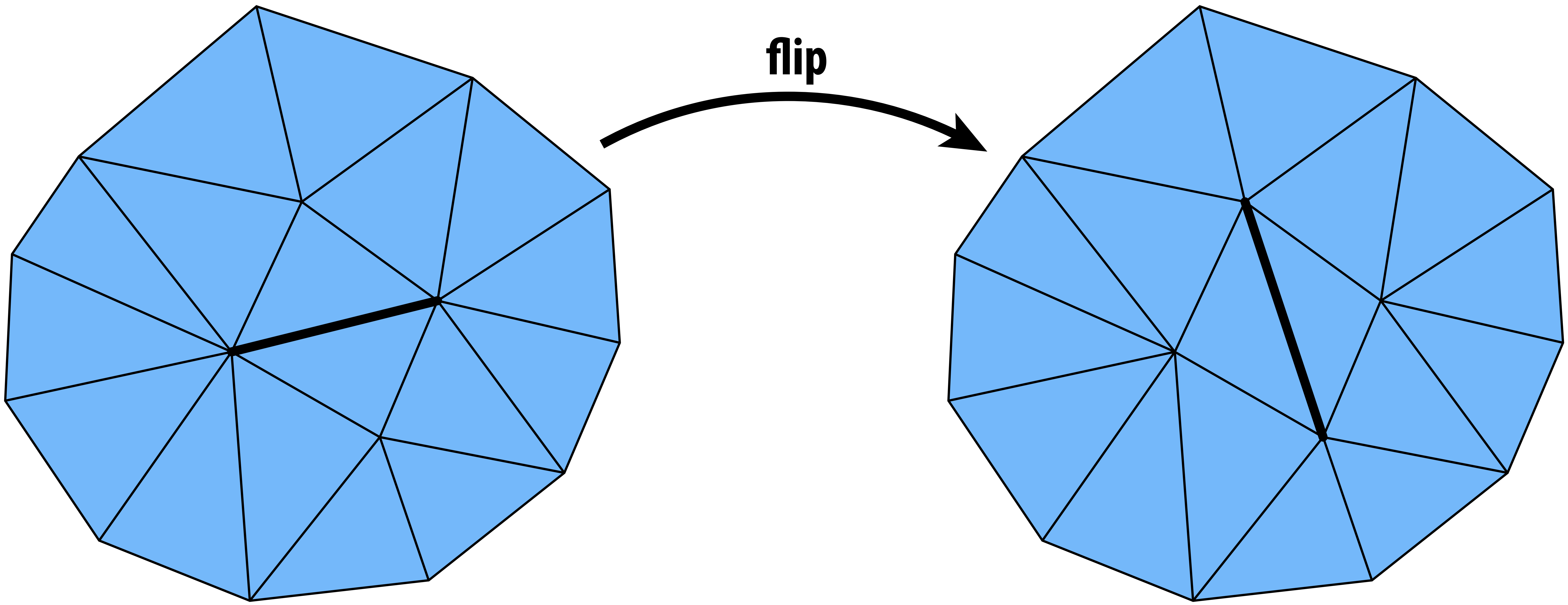
- Already have a good tool: edge flips!
- If $\alpha+\beta > \pi$, flip it!



- FACT: in 2D, flipping edges eventually yields Delaunay mesh
- Theory: worst case $O(n^2)$; no longer true for surfaces in 3D.
- Practice: simple, effective way to improve mesh quality

Alternatively: how do we improve degree?

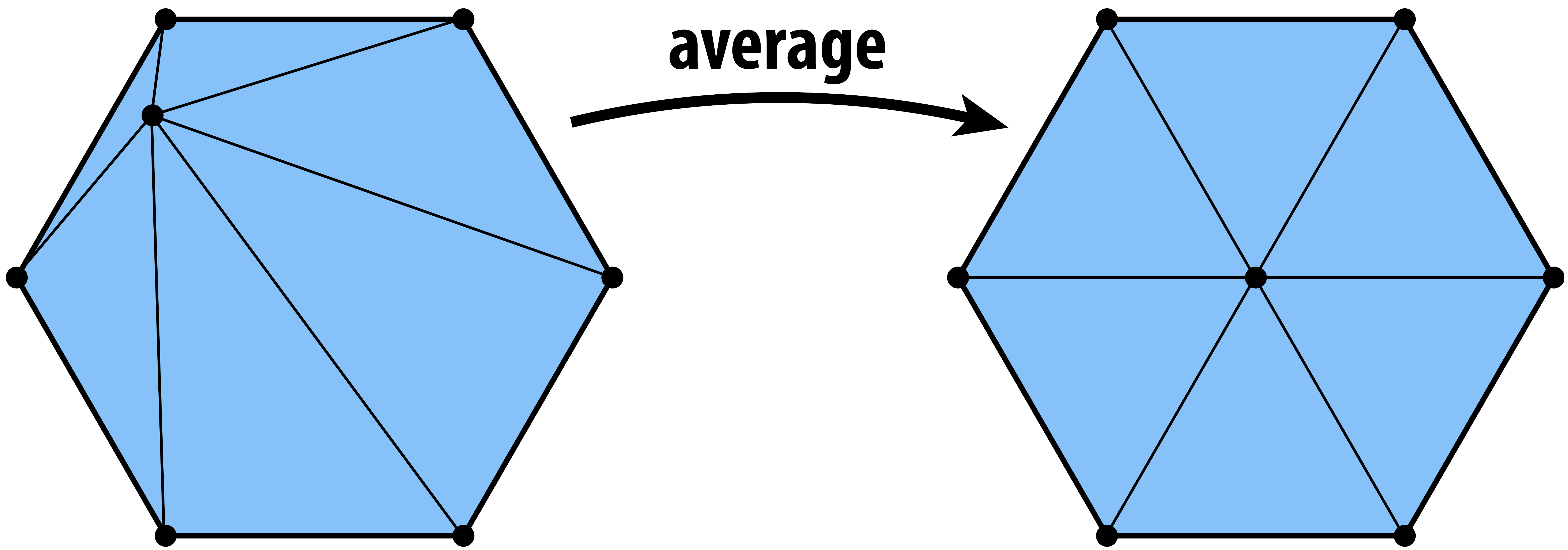
- Same tool: edge flips!
- If total deviation from degree-6 gets smaller, flip it!



