# Monte Carlo Rendering

Computer Graphics
CMU 15-462/15-662

# TODAY: Monte Carlo Rendering

- How do we render a photorealistic image?
- Put together many of the ideas we've studied:
  - color
  - materials
  - radiometry
  - numerical integration
  - geometric queries
  - spatial data structures
  - rendering equation

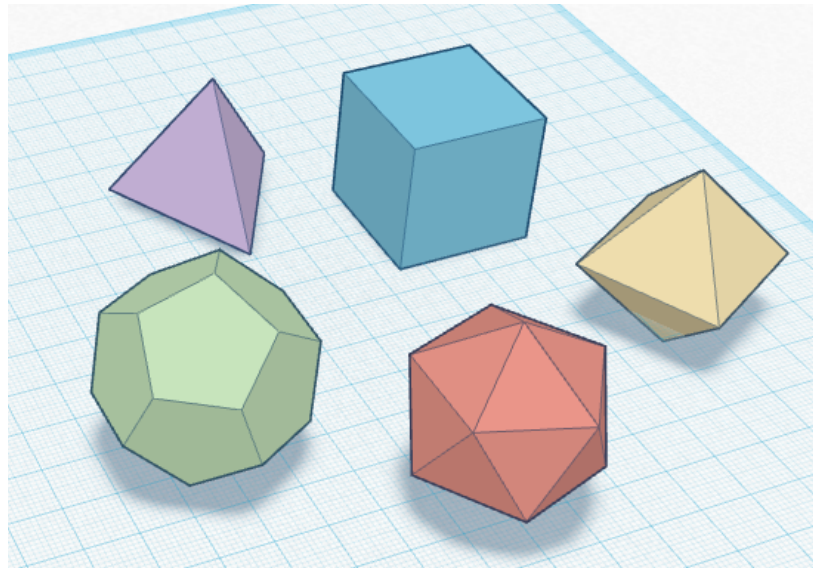
- Combine into final *Monte Carlo ray tracing* algorithm
- Alternative to rasterization, lets us generate *much* more realistic images (usually at much greater cost…)

# Photorealistic Rendering—Basic Goal

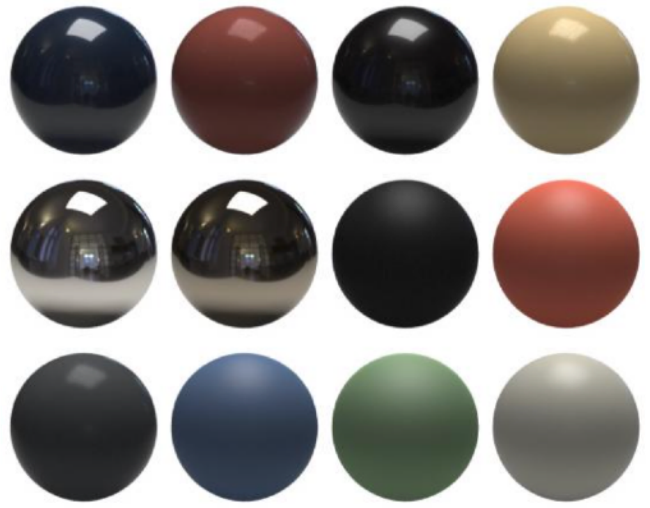
## What are the INPUTS and OUTPUTS?

camera


geometry



materials



lights

("scene")

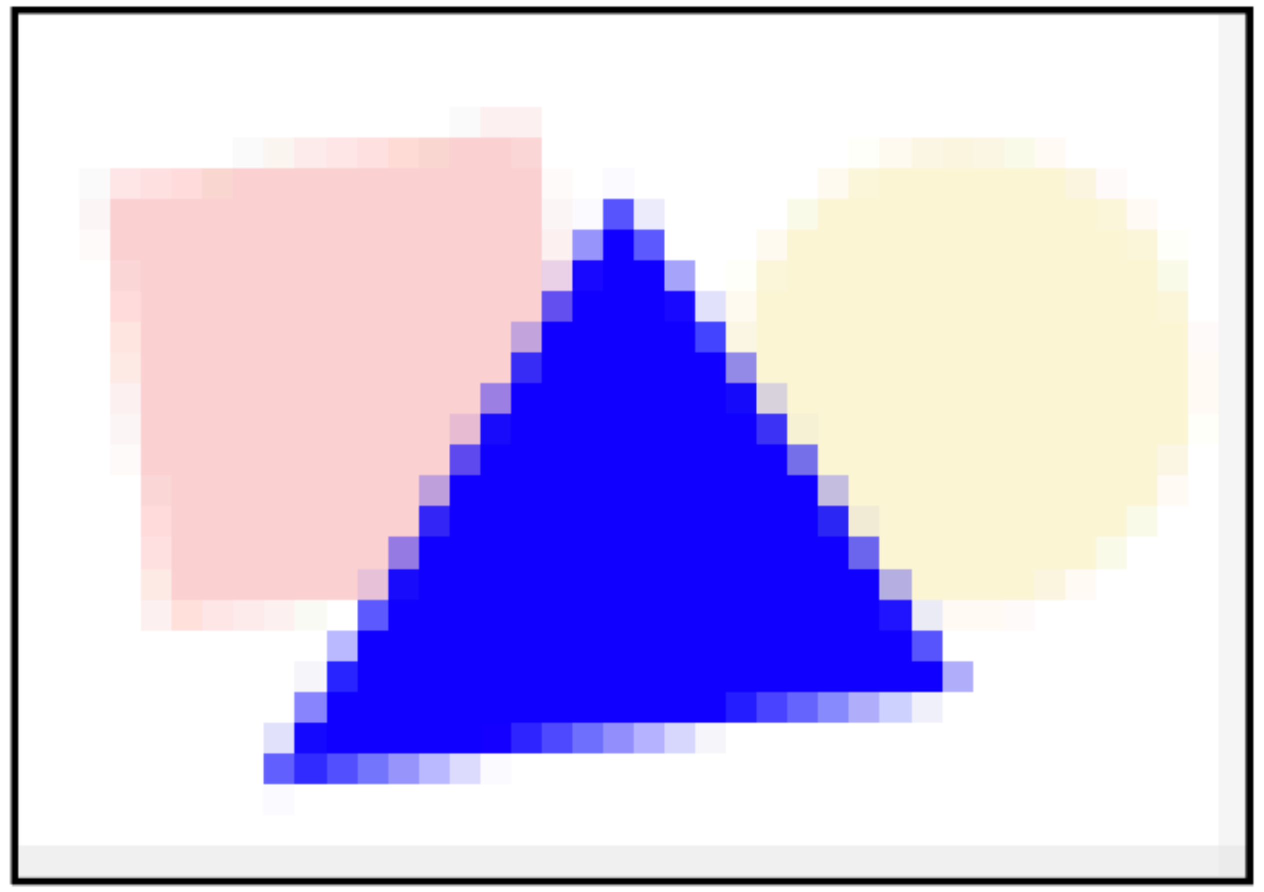
Ray Tracer

image

# Ray Tracing vs. Rasterization—Order

- Both rasterization & ray tracing will generate an image
- What's the difference?
- One basic difference: order in which we process samples

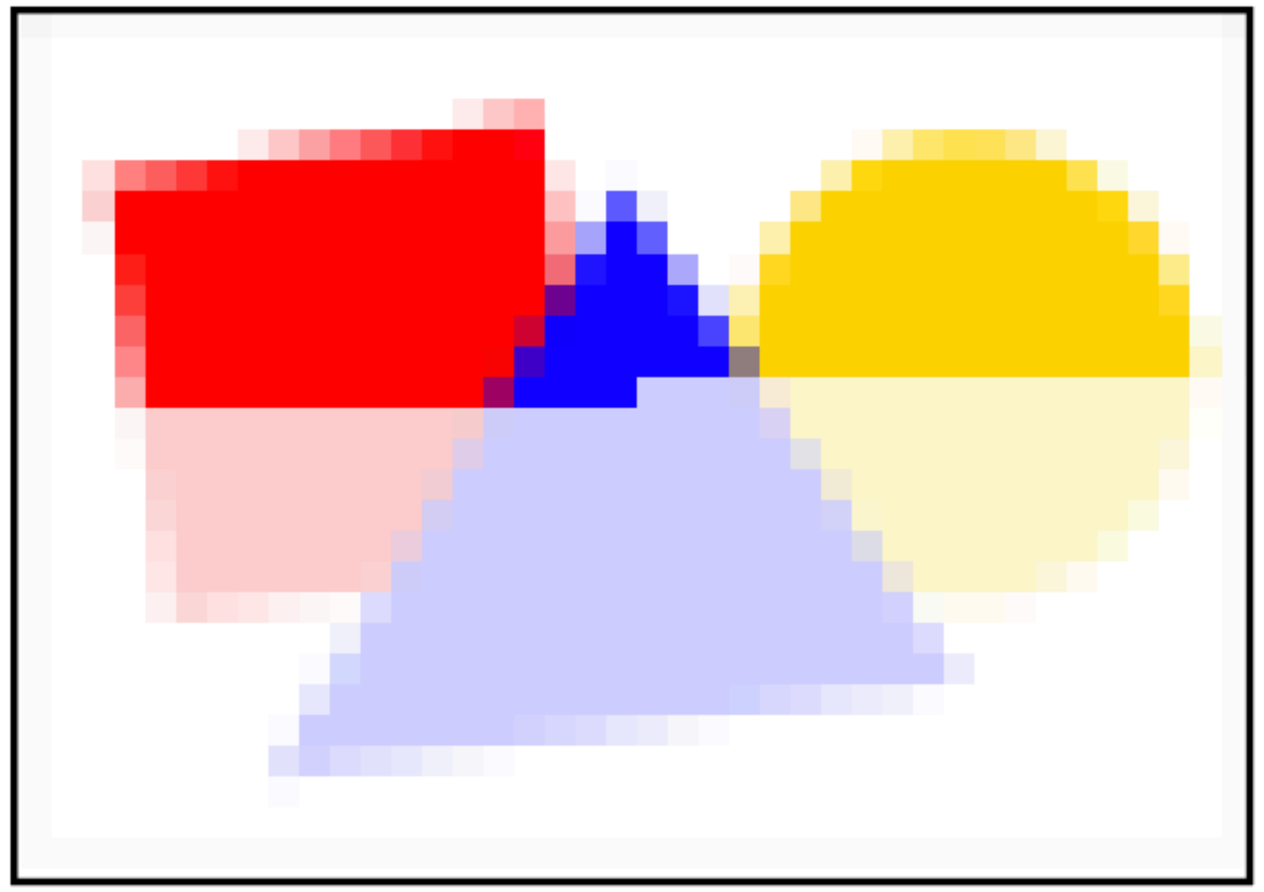
**RASTERIZATION**



```
for each primitive:
    for each sample:
        determine coverage
        evaluate color
```

(Use *Z-buffer* to determine which primitive is visible)

**RAY TRACING**



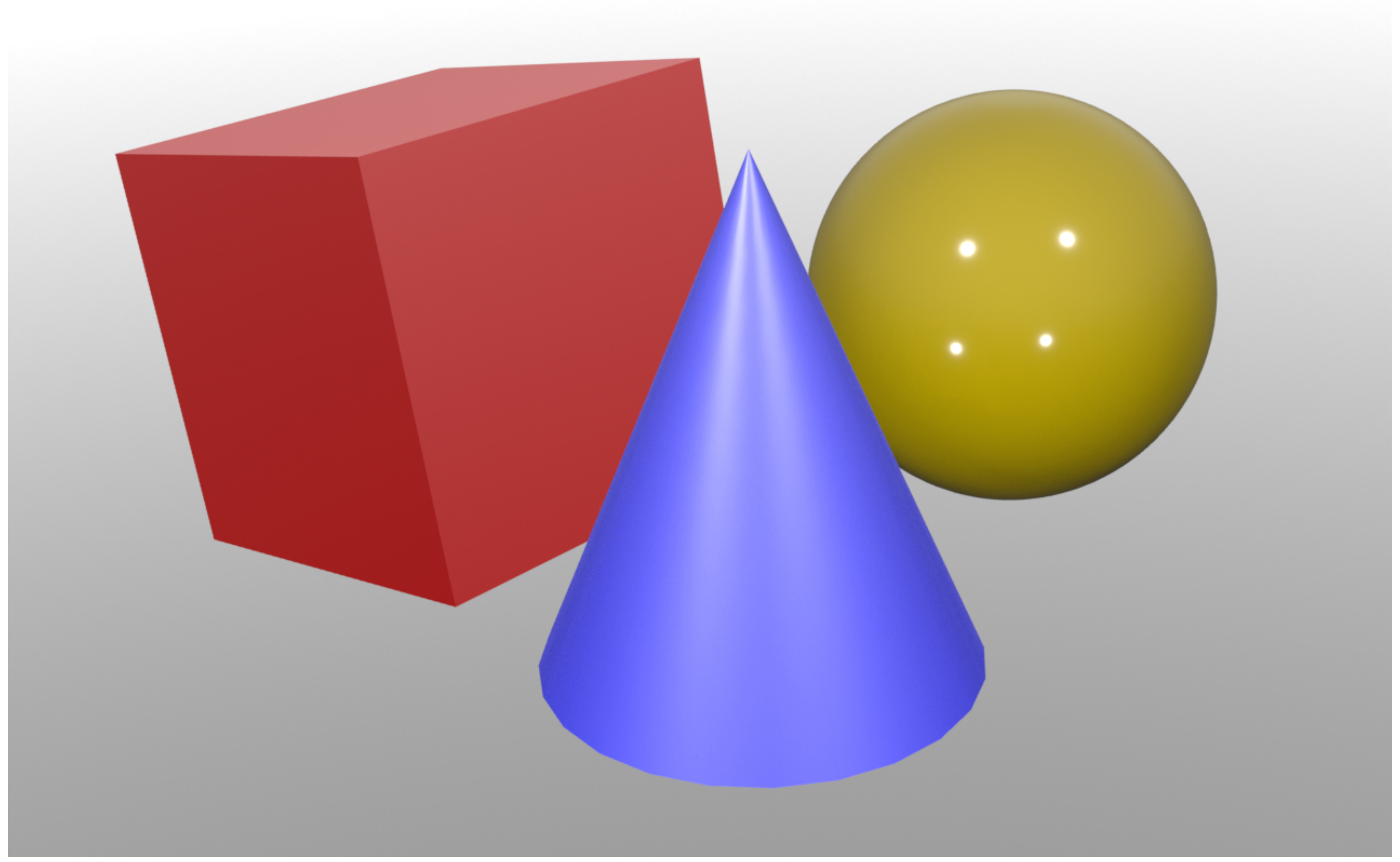
```
for each sample:
    for each primitive:
        determine coverage
        evaluate color
```

(Use spatial data structure like BVH to determine which primitive is visible)

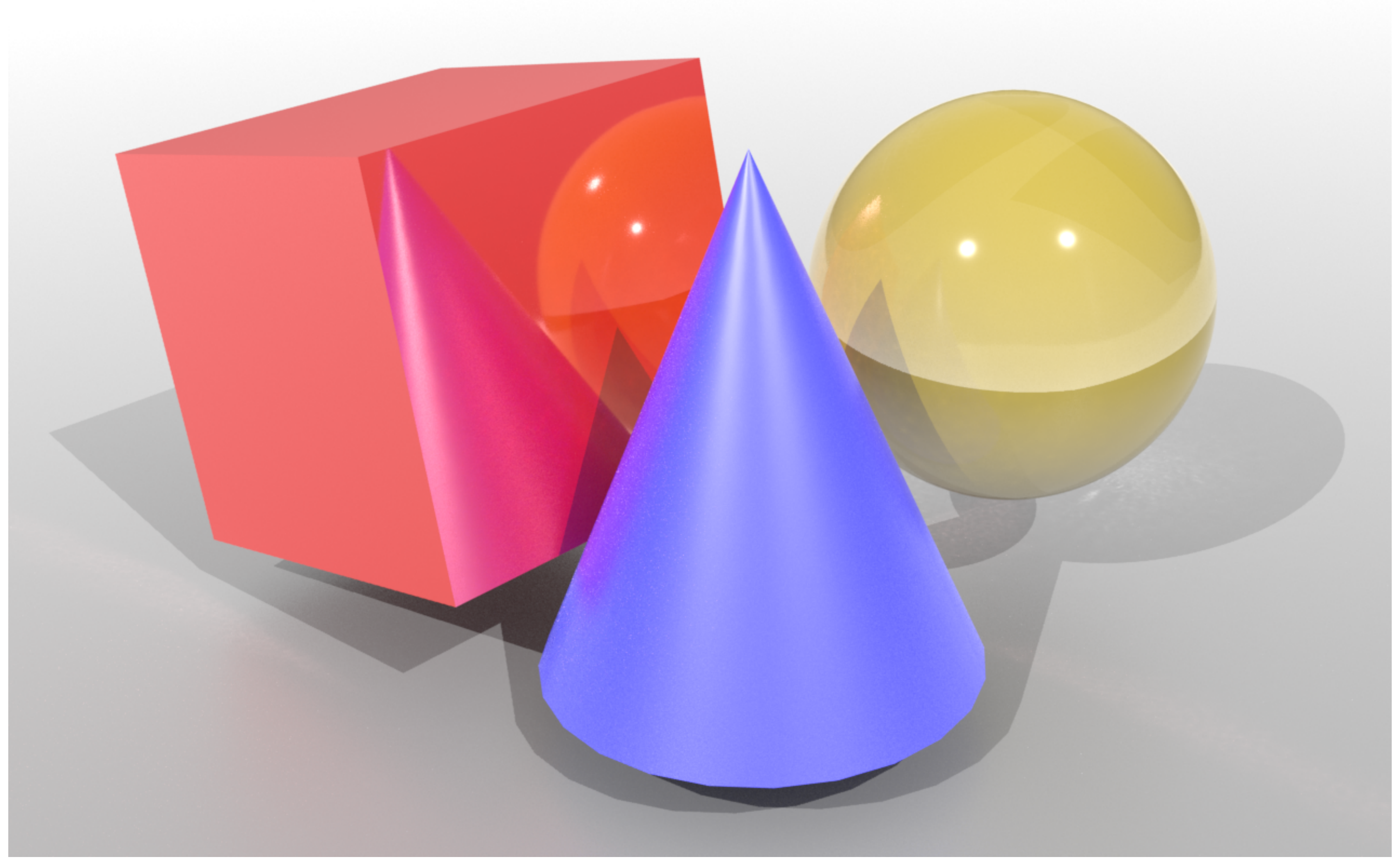
# Ray Tracing vs. Rasterization—Illumination

- More major difference: sophistication of illumination model
  - [LOCAL] rasterizer processes one *primitive* at a time; hard* to determine things like "A is in the shadow of B"
  - [GLOBAL] ray tracer processes on *ray* at a time; ray knows about everything it intersects, easy to talk about shadows & other "global" illumination effects

RASTERIZATION



RAY TRACING

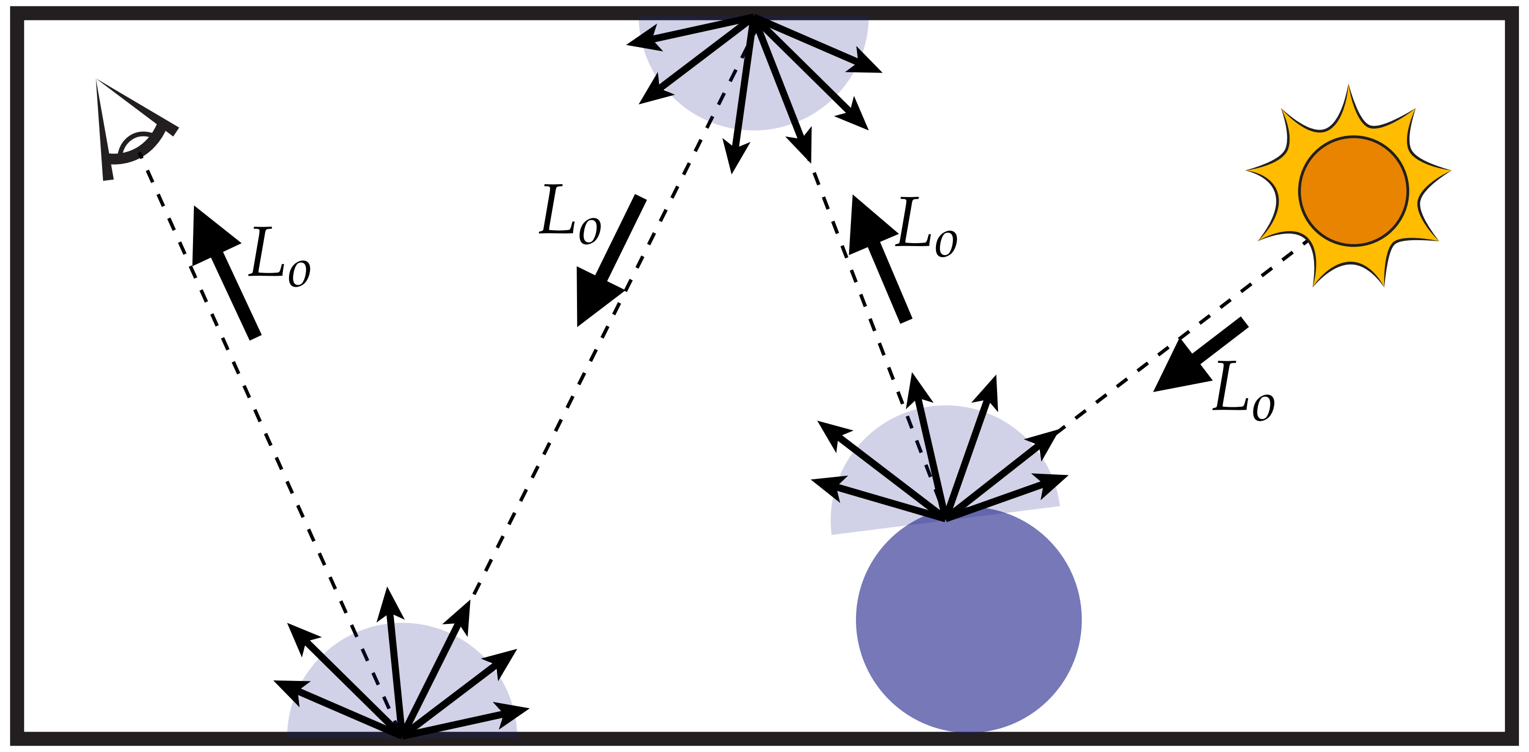


**Q: What illumination effects are missing from the image on the left?**

*But not *impossible* to do <u>some</u> things with rasterization (e.g., shadow maps)… just results in more complexity

# Monte Carlo Ray Tracing

- To develop a full-blown photorealistic ray tracer, will need to apply *Monte Carlo integration* to the *rendering equation*
- To determine color of each pixel, integrate incoming light
- What function are we integrating?
  - illumination along different paths of light
- What does a "sample" mean in this context?
  - each path we trace is a sample




$$L_o(\mathbf{p}, \omega_o) = L_e(\mathbf{p}, \omega_o) + \int_{\mathcal{H}^2} f_r(\mathbf{p}, \omega_i \to \omega_o) L_i(\mathbf{p}, \omega_i) \cos\theta \, d\omega_i$$

# Monte Carlo Integration

- **Started looking at Monte Carlo integration in our lecture on numerical integration**
- **Basic idea: take average of random samples**
- **Will need to flesh this idea out with some key concepts:**
  - **EXPECTED VALUE — what value do we get *on average*?**
  - **VARIANCE — what's the expected *deviation* from the average?**
  - **IMPORTANCE SAMPLING — how do we (correctly) take more samples in more important regions?**

$$\boxed{\lim_{N\to\infty} \frac{|\Omega|}{N} \sum_{i=1}^{N} f(X_i) = \int_{\Omega} f(x)\, dx}$$

# Expected Value

**Intuition: what value does a random variable take, on average?**

- E.g., consider a fair coin where heads = 1, tails = 0
- Equal probability of heads & is tails (1/2 for both)
- *Expected value* is then (1/2)•1 + (1/2)•0 = 1/2

expected value of random variable Y

number of possible outcomes

$$E(Y) := \sum_{i=1}^{k} p_i y_i$$

value of ith outcome

probability of ith outcome

**Properties of expectation:**

$$E\left[\sum_i Y_i\right] = \sum_i E[Y_i]$$

$$E[aY] = aE[Y]$$

(Can you show these are true?)

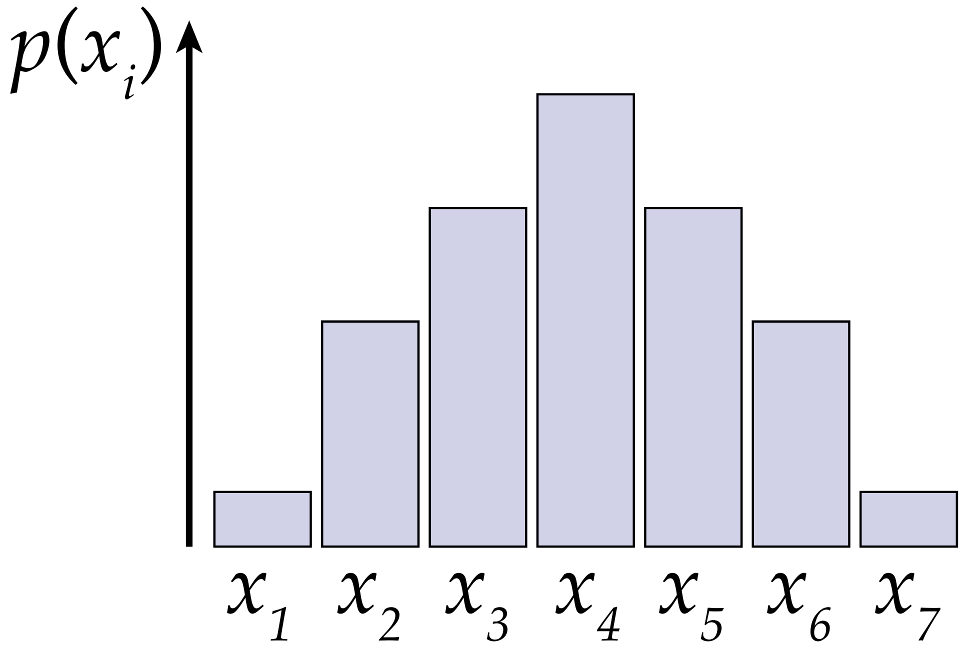
# Variance

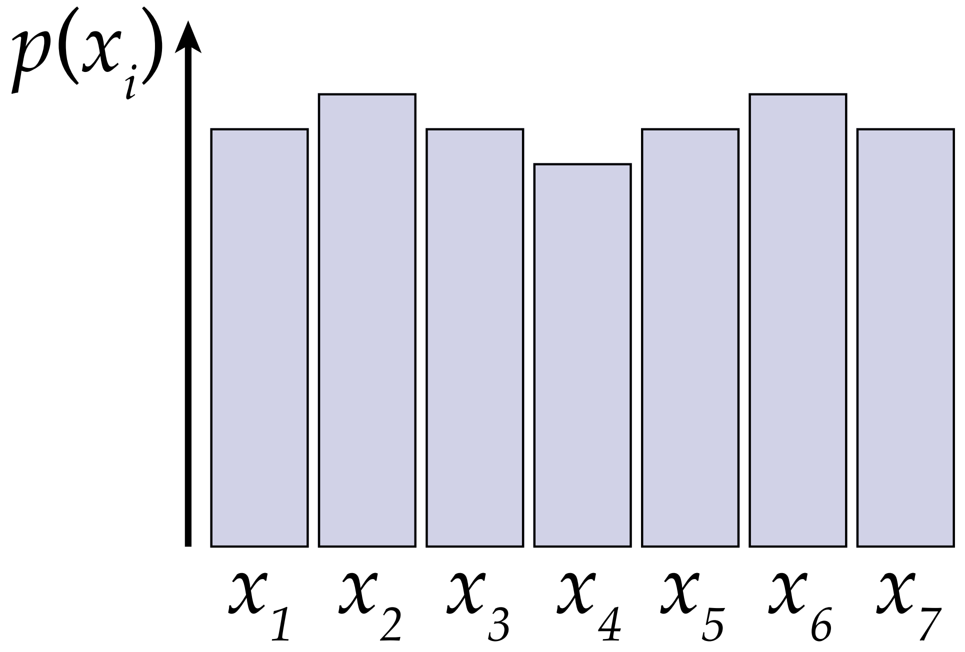
**Intuition: how far are our samples from the average, on average?**

## Definition

$$V[Y] = E[(Y - E[Y])^2]$$

**Q: Which of these has higher variance?**




**Properties of variance:**

$$V[Y] = E[Y^2] - E[Y]^2$$

$$V\left[\sum_{i=1}^{N} Y_i\right] = \sum_{i=1}^{N} V[Y_i]$$

$$V[aY] = a^2\, V[Y]$$

**(Can you show these are true?)**

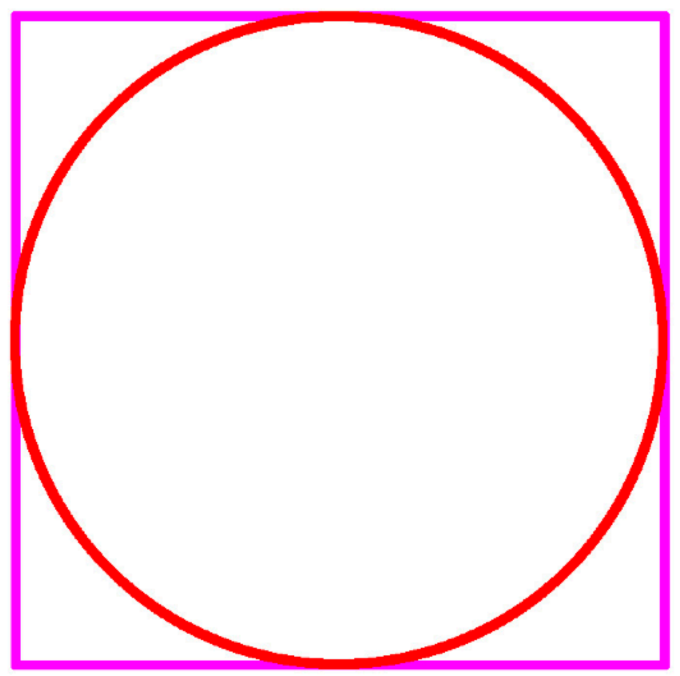
# Law of Large Numbers

- **Important fact: for any random variable, the average value of N trials approaches the expected value as we increase N**
- **Decrease in variance is always *linear* in N:**

$$V\left[\frac{1}{N}\sum_{i=1}^{N} Y_i\right] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N^2} N\, V[Y] = \frac{1}{N} V[Y]$$

**Consider a coconut…**

| nCoconuts | estimate of $\pi$ |
|---|---|
| 1 | 4.000000 |
| 10 | 3.200000 |
| 100 | 3.240000 |
| 1000 | 3.112000 |
| 10000 | 3.163600 |
| 100000 | 3.139520 |
| 1000000 | 3.141764 |



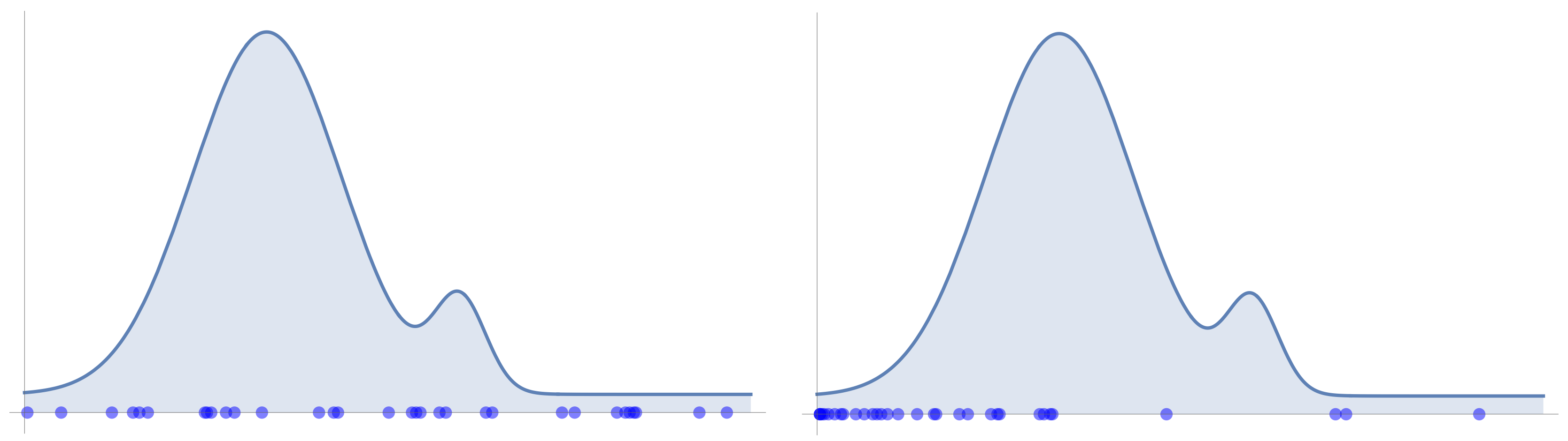
**Q: Why is the law of large numbers important for Monte Carlo ray tracing?**

**A: No matter how hard the integrals are (crazy lighting, geometry, materials, etc.), can *always** get the right image by taking more samples.**

*As long as we make sure to sample all possible kinds of light paths…

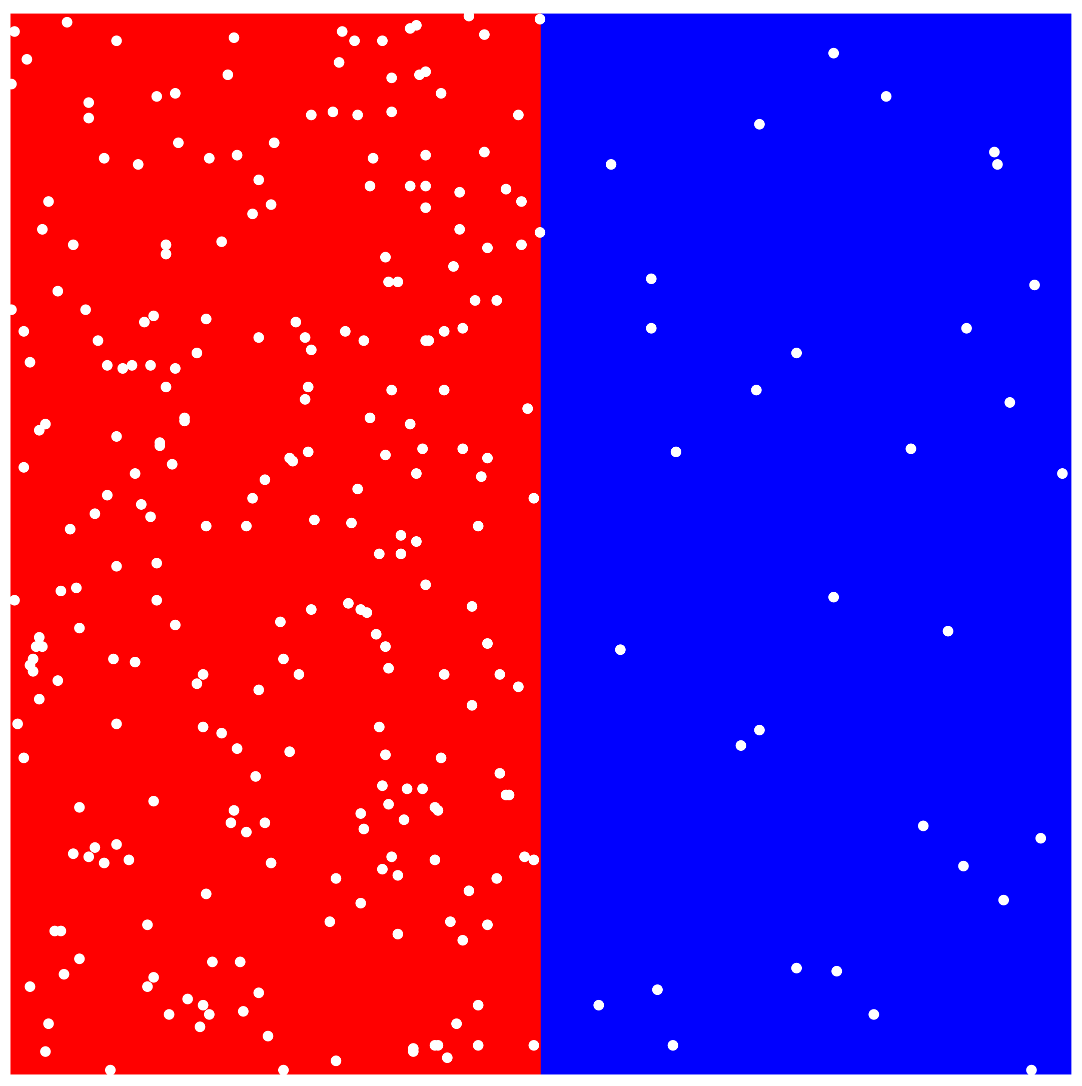
# Biasing

- So far, we've picked samples *uniformly* from the domain (every point is equally likely)
- Suppose we pick samples from some other distribution (more samples in one place than another)
- Q: Can we still use samples f(Xi) to get a (correct) estimate of our integral?
- A: Sure! Just weight contribution of each sample by how likely we were to pick it
- Q: Are we correct to *divide* by *p*? Or… should we *multiply* instead?
- A: Think about a simple example where we sample RED region 8x as often as BLUE region
  - average color over square should be purple
  - if we *multiply*, average will be TOO RED
  - if we divide, average will be JUST RIGHT



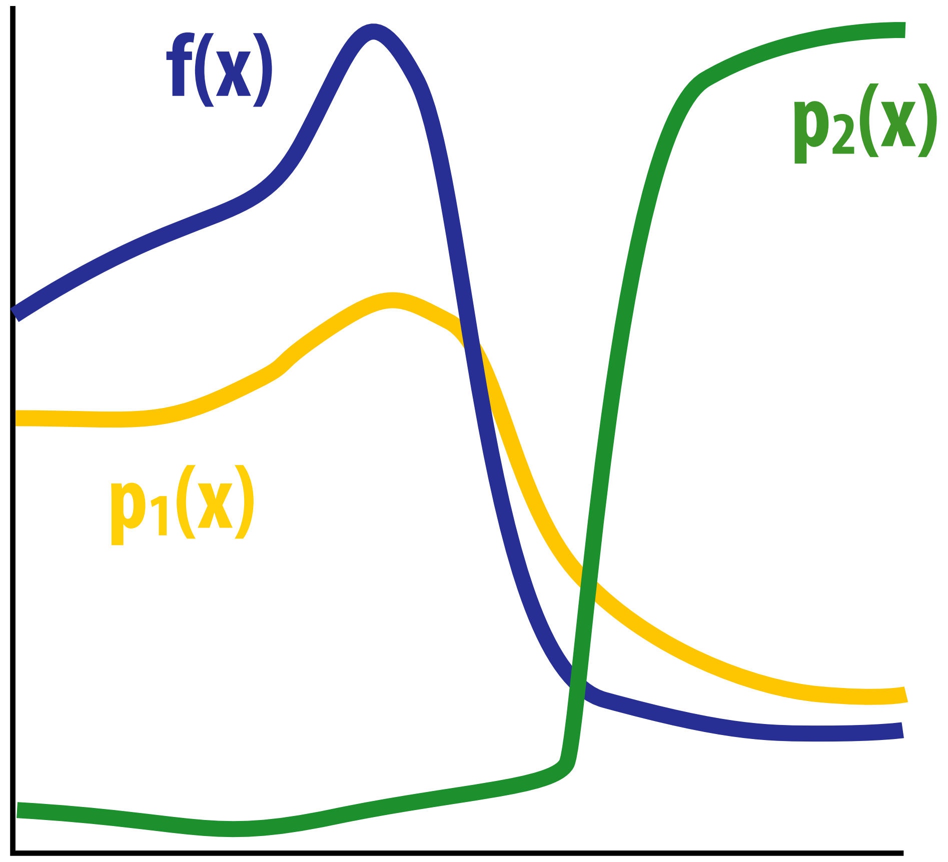
(uniform) $X_i \sim p$

$$\int_\Omega f(x)\,dx \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$



# Importance sampling

**Q: Ok, so then WHERE is the best place to take samples?**

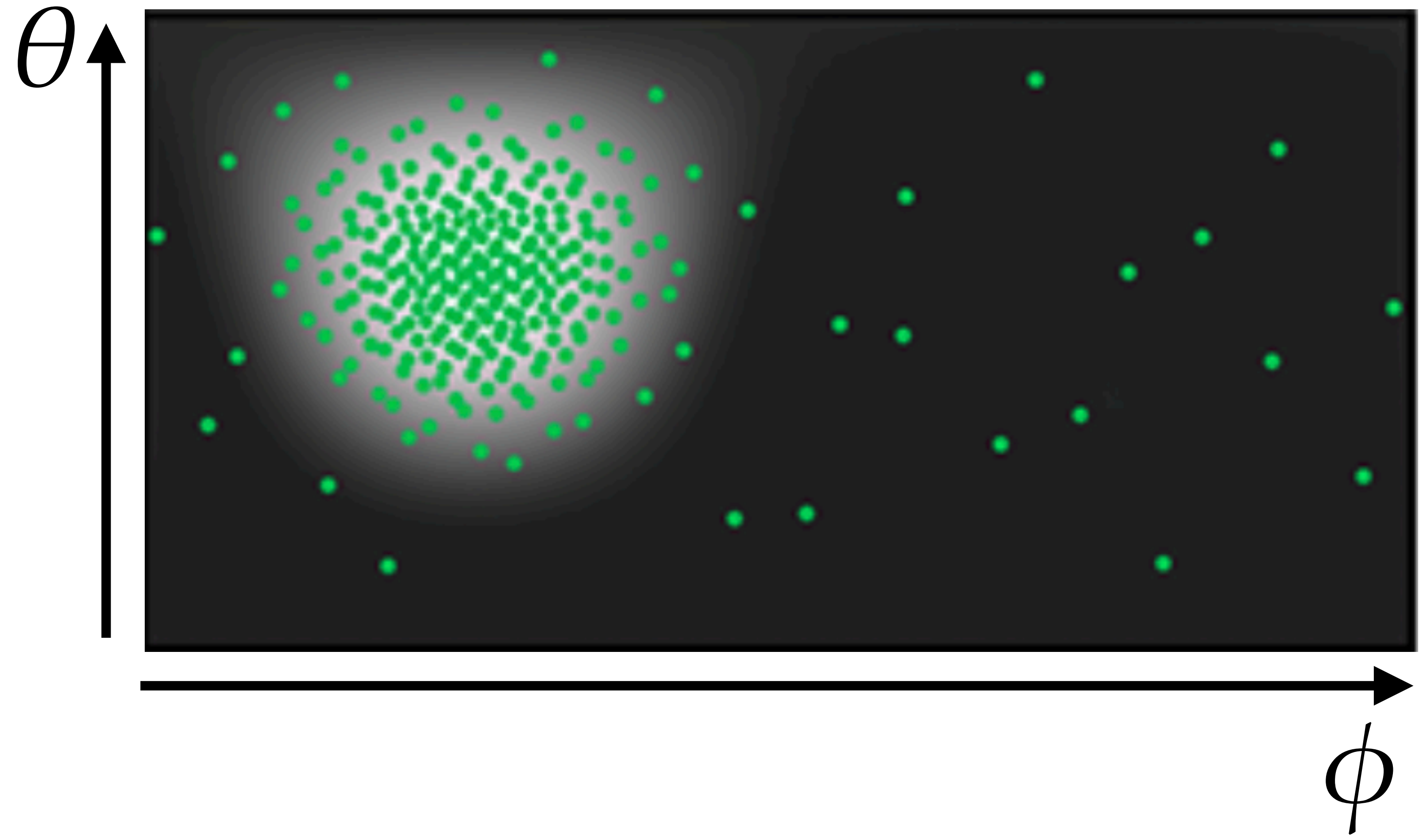



**Think:**

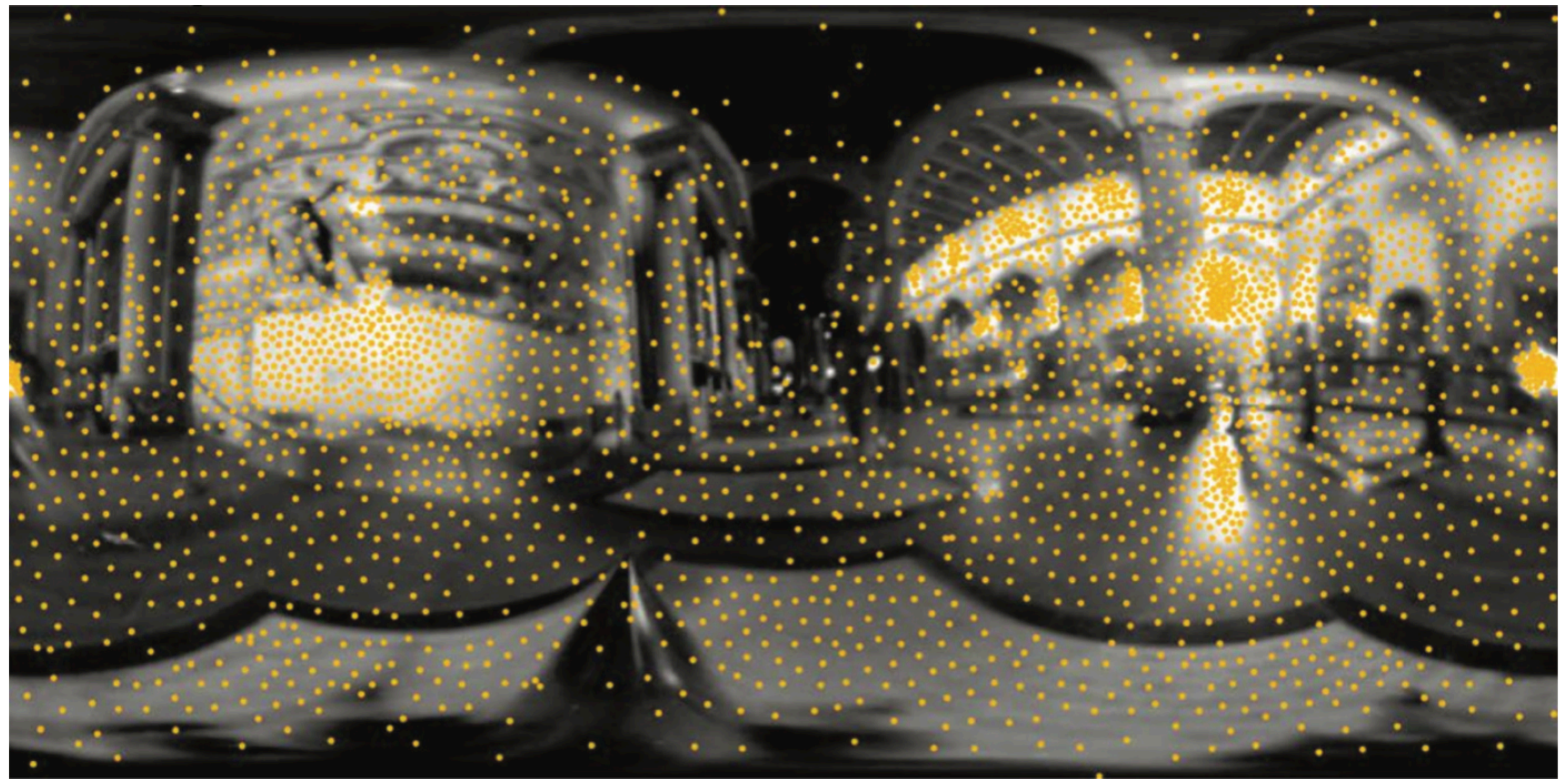
- **What is the behavior of $f(x)/p_1(x)$? $f(x)/p_2(x)$?**
- **How does this impact the variance of the estimator?**

**Idea: put *more* where integrand is *large* ("most useful samples"). E.g.:**

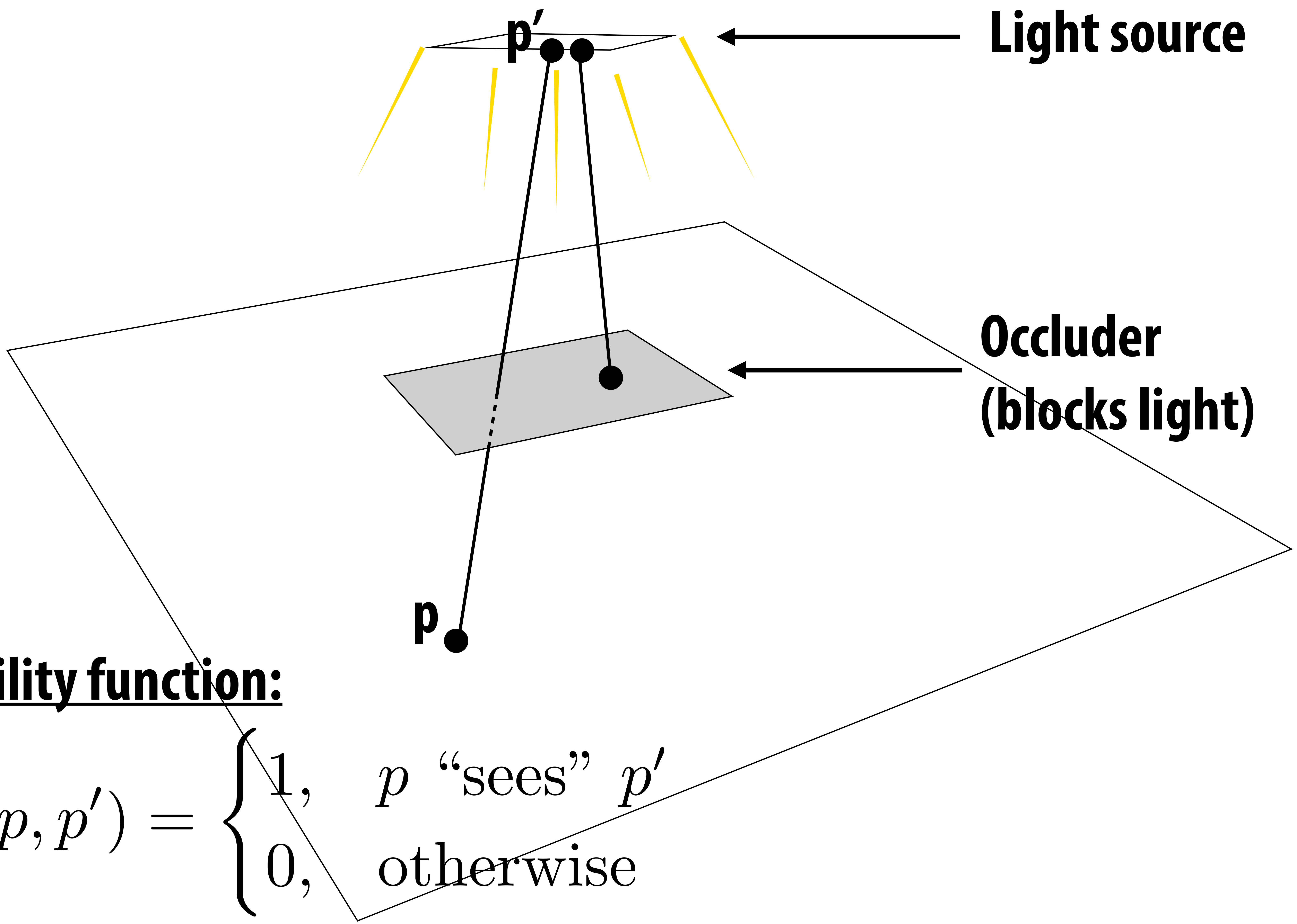
**(BRDF)**




**(image-based lighting)**



# Example: Direct Lighting




**Visibility function:**

$$V(p,p') = \begin{cases} 1, & p \text{ ``sees'' } p' \\ 0, & \text{otherwise} \end{cases}$$

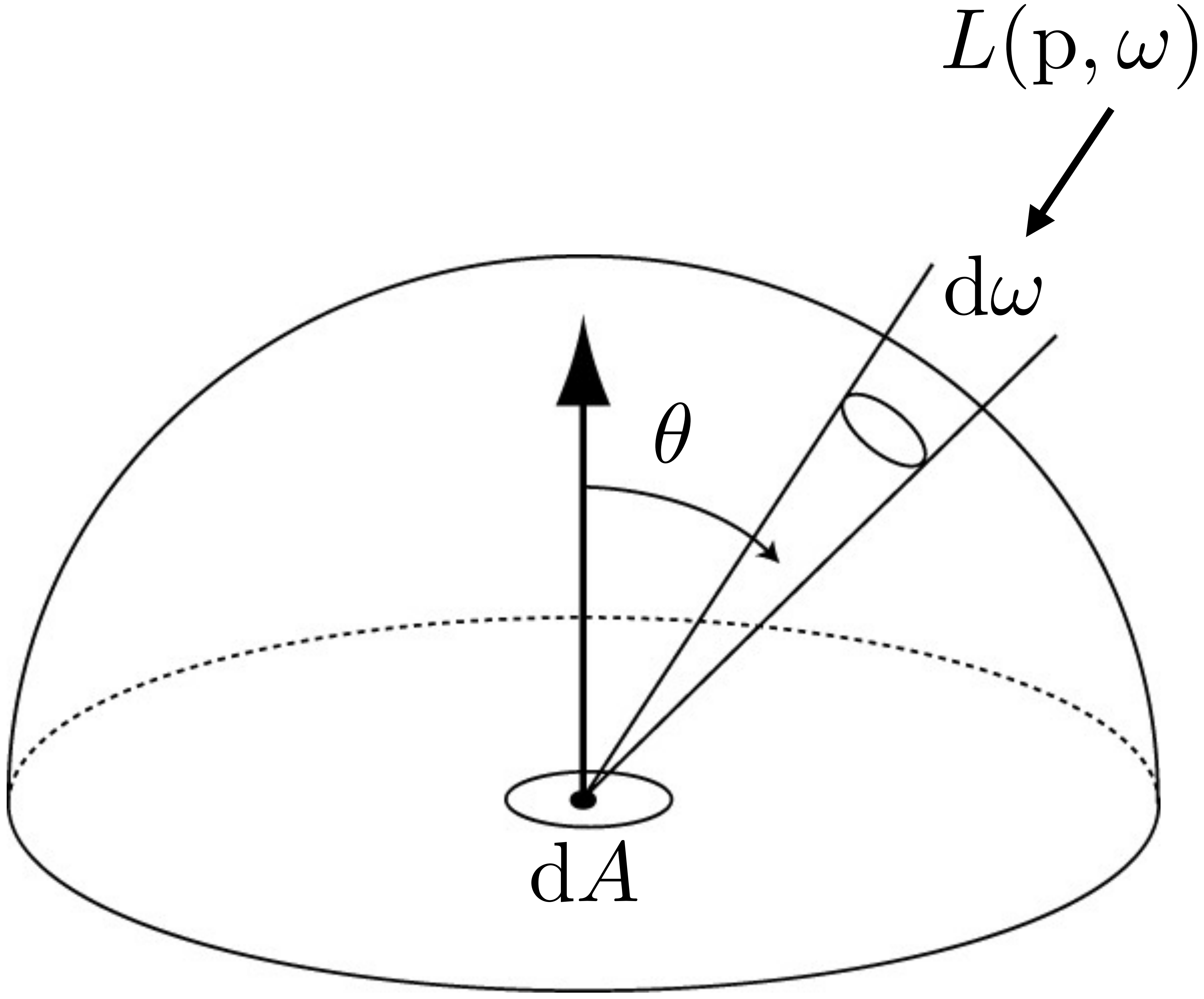
**How bright is each point on the ground?**

# Direct lighting—uniform sampling

**Uniformly-sample hemisphere of directions with respect to solid angle**

$$p(\omega) = \frac{1}{2\pi}$$

$$E(\mathrm{p}) = \int L(\mathrm{p}, \omega) \, \cos\theta \, \mathrm{d}\omega$$




**Estimator:**

$$X_i \sim p(\omega)$$

$$Y_i = f(X_i)$$

$$Y_i = L(\mathrm{p}, \omega_i) cos\,\theta_i$$

$$F_N = \frac{2\pi}{N} \sum_{i=1}^{N} Y_i$$

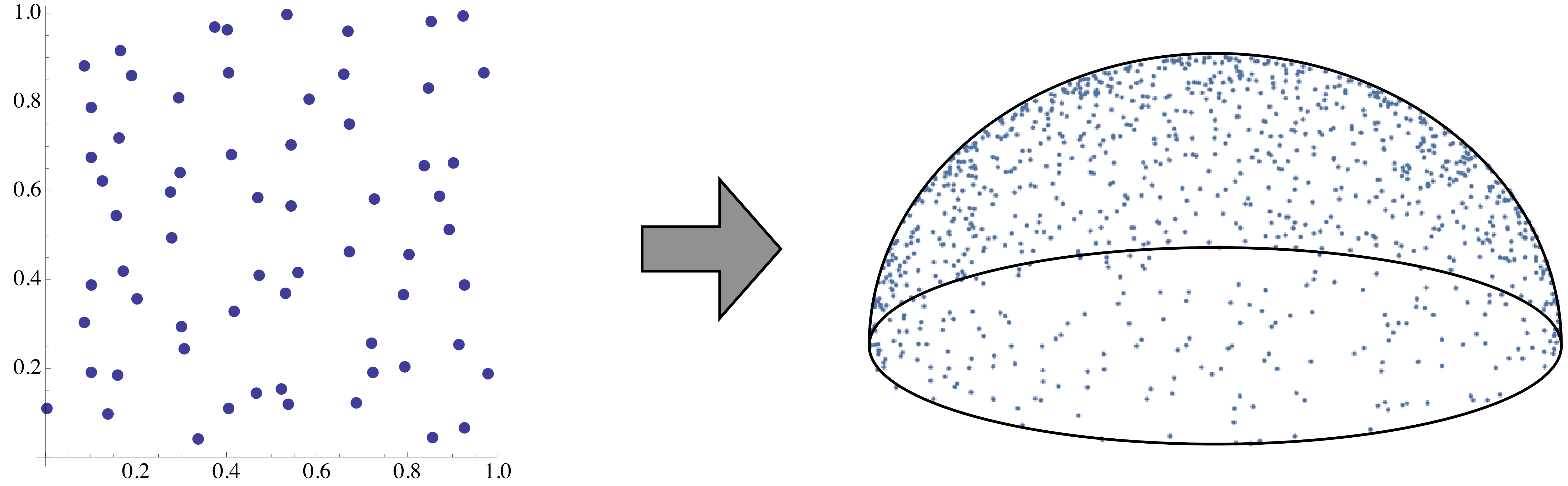
# Aside: Picking points on unit hemisphere

**How do we <u>uniformly</u> sample directions from the hemisphere?**

**One way: use *rejection sampling*. (How?)**

**Another way: "warp" two values in [0,1] via the inversion method:**

$$(\xi_1, \xi_2) = \left(\sqrt{1-\xi_1^2}\cos(2\pi\xi_2), \sqrt{1-\xi_1^2}\sin(2\pi\xi_2), \xi_1\right)$$




**Exercise: derive from the inversion method**

# Direct lighting—uniform sampling (algorithm)

**Uniformly-sample hemisphere of directions with respect to solid angle**

$$p(\omega) = \frac{1}{2\pi}$$

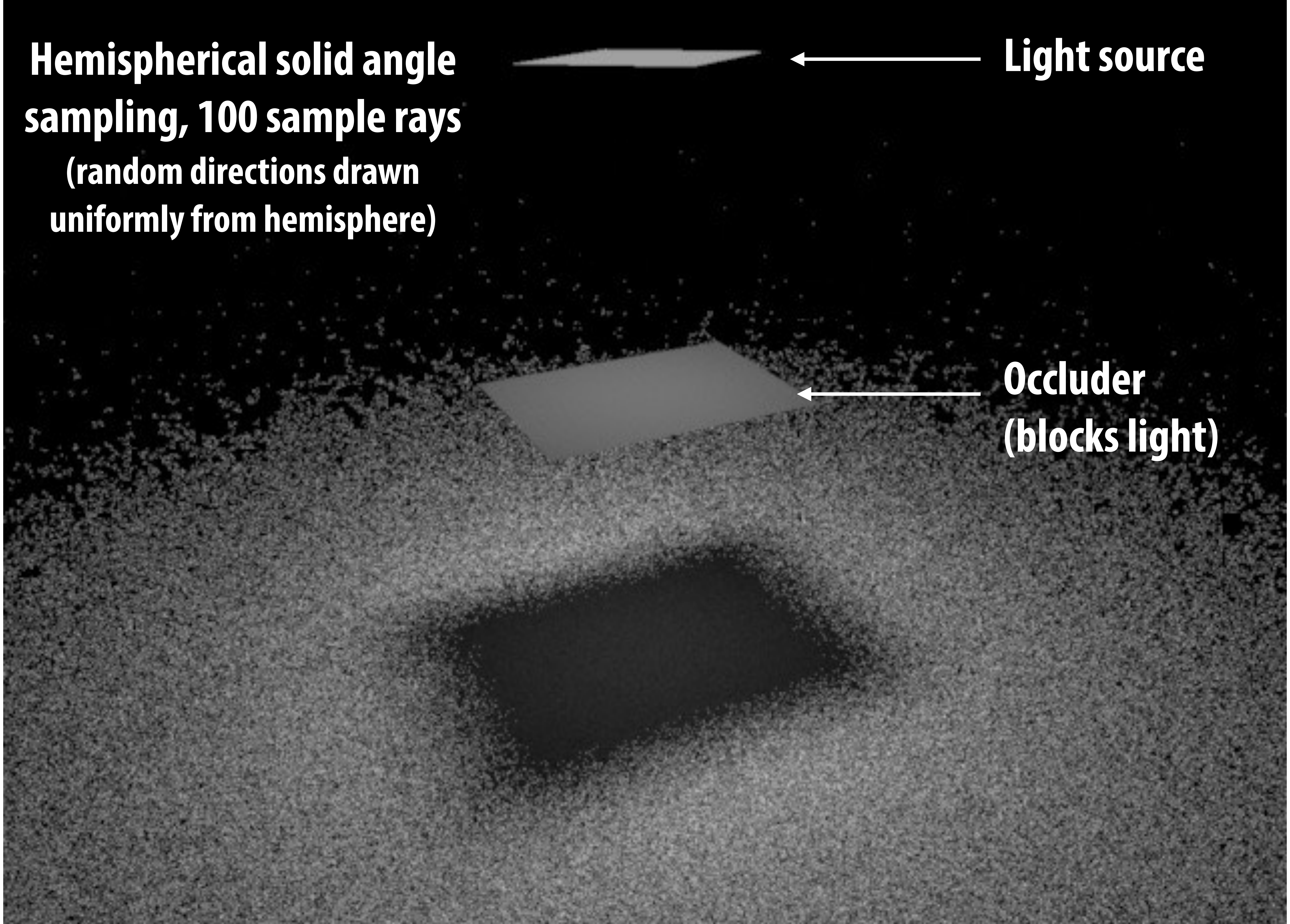
$$E(\mathrm{p}) = \int L(\mathrm{p}, \omega) \, \cos\theta \, \mathrm{d}\omega$$

**Given surface point *p***

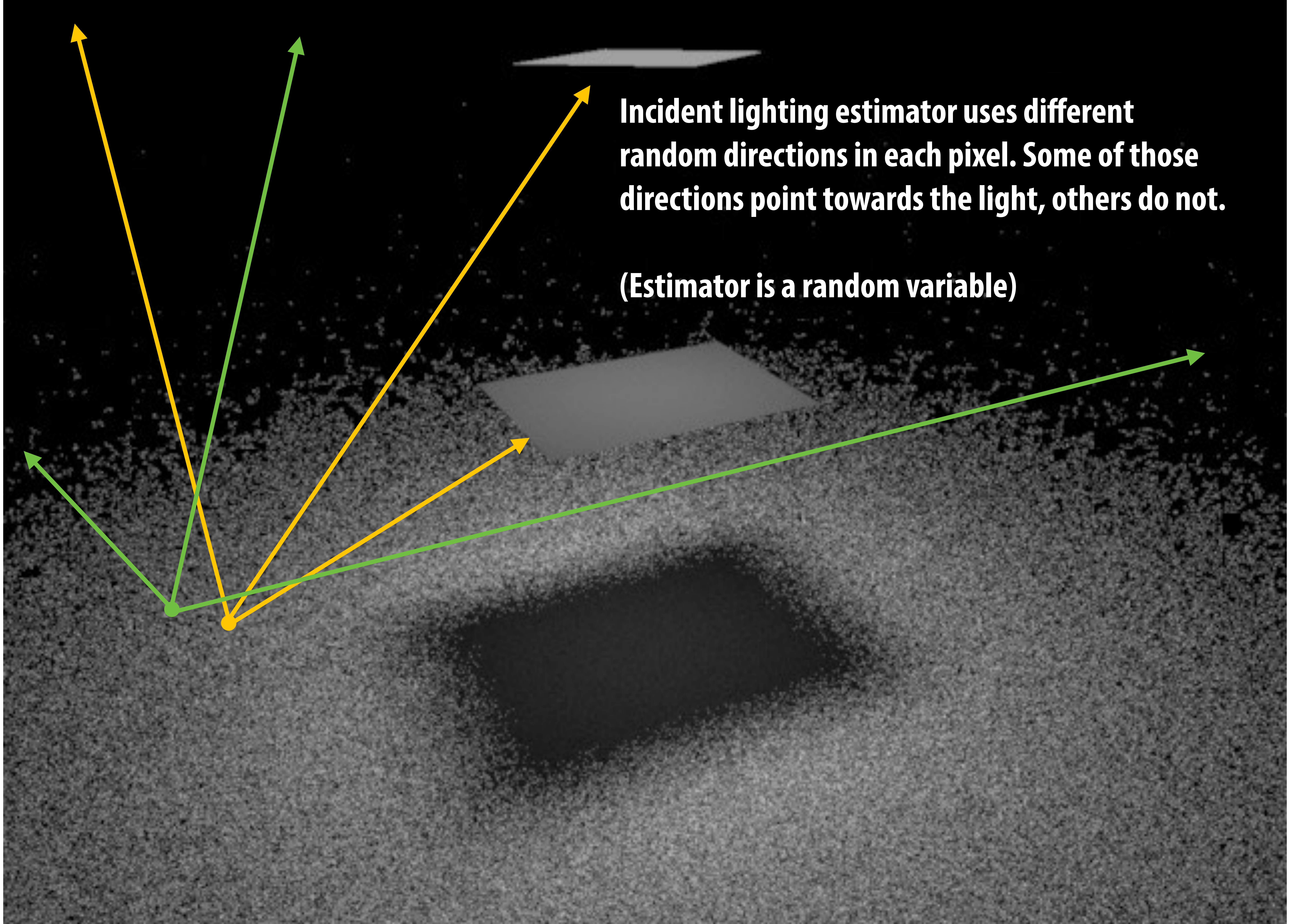
**A ray tracer evaluates radiance along a ray (see Pathtracer::trace() in pathtracer.cpp)**

**For each of N samples:**

- **Generate random direction:** $\omega_i$
- **Compute incoming radiance arriving** $L_i$ **at *p* from direction:** $\omega_i$
- **Compute incident irradiance due to ray:** $dE_i = L_i cos\,\theta_i$
- **Accumulate** $\frac{2\pi}{N} dE_i$ **into estimator**


Hemispherical solid angle sampling, 100 sample rays
(random directions drawn uniformly from hemisphere)
Light source
Occluder
(blocks light)

# Why is the image in the previous slide "noisy"?



Incident lighting estimator uses different random directions in each pixel. Some of those directions point towards the light, others do not.

(Estimator is a random variable)

# How can we reduce noise?

**One idea:** just take more samples!

**Another idea:**

- Don’t need to integrate over entire hemisphere of directions (incoming radiance is 0 from most directions).
- Just integrate over the area of the light (directions where incoming radiance is non-zero)and  weight appropriately

# Direct lighting: area integral

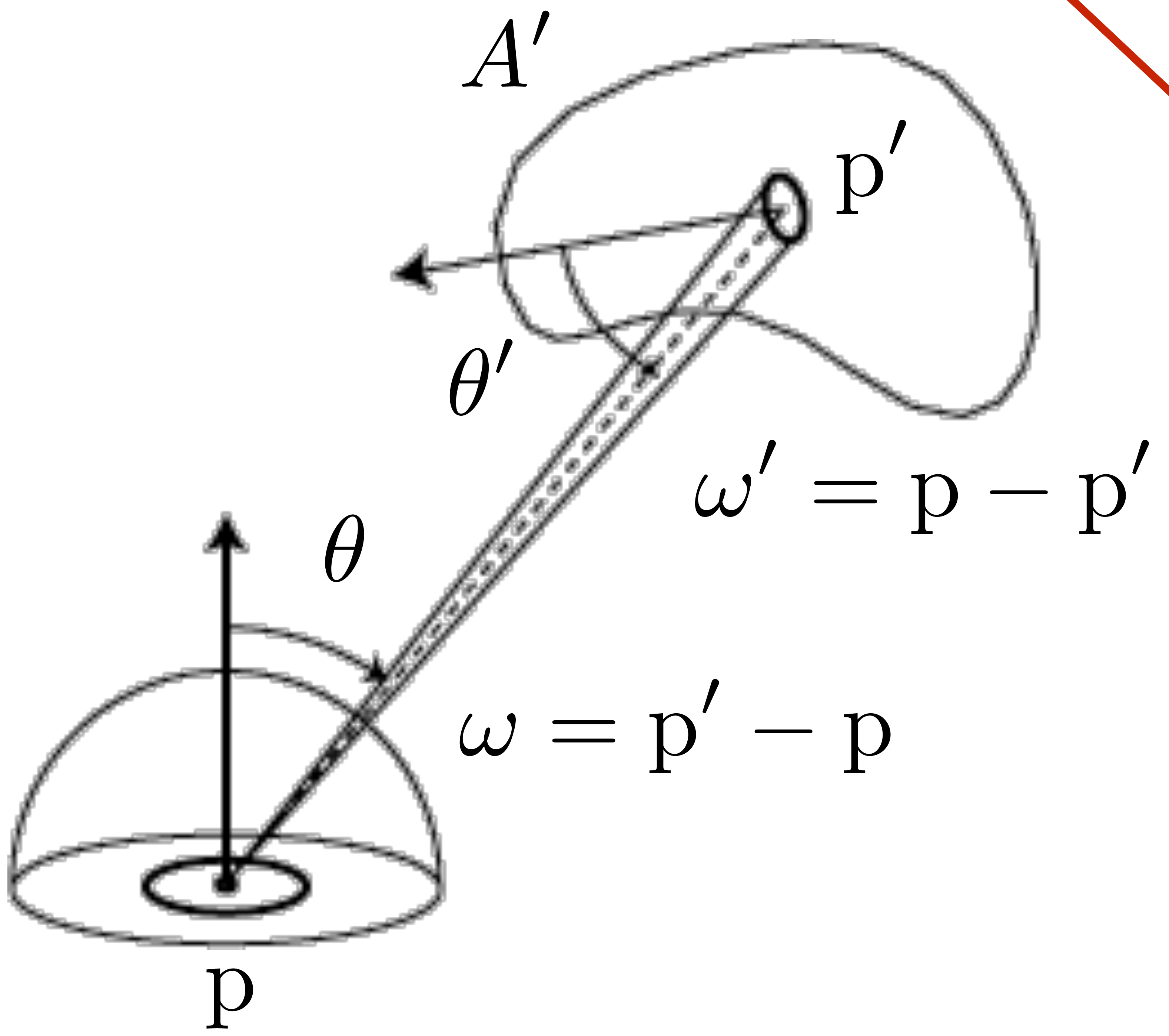
$$E(\mathrm{p}) = \int L(\mathrm{p}, \omega)\ \cos\theta\, \mathrm{d}\omega$$

Previously: just integrate over all directions

$$E(\mathrm{p}) = \int_{A'} L_o(\mathrm{p}', \omega')\, V(\mathrm{p}, \mathrm{p}')\, \frac{\cos\theta \cos\theta'}{|\mathrm{p} - \mathrm{p}'|^2}\, \mathrm{d}A'$$

Change of variables to integrate over area of light

$$d\omega = \frac{dA}{|\mathrm{p}' - \mathrm{p}|^2} = \frac{dA' cos\,\theta'}{|\mathrm{p}' - \mathrm{p}|^2}$$

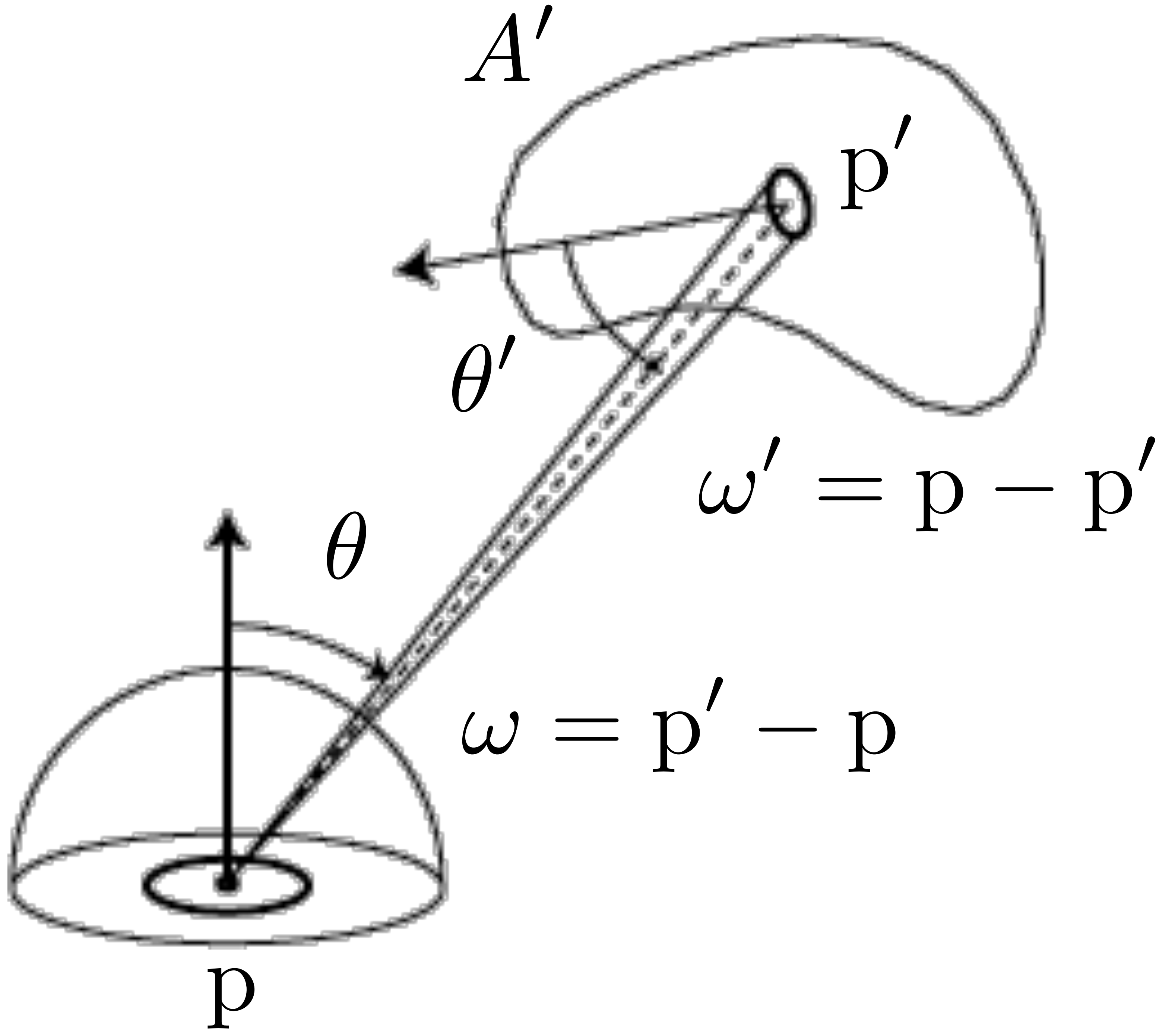



Binary visibility function:
1 if p′ is visible from p, 0 otherwise
(accounts for light occlusion)

Outgoing radiance from light point p, in direction w′ towards p

# Direct lighting: area integral

$$E(\mathrm{p}) = \int_{A'} L_o(\mathrm{p}', \omega')\, V(\mathrm{p}, \mathrm{p}')\, \frac{\cos\theta \cos\theta'}{|\mathrm{p} - \mathrm{p}'|^2}\, \mathrm{d}A'$$



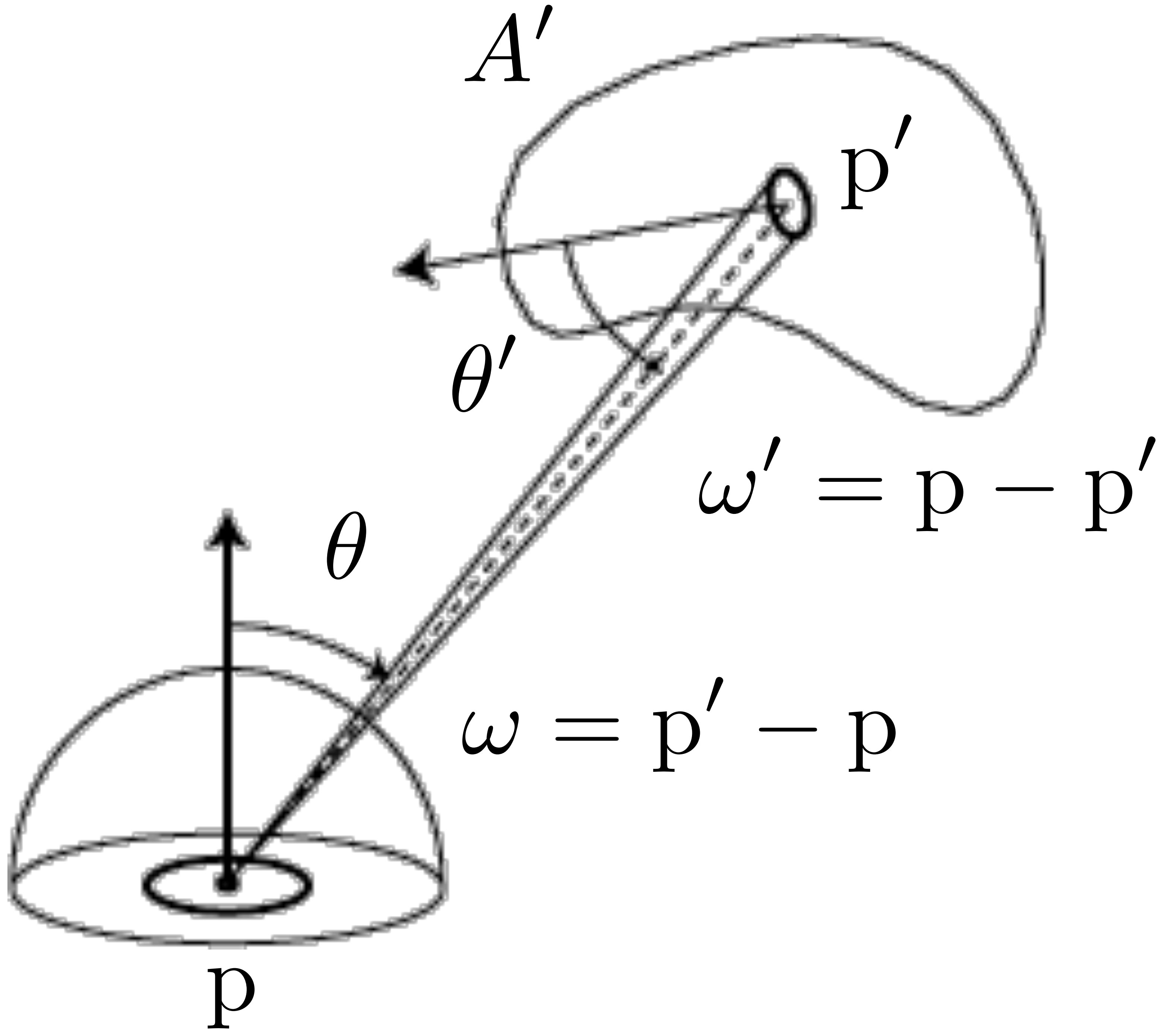

**Sample shape uniformly by area A'**

$$\int_{A'} p(\mathrm{p}')\, \mathrm{d}A' = 1$$

$$p(\mathrm{p}') = \frac{1}{A'}$$

# Direct lighting: area integral

$$E(\mathrm{p}) = \int_{A'} L_o(\mathrm{p}', \omega')\, V(\mathrm{p}, \mathrm{p}')\, \frac{\cos\theta \cos\theta'}{|\mathrm{p} - \mathrm{p}'|^2}\, \mathrm{d}A'$$




**Probability:**

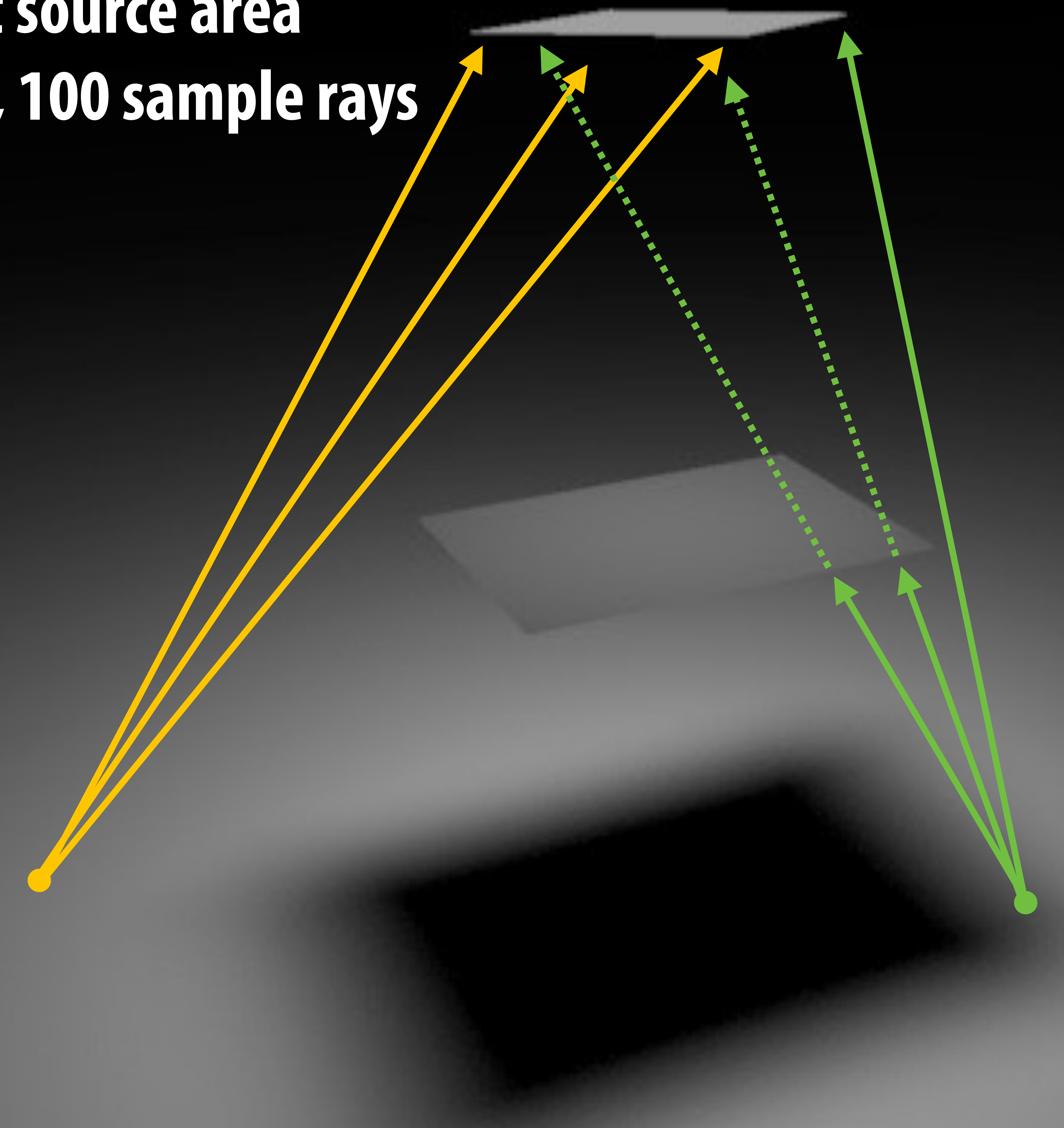
$$p(\mathrm{p}') = \frac{1}{A'}$$

**Estimator**

$$Y_i = L_o(\mathrm{p}'_i, \omega'_i)\, V(\mathrm{p}, \mathrm{p}'_i)\, \frac{\cos\theta_i \cos\theta'_i}{|\mathrm{p} - \mathrm{p}'_i|^2}$$

$$F_N = \frac{A'}{N} \sum_{i=1}^{N} Y_i$$

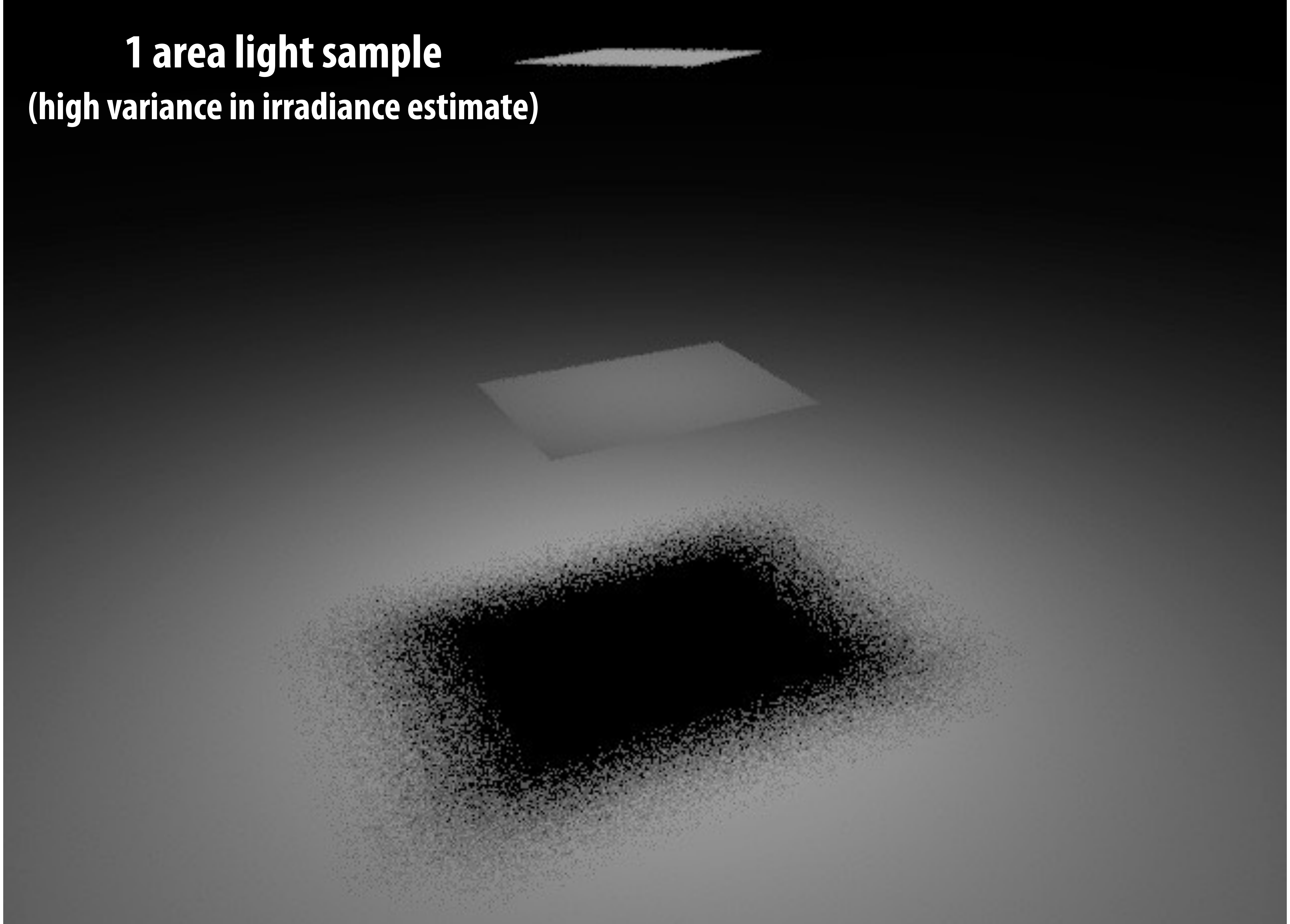
## Light source area sampling, 100 sample rays



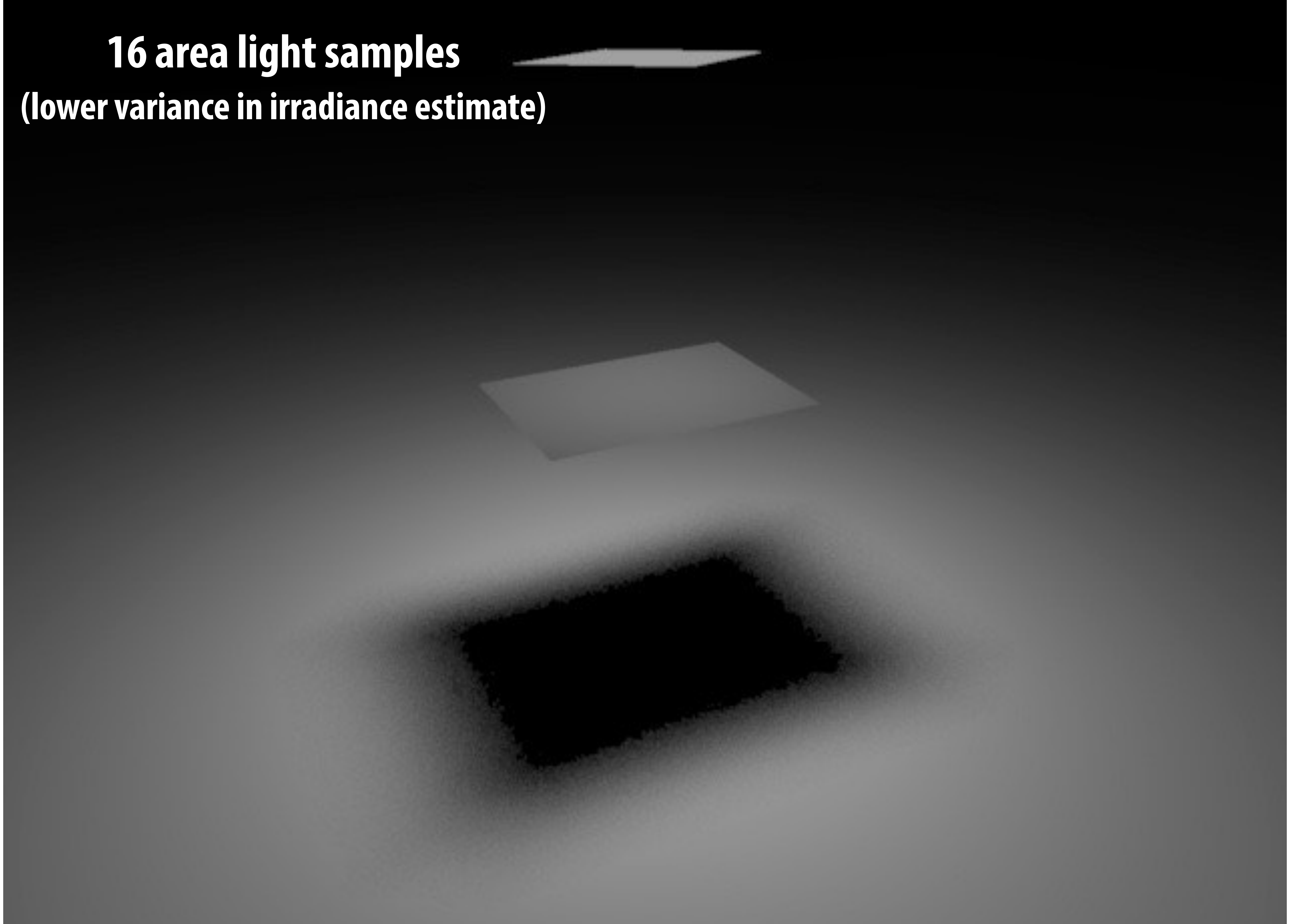
If no occlusion is present, all directions chosen in computing estimate "hit" the light source.
(Choice of direction only matters if portion of light is occluded from surface point *p*.)

# 1 area light sample

## (high variance in irradiance estimate)



## 16 area light samples
(lower variance in irradiance estimate)



# Comparing different techniques

- **Variance in an estimator manifests as noise in rendered images**
- **Estimator efficiency measure:**

$$\text{Efficiency} \propto \frac{1}{\text{Variance} \times \text{Cost}}$$

- **If one integration technique has twice the variance of another, then it takes twice as many samples to achieve the same variance**

- **If one technique has twice the cost of another technique with the same variance, then it takes twice as much time to achieve the same variance**

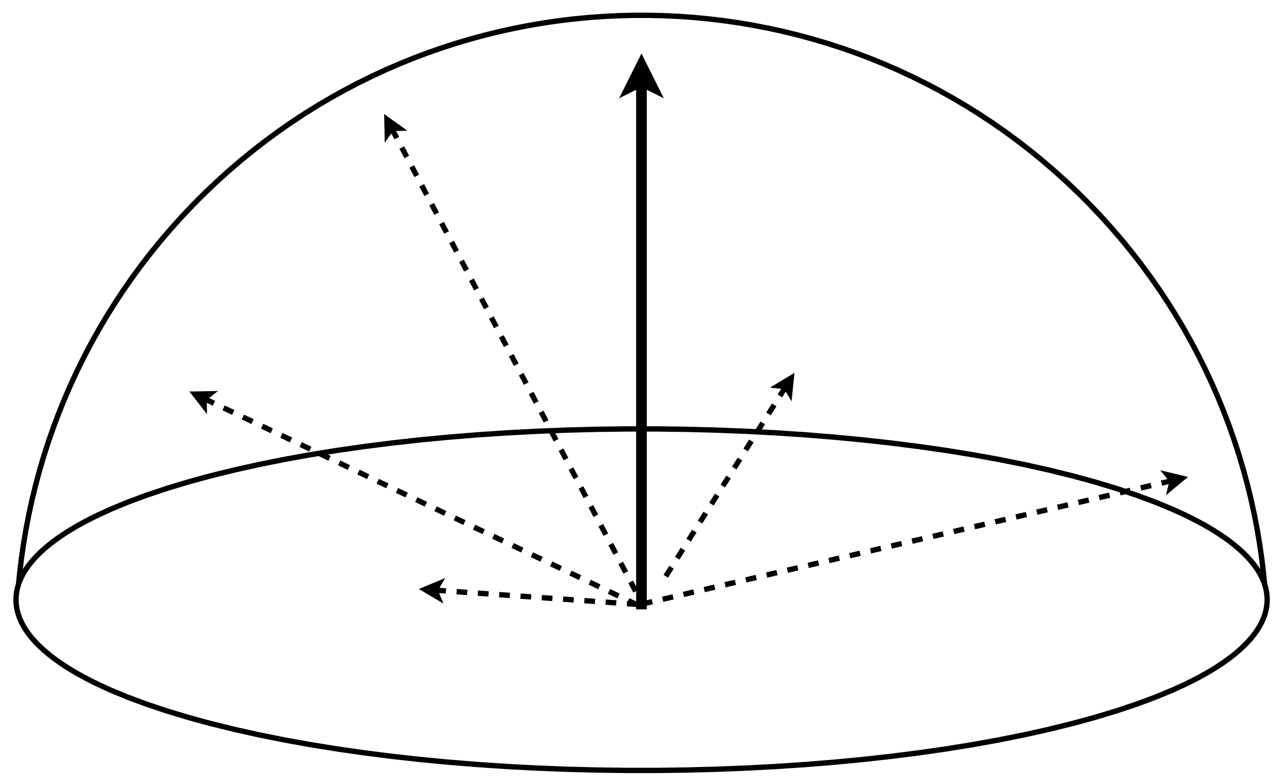
# Example—Cosine-Weighted Sampling

**Consider <u>uniform</u> hemisphere sampling in irradiance estimate:**

$$f(\omega) = L_i(\omega)\cos\theta \qquad\qquad p(\omega) = \frac{1}{2\pi}$$

$$(\xi_1, \xi_2) = (\sqrt{1-\xi_1^2}\cos(2\pi\xi_2), \sqrt{1-\xi_1^2}\sin(2\pi\xi_2), \xi_1)$$

$$\int_\Omega f(\omega)\,\mathrm{d}\omega \approx \frac{1}{N}\sum_i^N \frac{f(\omega)}{p(\omega)} = \frac{1}{N}\sum_i^N \frac{L_i(\omega)\ \cos\theta}{1/2\pi} = \frac{2\pi}{N}\sum_i^N L_i(\omega)\cos\theta$$



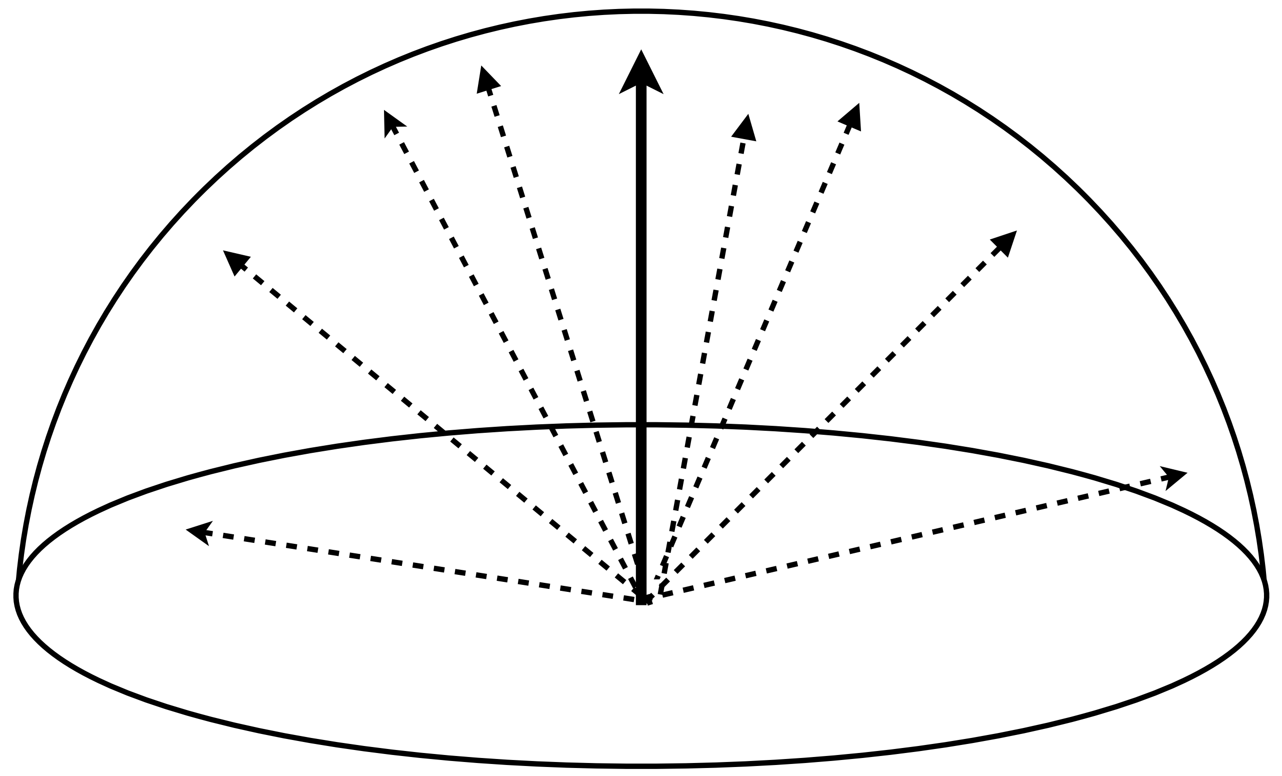
# Example—Cosine-Weighted Sampling

**<u>Cosine-weighted</u> hemisphere sampling in irradiance estimate:**

$$f(\omega) = L_i(\omega)\cos\theta \qquad\qquad p(\omega) = \frac{\cos\theta}{\pi}$$

$$\int_\Omega f(\omega)\,\mathrm{d}\omega \approx \frac{1}{N}\sum_i^N \frac{f(\omega)}{p(\omega)} = \frac{1}{N}\sum_i^N \frac{L_i(\omega)\;\cos\theta}{\cos\theta/\pi} = \frac{\pi}{N}\sum_i^N L_i(\omega)$$

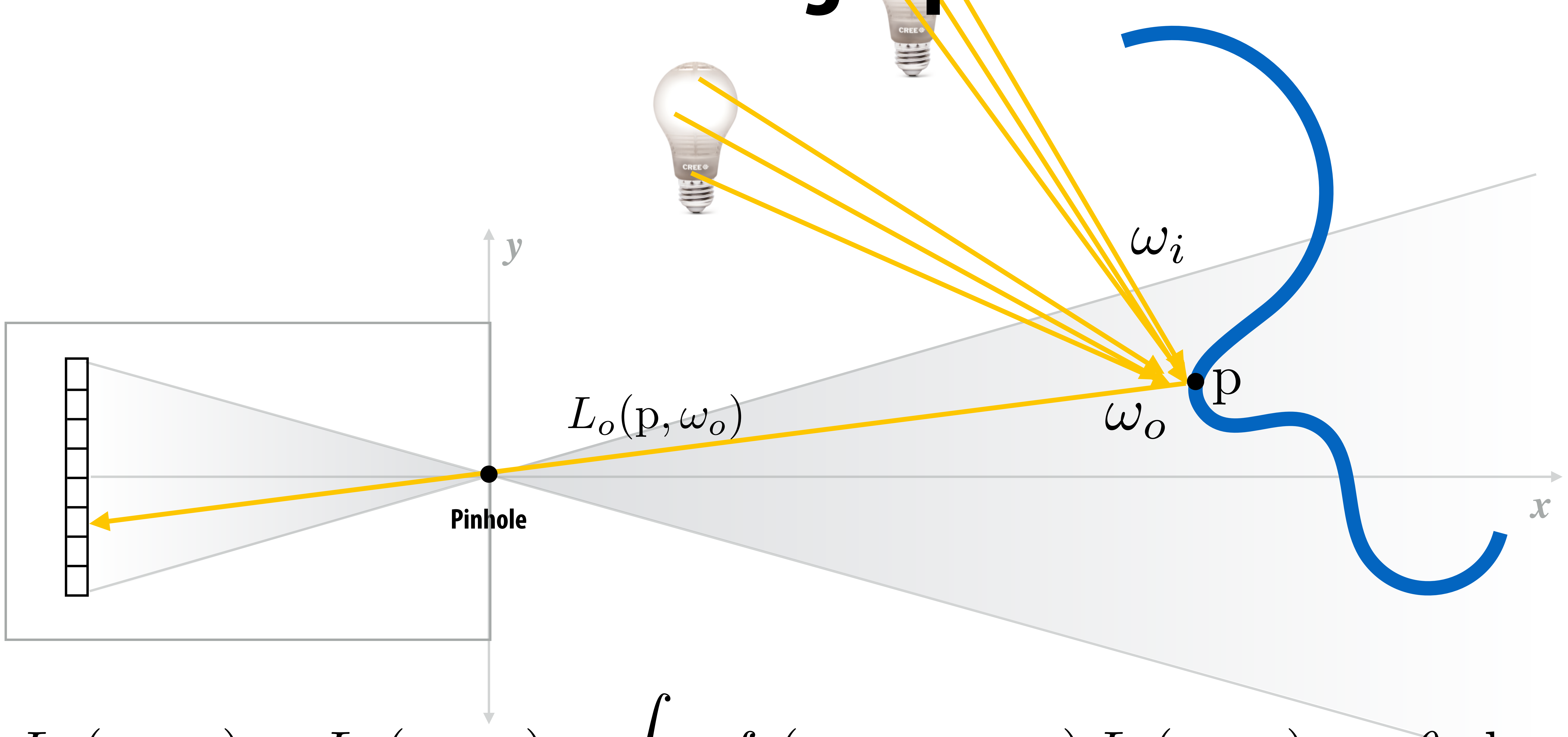
**Idea: bias samples toward directions where $\cos\theta$ is large**
**(if L is constant, then these are the directions that contribute most)**



**So far we've considered light coming directly from light sources, scattered once.**

**How do we use Monte Carlo integration to get the final color values for each pixel?**

# Monte Carlo + Rendering Equation

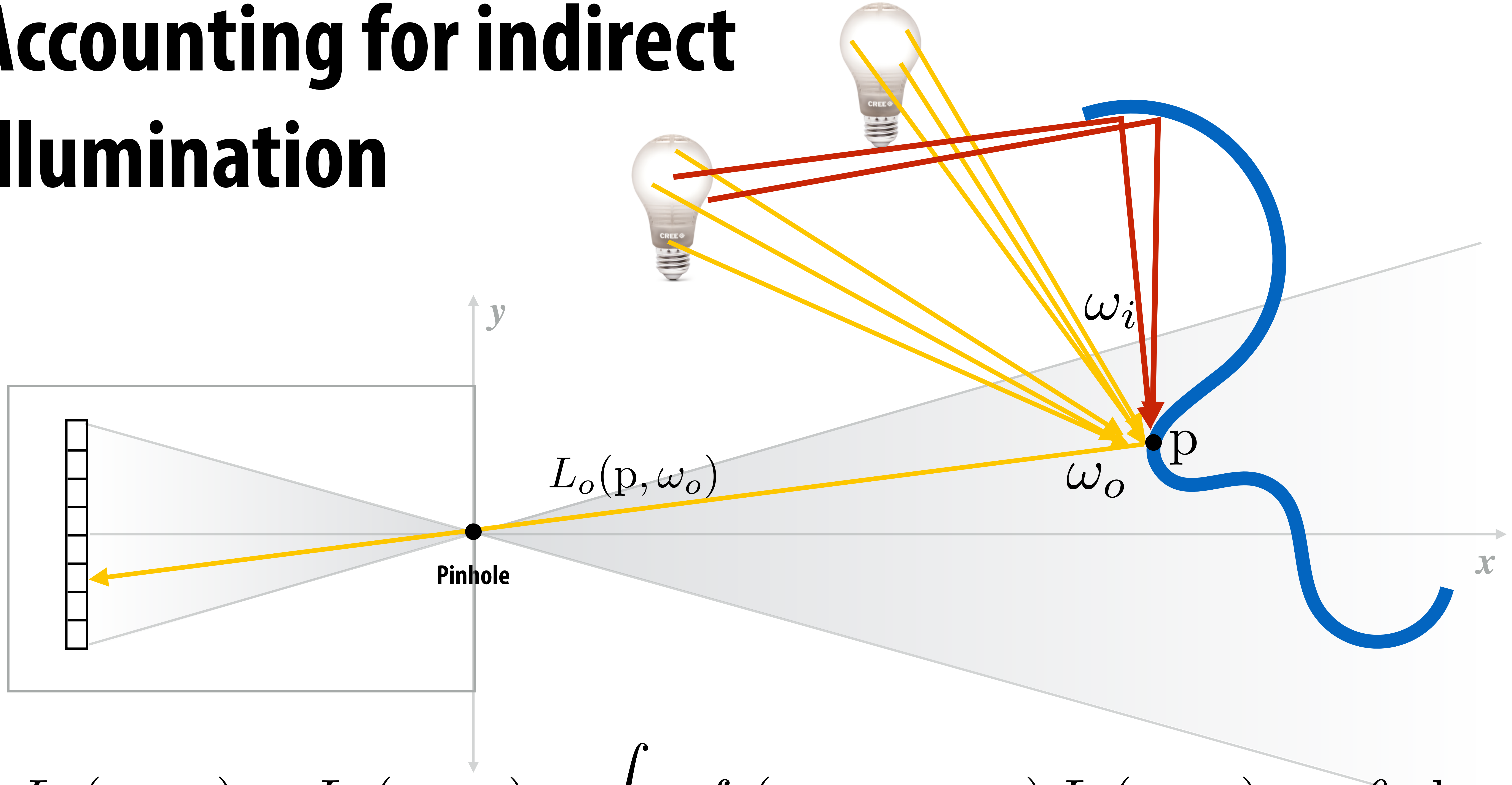



$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \rightarrow \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \,\mathrm{d}\omega_i$$

**Need to know incident radiance.**

**So far, have only computed incoming radiance from scene light sources.**

# Accounting for indirect illumination




$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \rightarrow \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \, \mathrm{d}\omega_i$$

**Incoming light energy from direction $\omega_i$ may be due to light reflected off another surface in the scene (not an emitter)**

# Path tracing: indirect illumination

$$\int_{H^2} f_r(\omega_i \to \omega_o)\, L_{o,i}(tr(\mathrm{p}, \omega_i), -\omega_i)\ \cos\theta_i\, \mathrm{d}\omega_i$$

- **Sample incoming direction from some distribution (e.g. proportional to BRDF):**

$$\omega_i \sim p(\omega)$$

- **Recursively call path tracing function to compute incident indirect radiance**

•p
Direct illumination


p
One-bounce global illumination

•p
Two-bounce global illumination


p
Four-bounce global illumination


p
Eight-bounce global illumination

•p
Sixteen-bounce global illumination

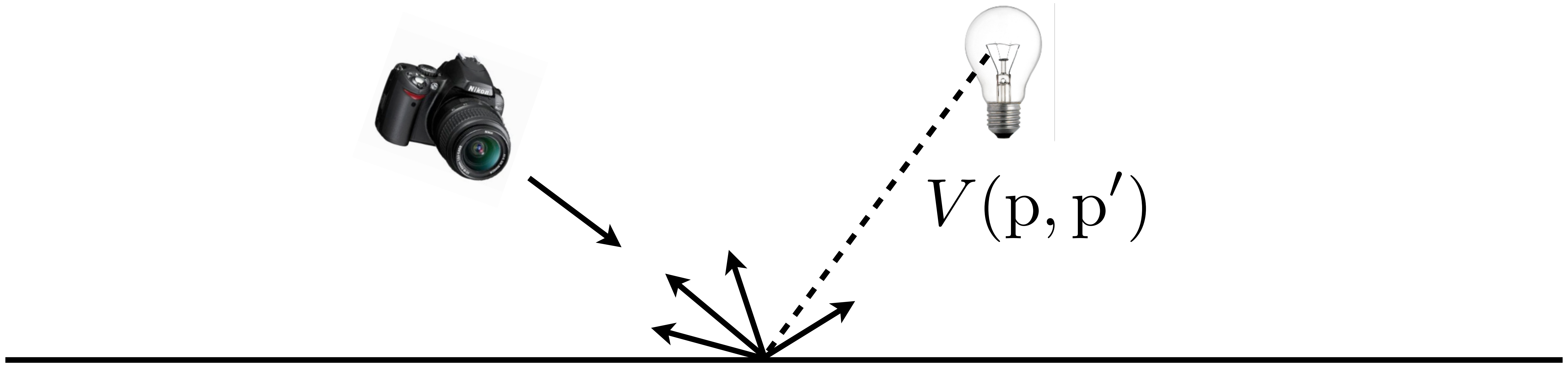
**Wait a minute…**

**When do we *stop*?!**

# Russian roulette

- **Idea: want to avoid spending time evaluating function for samples that make a small contribution to the final result**
- **Consider a low-contribution sample of the form:**

$$L = \frac{f_r(\omega_i \to \omega_o)\, L_i(\omega_i)\, V(\mathrm{p}, \mathrm{p}')\ \cos\theta_i}{p(\omega_i)}$$



# Russian roulette

$$L = \frac{f_r(\omega_i \to \omega_o)\, L_i(\omega_i)\, V(\mathrm{p}, \mathrm{p}')\, \cos\theta_i}{p(\omega_i)}$$

$$\downarrow$$

$$L = \left[ \frac{f_r(\omega_i \to \omega_o)\, L_i(\omega_i)\, \cos\theta_i}{p(\omega_i)} \right] V(\mathrm{p}, \mathrm{p}')$$

- **If tentative contribution (in brackets) is small, total contribution to the image will be small regardless of $V(\mathrm{p}, \mathrm{p}')$**
- **Ignoring low-contribution samples introduces systematic error**
  - **No longer converges to correct value!**
- **Instead, randomly discard low-contribution samples in a way that leaves estimator unbiased**

# Russian roulette

- **New estimator: evaluate original estimator with probability $p_{\text{rr}}$, reweight. