Meshes and Manifolds
Last time: overview of geometry

- Many types of geometry in nature
- Demand sophisticated representations
- Two major categories:
  - IMPLICIT - “tests” if a point is in shape
  - EXPLICIT - directly “lists” points
- Lots of representations for both
- Today:
  - what is a surface, anyway?
  - nuts & bolts of polygon meshes
  - geometry processing / resampling
Manifold Assumption

- Today we’re going to introduce the idea of manifold geometry
- Can be hard to understand motivation at first!
- So first, let’s revisit a more familiar example...
Bitmap Images, Revisited

To encode images, we used a regular grid of pixels:
But images are not fundamentally made of little squares:

Goyō Hashiguchi, *Kamisuki* (ca 1920)
So why did we choose a square grid?

...rather than dozens of alternatives?
Regular grids make life easy

- One reason: SIMPLICITY / EFFICIENCY
  - E.g., always have four neighbors
  - Easy to index, easy to filter...
  - Storage is just a list of numbers

- Another reason: GENERALITY
  - Can encode basically any image

Are regular grids always the best choice for bitmap images?
- No! E.g., suffer from anisotropy, don’t capture edges, ...
- But more often than not are a pretty good choice

Will see a similar story with geometry...
So, how should we encode surfaces?
Smooth Surfaces

- Intuitively, a surface is the boundary or “shell” of an object
  (Think about the candy shell, not the chocolate.)
- Surfaces are manifold:
  - If you zoom in far enough (at any point) looks like a plane*
  - E.g., the Earth from space vs. from the ground

*...or can easily be flattened into the plane, without cutting or ripping.
Isn’t every shape manifold?

- No, for instance:

Center point never looks like the plane, no matter how close we get.
More Examples of Smooth Surfaces

- Which of these shapes are manifold?
A manifold polygon mesh has fans, not fins

- For polygonal surfaces just two easy conditions to check:
  1. Every edge is contained in only two polygons (no “fins”)
  2. The polygons containing each vertex make a single “fan”
What about boundary?

- The boundary is where the surface “ends.”
- E.g., waist & ankles on a pair of pants.
- Locally, looks like a half disk
- Globally, each boundary forms a loop

- Polygon mesh:
  - one polygon per boundary edge
  - boundary vertex looks like “pacman”
Ok, but why is the manifold assumption useful?
Keep it Simple!

- Same motivation as for images:
  - make some assumptions about our geometry to keep data structures/algorithms simple and efficient
  - in many common cases, doesn’t fundamentally limit what we can do with geometry
How do we actually encode all this data?
Q: What data structures can we use to store a list of numbers?
One idea: use an array (constant time lookup, coherent access)

Alternative: use a linked list (linear lookup, incoherent access)

Q: Why bother with the linked list?
A: For one, we can easily insert numbers wherever we like...
Polygon Soup (Array-like)

- Store triples of coordinates \((x,y,z)\), tuples of indices
- E.g., tetrahedron:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>i</th>
<th>j</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>1</td>
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<tr>
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<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
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<tr>
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<td>1</td>
</tr>
<tr>
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<td>-1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Q: How do we find all the polygons touching vertex 2?

Ok, now consider a more complicated mesh:

~1 billion polygons

Very expensive to find the neighboring triangles! (What’s the cost?)
Incidence Matrices

- If we want to answer neighborhood queries, why not simply store a list of neighbors?
- Can encode all neighbor information via incidence matrices
- E.g., tetrahedron: VERTEX ↔ EDGE

<table>
<thead>
<tr>
<th></th>
<th>v0</th>
<th>v1</th>
<th>v2</th>
<th>v3</th>
</tr>
</thead>
<tbody>
<tr>
<td>e0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>e1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>e2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>e3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>e4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>e5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

- 1 means “touches”; 0 means “does not touch”
- Instead of storing lots of 0’s, use sparse matrices
- Still large storage cost, but finding neighbors is now $O(1)$
- Hard to change connectivity, since we used fixed indices
- Bonus feature: mesh does not have to be manifold
**Halfedge Data Structure (Linked-list-like)**

- Store some information about neighbors
- Don't need an exhaustive list; just a few key pointers
- Key idea: two halfedges act as "glue" between mesh elements:

```c
struct Halfedge
{
    Halfedge* twin;
    Halfedge* next;
    Vertex* vertex;
    Edge* edge;
    Face* face;
};
```

```
struct Vertex
{
    Halfedge* halfedge;
}
```

```
struct Edge
{
    Halfedge* halfedge;
};
```

```
struct Face
{
    Halfedge* halfedge;
};
```

- Each vertex, edge face points to just one of its halfedges.
Halfedge makes mesh traversal easy