- FACT: average valence of any triangle mesh is 6
- Iterative edge flipping acts like "discrete diffusion" of degree
- Again, no (known) guarantees; works well in practice

How do we make a triangles “more round”?

- Delaunay doesn’t mean triangles are “round” (angles near 60°)
- Can often improve shape by centering vertices:

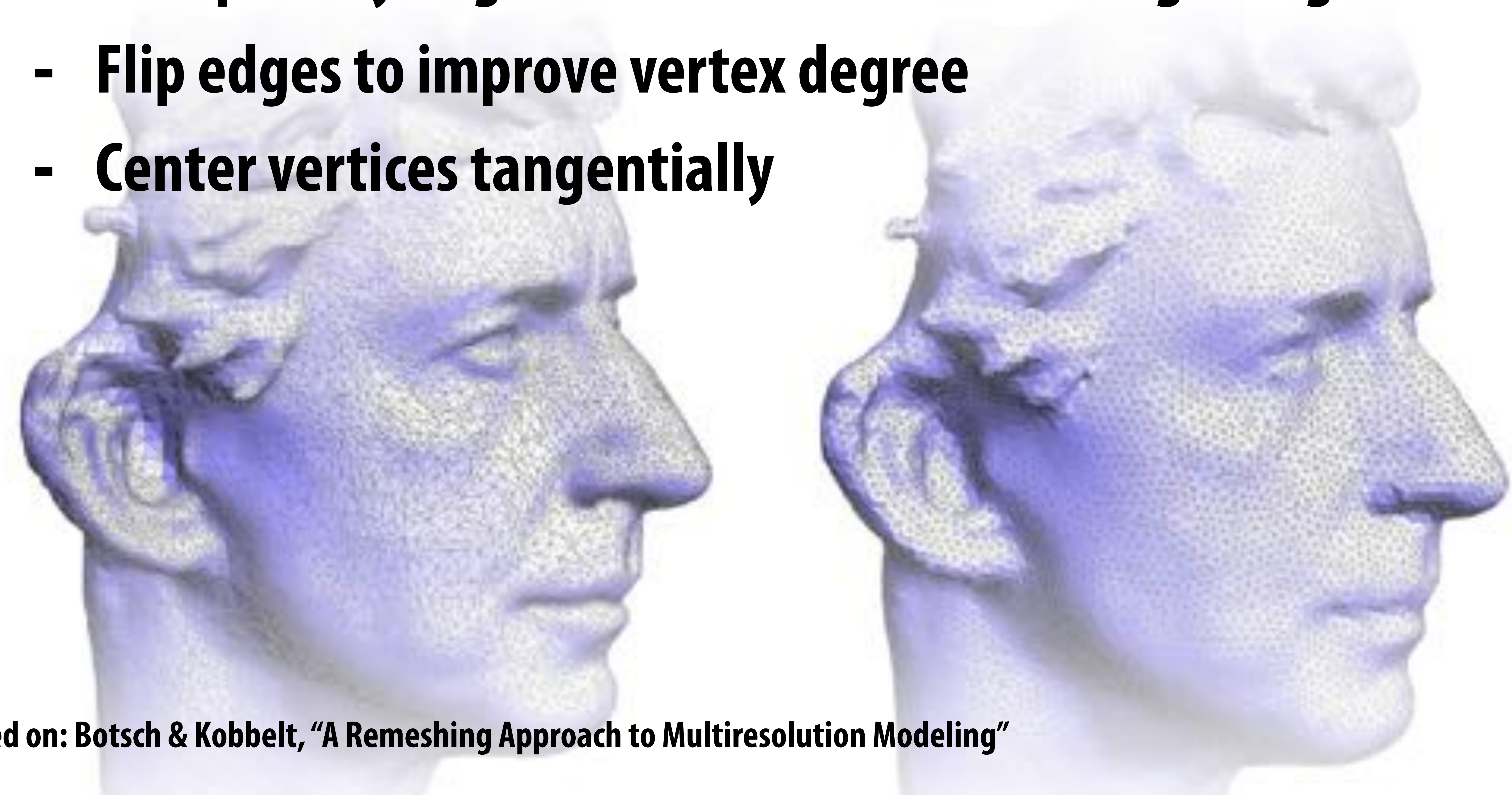


- Simple version of technique called “Laplacian smoothing”.*
- On surface: move only in tangent direction
- How? Remove normal component from update vector.

*See Crane, “Digital Geometry Processing with Discrete Exterior Calculus” http://keenan.is/ddg

Isotropic Remeshing Algorithm

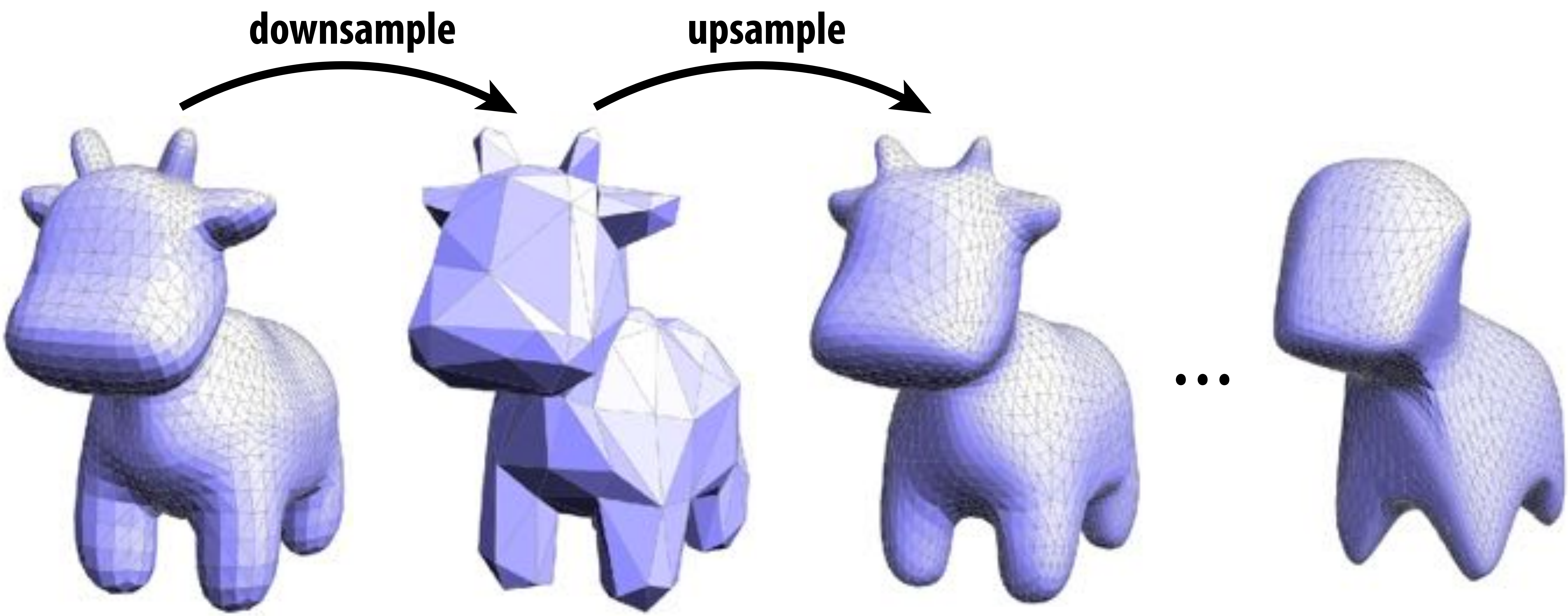
- Try to make triangles uniform shape & size
- Repeat four steps:
  - Split any edge over 4/3rds mean edge length
  - Collapse any edge less than 4/5ths mean edge length
  - Flip edges to improve vertex degree
  - Center vertices tangentially



Based on: Botsch & Kobbelt, "A Remeshing Approach to Multiresolution Modeling"

What can go wrong when you resample a signal?

Danger of Resampling



(Q: What happens with an image?)

But wait: we have the original mesh. Why not just project each new sample point onto the closest point of the original mesh?

Next Time: Geometric Queries

- Q: Given a point, in space, how do we find the closest point on a surface? Are we inside or outside the surface? How do we find intersection of two triangles? Etc.
- Q: Do implicit/explicit representations make such tasks easier?
- Q: What's the cost of the naïve algorithm, and how do we accelerate such queries for large meshes?
- So many questions!