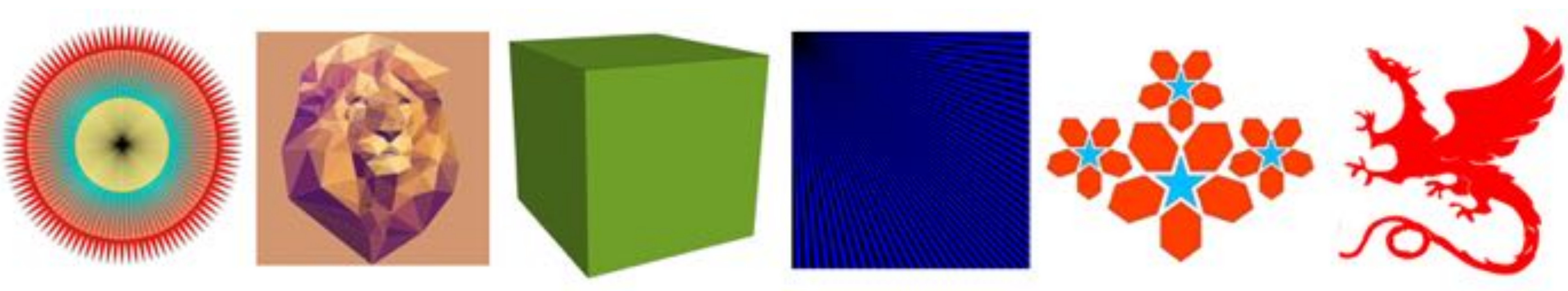
Lecture 4:

Drawing a Triangle (and an Intro to Sampling)

Computer Graphics
CMU 15-462/15-662, Spring 2018

HW 0.5 Due, HW 1 Out Today!

- GOAL: Implement a basic "rasterizer"
  - (Topic of today's lecture)
  - We hand you a bunch of lines, triangles, etc.
  - You draw them by lighting up pixels on the screen!
- Code skeleton available (later today) from course webpage
- DUE February 12



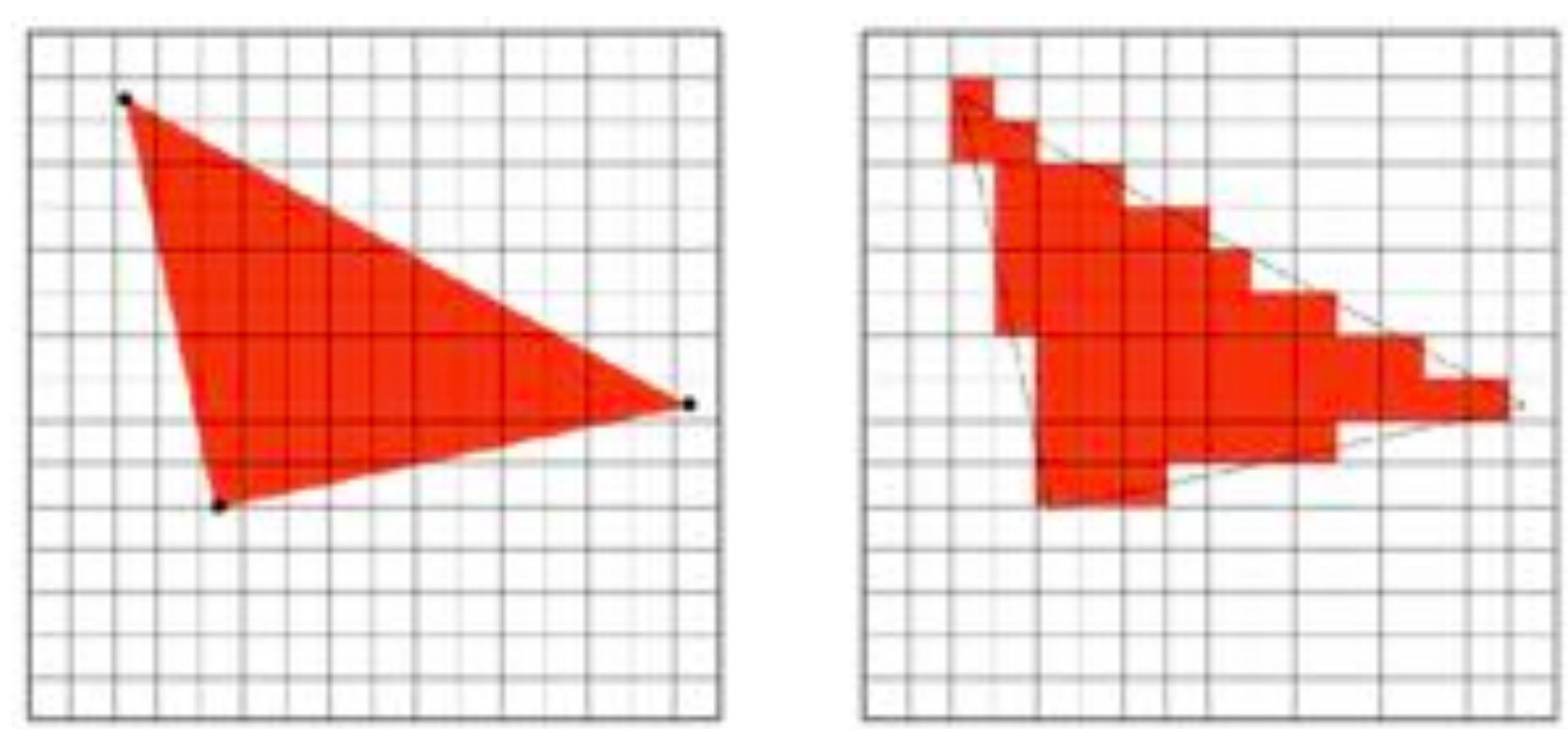
OpenGL tutorial session tomorrow

- You will use a very small amount of OpenGL in this project
  - longstanding and standard rendering library
  - get something up on the screen quickly
  - have some understanding of how OpenGL "does things for you"
  - you will rewrite some of its functionality in the remainder of the assignment

- Tutorial TOMORROW January 30
  - 8-9pm
  - location will be announced on Piazza

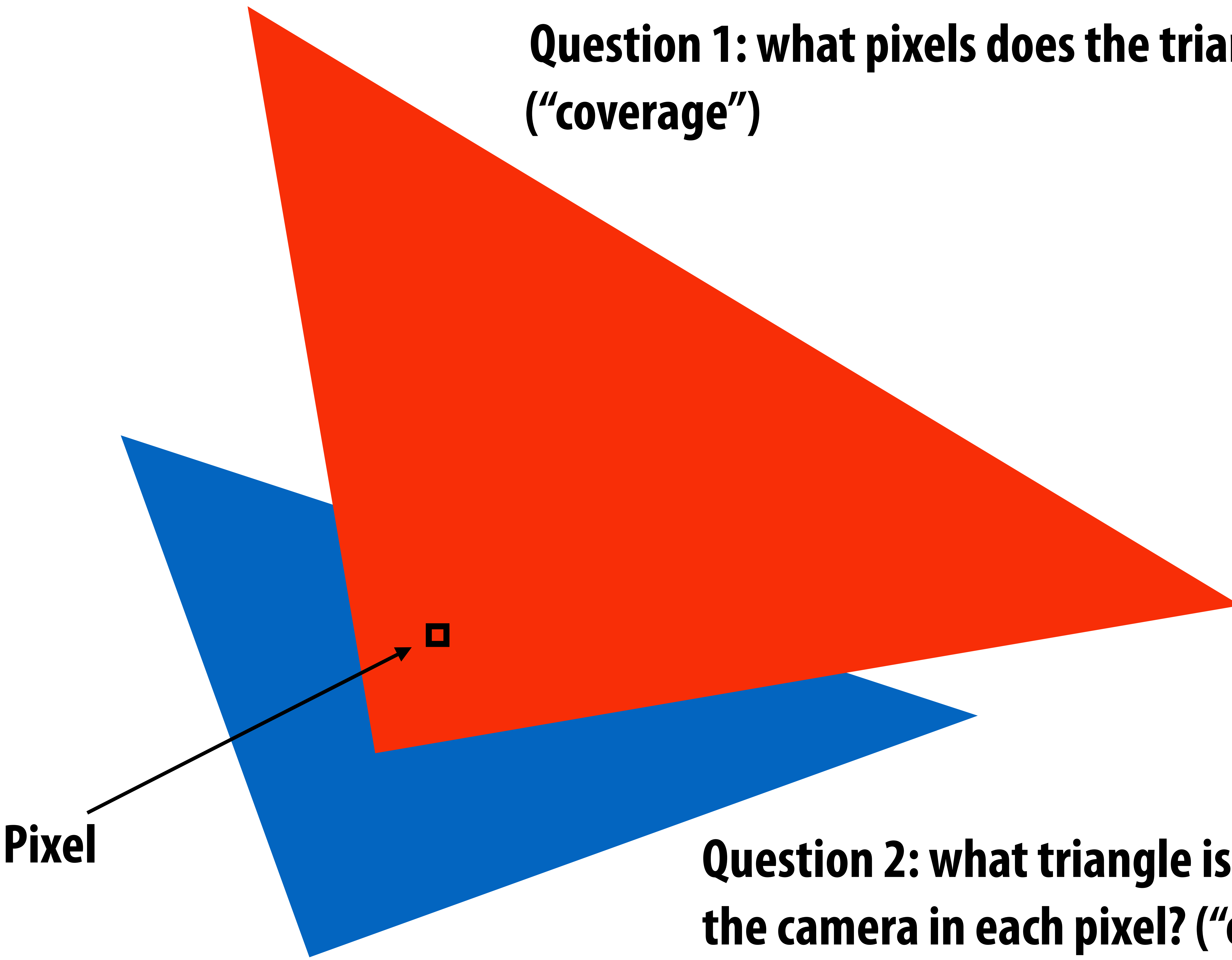
TODAY: Rasterization

- Two major techniques for “getting stuff on the screen”
- Rasterization
  - for each primitive (e.g., triangle), which pixels light up?
  - extremely fast (BILLIONS of triangles per second on GPU)
  - harder (but not impossible) to achieve photorealism
  - perfect match for 2D vector art, fonts, quick 3D preview, …
- Ray Tracing
  - for each pixel, which primitives are seen?
  - easier to get photorealism
  - generally slower
  - much more later in the semester!



Let's draw some triangles on the screen

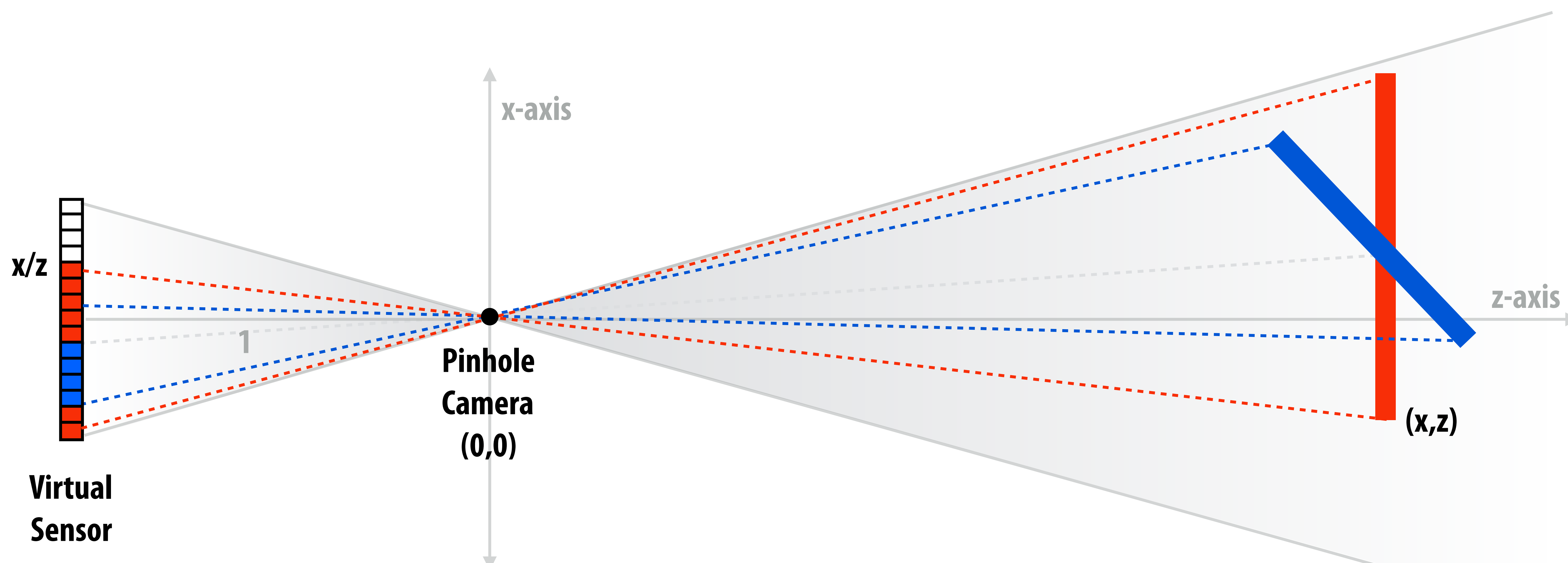
Question 1: what pixels does the triangle overlap? ("coverage")



Question 2: what triangle is closest to the camera in each pixel? ("occlusion")

The visibility problem

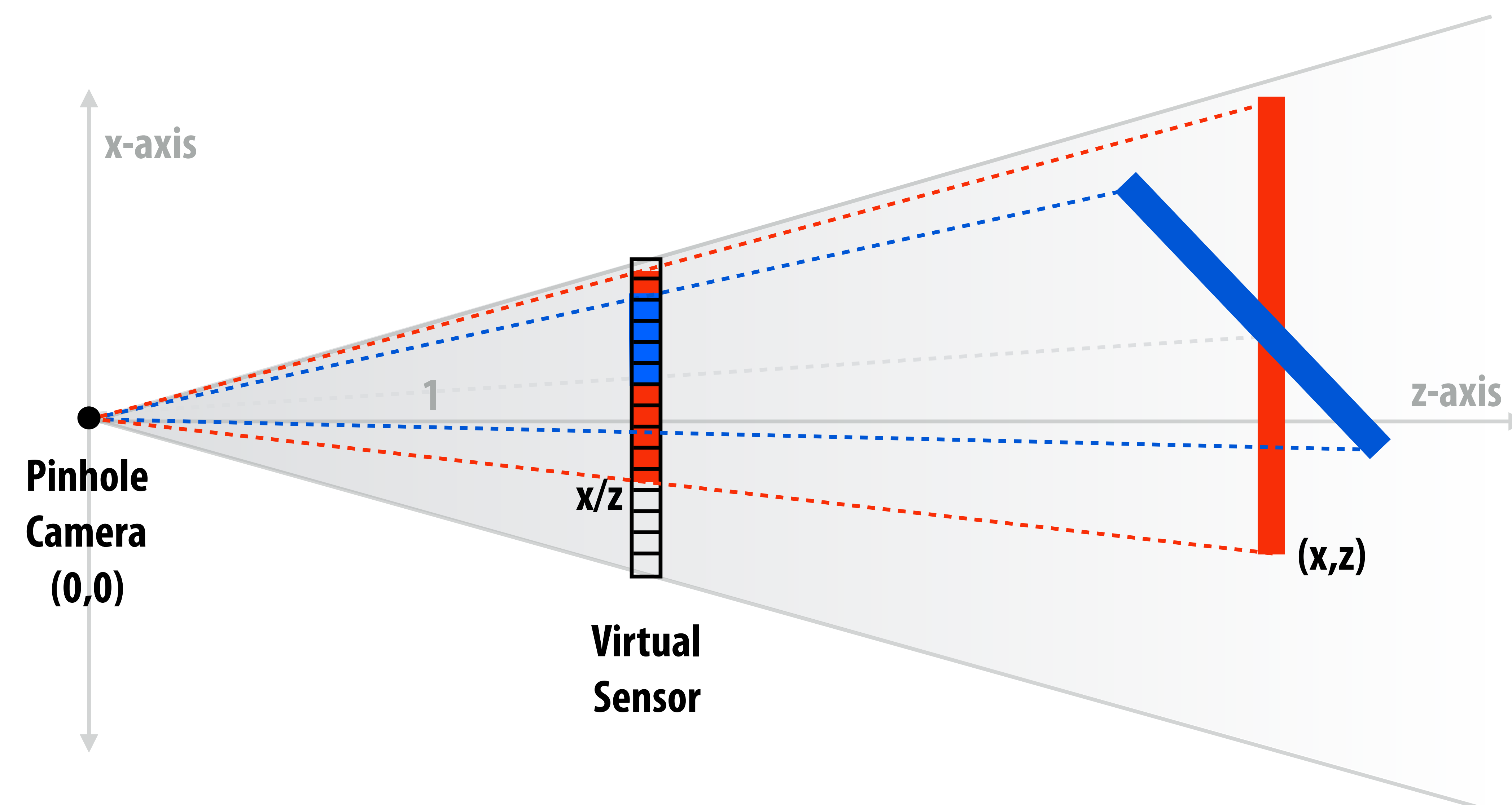
- An informal definition: what scene geometry is visible within each screen pixel?
  - What scene geometry projects into a screen pixel? (coverage)
  - Which geometry is visible from the camera at that pixel? (occlusion)



(Recall pinhole camera from first lecture)

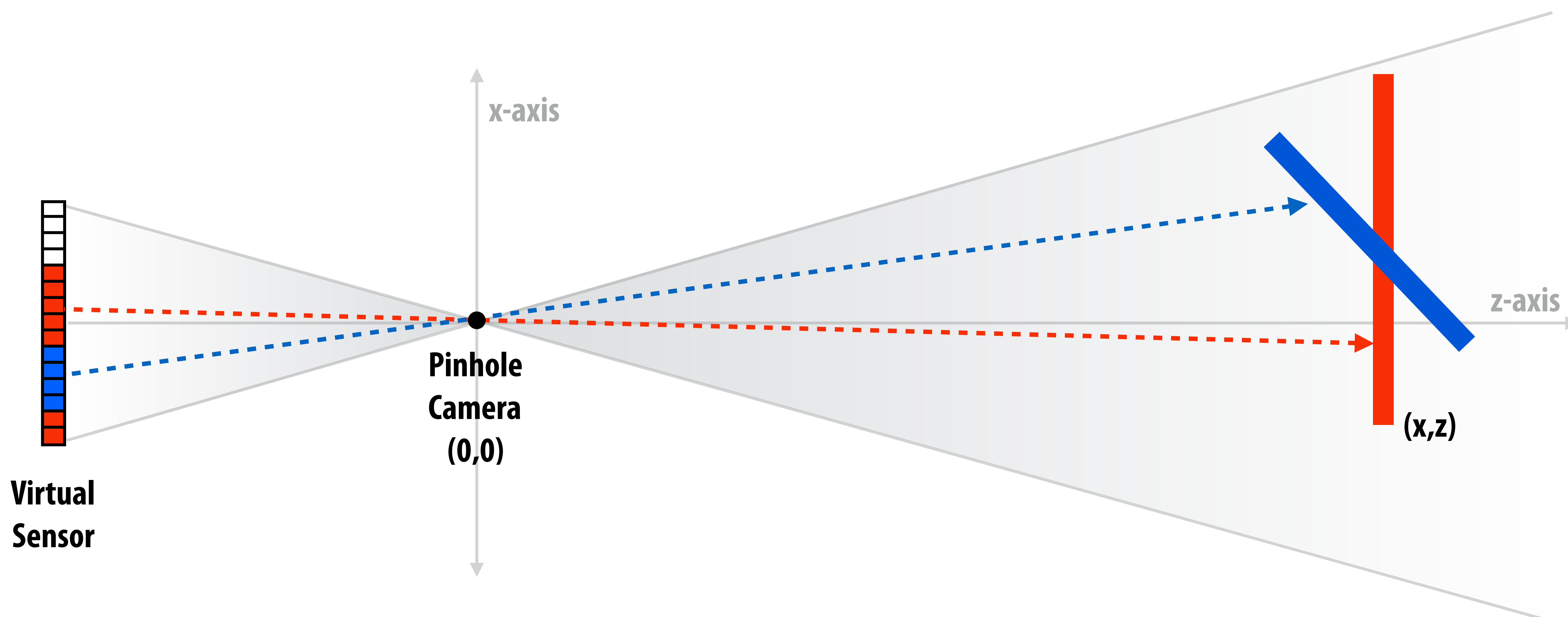
The visibility problem

- **An informal definition: what scene geometry is visible within each screen pixel?**
  - What scene geometry projects into a screen pixel? (coverage)
  - Which geometry is visible from the camera at that pixel? (occlusion)



The visibility problem (said differently)

- **In terms of rays:**
  - What scene geometry is hit by a ray from a pixel through the pinhole? (coverage)
  - What object is the first hit along that ray? (occlusion)

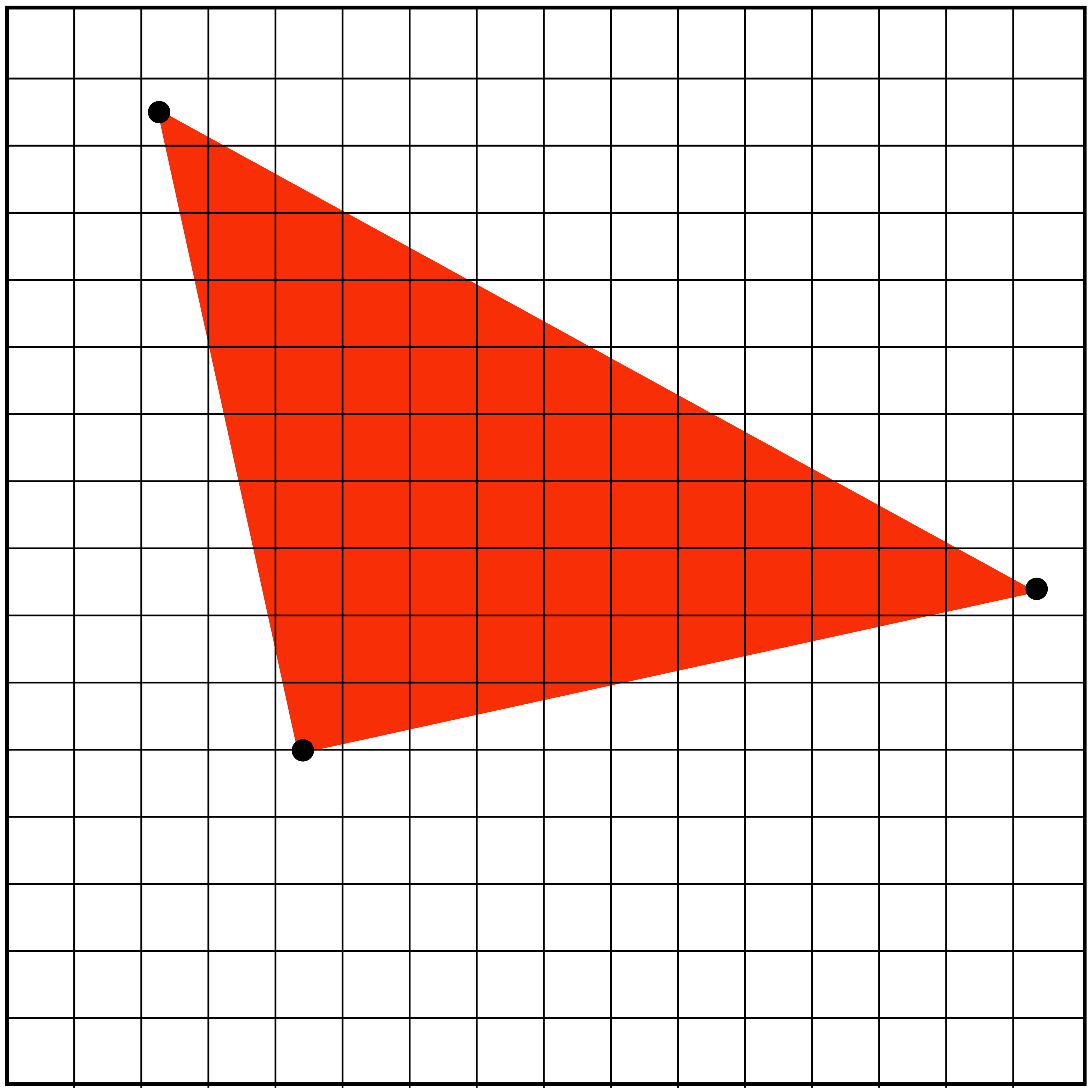


Hold onto this thought for later in the semester.

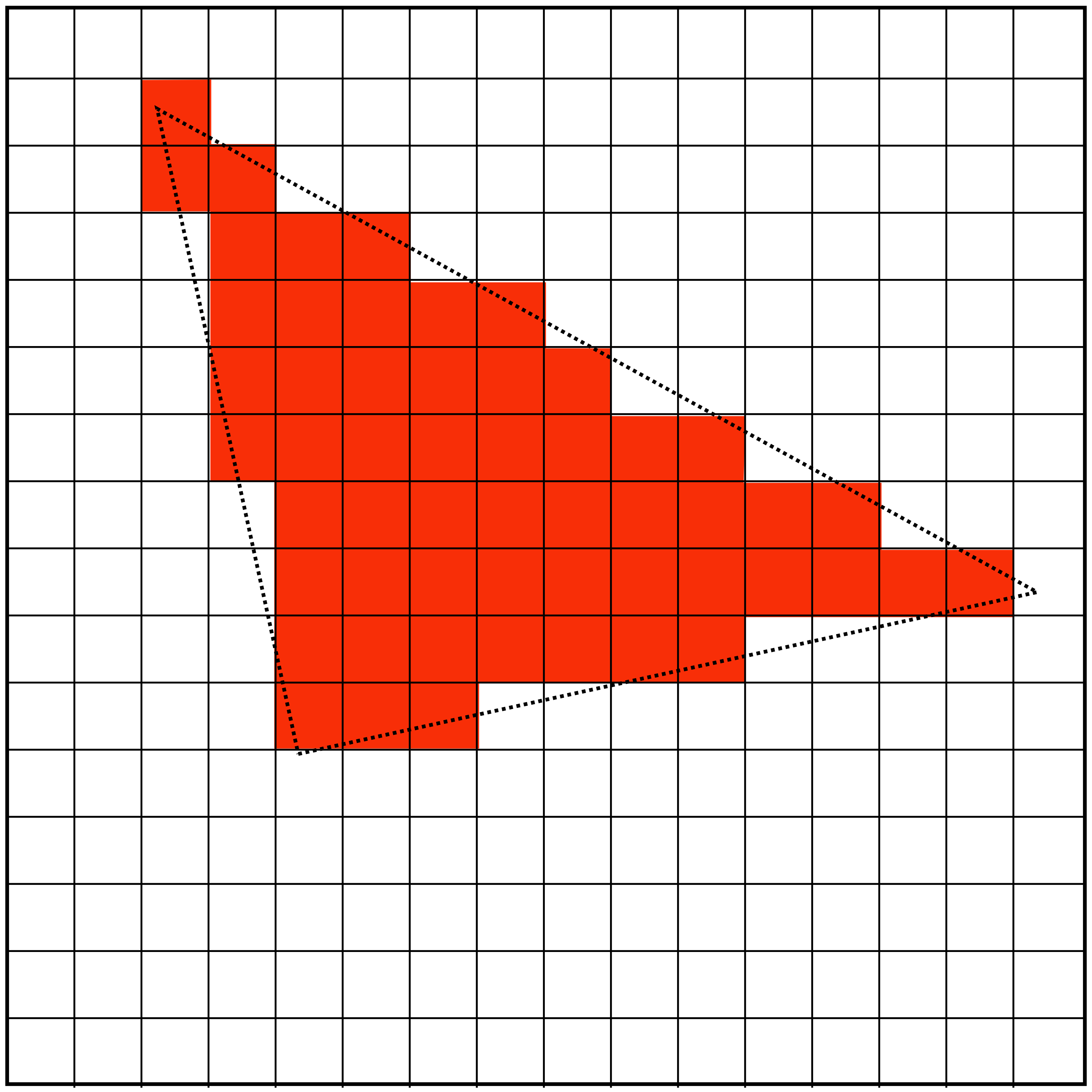
Computing triangle coverage

What pixels does the triangle overlap?

Input:
projected position of triangle vertices: P_0, P_1, P_2

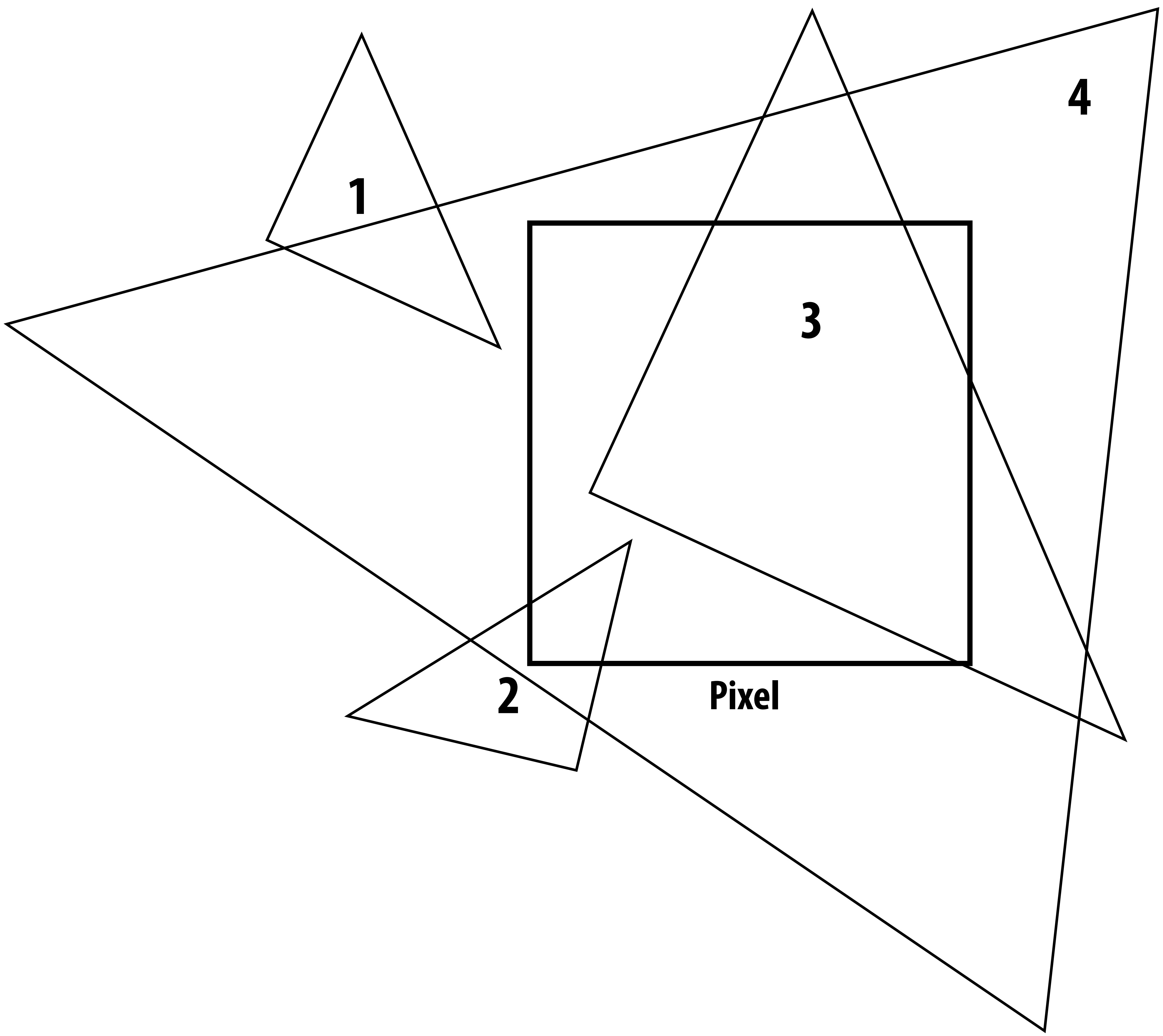


Output:
set of pixels "covered" by the triangle



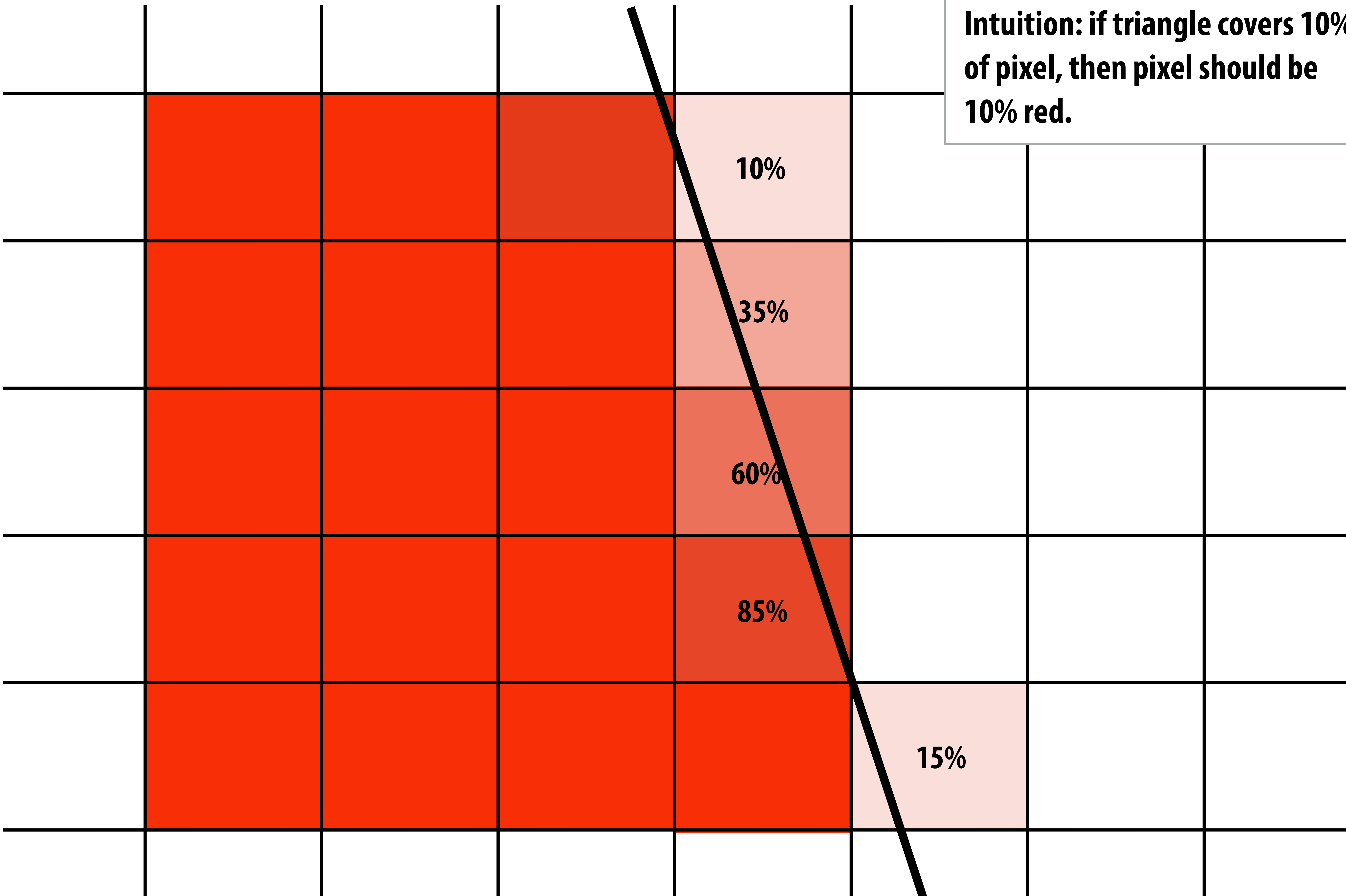
What does it mean for a pixel to be covered by a triangle?