- Use “twin” and “next” pointers to move around mesh
- Use “vertex”, “edge”, and “face” pointers to grab element
- Example: visit all vertices of a face:
  
  ```c
  Halfedge* h = f->halfedge;
  do {
    h = h->next;
    // do something w/ h->vertex
  } while( h != f->halfedge );
  ```

- Example: visit all neighbors of a vertex:

  ```c
  Halfedge* h = v->halfedge;
  do {
    h = h->twin->next;
  } while( h != v->halfedge );
  ```

- Note: only makes sense if mesh is manifold!
Halfedge meshes are always manifold

- Consider simplified halfedge data structure
- Require only “common-sense” conditions

```c
struct Halfedge {
    Halfedge *next, *twin;
};
twin->twin == this
next != this
twin != this
```

- Keep following `next`, and you’ll get faces.
- Keep following `twin` and you’ll get edges.
- Keep following `next->twin` and you’ll get vertices.

Q: Why, therefore, is it impossible to encode the red figures?
Halfedge meshes are easy to edit

- Remember key feature of linked list: insert/delete elements
- Same story with halfedge mesh (“linked list on steroids”)
- E.g., for triangle meshes, several atomic operations:

  How? Allocate/delete elements; reassigning pointers.
  Must be careful to preserve manifoldness!
Edge Flip (Triangles)

- Triangles (a, b, c), (b, d, c) become (a, d, c), (a, b, d):

![Diagram showing triangle flip]

- Long list of pointer reassignments (`edge->halfedge = ...`)
- However, no elements created/destroyed.
- Q: What happens if we flip twice?
- Challenge: can you implement edge flip such that pointers are unchanged after two flips?
Edge Split (Triangles)

- Insert midpoint m of edge (c,b), connect to get four triangles:

  This time, have to add new elements.

  Lots of pointer reassignments.

  Q: Can we “reverse” this operation?
Edge Collapse (Triangles)

- Replace edge \((b, c)\) with a single vertex \(m\):

- Now have to delete elements.
- Still lots of pointer assignments!
- Q: How would we implement this with a polygon soup?
- Any other good way to do it? (E.g., different data structure?)
## Comparison of Polygon Mesh Data Structures

<table>
<thead>
<tr>
<th>Case study: triangles.</th>
<th>Polygon Soup</th>
<th>Incidence Matrices</th>
<th>Halfedge Mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>storage cost*</td>
<td>~3 x #vertices</td>
<td>~33 x #vertices</td>
<td>~36 x #vertices</td>
</tr>
<tr>
<td>constant-time neighborhood access?</td>
<td>NO</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>easy to add/remove mesh elements?</td>
<td>NO</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>nonmanifold geometry?</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
</tr>
</tbody>
</table>

**Conclusion:** pick the right data structure for the job!

*number of integer values and/or pointers required to encode connectivity (all data structures require same amount of storage for vertex positions)*
Alternatives to Halfedge

- Many very similar data structures:
  - winged edge
  - corner table
  - quadedge
  - ...

- Each stores local neighborhood information

- Similar tradeoffs relative to simple polygon list:
  - **CONS**: additional storage, incoherent memory access
  - **PROS**: better access time for individual elements, intuitive traversal of local neighborhoods

- (Food for thought: can you design a halfedge-like data structure with reasonably coherent data storage?)
Ok, but what can we actually do with our fancy new data structure?
Subdivision Modeling
Subdivision Modeling

- Common modeling paradigm in modern 3D tools:
  - Coarse “control cage”
  - Perform local operations to control/edit shape
  - Global subdivision process determines final surface
Subdivision Modeling—Local Operations

- For general polygon meshes, we can dream up lots of local mesh operations that might be useful for modeling:

- EdgeBevel
- FaceBevel
- VertexBevel
- EdgeCollapse
- EdgeFlip
- EraseVertex
- EdgeVertexSplit
- EraseEdge
- FaceCollapse

...and many, many more!
Global Subdivision

- Start with coarse polygon mesh ("control cage")
- Subdivide each element
- Update vertices via local averaging
- Many possible rule:
  - Catmull-Clark (quads)
  - Loop (triangles)
  - ...
- Common issues:
  - interpolating or approximating?
  - continuity at vertices?
- Easier than splines for modeling; harder to evaluate pointwise
Next Time: Digital Geometry Processing

- Extend traditional digital signal processing (audio, video, etc.) to deal with geometric signals:
  - upsampling / downsampling / resampling / filtering ...
  - aliasing (reconstructed surface gives “false impression”)

- Also some new challenges (very recent field!):
  - over which domain is a geometric signal expressed?
  - no terrific sampling theory, no fast Fourier transform, ...

- Often need new data structures & new algorithms