Question: which triangles "cover" this pixel?

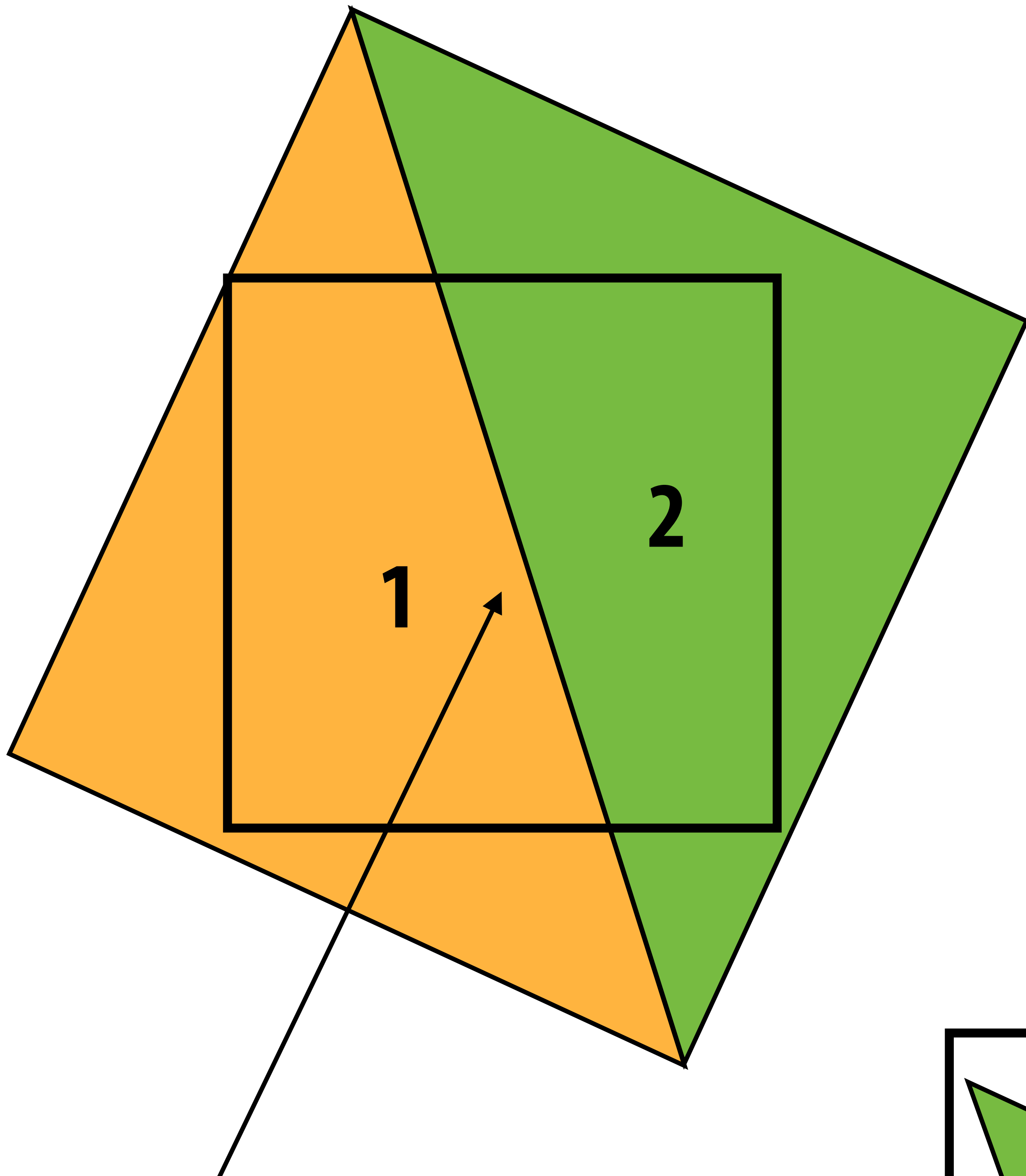


One option: compute fraction of pixel area covered by triangle, then color pixel according to this fraction.

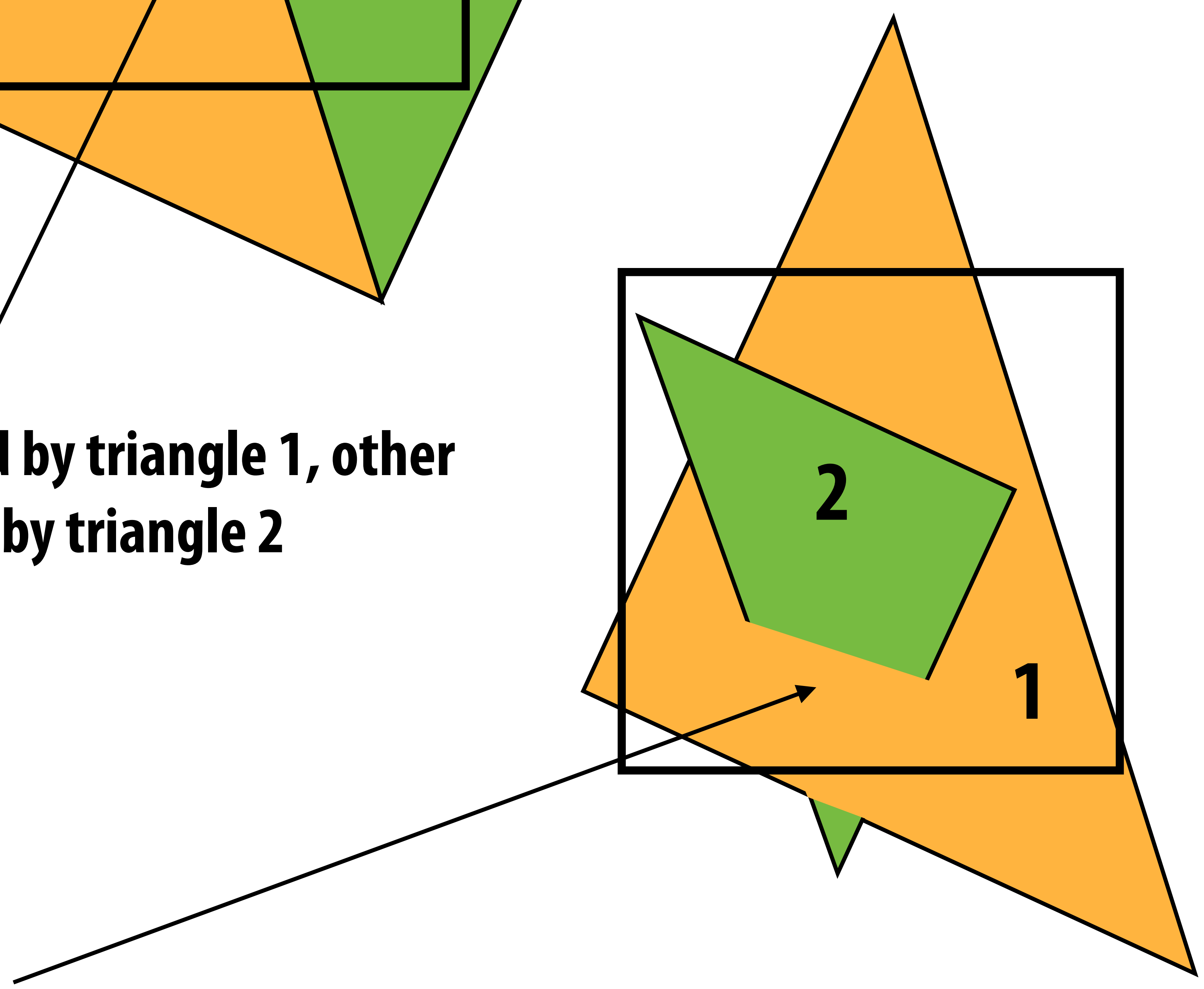
Intuition: if triangle covers 10% of pixel, then pixel should be 10% red.



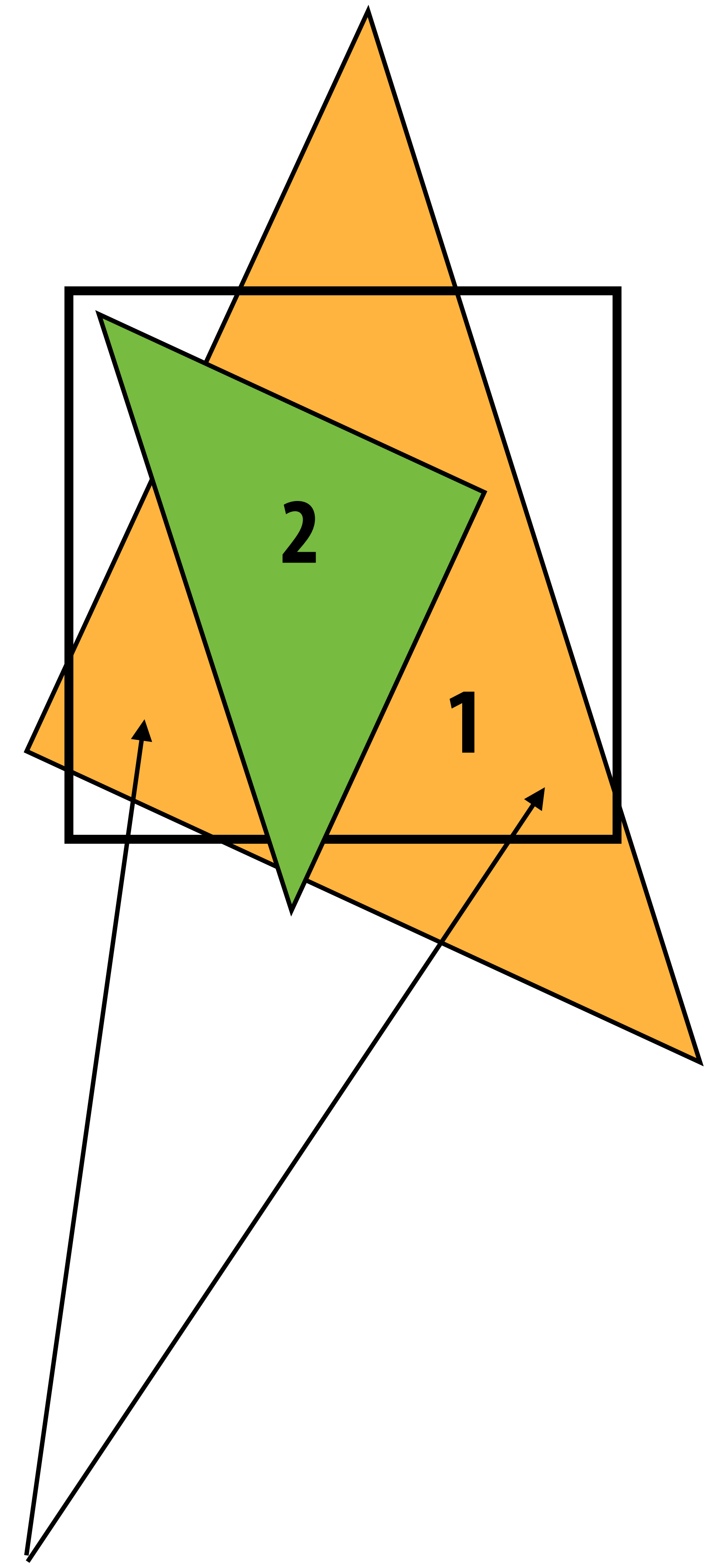
Coverage gets tricky when considering occlusion



Pixel covered by triangle 1, other
half covered by triangle 2



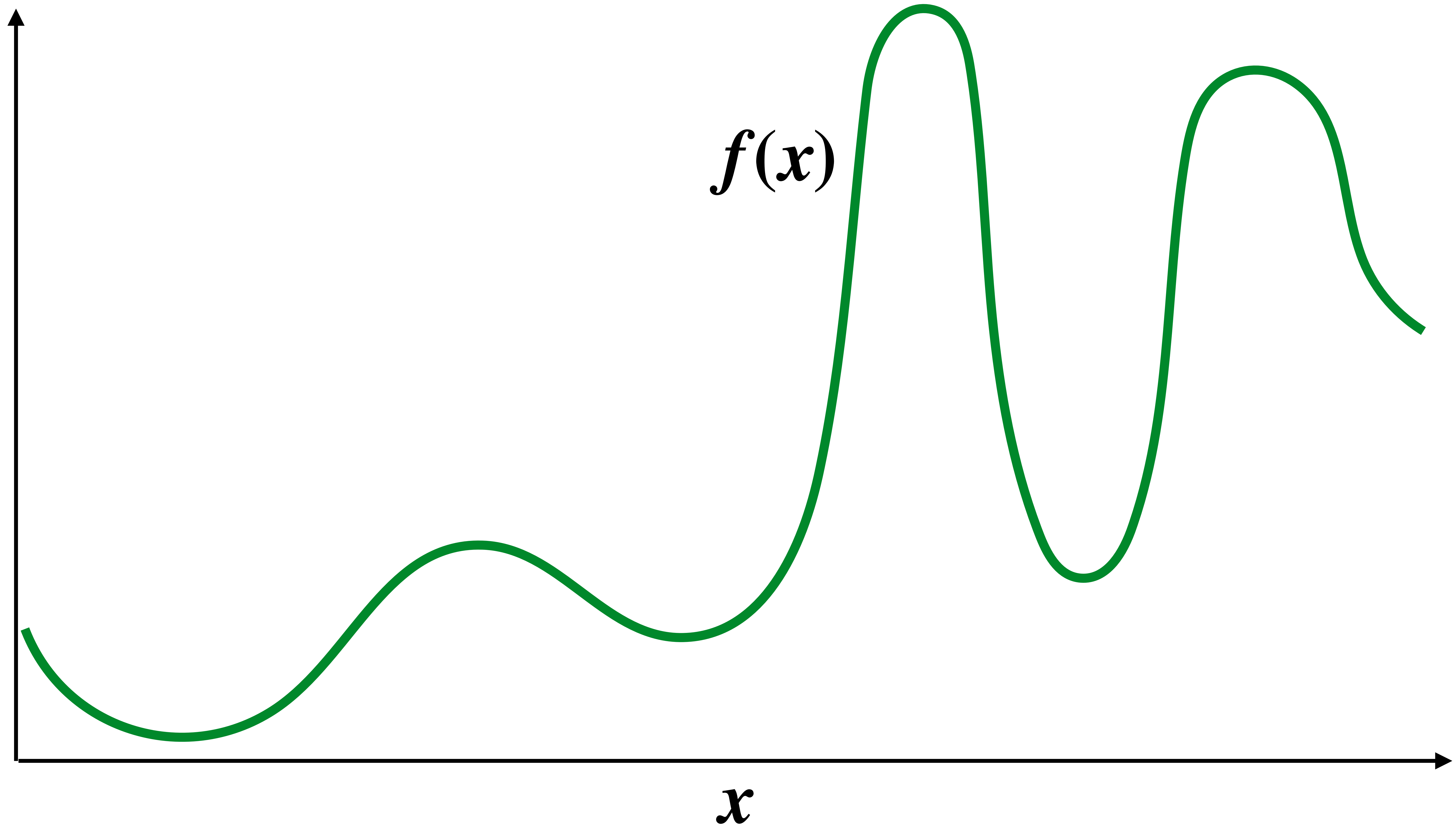
Interpenetration of triangles: even trickier



Two regions of triangle 1 contribute to pixel.
One of these regions is not even convex.

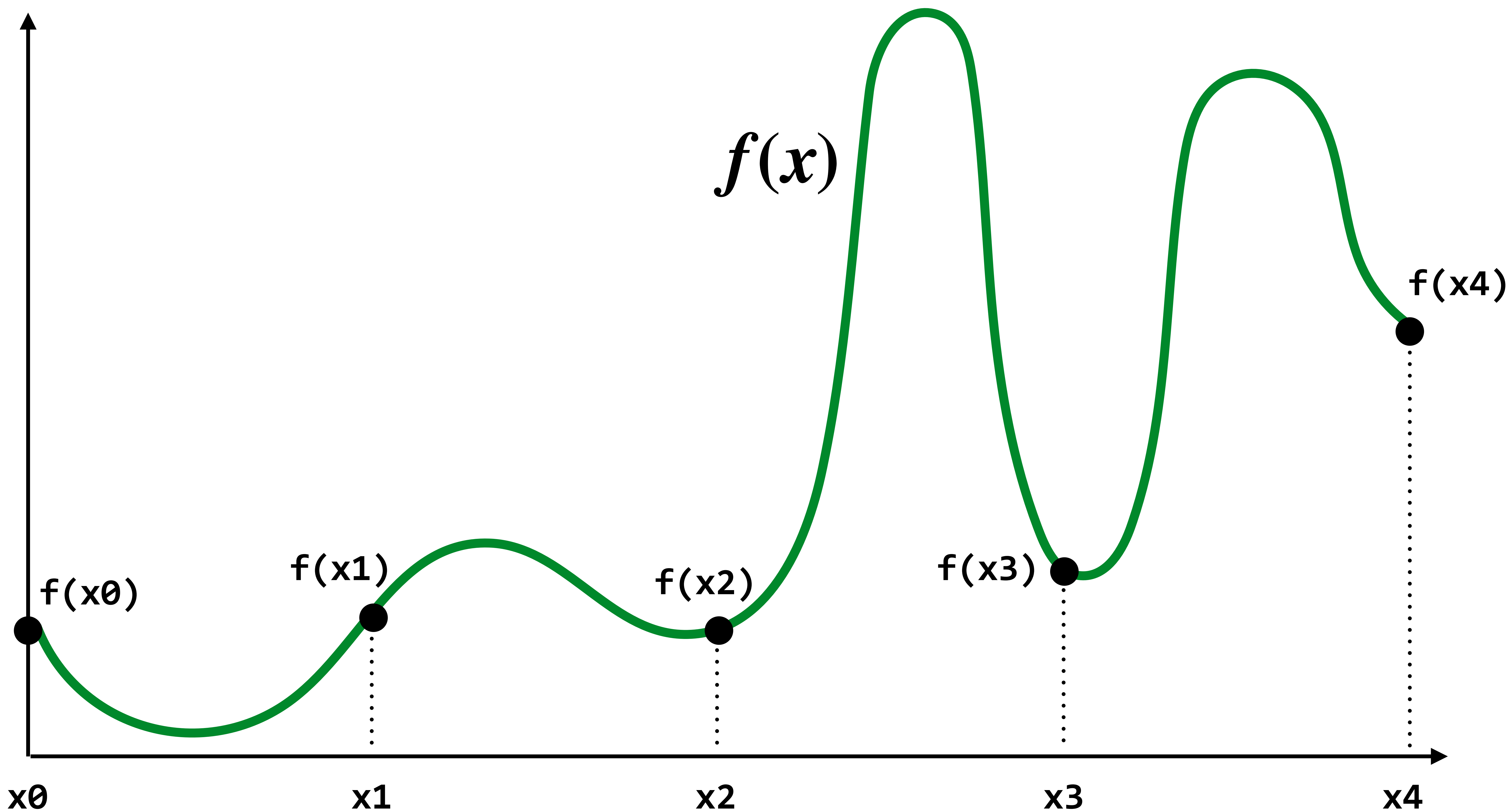
Sampling 101

1D signal



Sampling: taking measurements a signal

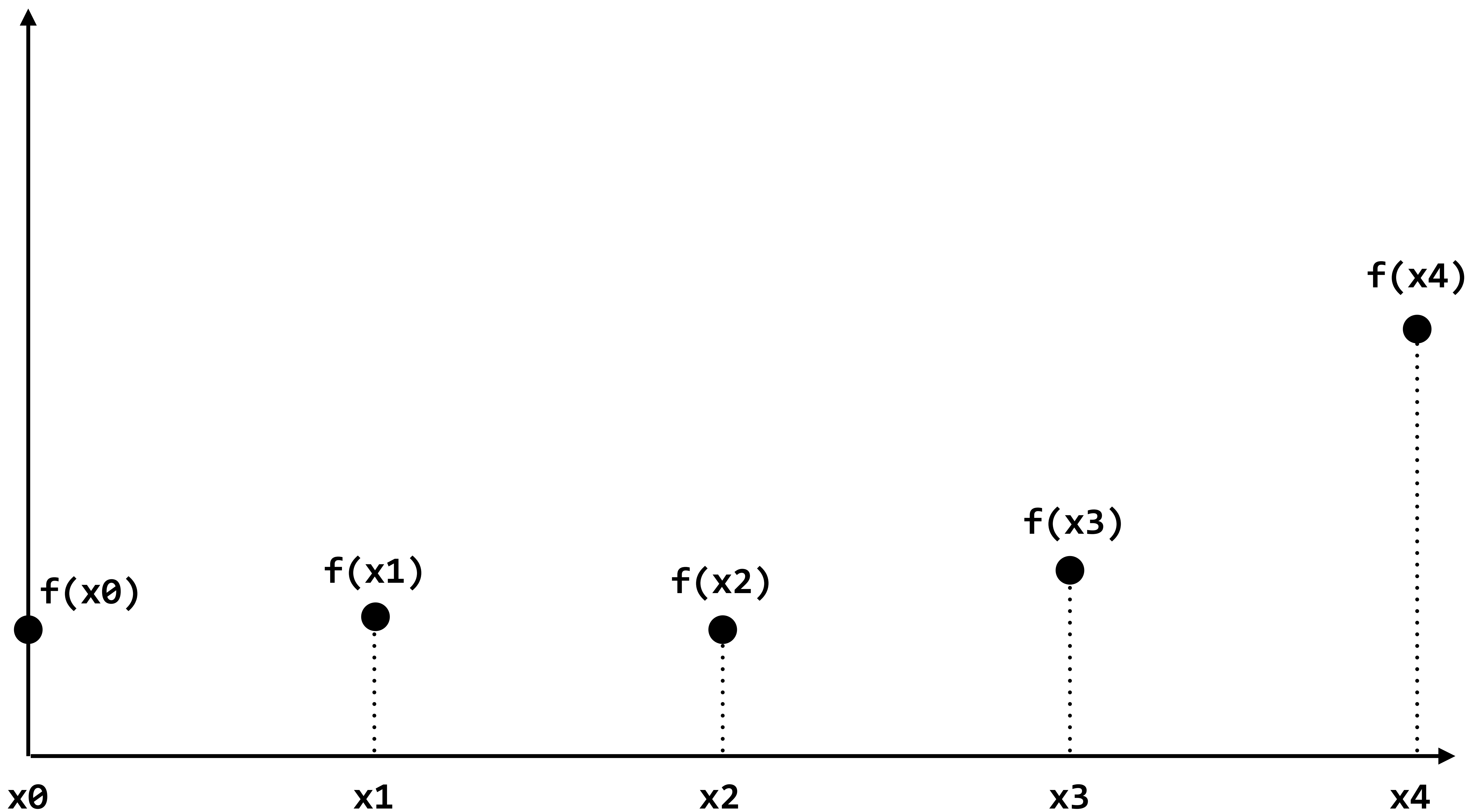
Below: 5 measurements ("samples") of $f(x)$



Audio file: stores samples of a 1D signal

Most consumer audio is sampled at 44.1 KHz

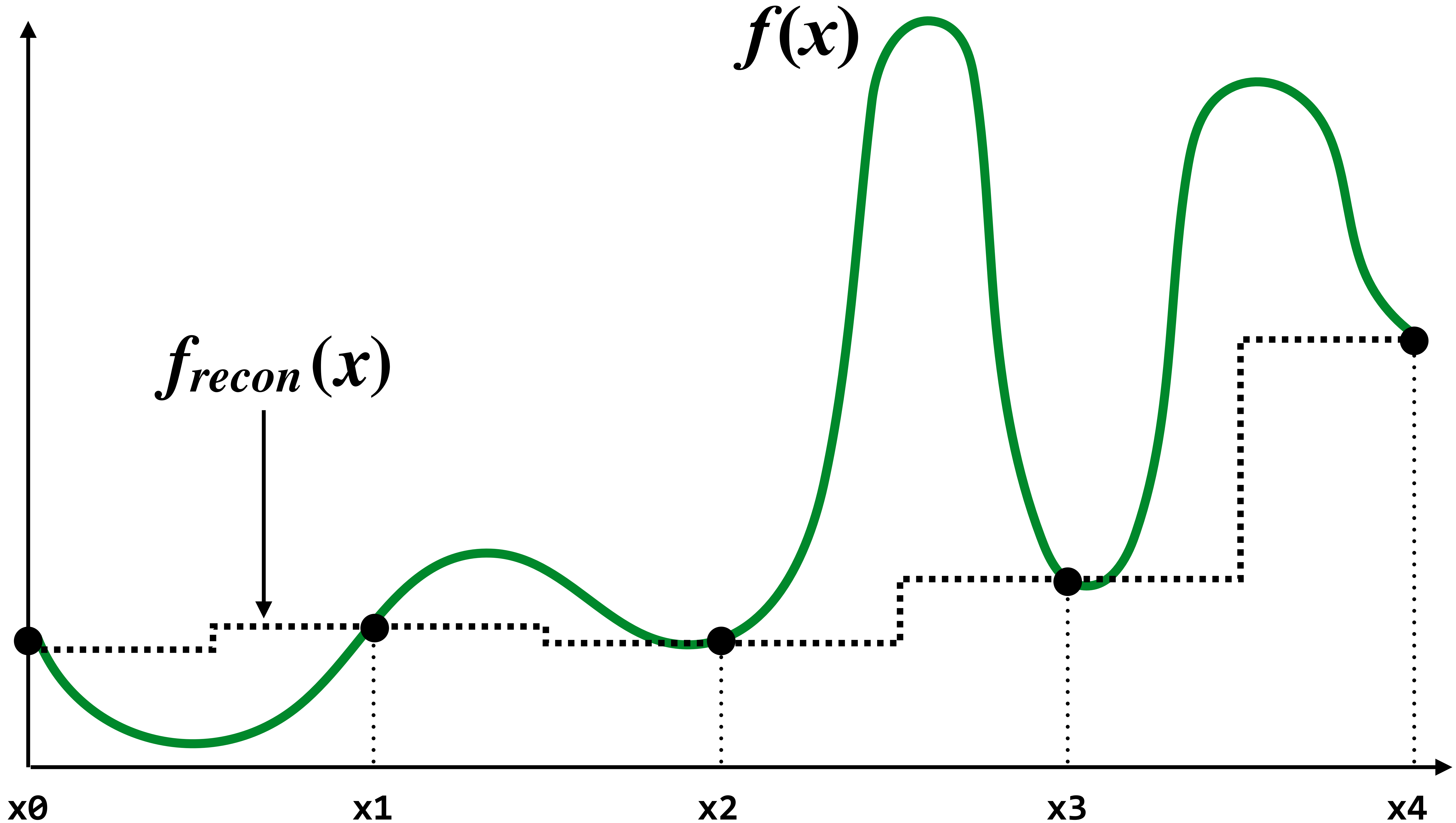
Reconstruction: given a set of samples, how might we attempt to reconstruct the original signal $f(x)$?



Piecewise constant approximation

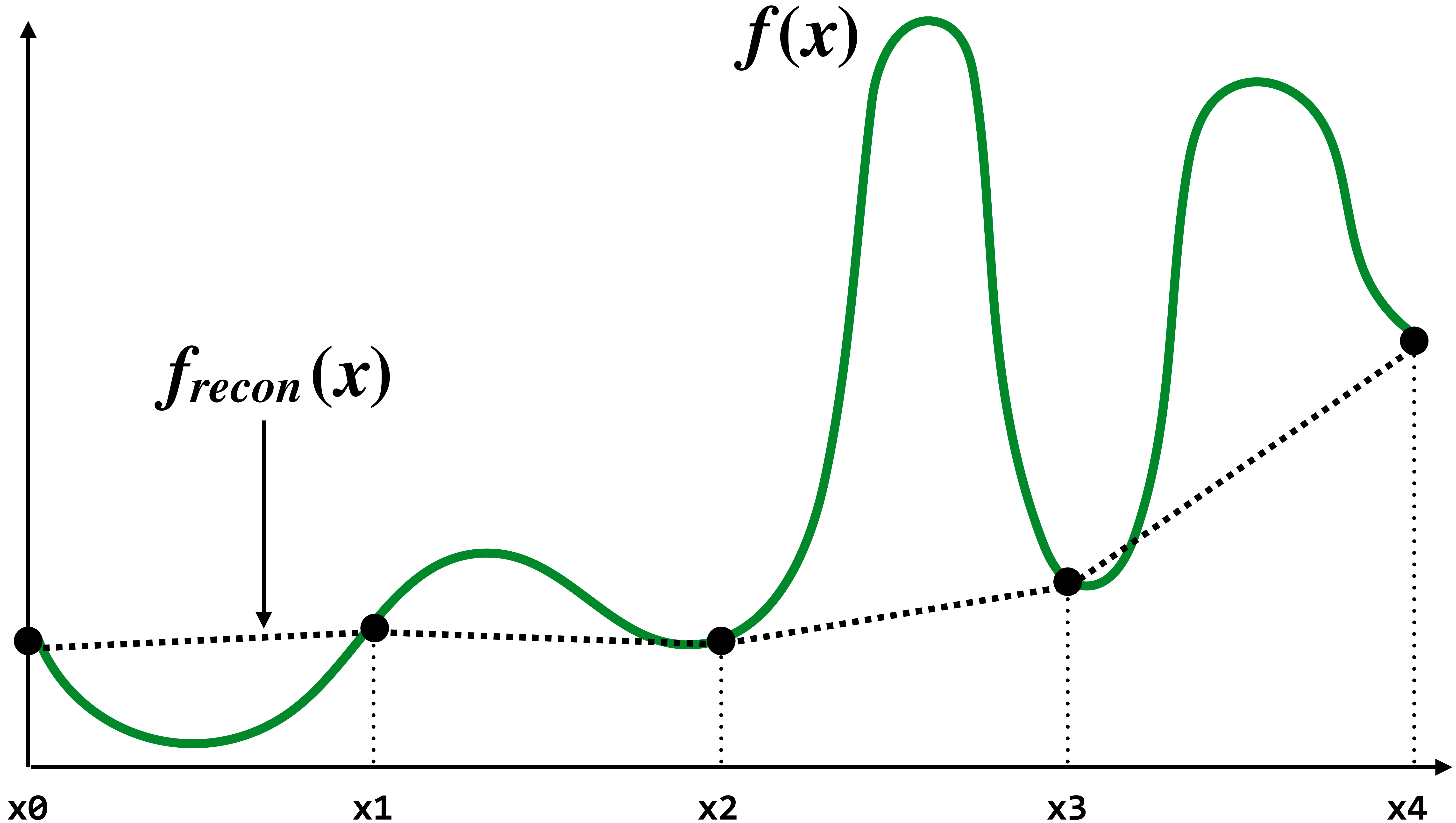
$f_{recon}(x)$ = value of sample closest to x

$f_{recon}(x)$ approximates $f(x)$

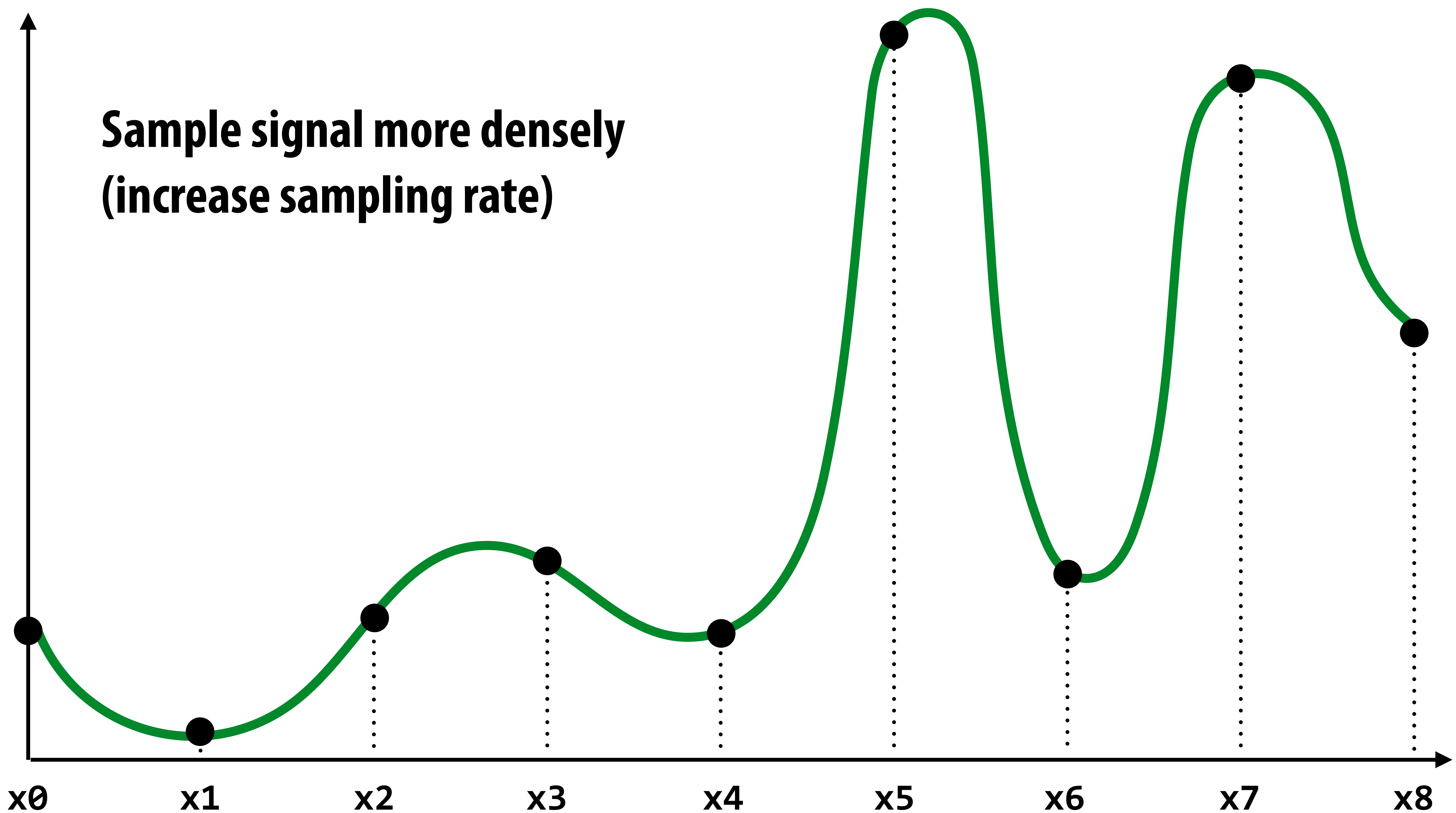


Piecewise linear approximation

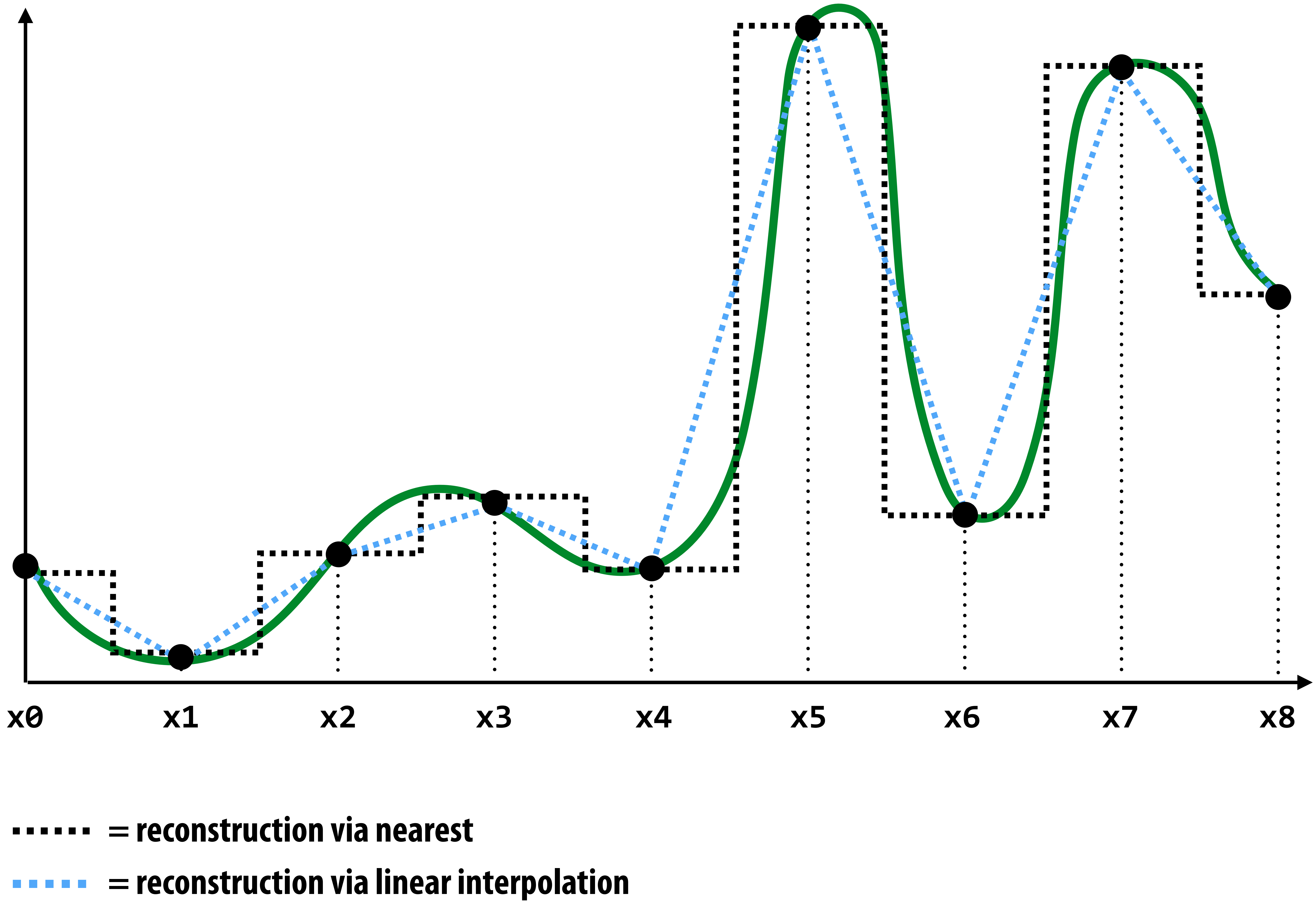
$f_{recon}(x)$ = **linear interpolation between values of two closest samples to x**



How can we represent the signal more accurately?



Reconstruction from denser sampling



Mathematical representation of sampling

Consider the Dirac delta: $\delta(x)$

where for all $x \neq 0, \delta(x) = 0$ **and** $\int_{-\infty}^{\infty} \delta(x) dx = 1$

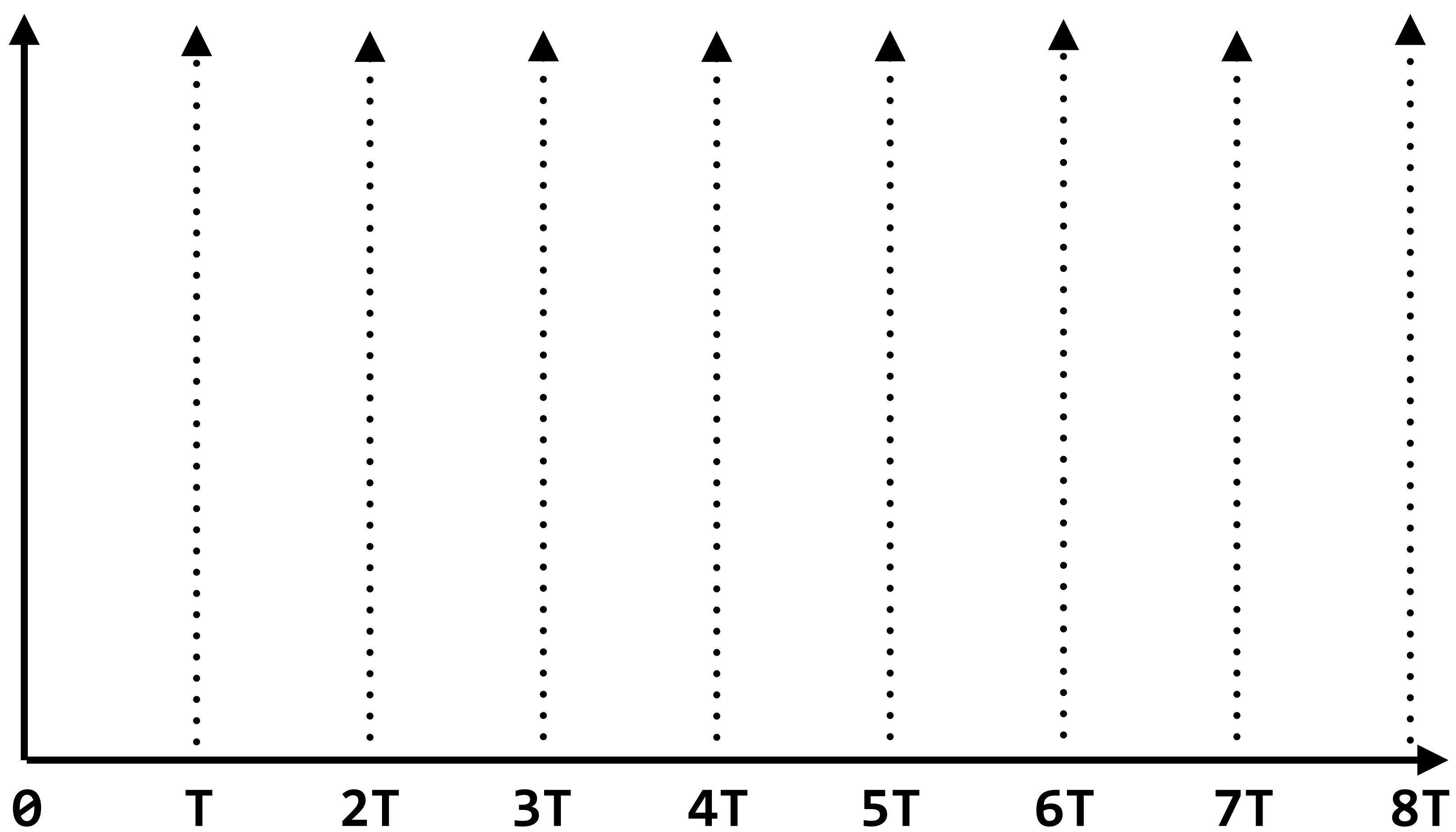
When applied to a function f**,** $\delta(x)$ **acts to pull out the value of** f **at** $x = 0$**:**

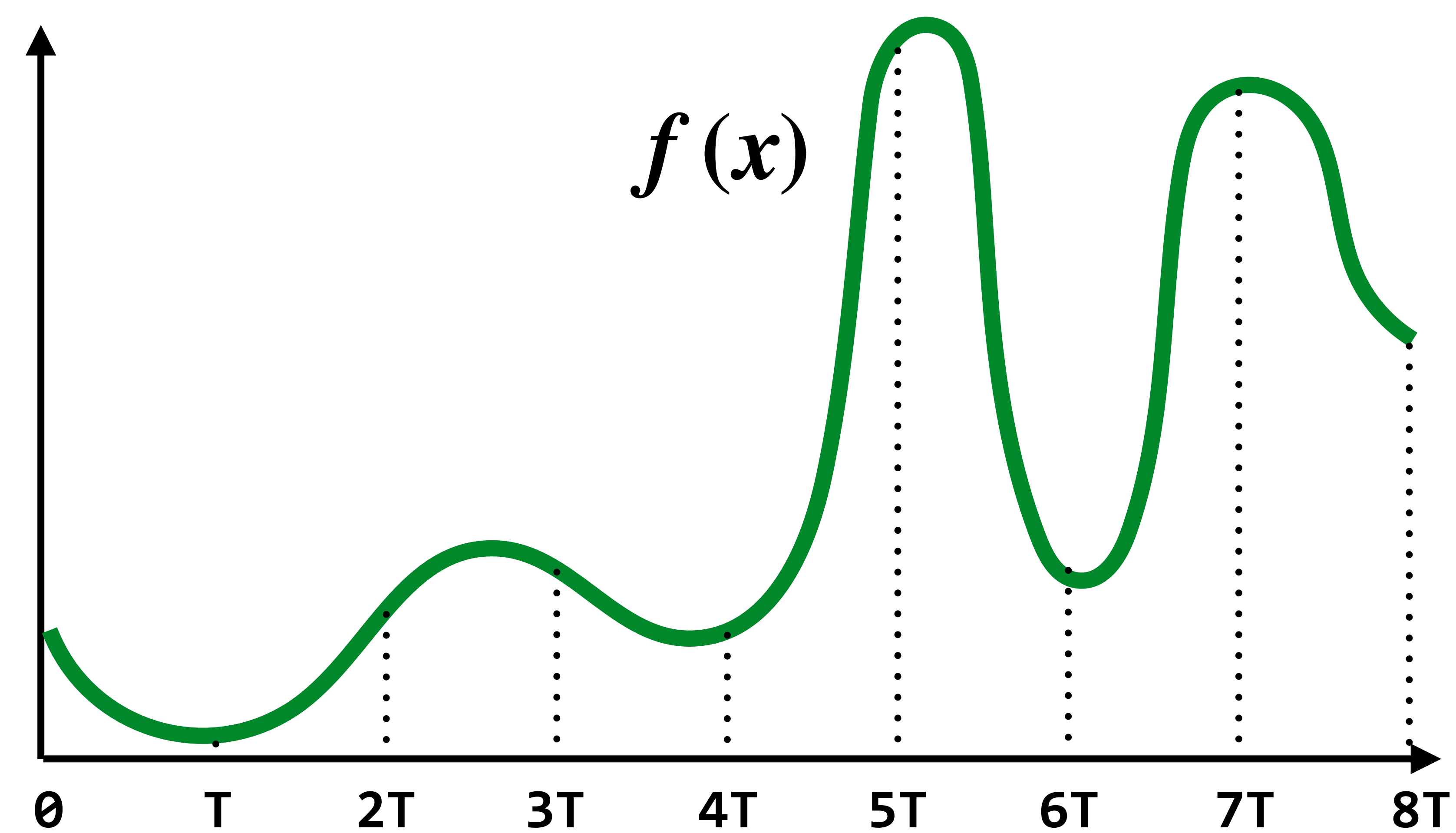
$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

Sampling via "Dirac Comb"

Consider a sequence of impulses with period T:

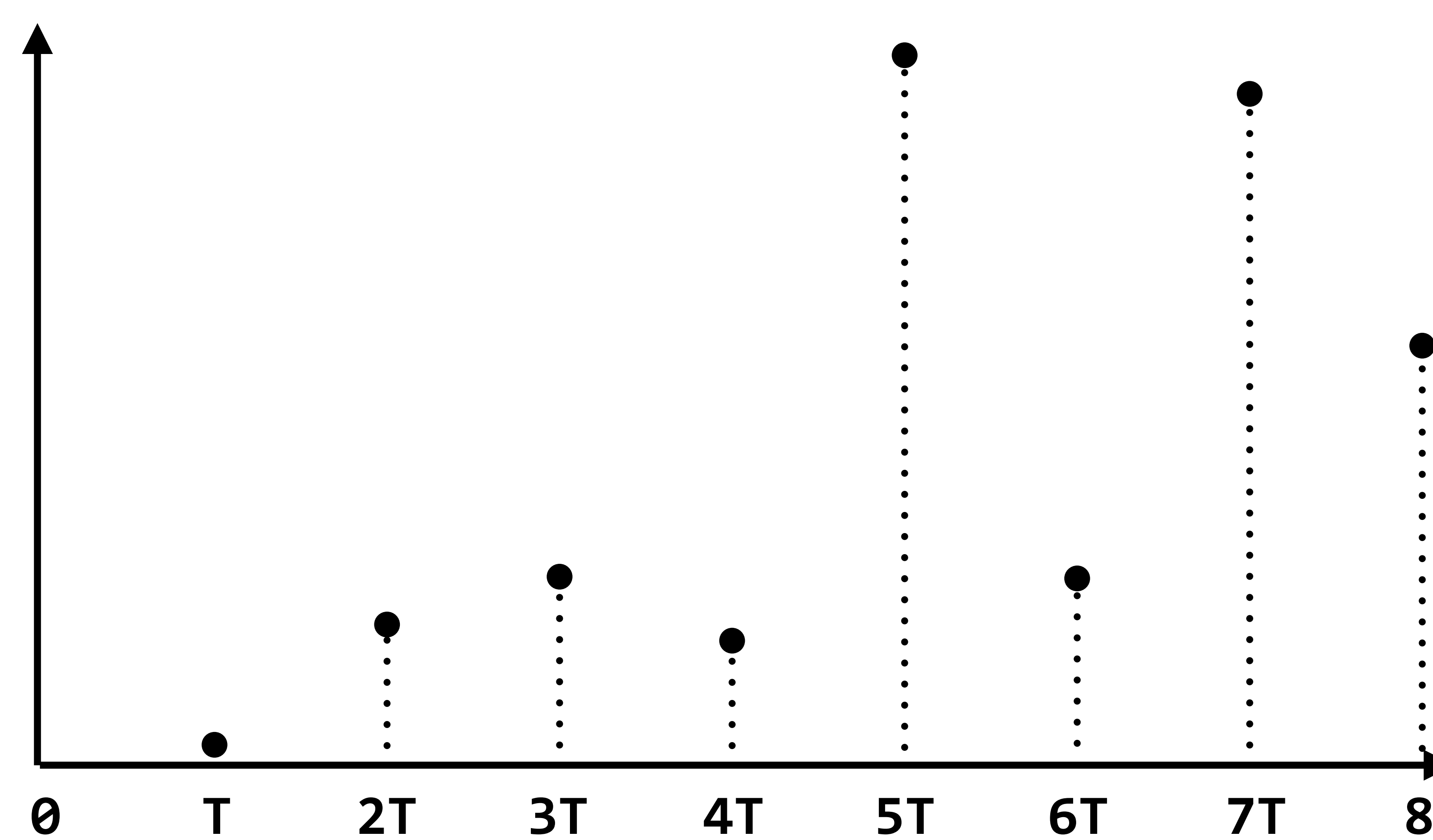
$$Ш_T(x) := \sum_{k=-\infty}^{\infty} \delta(x - kT)$$





Discrete sampling of a continuous function f can be expressed as product between f and Dirac comb:

$$(Ш_T f)(x) = \sum_{k=-\infty}^{\infty} f(kT)\delta(x - kT)$$



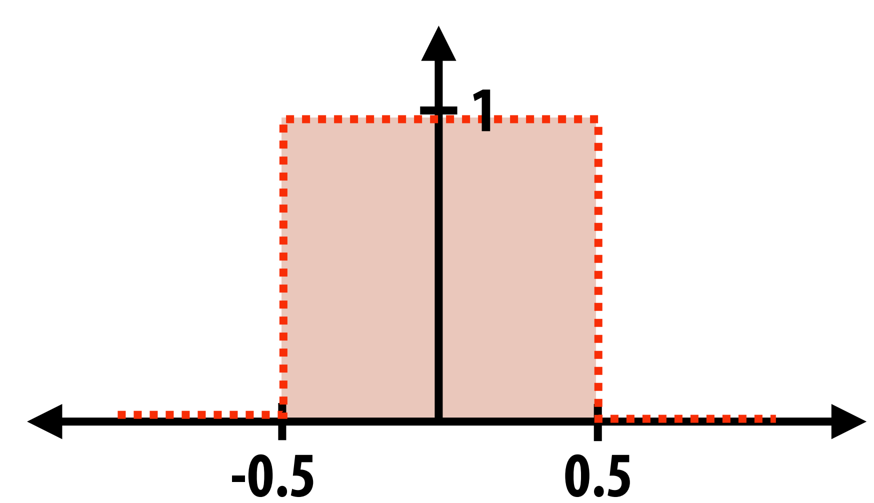
(Ш = Cyrillic "sha")

Reconstruction via Convolution

$$\underbrace{(f*g)(x)}_{\text{output signal}} = \int_{-\infty}^{\infty} \underbrace{f(y)}_{\text{filter}}\underbrace{g(x-y)}_{\text{input signal}}dy$$

It may be helpful to consider the effect of convolution with the simple unit-area "box" function:

$$f(x) = \begin{cases} 1 & |x| \leq 0.5 \\ 0 & otherwise \end{cases}$$



$$(f*g)(x) = \int_{-0.5}^{0.5} g(x-y)dy$$

f* * *g* is a "smoothed" version of *g

Piecewise Constant Reconstruction: Box Filter

Sampled signal:

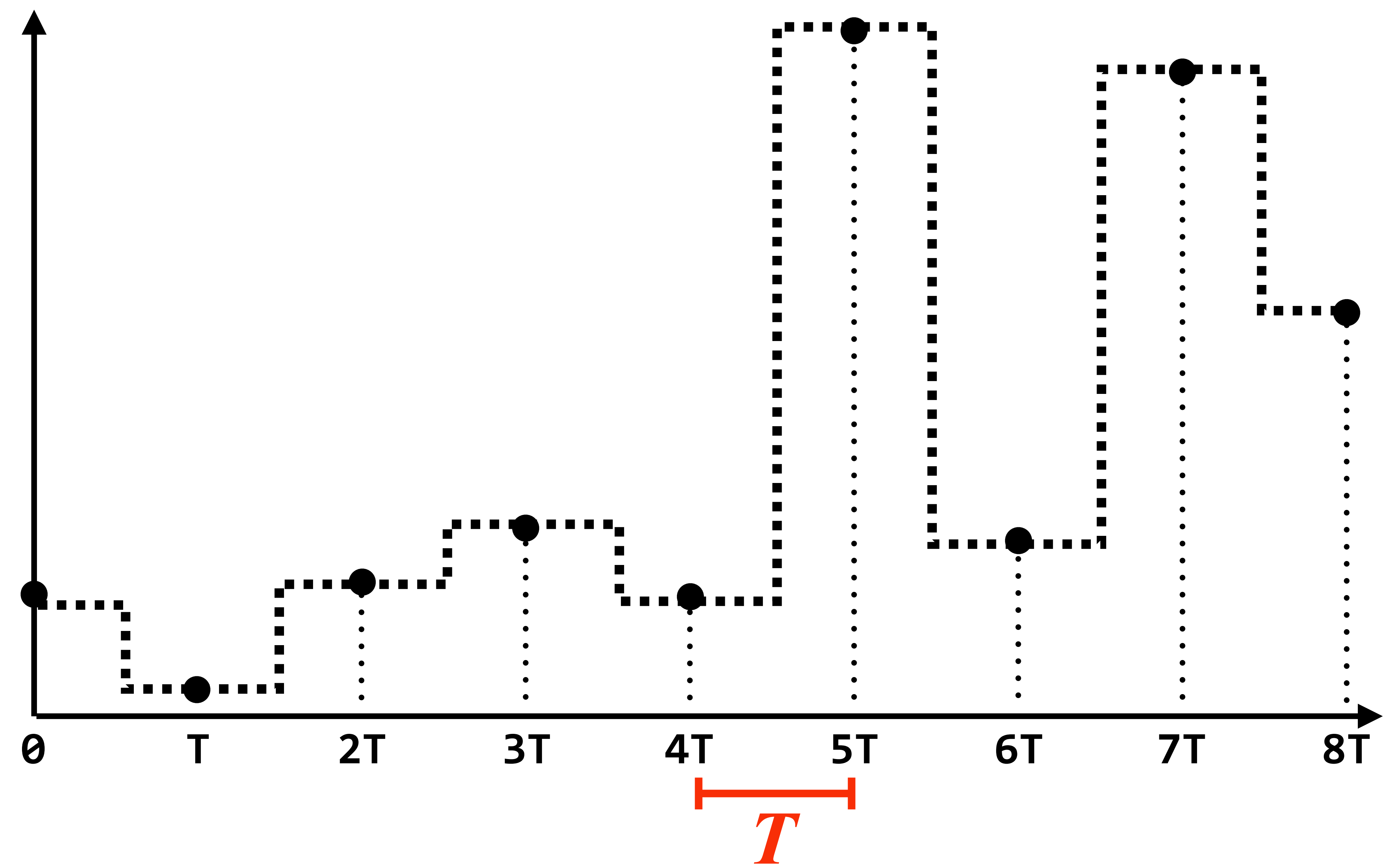
(with period T)

$$g(x) = \text{Ш}_T(x)f(x) = T\sum_{i=-\infty}^{\infty} f(iT)\delta(x - iT)$$

Reconstruction filter:

(unit area box of width T)

$$h(x) = \begin{cases} 1/T & |x| \le T/2 \\ 0 & otherwise \end{cases}$$



Reconstructed signal:

(chooses nearest sample)

$$f_{recon}(x) = (h * g)(x) = T\int_{-\infty}^{\infty} h(y) \sum_{i=-\infty}^{\infty} f(iT)\delta(x - y - iT)dy = \int_{-T/2}^{T/2} \sum_{i=-\infty}^{\infty} f(iT)\delta(x - y - iT)$$

non-zero only for iT closest to x

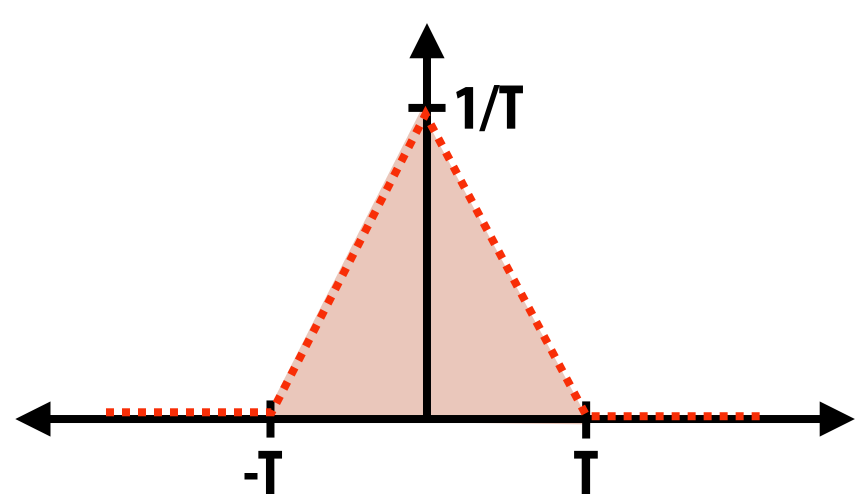
Piecewise Linear Reconstruction: Triangle Filter

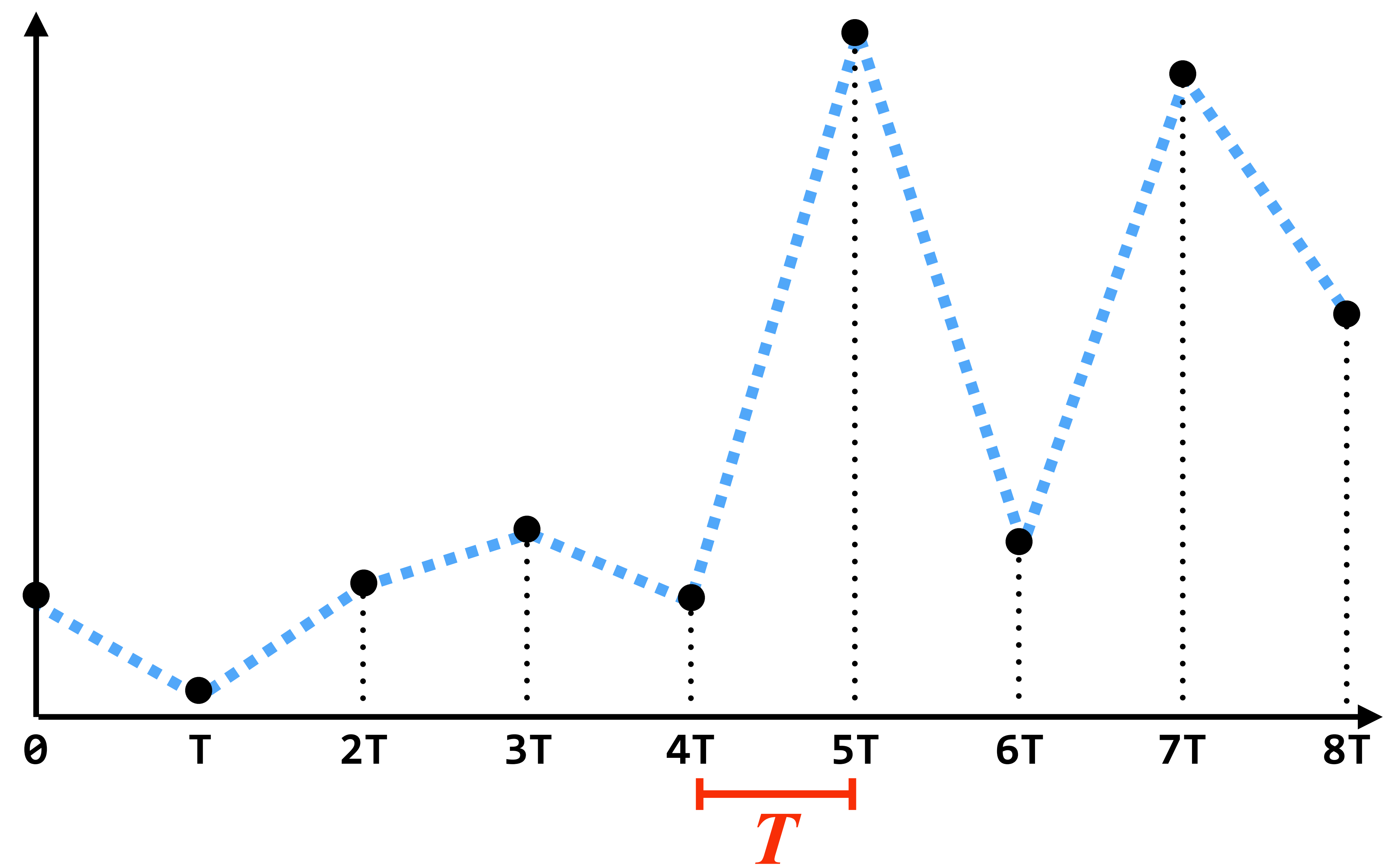
Sampled signal:
(with period T)

$$g(x) = Ш_T(x)f(x) = T\sum_{i=-\infty}^{\infty} f(iT)\delta(x - iT)$$

Reconstruction filter:
(unit area triangle of width T)

$$h(x) = \begin{cases} (1 - \frac{|x|}{T})/T & |x| \leq T \\ 0 & otherwise \end{cases}$$

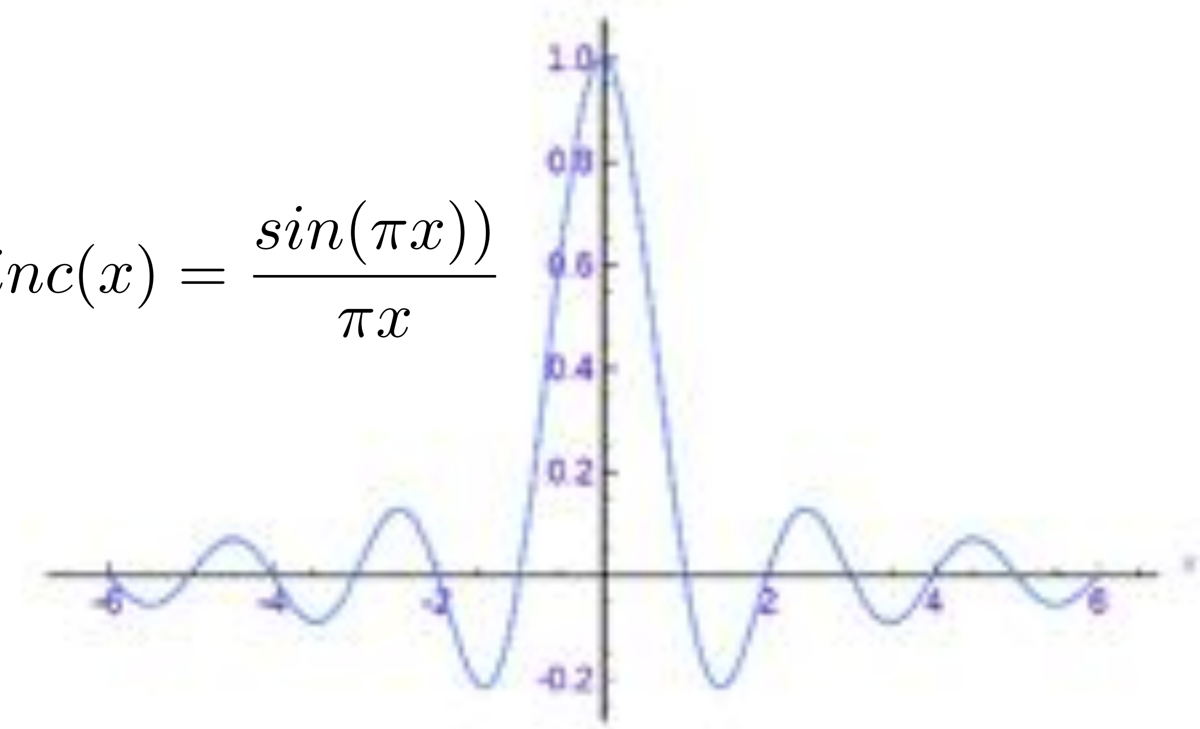




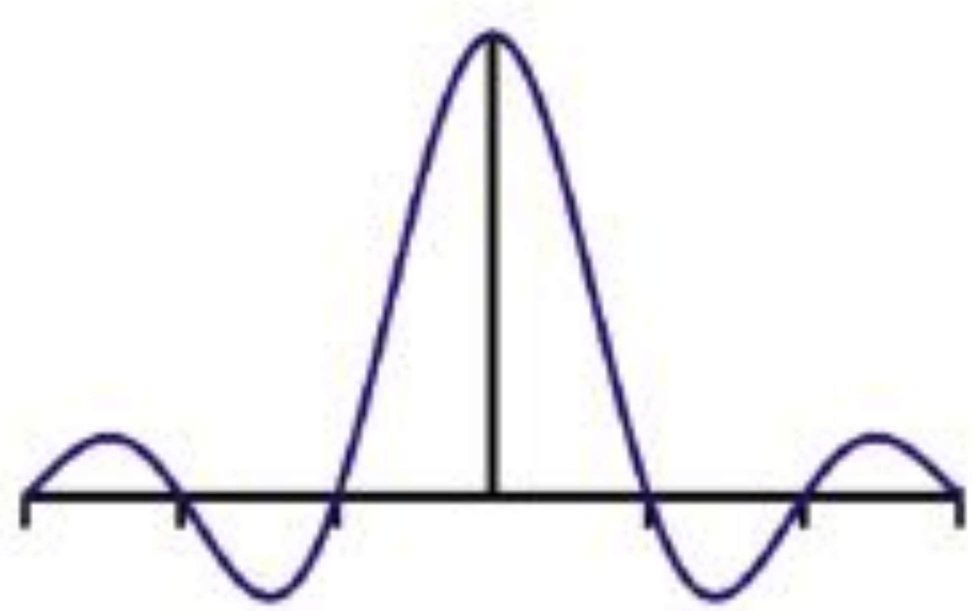
Summary

- Sampling = measurement of a signal
  - Represent signal as discrete set of samples
  - Mathematically described by multiplication by impulse train
- Reconstruction = generating signal from a discrete set of samples
  - Convolution of sampled signal with a reconstruction filter
  - Intuition: value of reconstructed function at any point in domain is a weighted combination of sampled values
  - We discussed simple box, triangle filters, but much higher quality filters exist

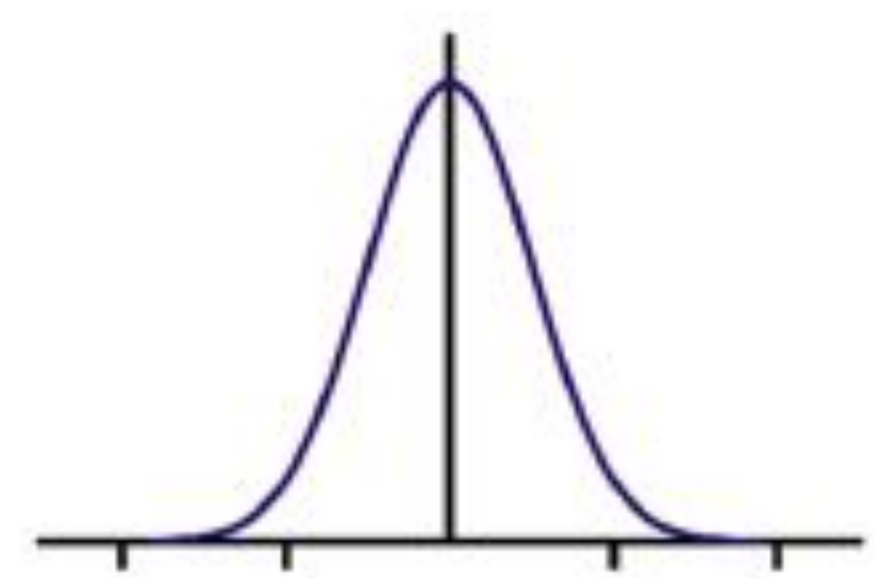
$$sinc(x) = \frac{sin(\pi x))}{\pi x}$$



Normalized sinc filter



Truncated sinc filter



Truncated gaussian filter

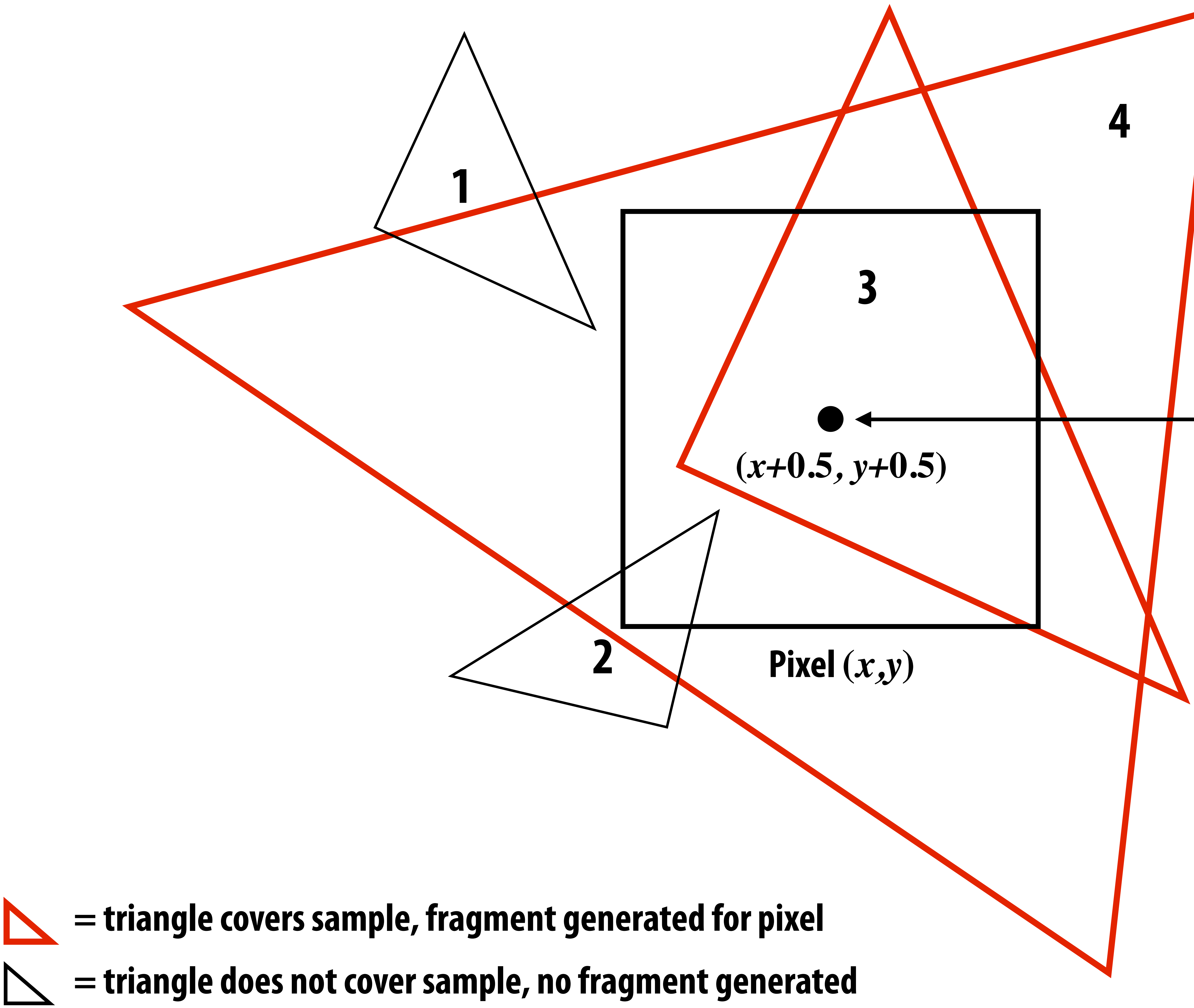
[Image credit: Wikipedia]

Now back to computing coverage

Think of coverage as a 2D signal

$$\texttt{coverage(x,y)} = \begin{cases} \texttt{1} & \texttt{if the triangle contains point (x,y)} \\ \texttt{0} & \texttt{otherwise} \end{cases}$$

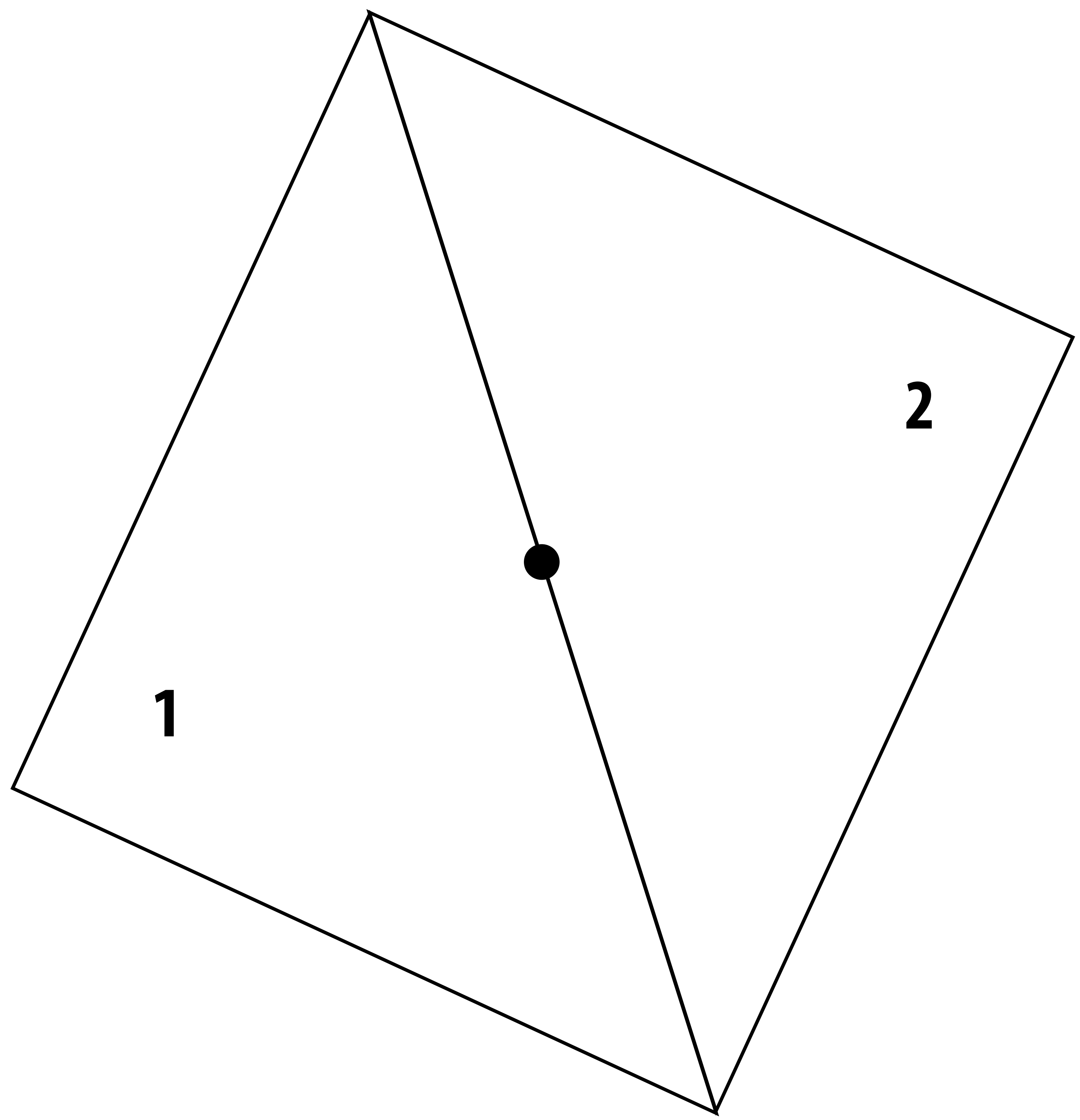
Estimate triangle-screen coverage by sampling the binary function: `coverage(x,y)`



Example:
Here I chose the coverage sample point to be at a point corresponding to the pixel center.

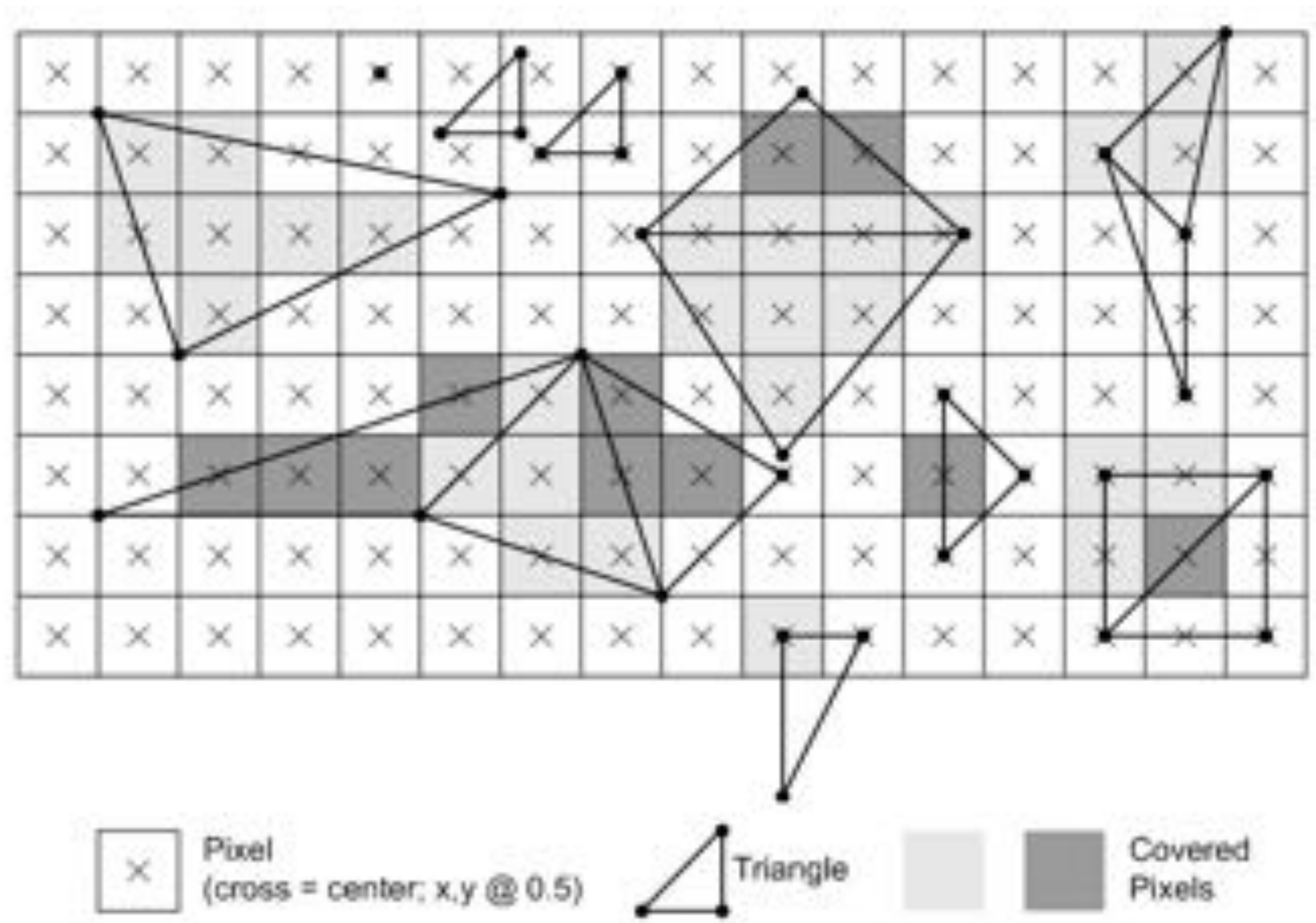
Edge cases (literally)

Is this sample point covered by triangle 1? or triangle 2? or both?



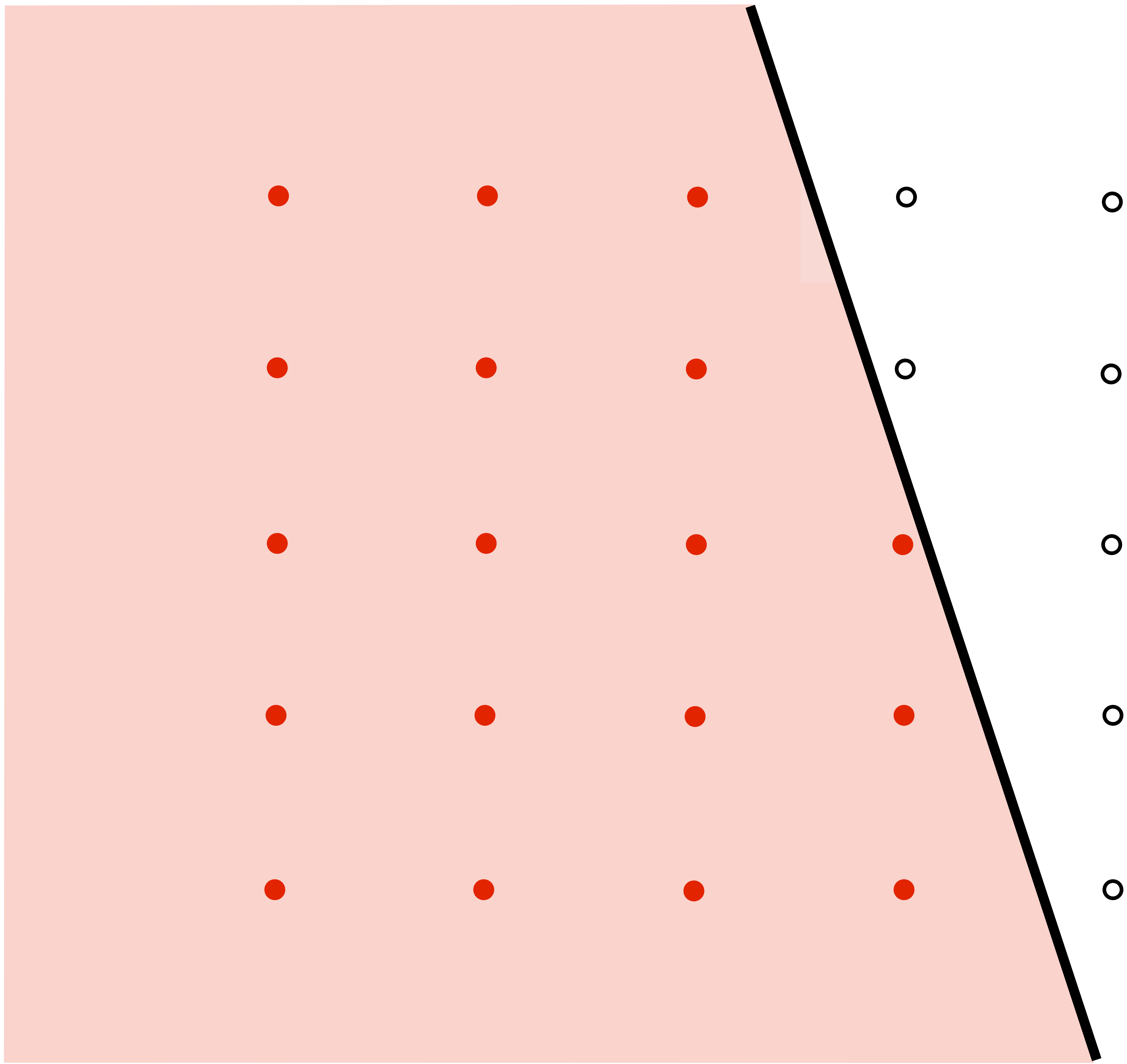
Breaking Ties*

- When edge falls directly on a screen sample point, the sample is classified as within triangle if the edge is a “top edge” or “left edge”
  - Top edge: horizontal edge that is above all other edges
  - Left edge: an edge that is not exactly horizontal and is on the left side of the triangle. (triangle can have one or two left edges)

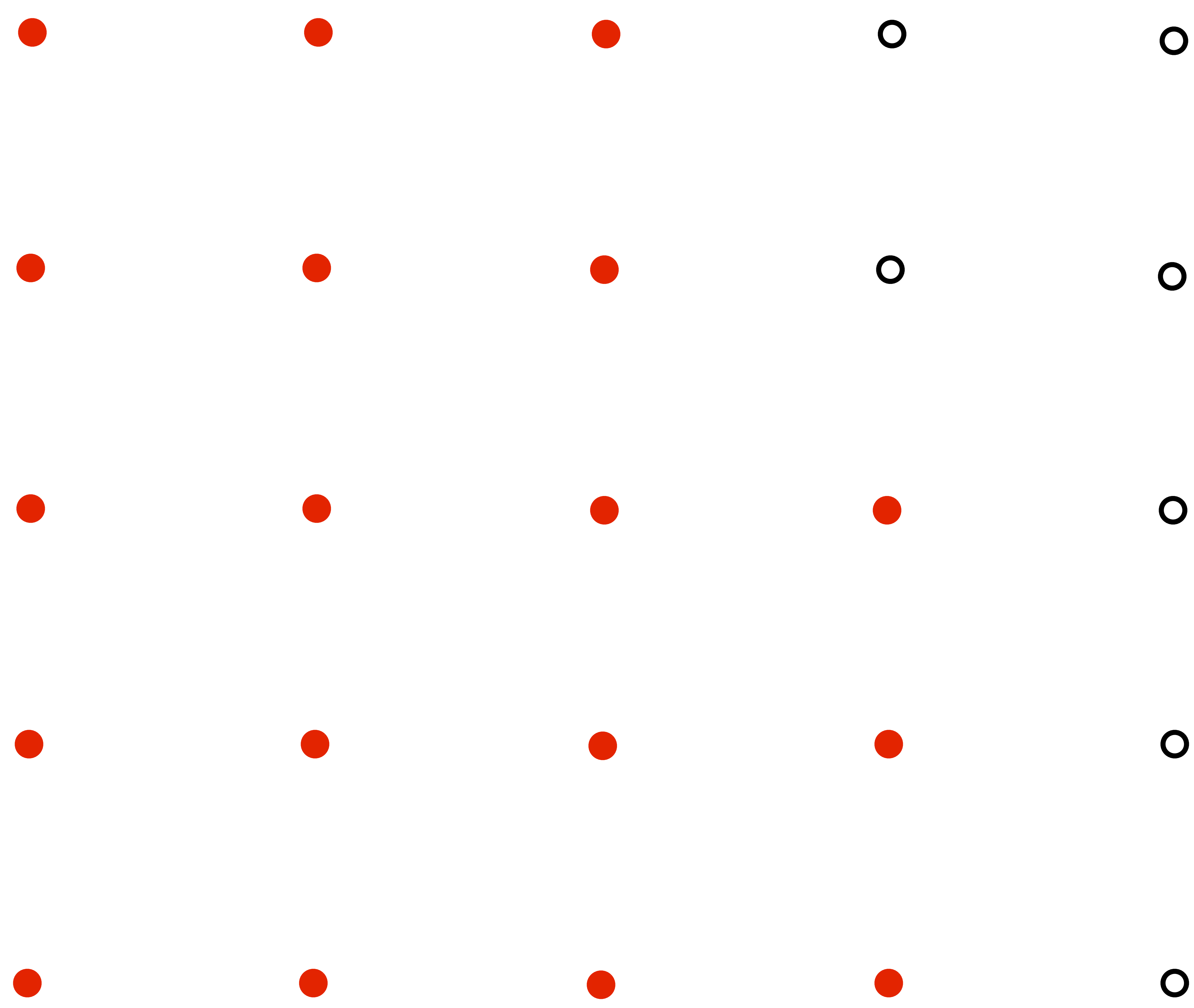


*These are the rules used in OpenGL/Direct3D, i.e., in modern GPUs. Source: Direct3D Programming Guide, Microsoft

Results of sampling triangle coverage



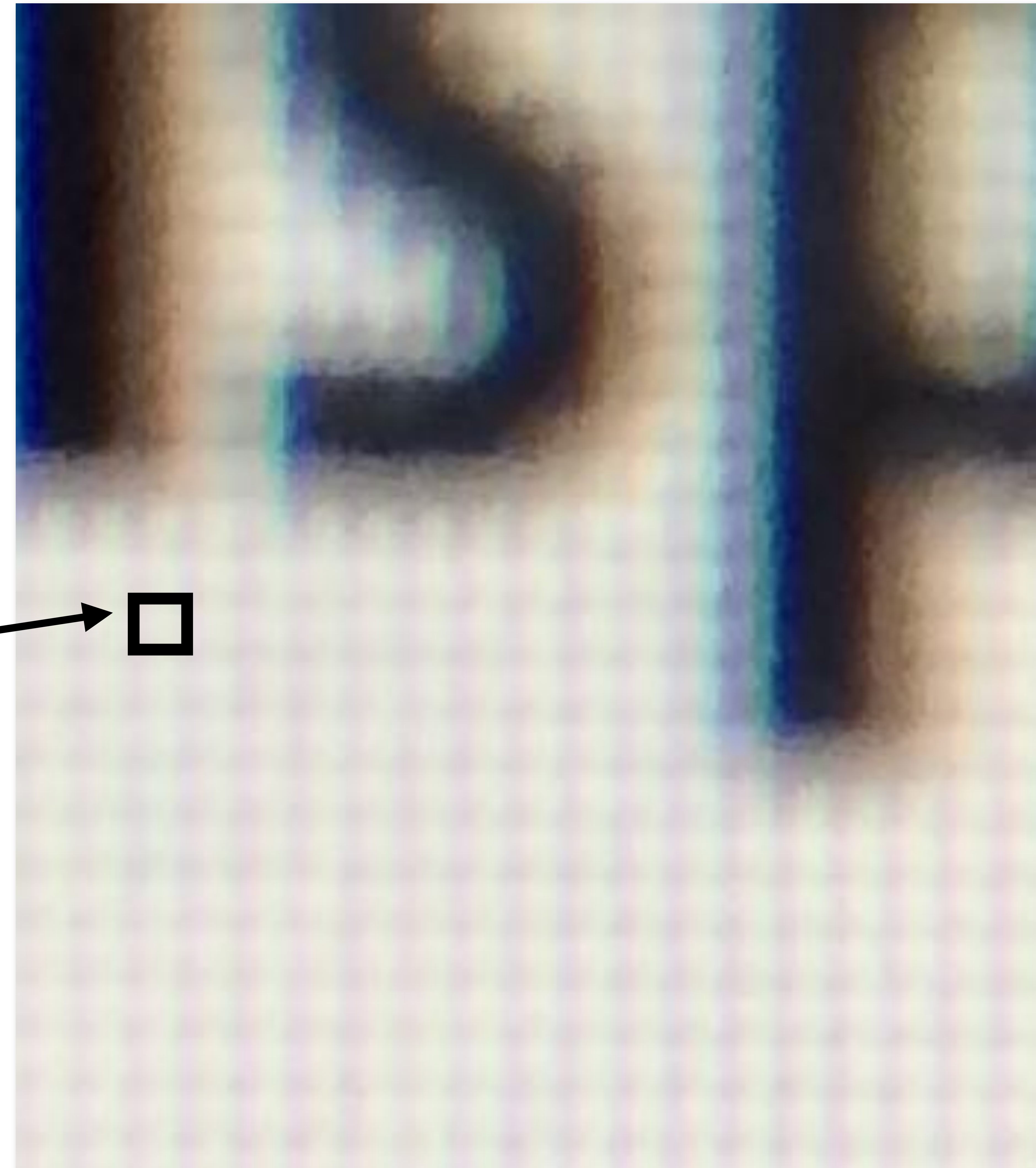
I have a sampled signal, now I want to display it on a screen



Pixels on a screen

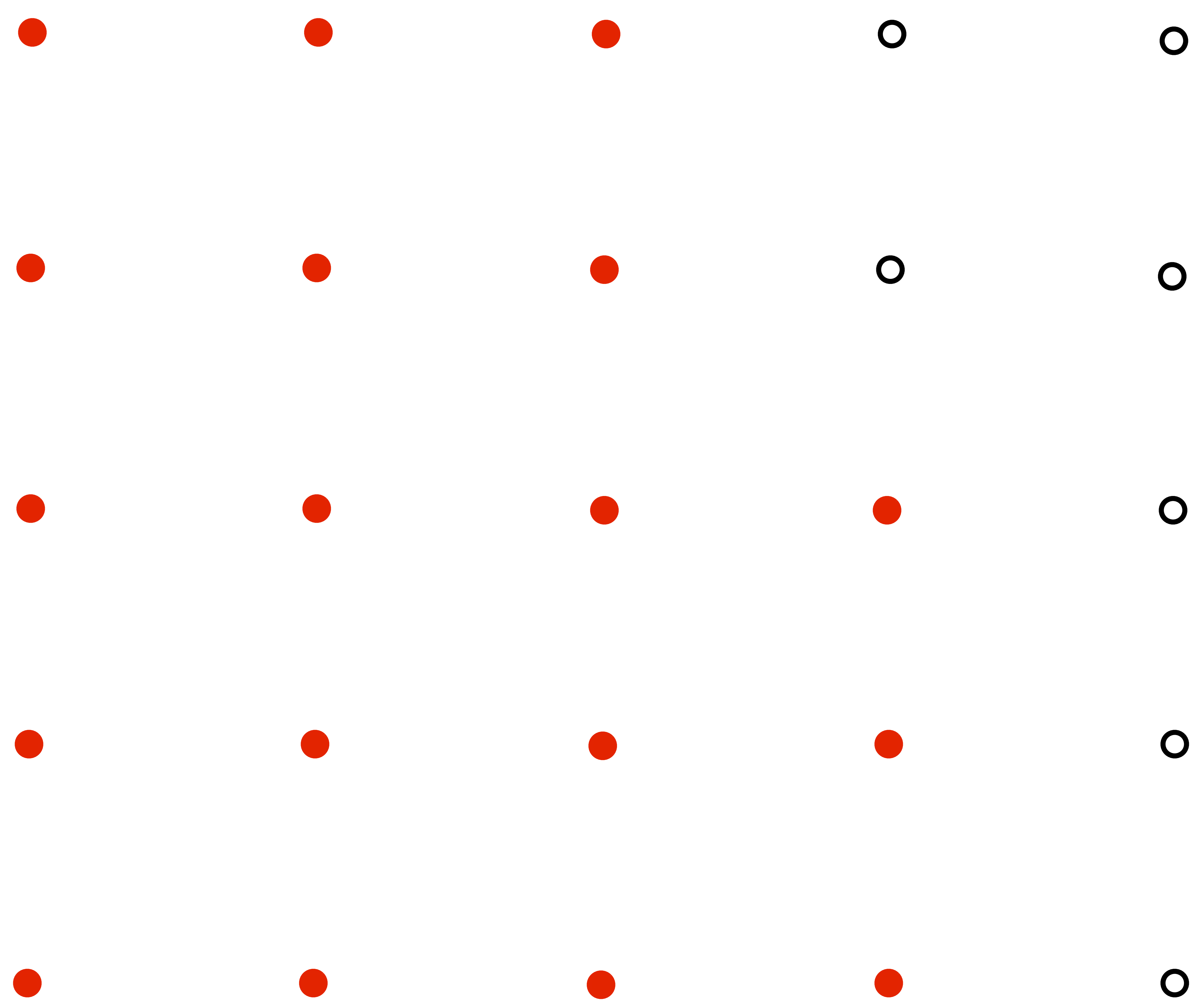
Each image sample sent to the display is converted into a little square of light of the appropriate color:
(a pixel = picture element)

LCD display pixel on my laptop



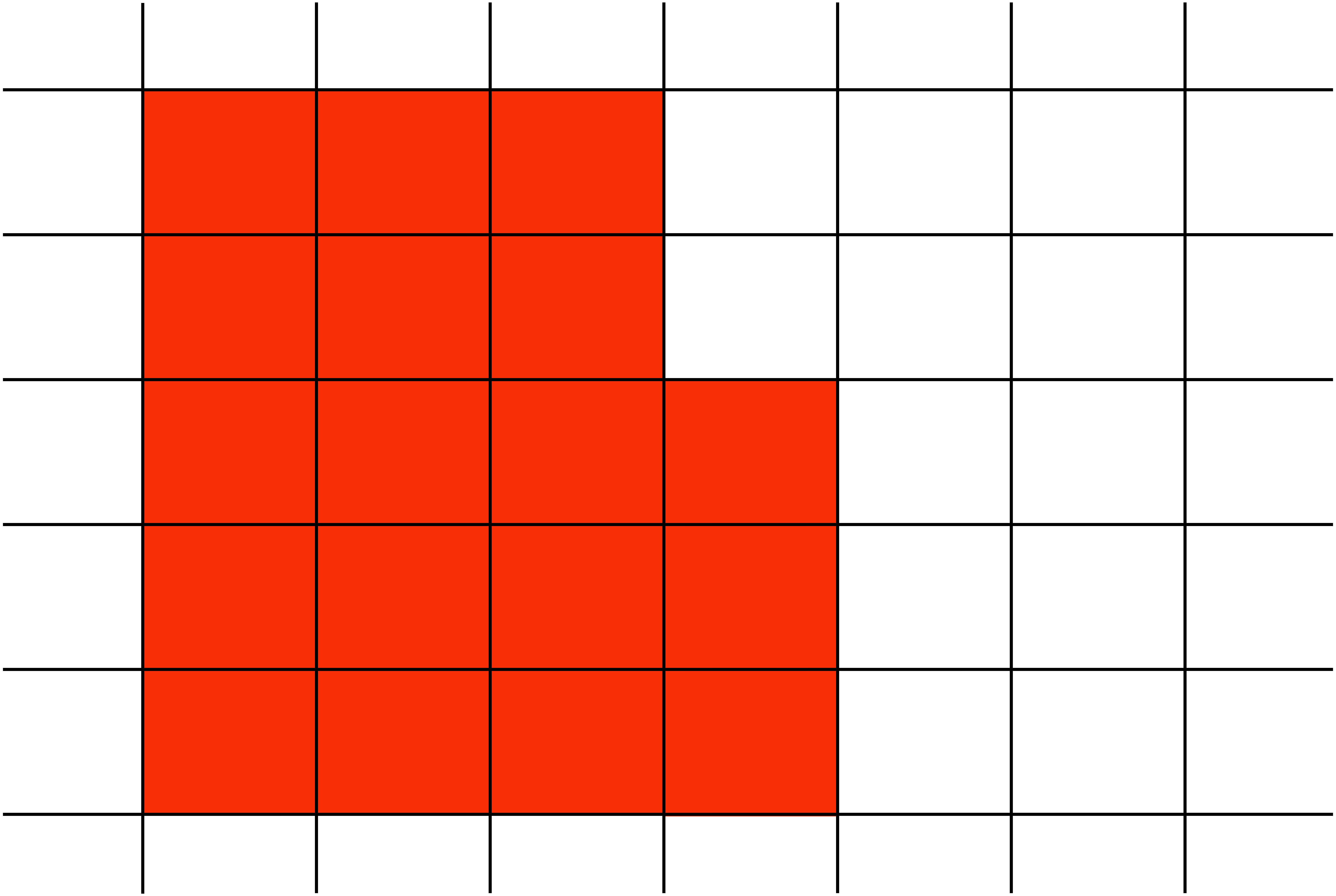
*** Thinking of each LCD pixel as emitting a square of uniform intensity light of a single color is a bit of an approximation to how real displays work, but it will do for now.**

So if we send the display this:



We see this when we look at the screen

(assuming a screen pixel emits a square of perfectly uniform intensity of light)

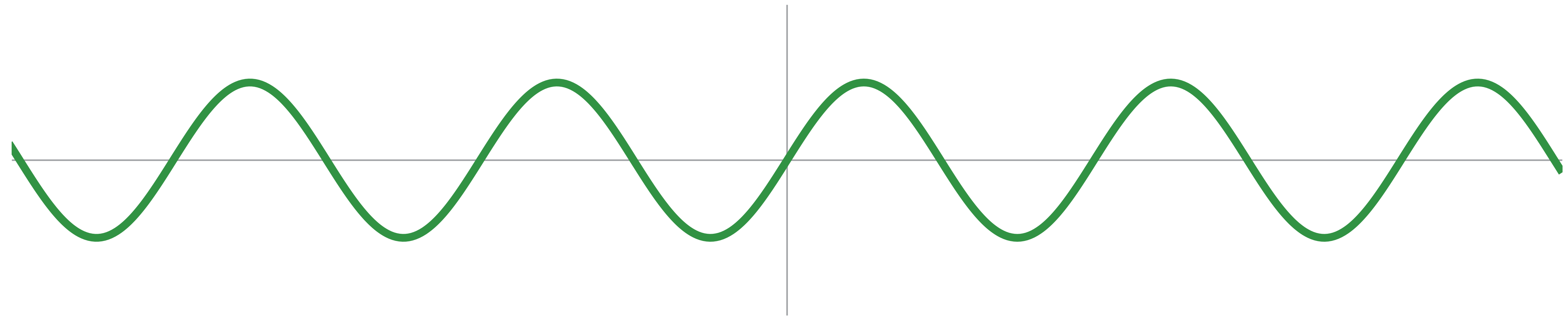


Recall: the real coverage signal was this

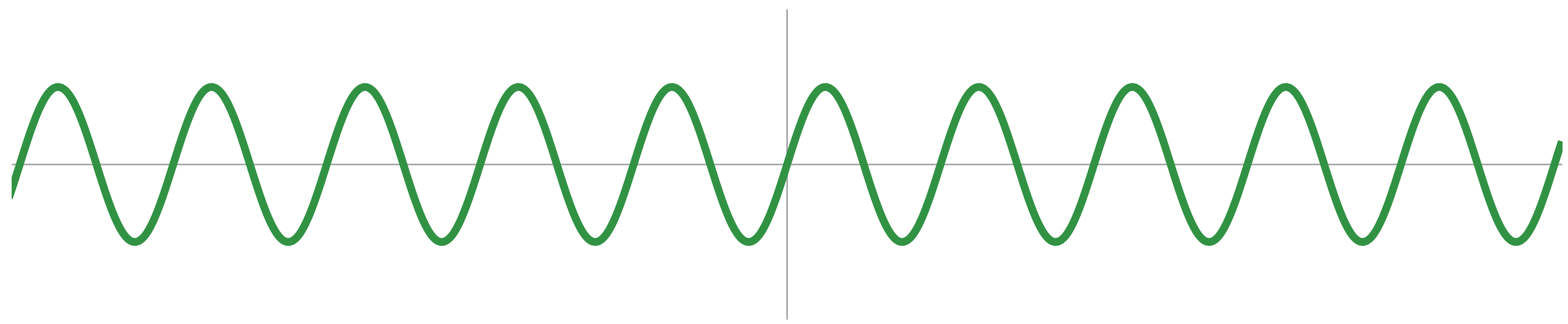
Aliasing

Sound can be expressed as a superposition of frequencies

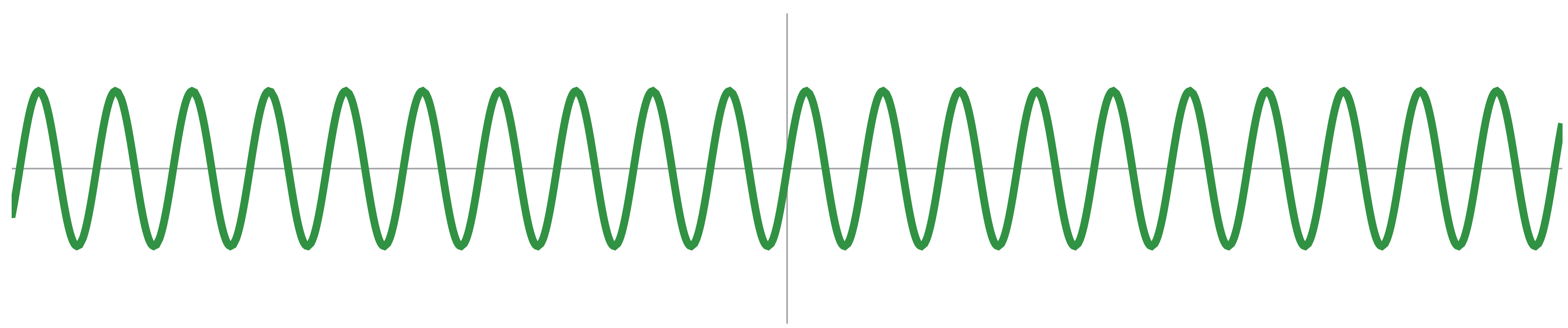
$f_1(x) = sin(\pi x)$



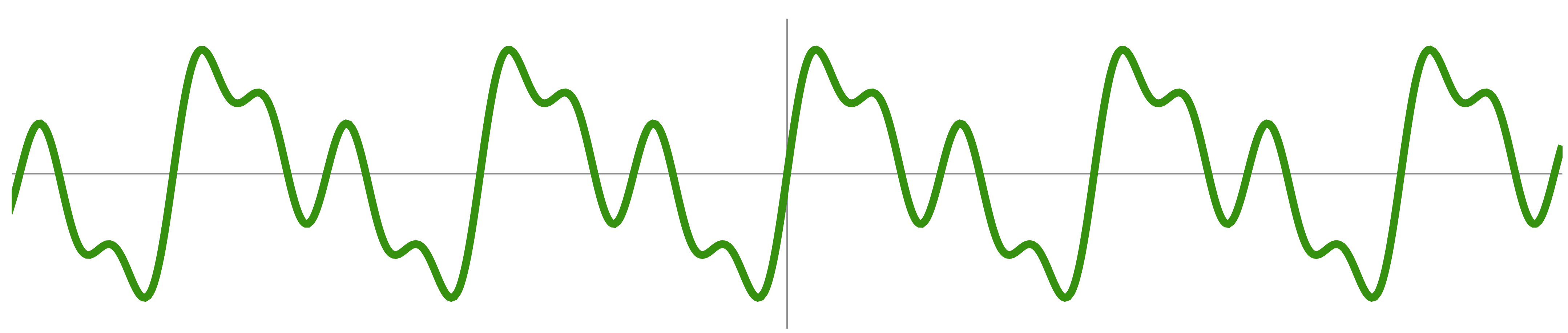
$f_2(x) = sin(2\pi x)$



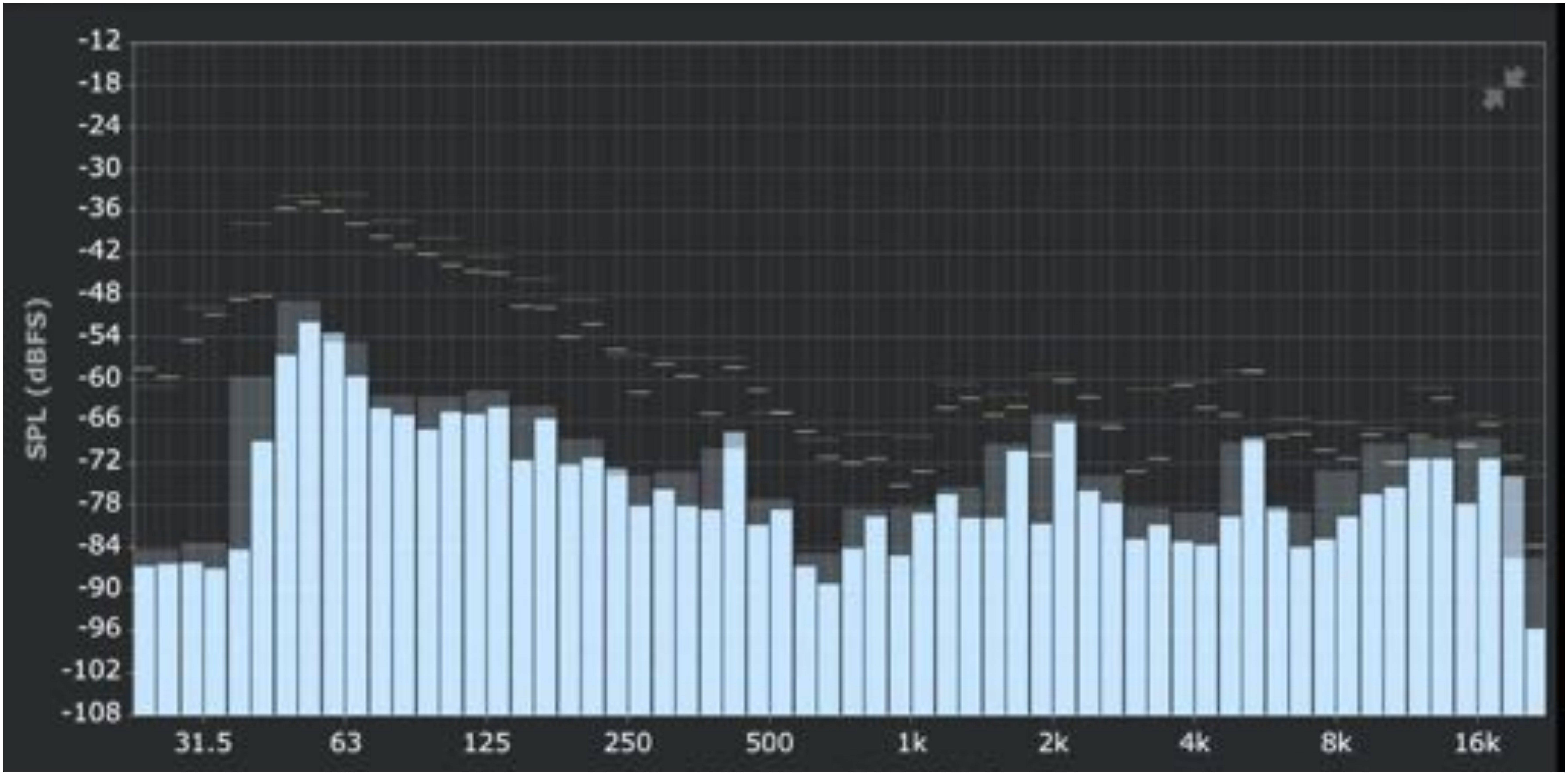
$f_4(x) = sin(4\pi x)$



$f(x) = f_1(x) + 0.75\, f_2(x) + 0.5\, f_4(x)$



An audio spectrum analyzer shows the amplitude of each frequency



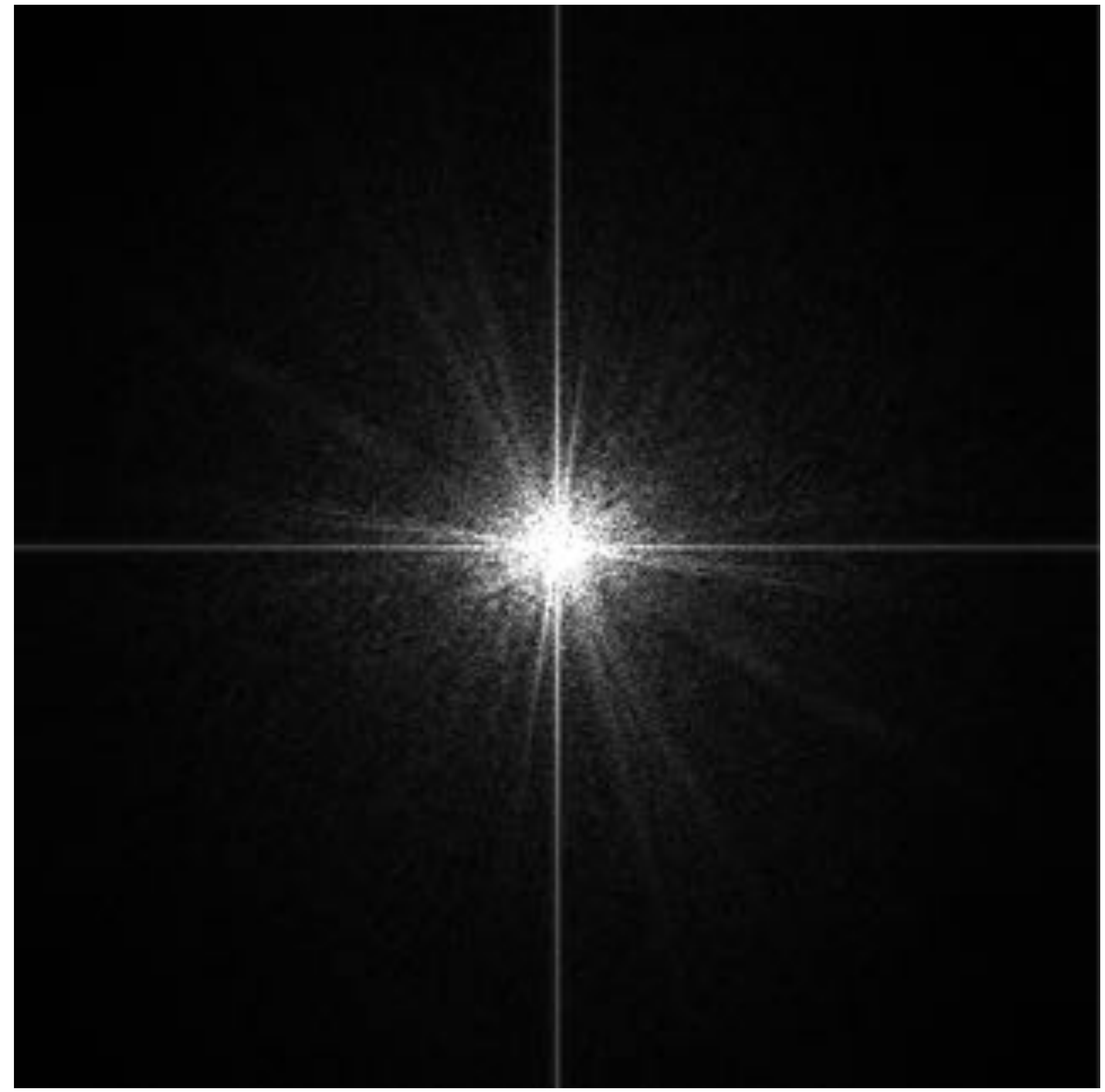
Intensity of
low-frequencies (bass)

Intensity of
high frequencies

Image credit: ONYX Apps

Visualizing the frequency content of images

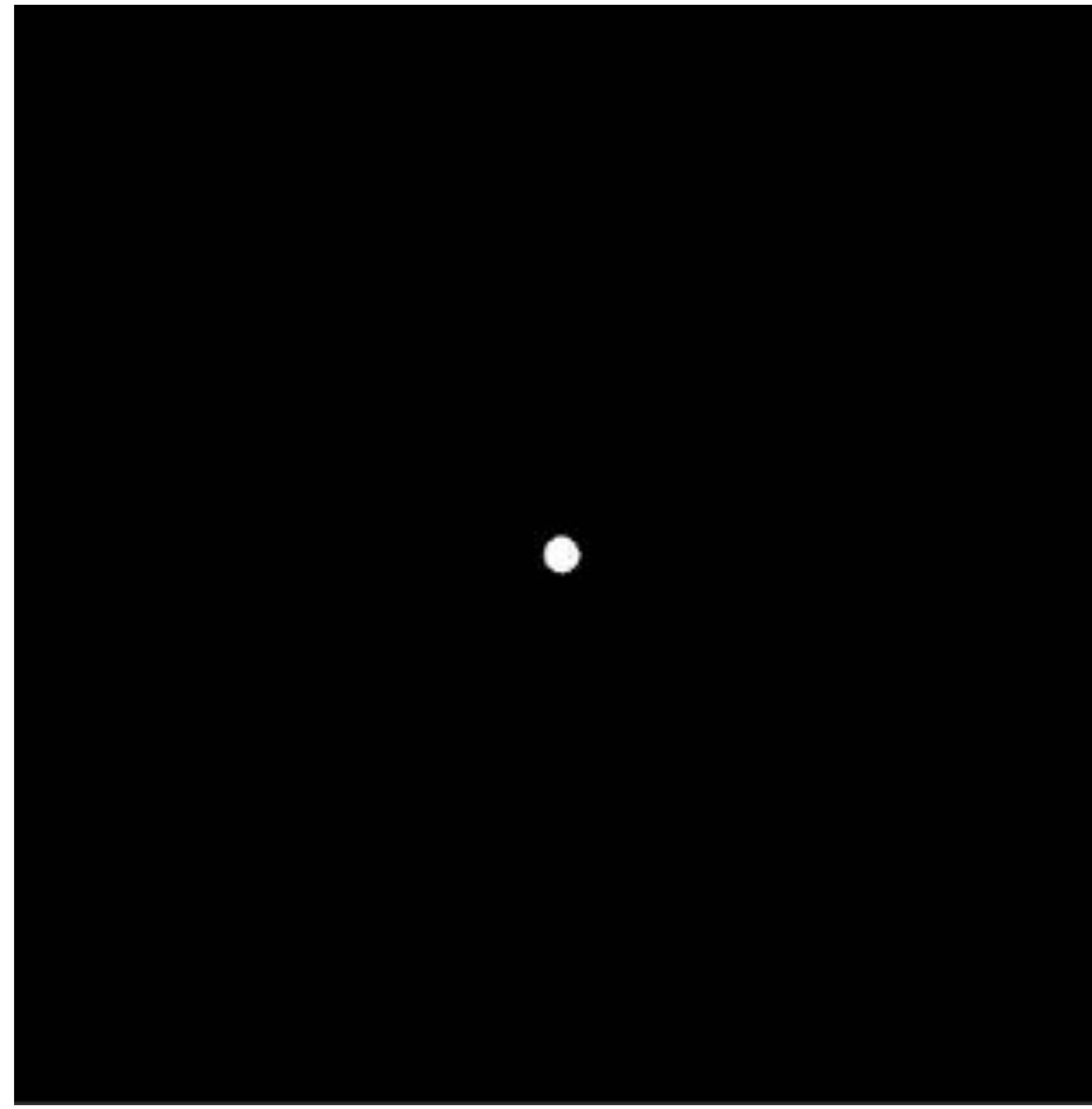
Spatial domain result



Spectrum

Low frequencies only (smooth gradients)

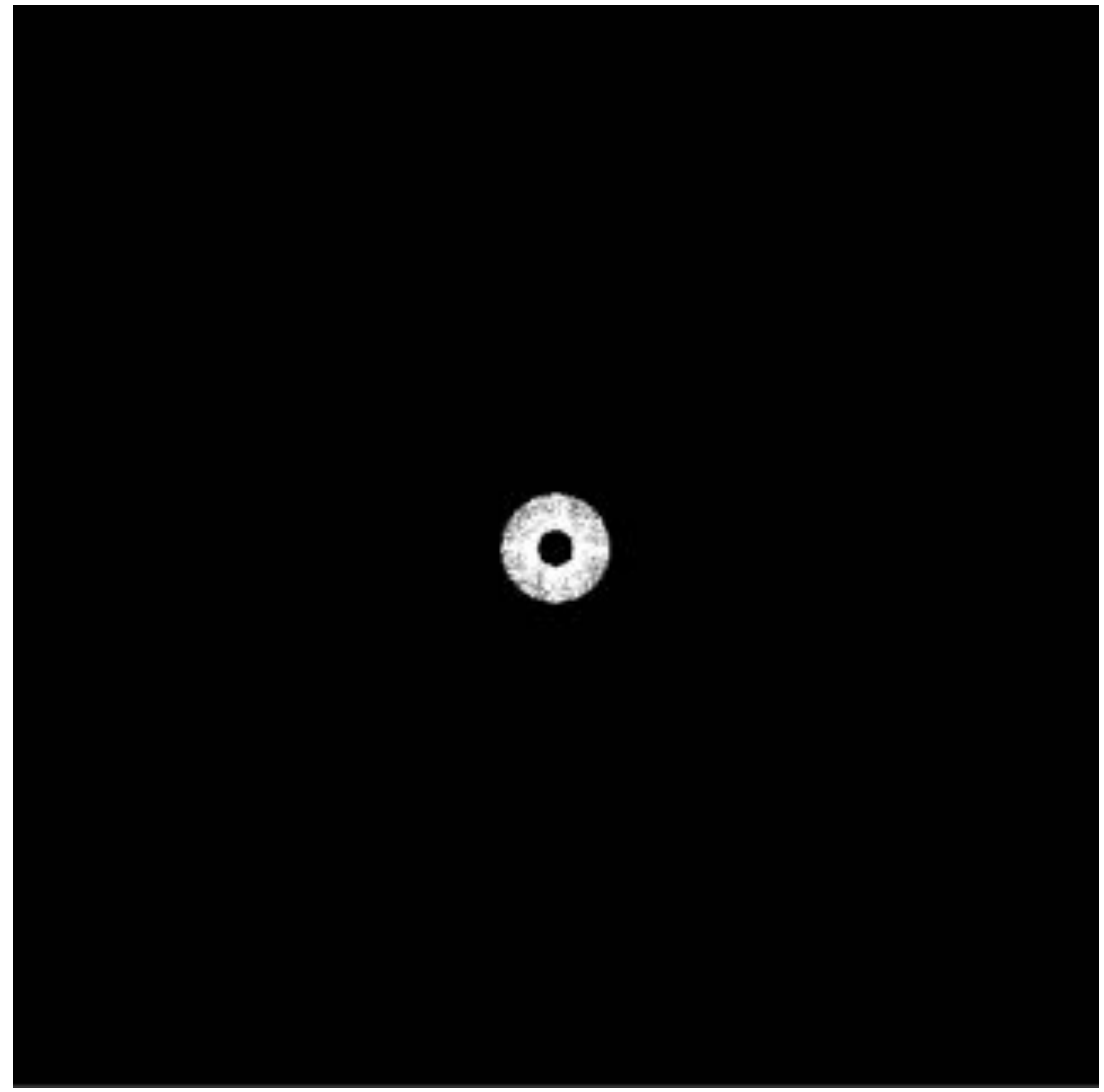
Spatial domain result



Spectrum (after low-pass filter)
All frequencies above cutoff have 0 magnitude

Mid-range frequencies

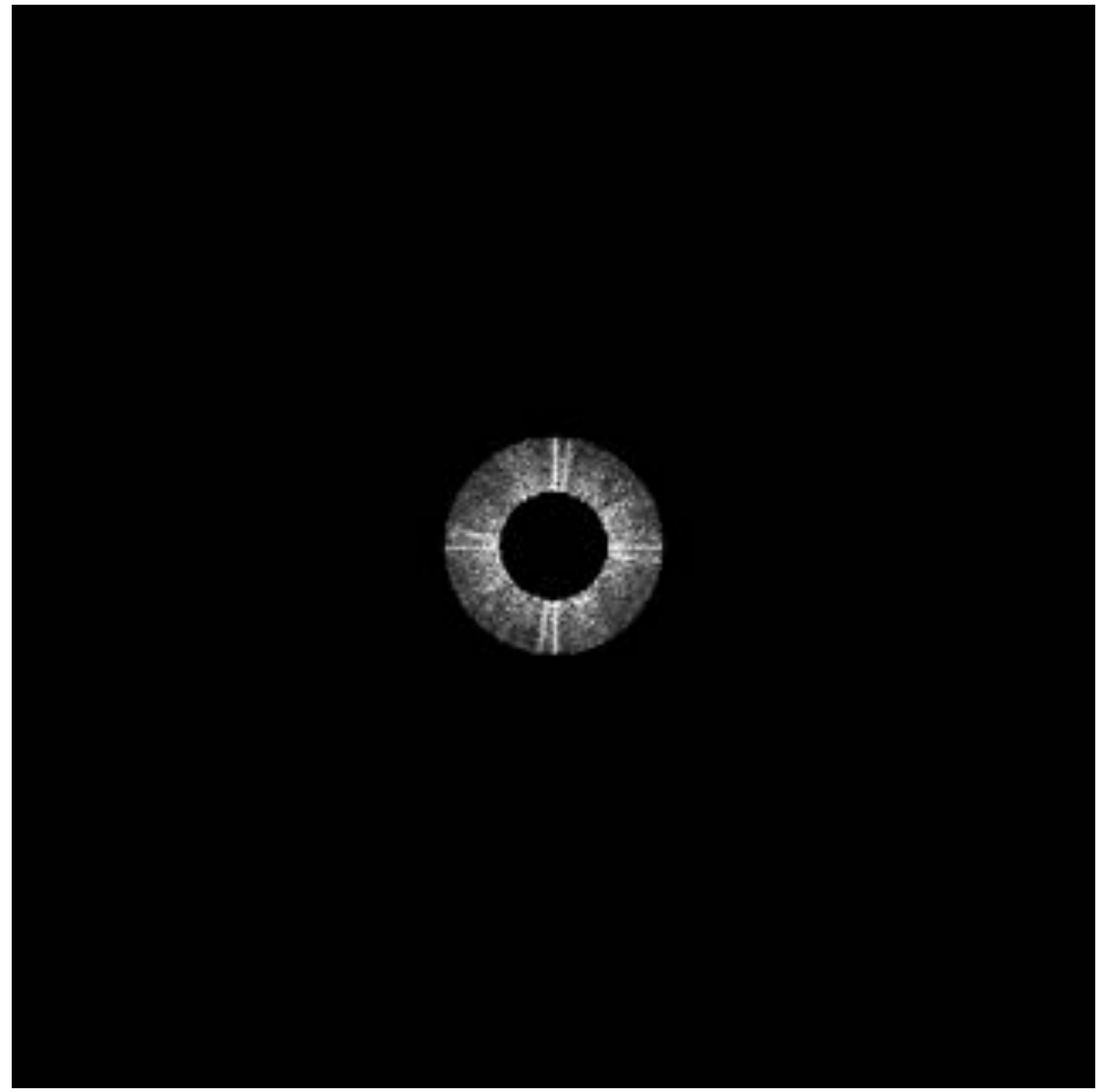
Spatial domain result



Spectrum (after band-pass filter)

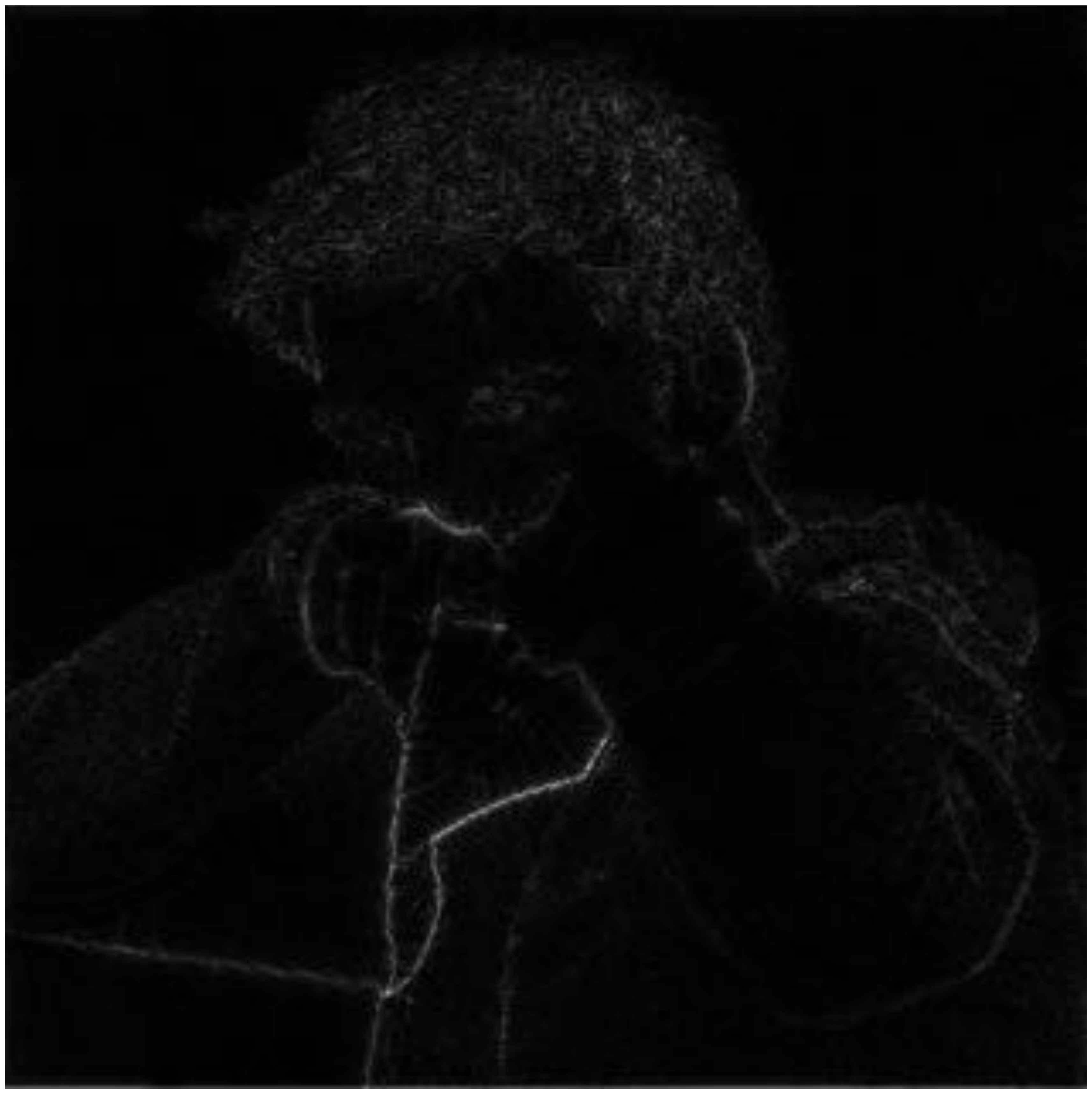
Mid-range frequencies

Spatial domain result

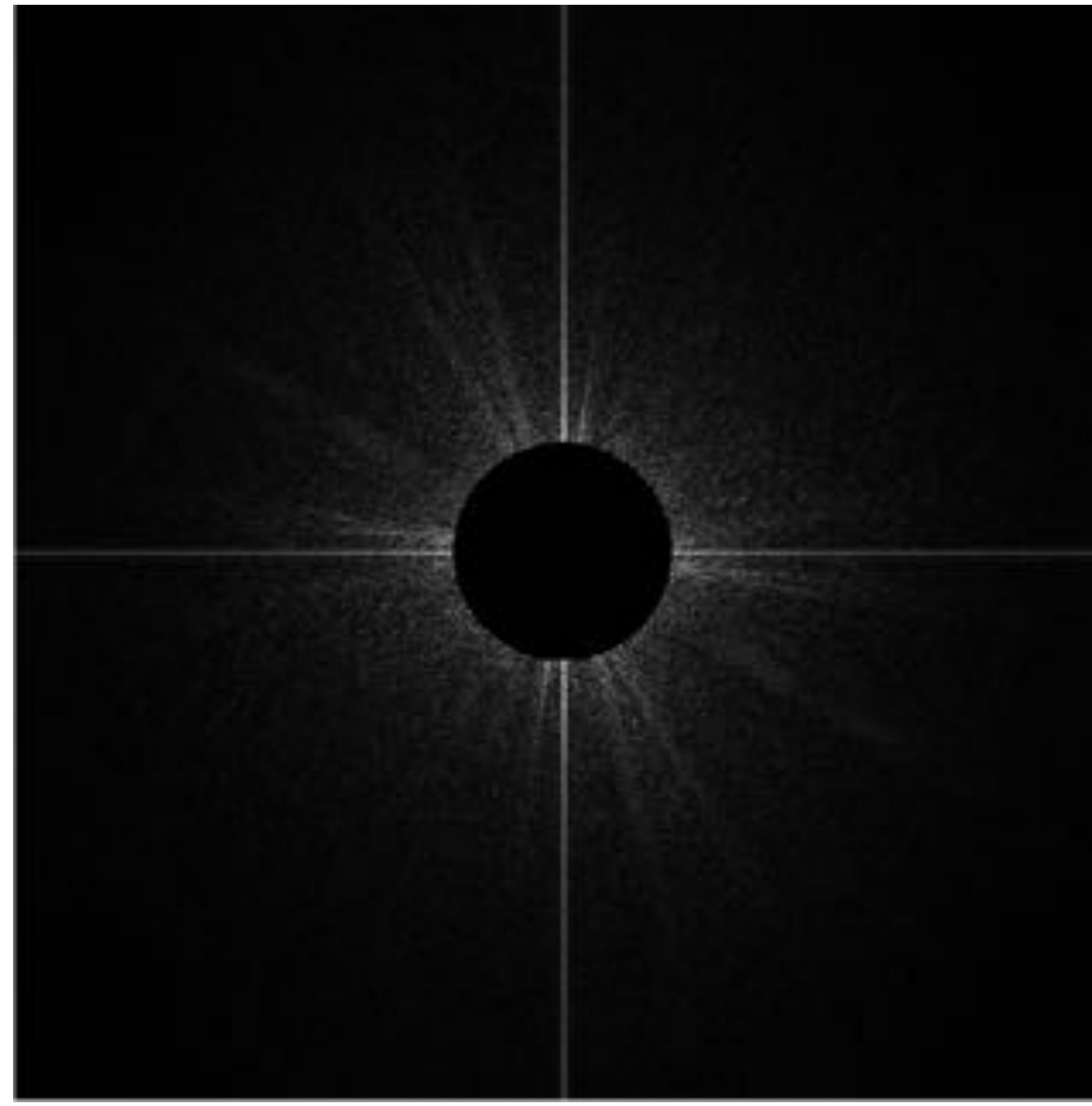


Spectrum (after band-pass filter)

High frequencies (edges)

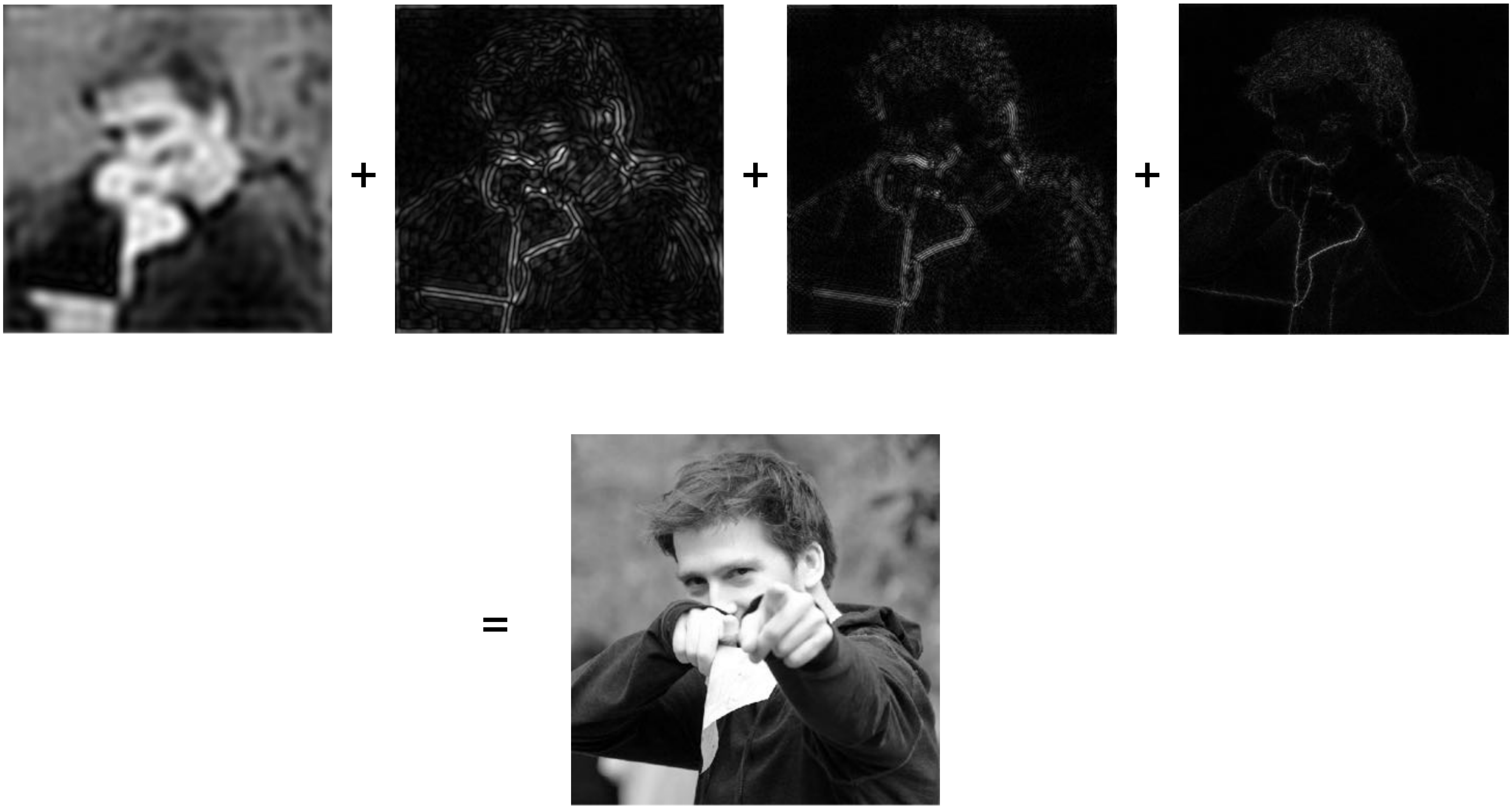


Spatial domain result
(strongest edges)

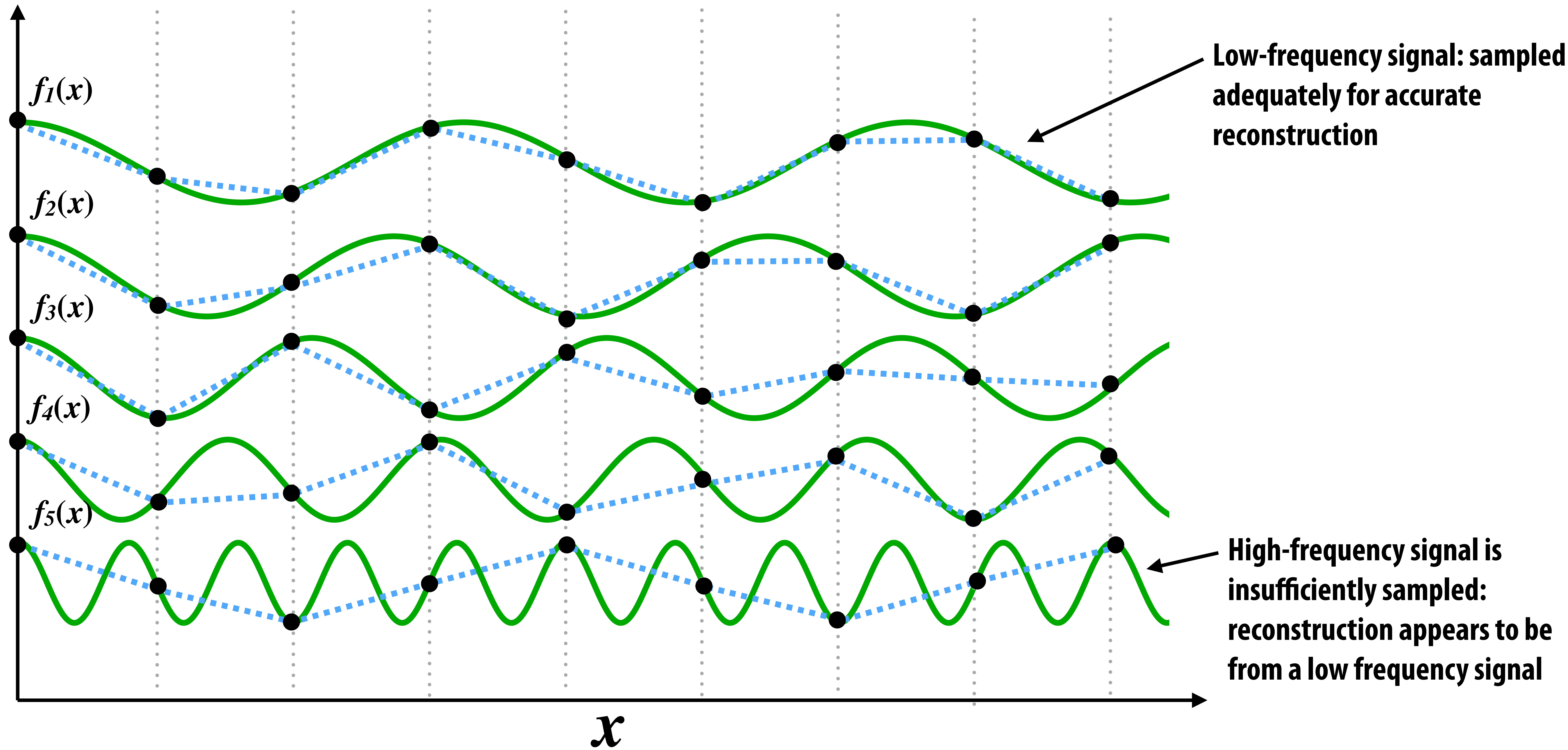


Spectrum (after high-pass filter)
All frequencies below threshold
have 0 magnitude

An image as a sum of its frequency components



1D example:
Undersampling high-frequency signals results in aliasing



"Aliasing": high frequencies in the original signal masquerade as low frequencies after reconstruction (due to undersampling)

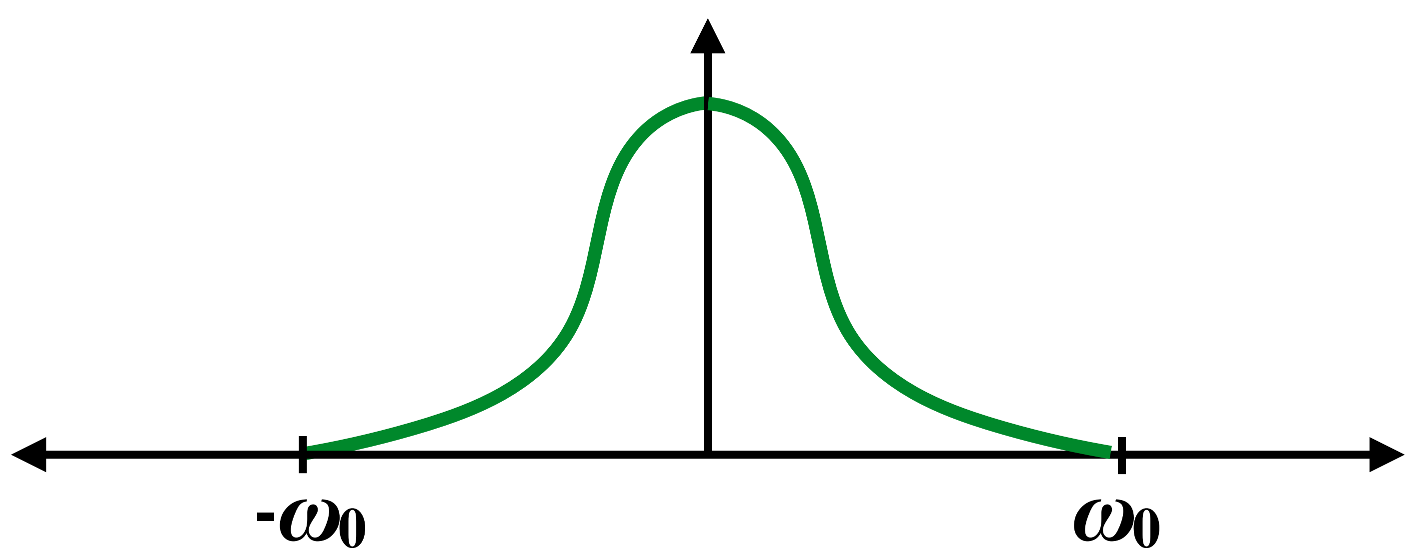
Temporal aliasing: wagon wheel effect

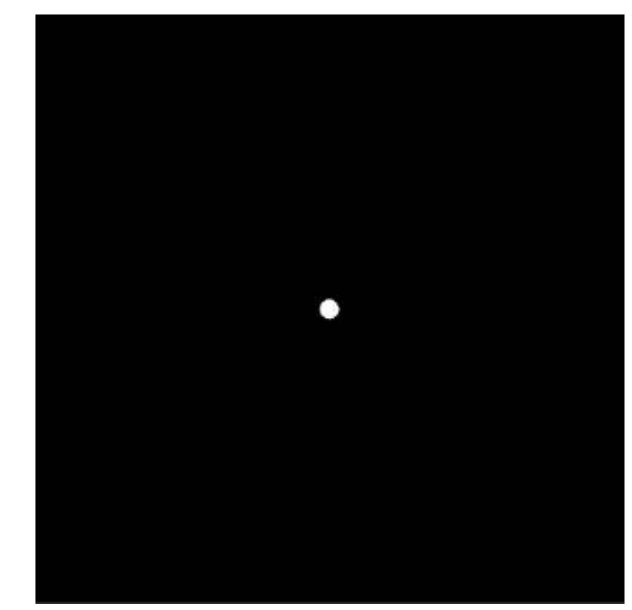
Camera's frame rate (temporal sampling rate) is too low for rapidly spinning wheel.

https://www.youtube.com/watch?v=VNftf5qLpiA

Nyquist-Shannon theorem

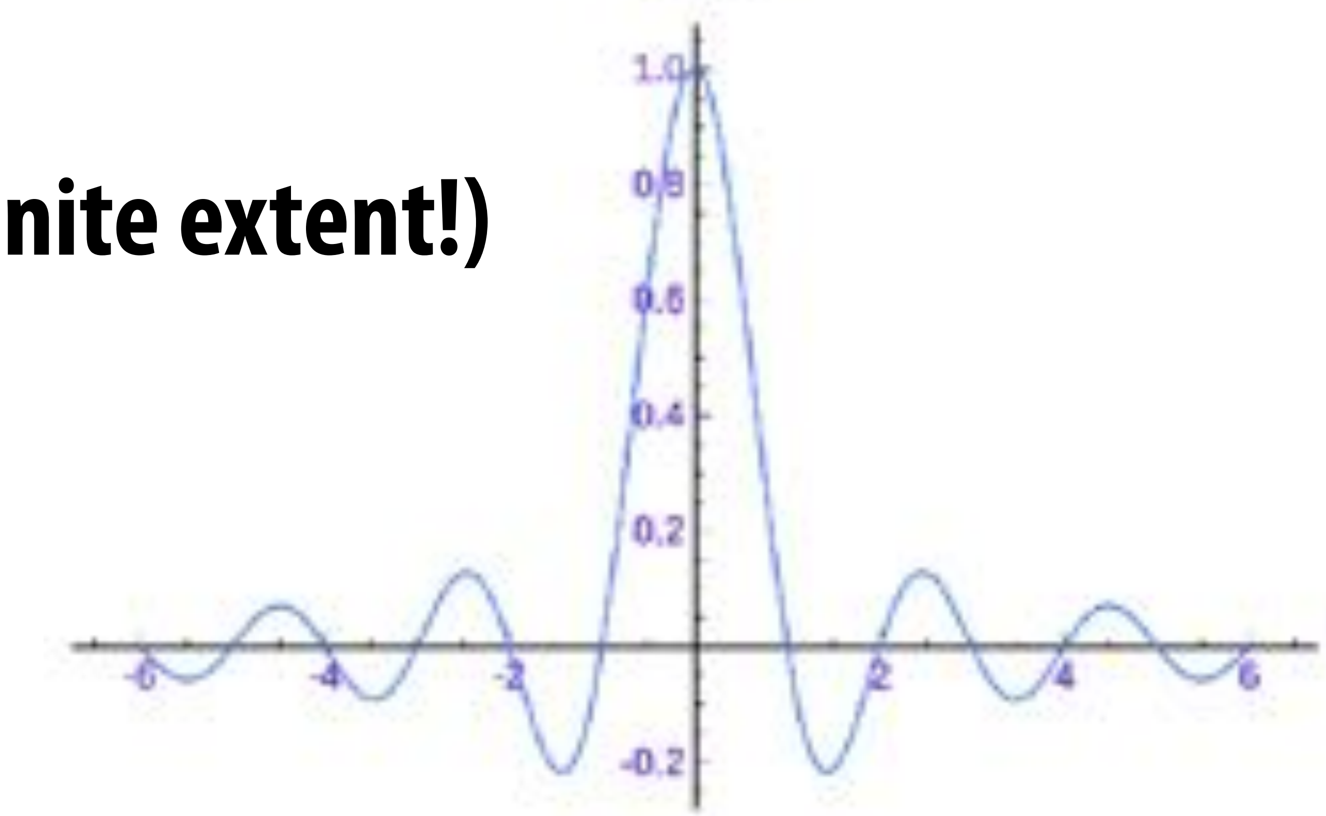
- Consider a band-limited signal: has no frequencies above ω_0
  - 1D: consider low-pass filtered audio signal
  - 2D: recall the blurred image example from a few slides ago





- The signal can be perfectly reconstructed if sampled with period $T = 1 / 2\omega_0$
- And reconstruction is performed using a "sinc filter"
  - Ideal filter with no frequencies above cutoff (infinite extent!)

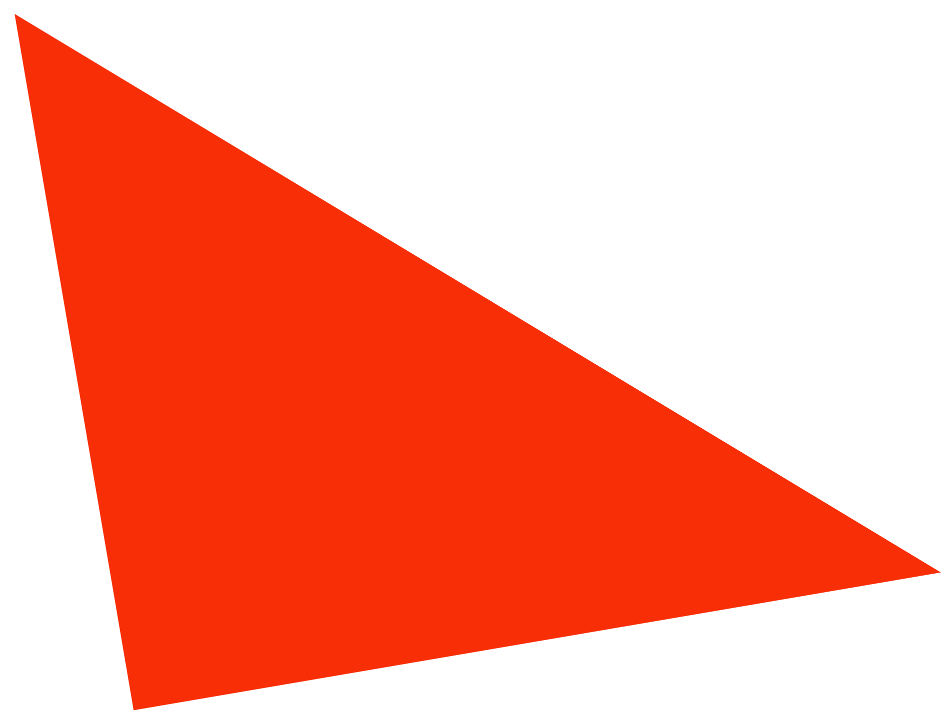
$$sinc(x) = \frac{sin(\pi x))}{\pi x}$$

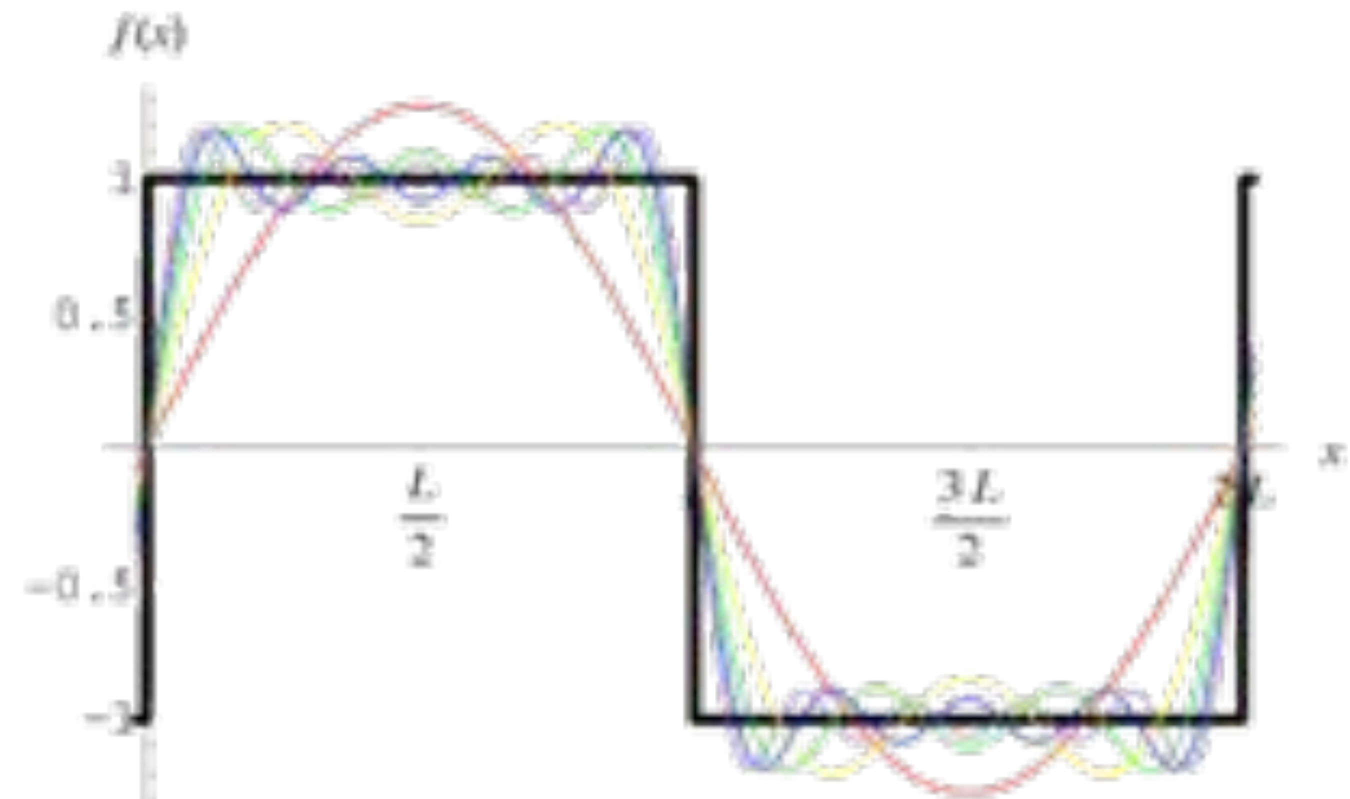


Challenges of sampling-based approaches in graphics

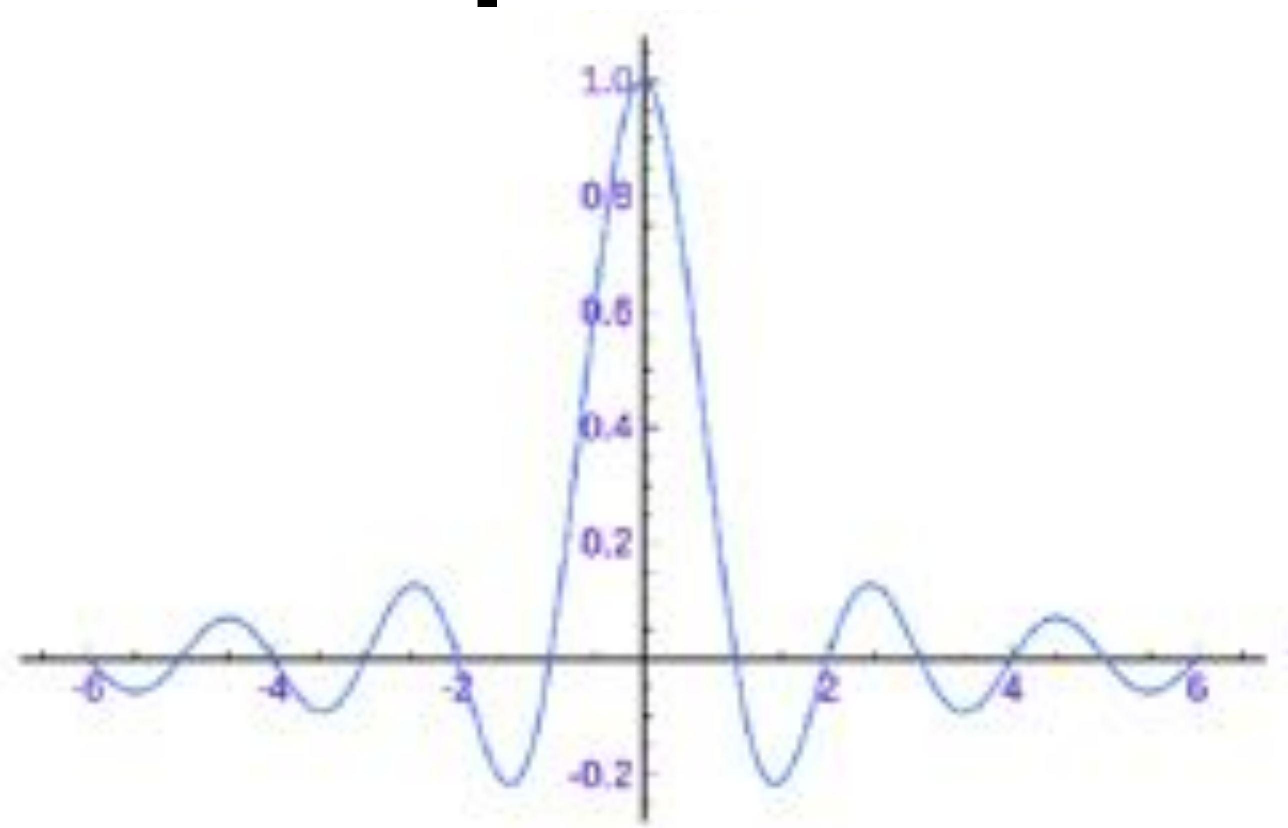
- **Our signals are not always band-limited in computer graphics. Why?**

Hint:



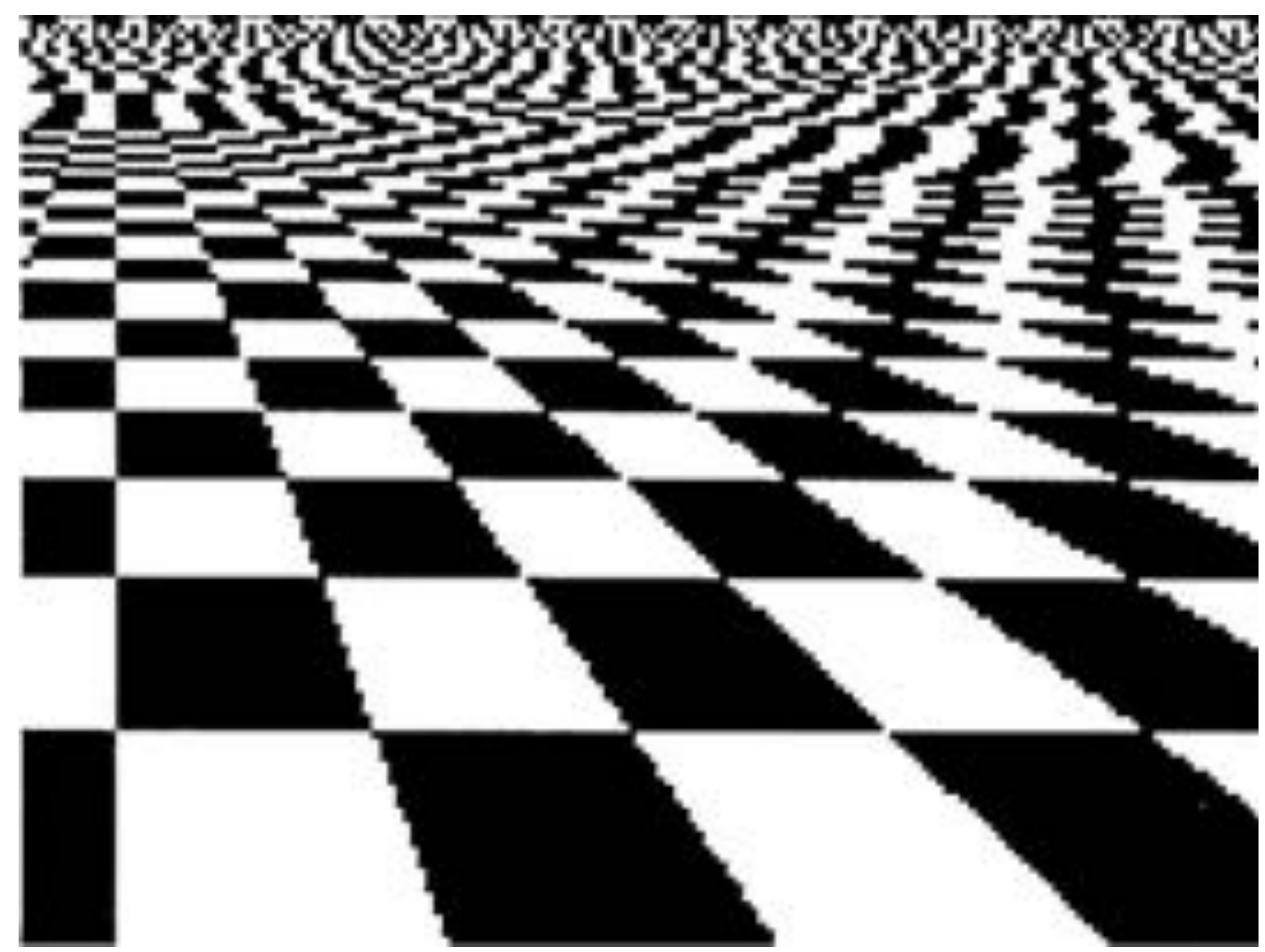


- **Also, infinite extent of "ideal" reconstruction filter (sinc) is impractical for efficient implementations. Why?**



Aliasing artifacts in images

- **Undersampling high-frequency signals and the use of non-ideal resampling filters yields image artifacts**
  - “Jaggies” in a single image
  - “Roping” or “shimmering” of images when animated
  - Moiré patterns in high-frequency areas of images



Sampling a zone plate: $\sin(x^2 + y^2)$

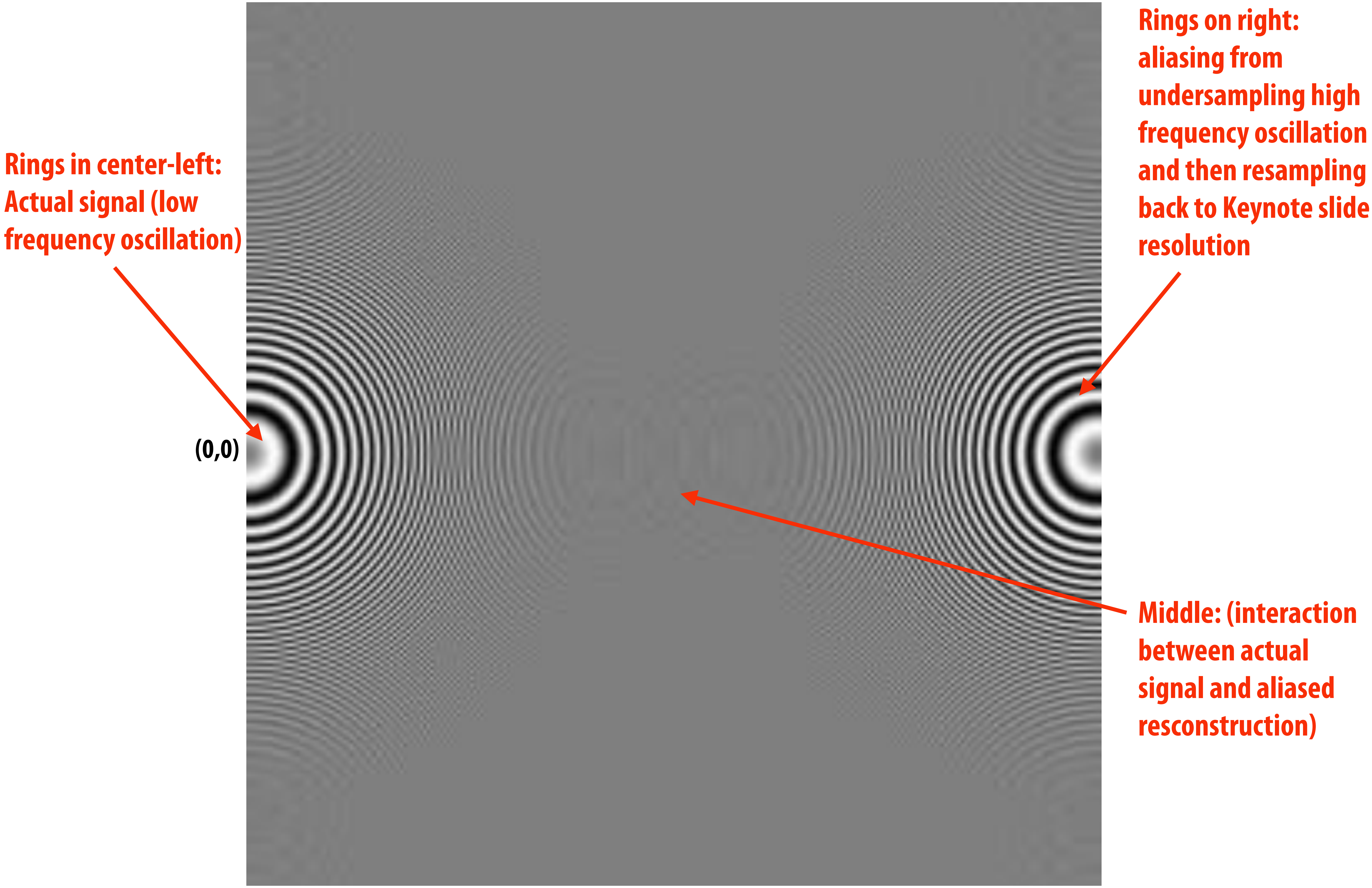
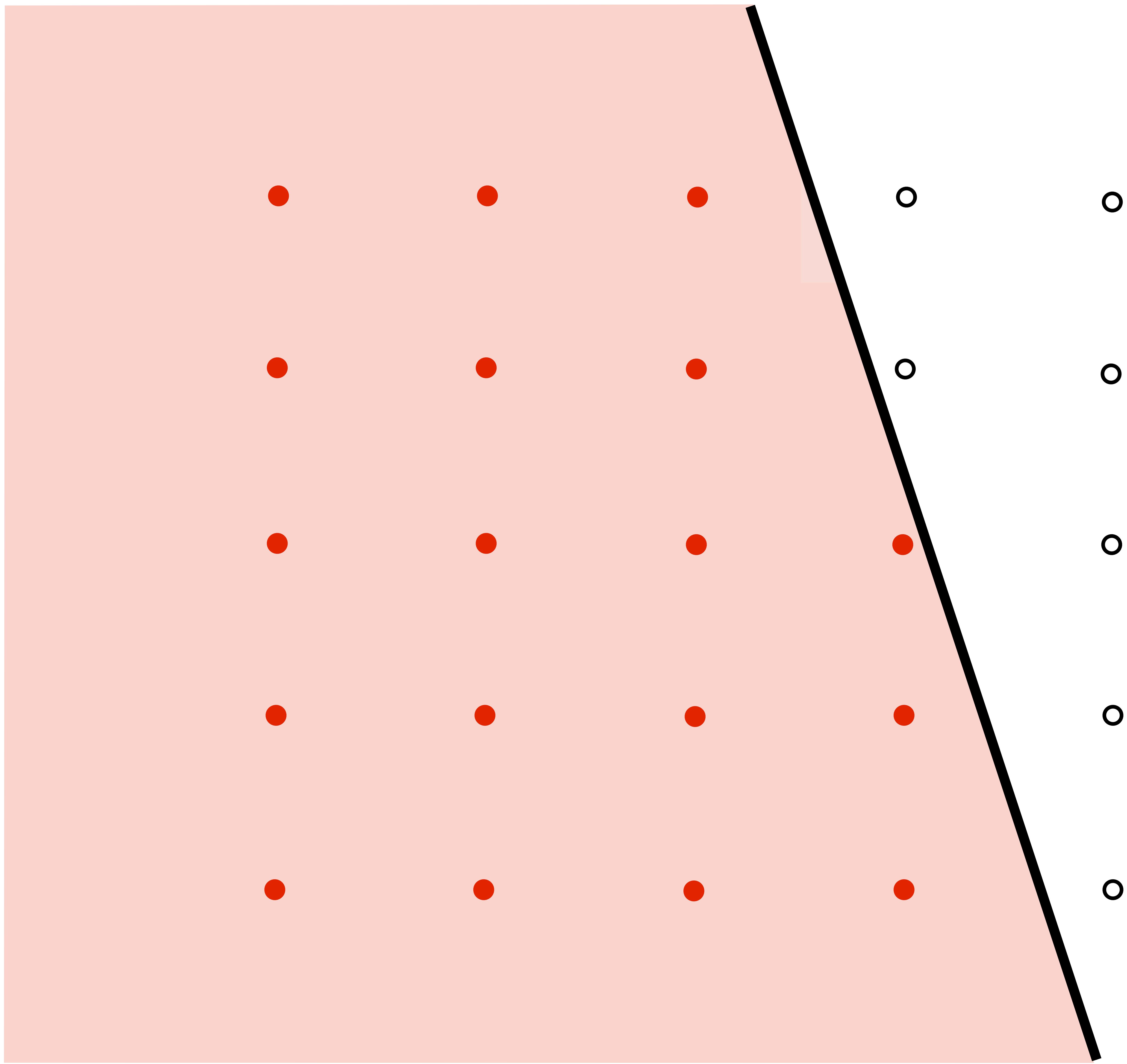


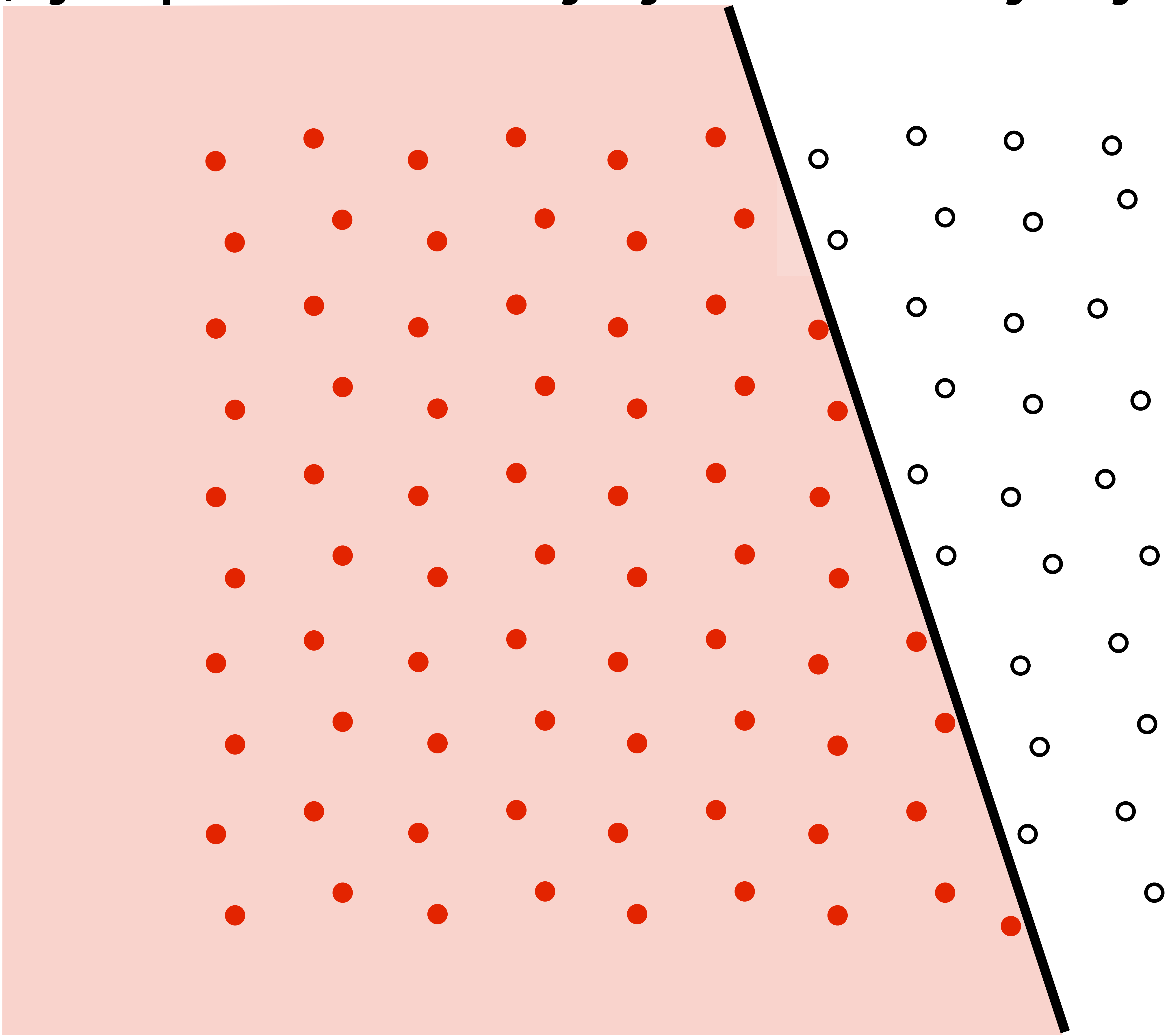
Figure credit: Pat Hanrahan and Bryce Summers

Initial coverage sampling rate (1 sample per pixel)



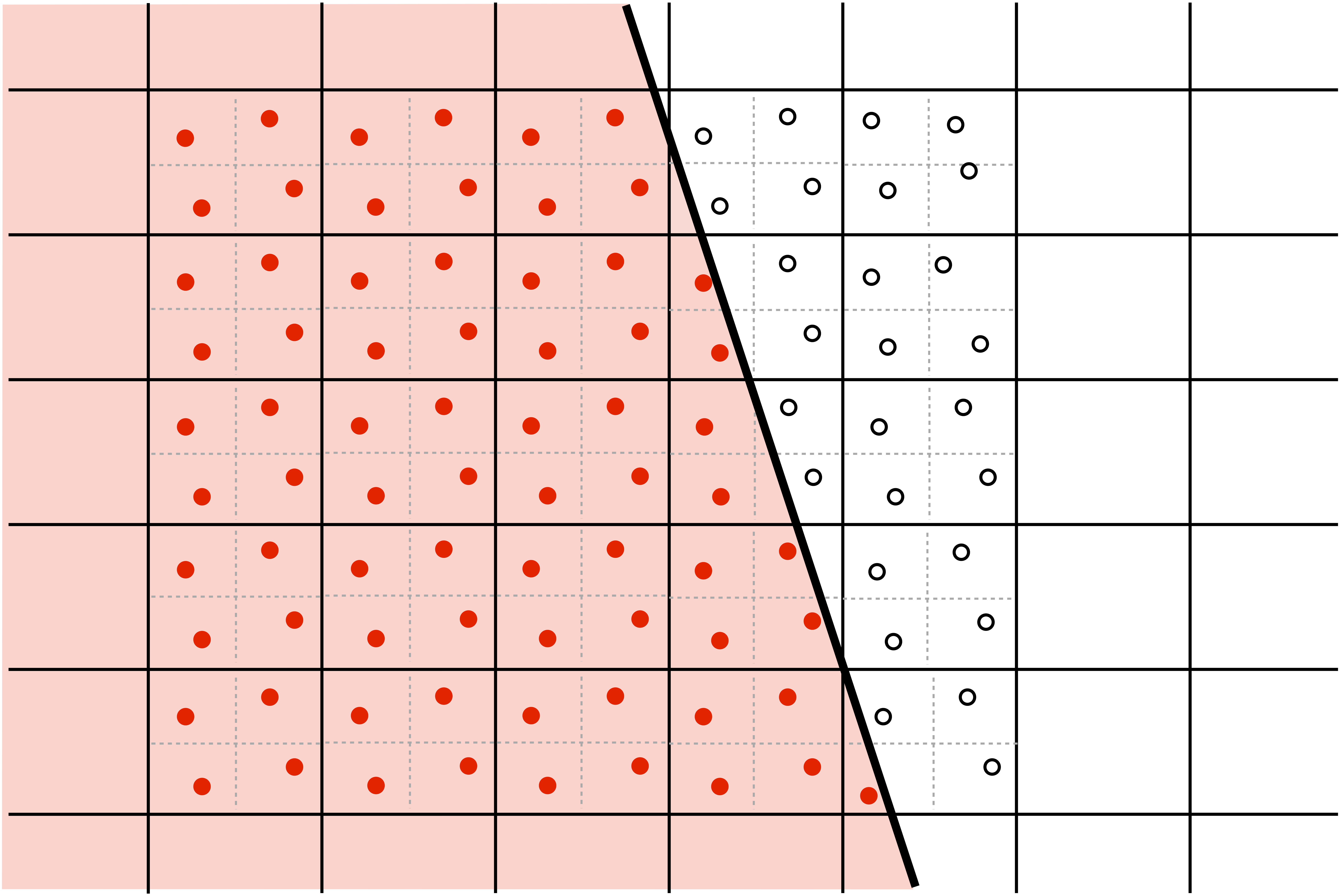
Increase density of sampling coverage signal

(high frequencies exist in coverage signal because of triangle edges)



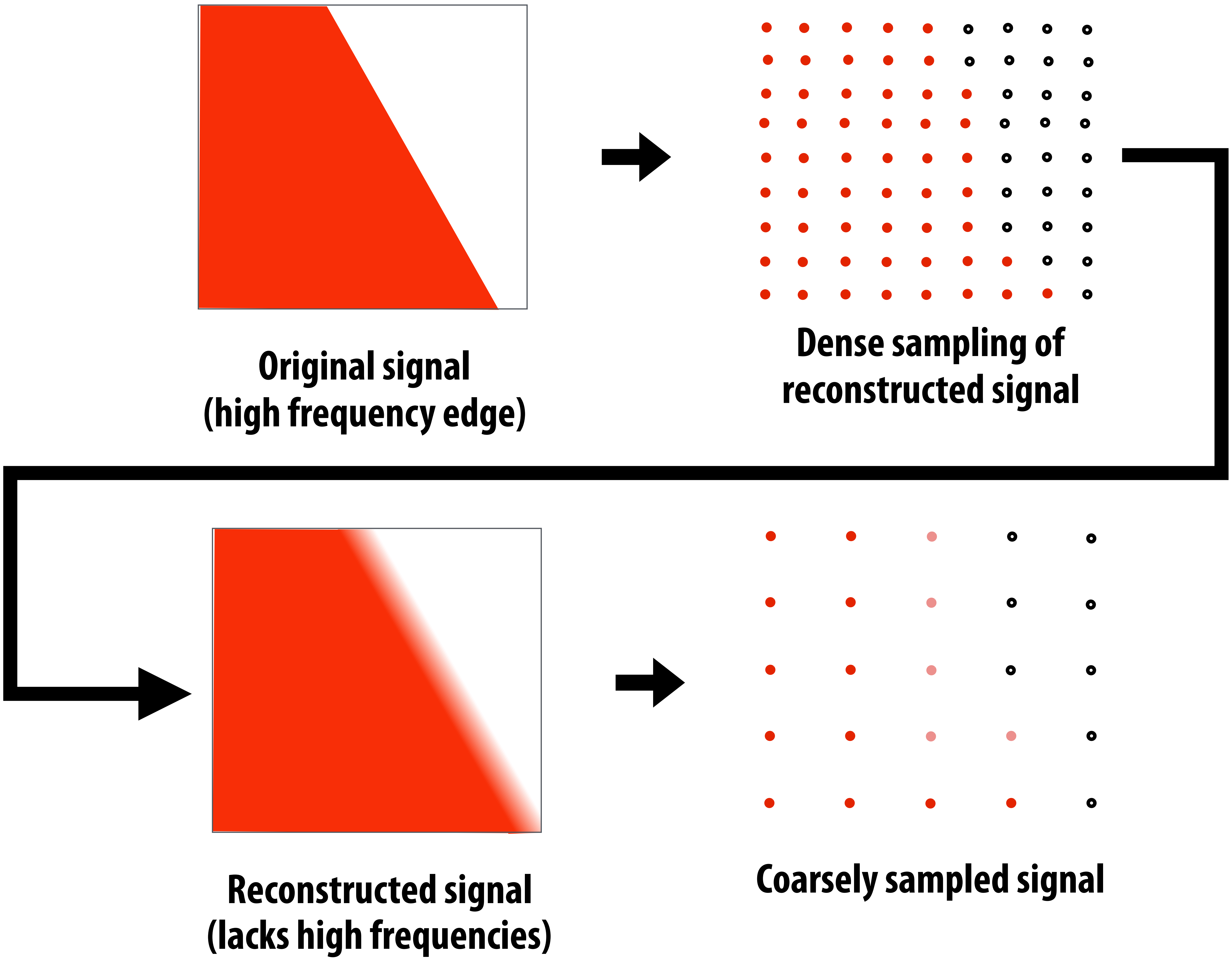
Supersampling

Example: stratified sampling using four samples per pixel



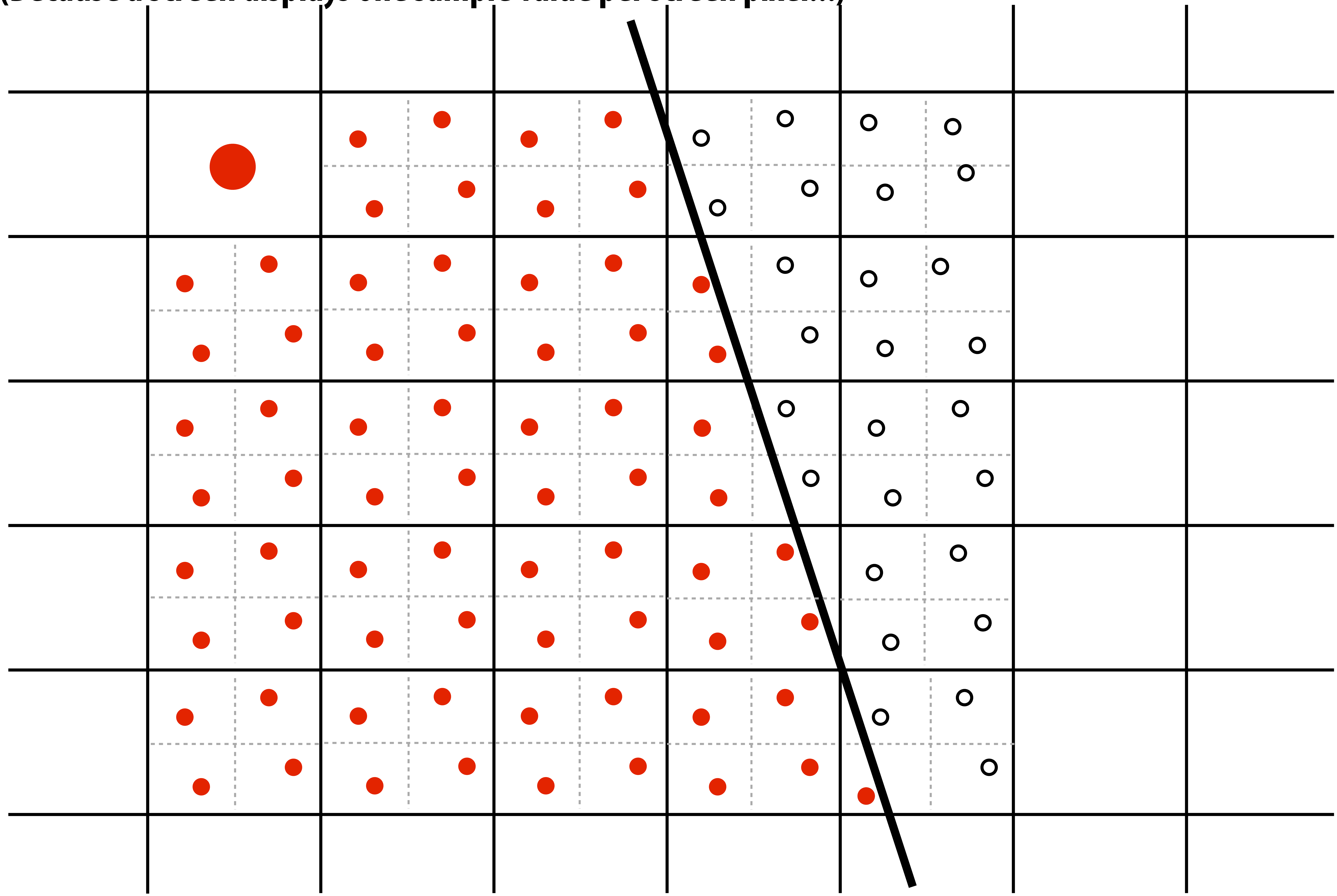
Resampling

Converting from one discrete sampled representation to another

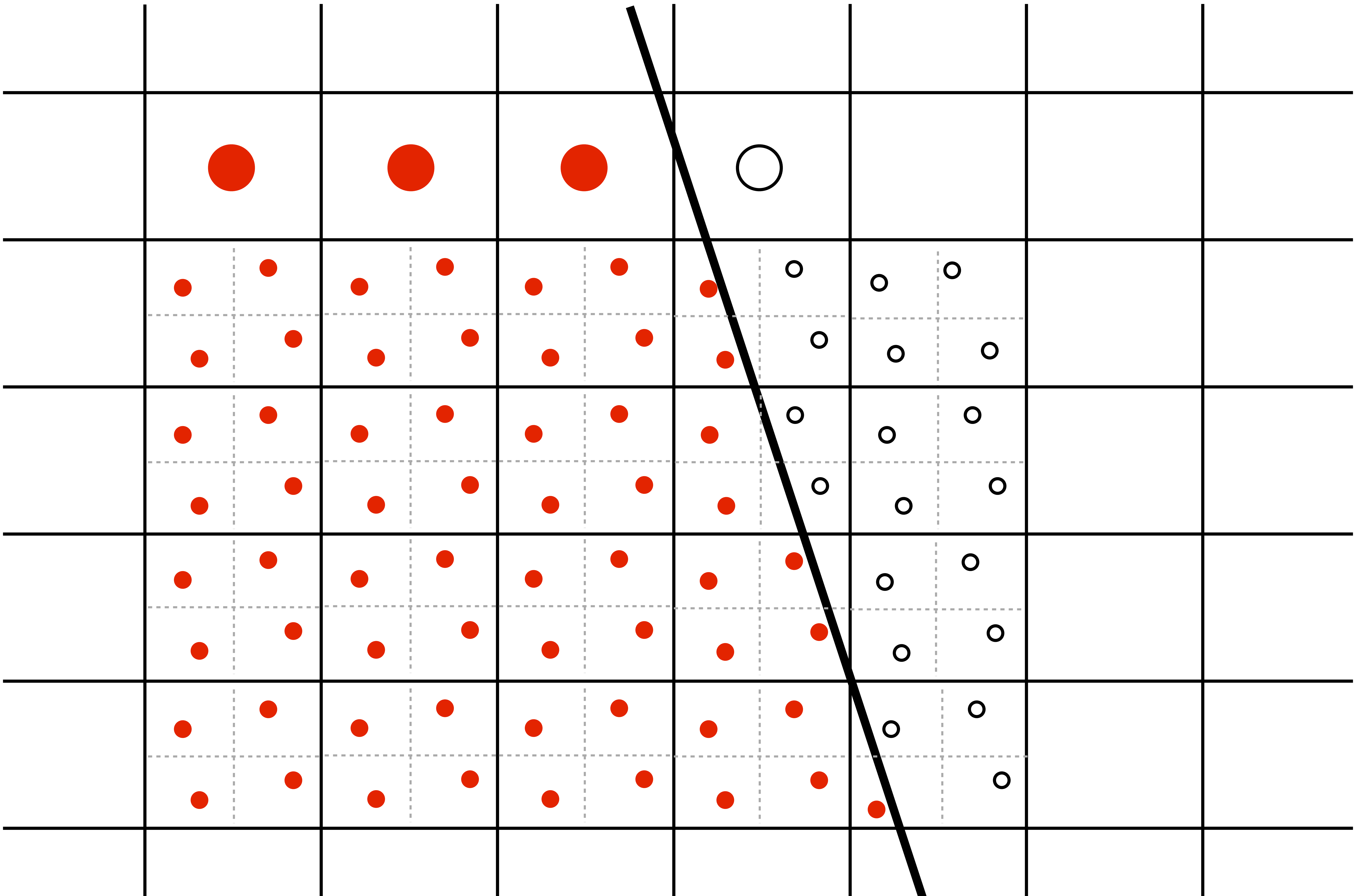


Resample to display's pixel resolution

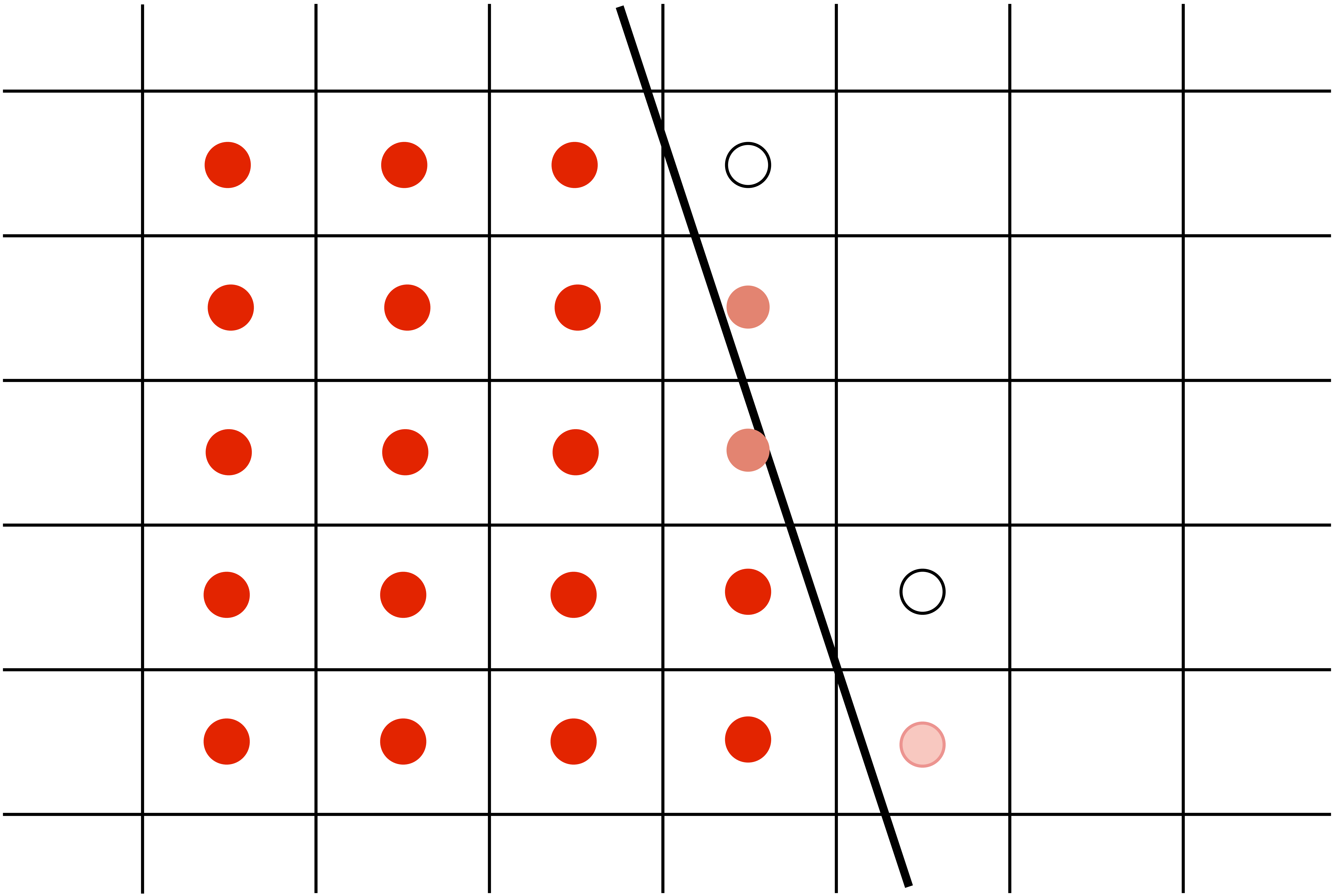
(Because a screen displays one sample value per screen pixel...)



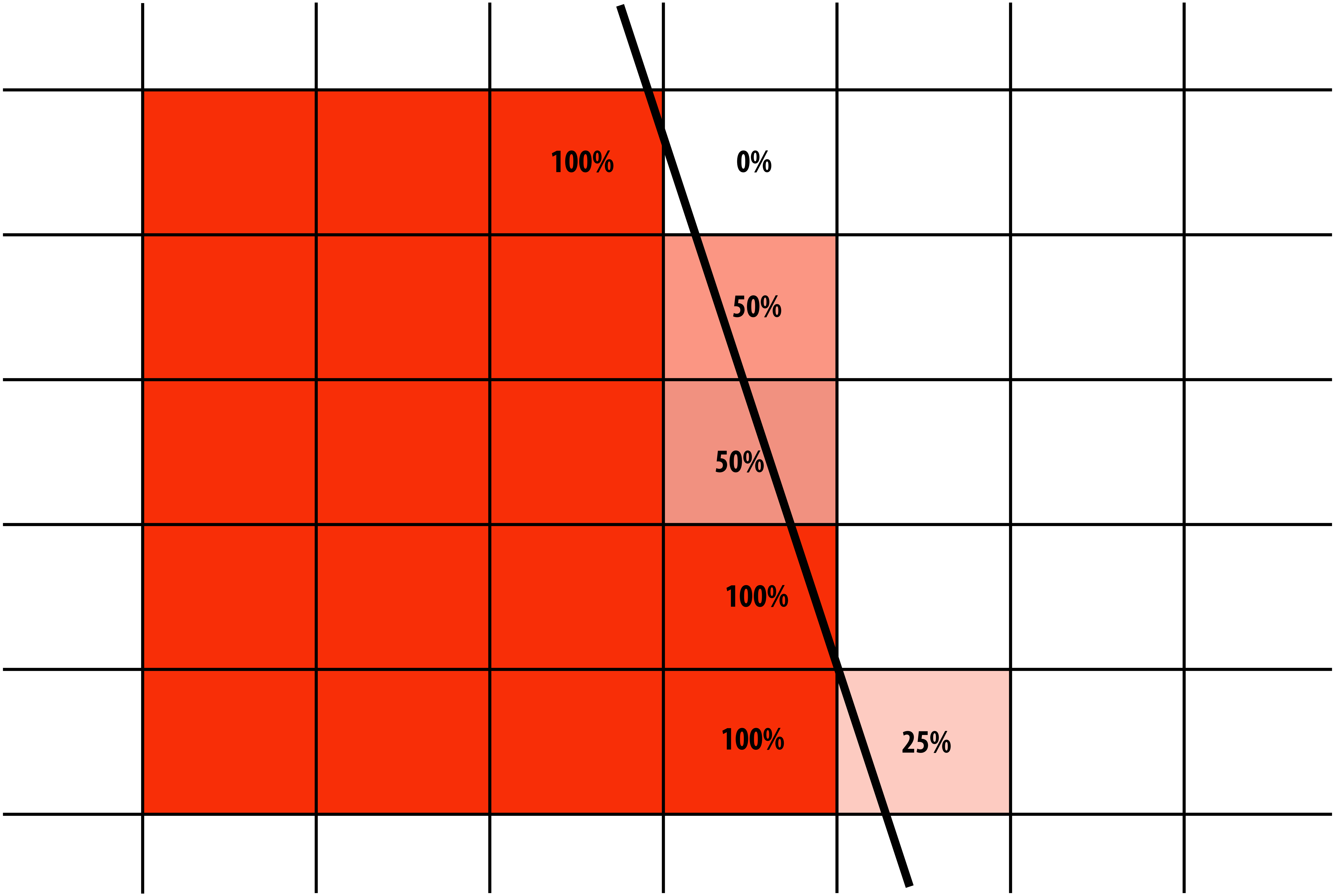
Resample to display's pixel rate (box filter)



Resample to display's pixel rate (box filter)



Displayed result (note anti-aliased edges)



Recall: the real coverage signal was this

Sampling coverage

- We want the light emitted from a display to be an accurate to match the ground truth signal: `coverage(x,y)`)

- Resampling a densely sampled signal (supersampled) integrates coverage values over the entire pixel region. The integrated result is sent to the display (and emitted by the pixel) so that the light emitted by the pixel is similar to what would be emitted in that screen region by an "infinite resolution display"

Sampling triangle coverage
(evaluating coverage(x,y) for a triangle)

Point-in-triangle test

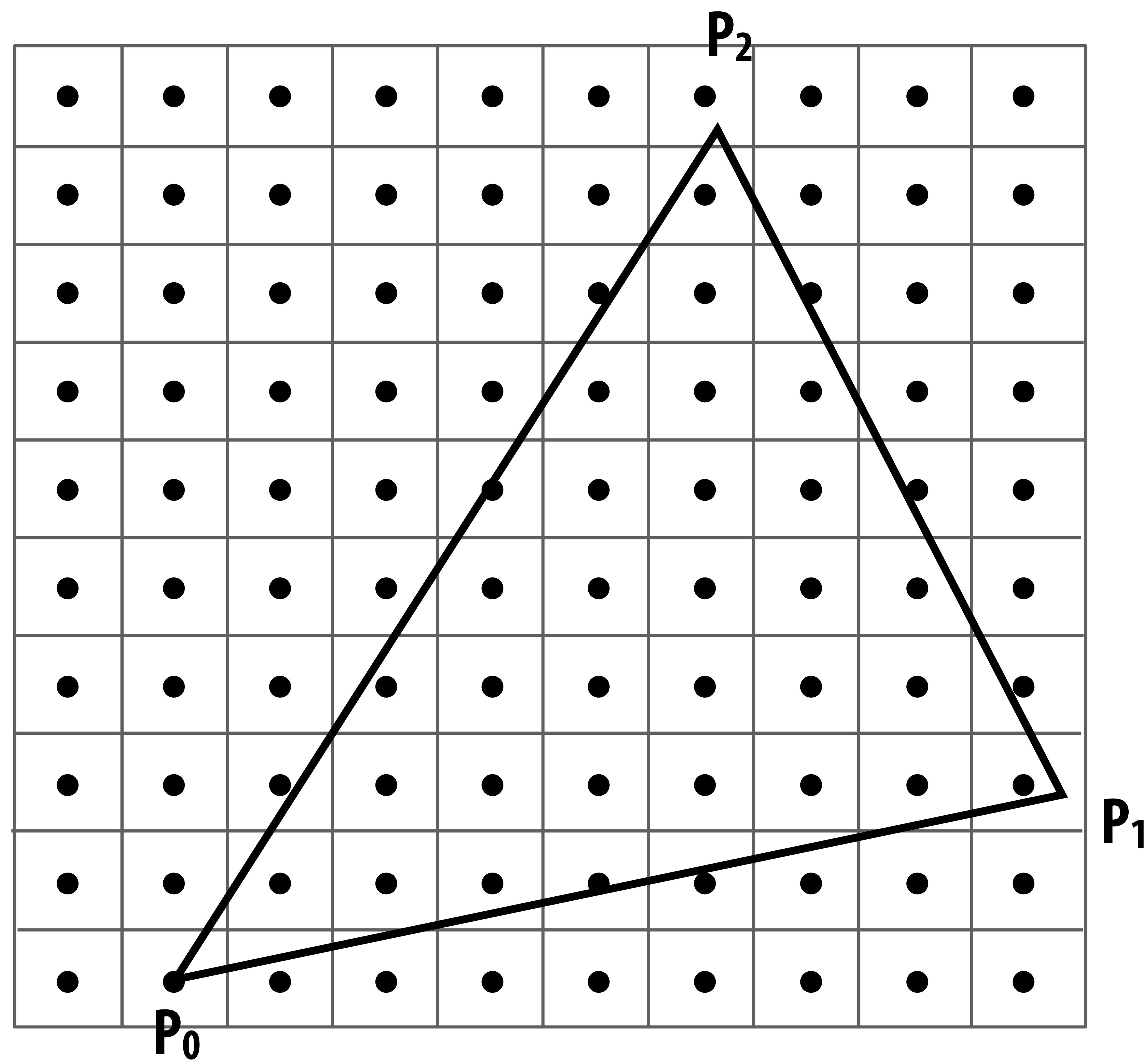
Compute triangle edge equations from projected positions of vertices

$P_i = (X_i, Y_i)$

$dX_i = X_{i+1} - X_i$
$dY_i = Y_{i+1} - Y_i$

$$\begin{aligned} E_i(x, y) &= (x - X_i)\, dY_i - (y - Y_i)\, dX_i \\ &= A_i x + B_i y + C_i \end{aligned}$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge



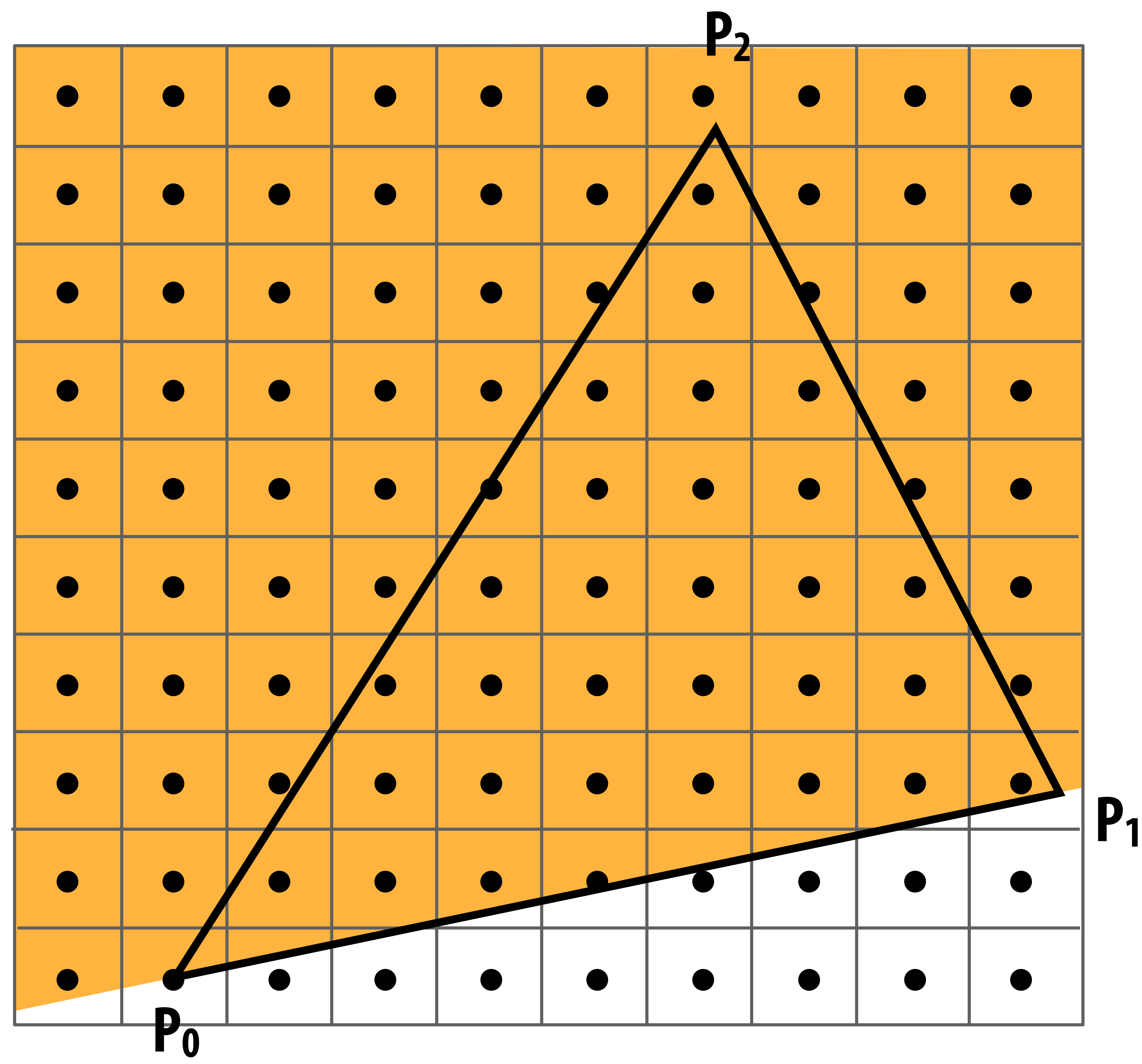
Point-in-triangle test

$P_i = (X_i, Y_i)$

$dX_i = X_{i+1} - X_i$
$dY_i = Y_{i+1} - Y_i$

$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$
$\quad = A_i x + B_i y + C_i$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge



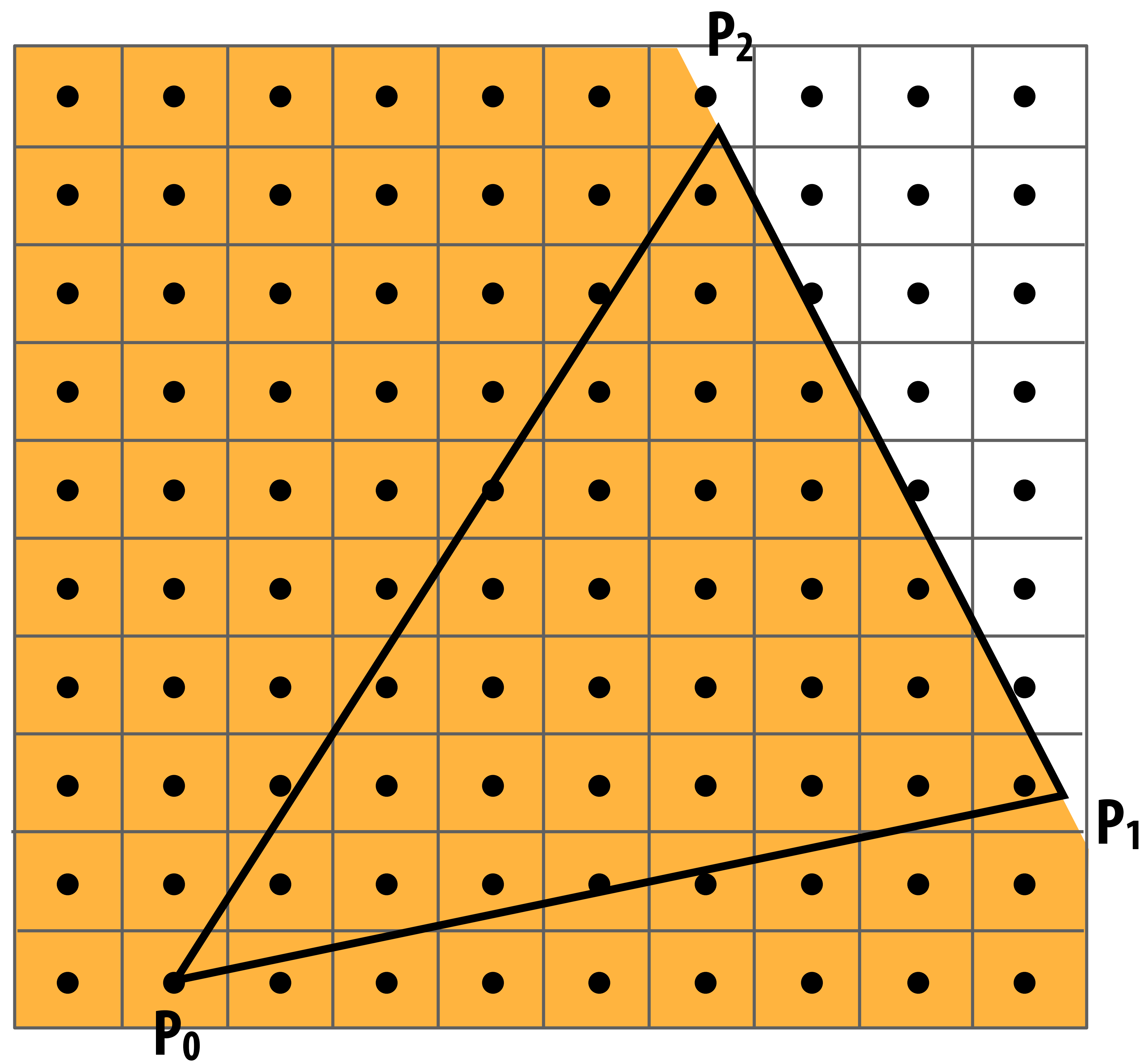
Point-in-triangle test

$P_i = (X_i, Y_i)$

$dX_i = X_{i+1} - X_i$
$dY_i = Y_{i+1} - Y_i$

$$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$$
$$= A_i\, x + B_i\, y + C_i$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge



Point-in-triangle test

$P_i = (X_i, Y_i)$

$dX_i = X_{i+1} - X_i$
$dY_i = Y_{i+1} - Y_i$

$$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$$
$$= A_i x + B_i y + C_i$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
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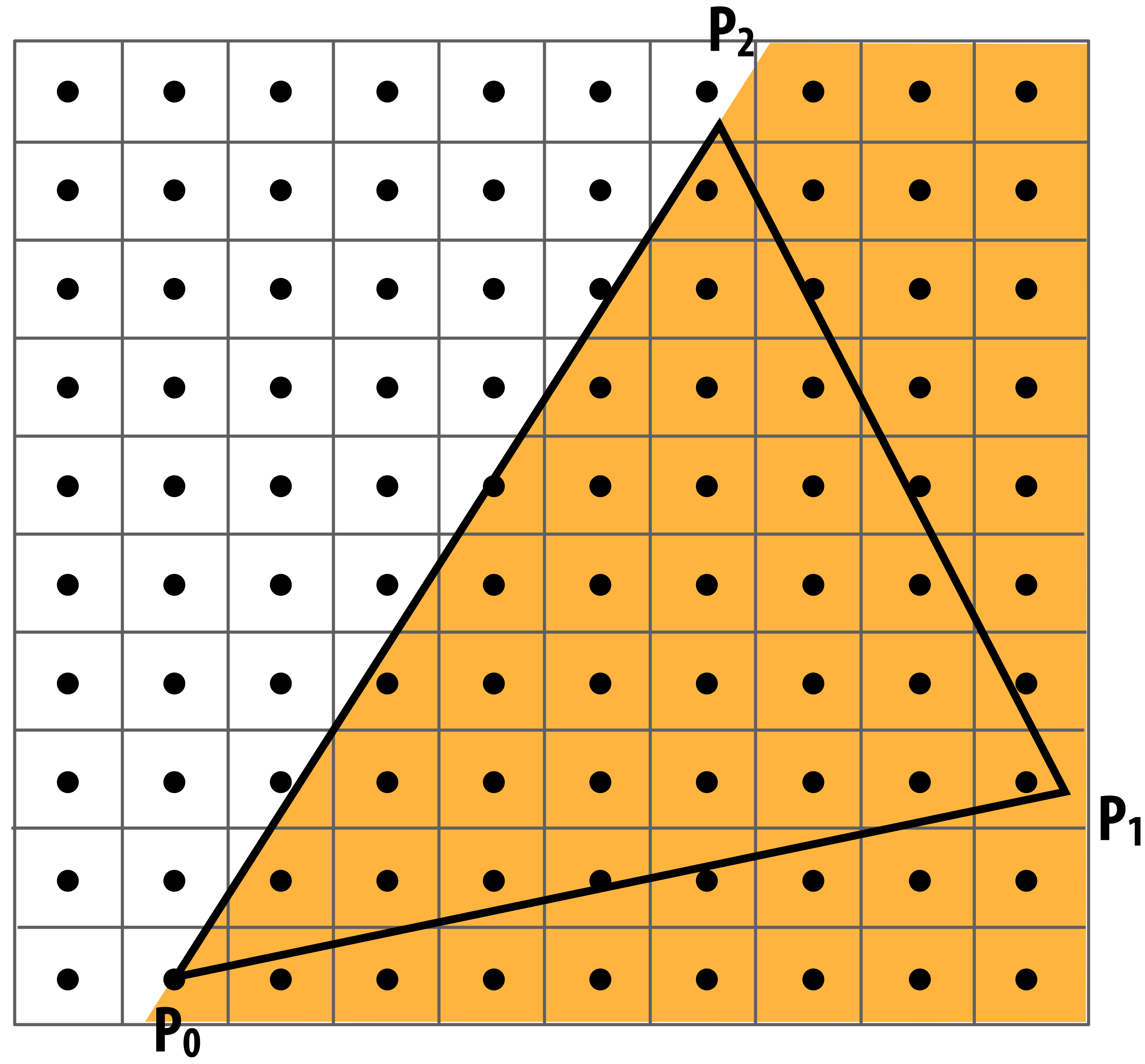
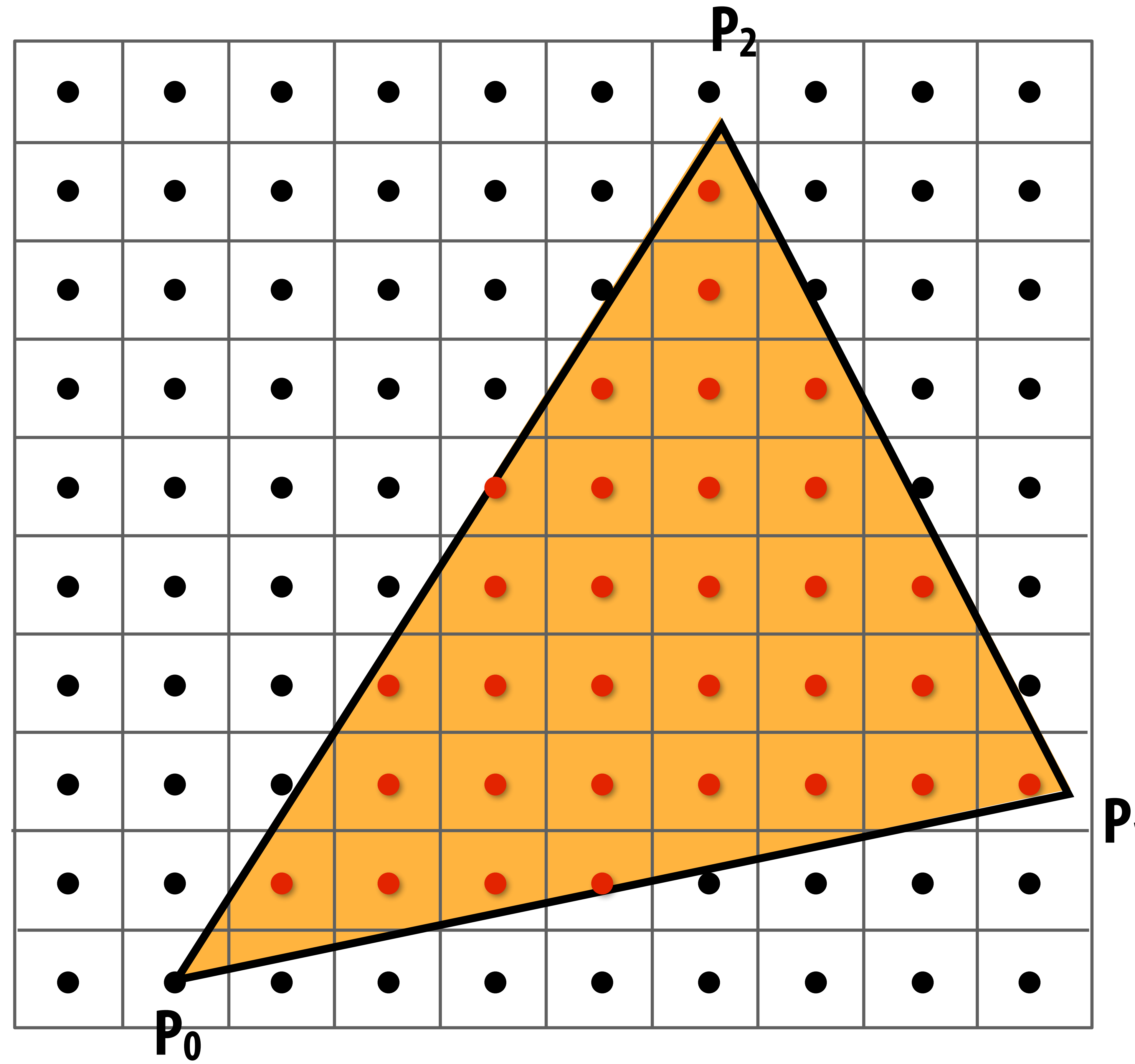


Point-in-triangle test

Sample point $s = (sx, sy)$ **is inside the triangle if it is inside all three edges.**

$$
\begin{aligned}
&inside(sx, sy) = \\
&\quad E_0(sx, sy) < 0 \ \&\& \\
&\quad E_1(sx, sy) < 0 \ \&\& \\
&\quad E_2(sx, sy) < 0;
\end{aligned}
$$

Note: actual implementation of $inside(sx,sy)$ **involves ≤ checks based on the triangle coverage edge rules (see beginning of lecture)**



Sample points inside triangle are highlighted red.

Incremental triangle traversal

$P_i = (X_i, Y_i)$

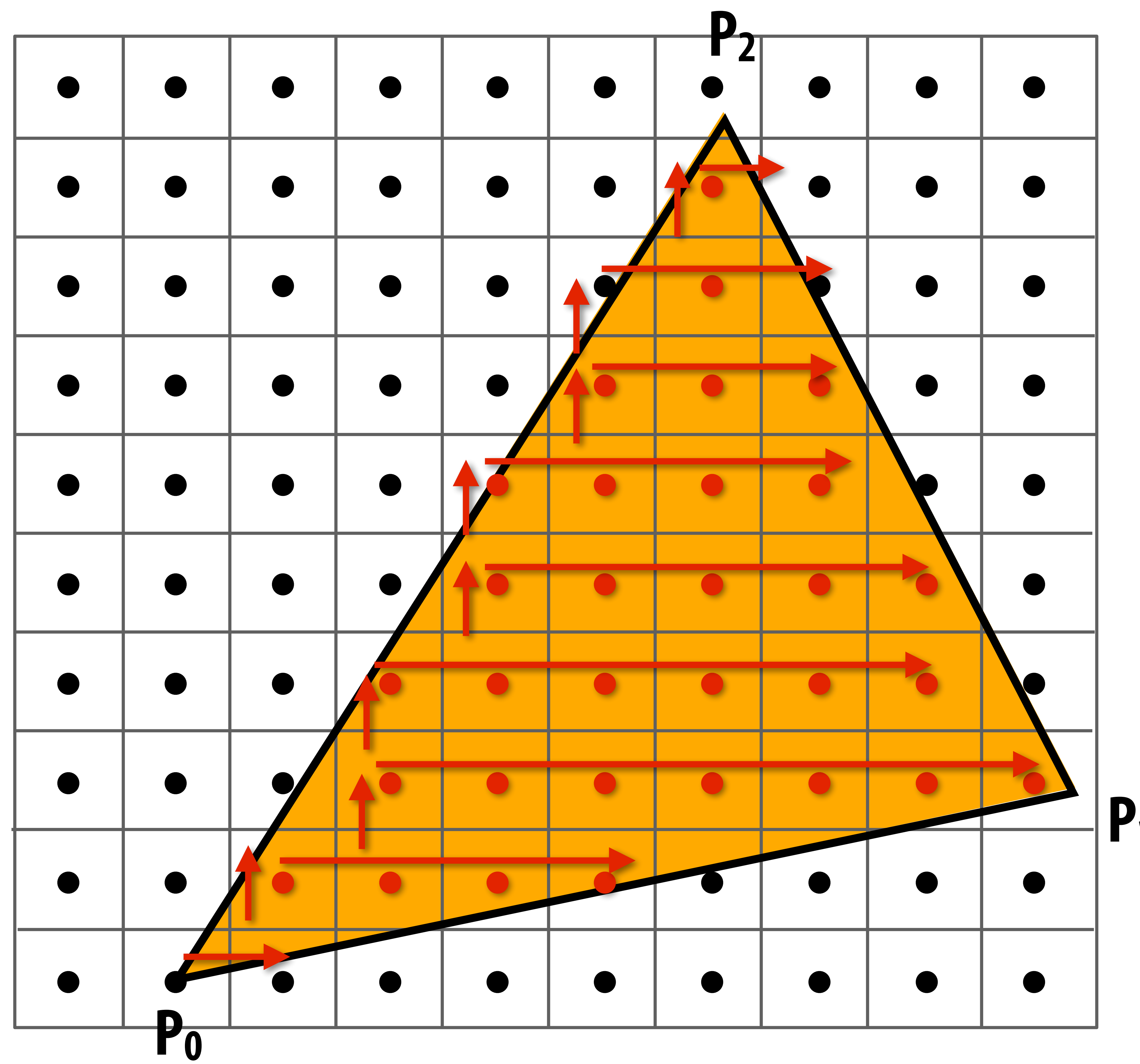
$dX_i = X_{i+1} - X_i$
$dY_i = Y_{i+1} - Y_i$

$$E_i(x, y) = (x - X_i)\, dY_i - (y - Y_i)\, dX_i$$
$$= A_i\, x + B_i\, y + C_i$$

$E_i(x, y) = 0$: point on edge
> 0 : outside edge
< 0 : inside edge

Efficient incremental update:

$$dE_i(x+1,y) = E_i(x,y) + dY_i = E_i(x,y) + A_i$$
$$dE_i(x,y+1) = E_i(x,y) + dX_i = E_i(x,y) + B_i$$



Incremental update saves computation:
Only one addition per edge, per sample test

Many traversal orders are possible: backtrack, zig-zag, Hilbert/Morton curves (locality maximizing)

Modern approach: tiled triangle traversal

Traverse triangle in blocks

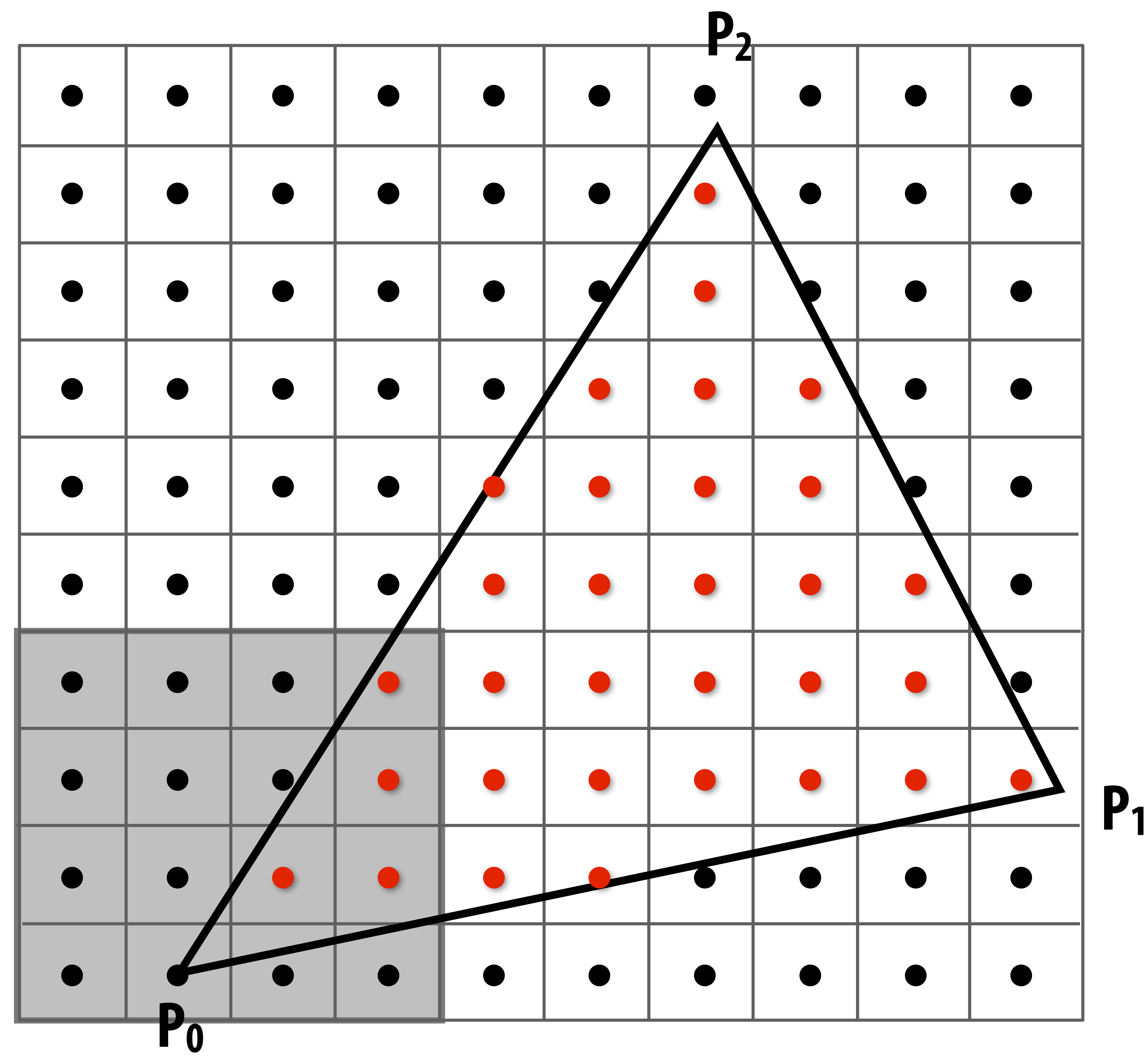
Test all samples in block against triangle in parallel

Advantages:

- Simplicity of wide parallel execution overcomes cost of extra point-in-triangle tests (most triangles cover many samples, especially when super-sampling coverage)

- Can skip sample testing work: entire block not in triangle (“early out”), entire block entirely within triangle (“early in”)

- Additional advantaged related to accelerating occlusion computations (not discussed today)



All modern GPUs have special-purpose hardware for efficiently performing point-in-triangle tests

Summary

- **We formulated computing triangle-screen coverage as a sampling problem**
  - Triangle-screen coverage is a 2D signal
  - Undersampling and the use of simple (non-ideal) reconstruction filters may yield aliasing
  - In today's example, we reduced aliasing via supersampling

- **Image formation on a display**
  - When samples are 1-to-1 with display pixels, sample values are handed directly to display
  - When "supersampling", resample densely sampled signal down to display resolution

- **Sampling screen coverage of a projected triangle:**
  - Performed via three point-inside-edge tests
  - Real-world implementation challenge: balance conflicting goals of avoiding unnecessary point-in-triangle tests and maintaining parallelism in algorithm implementation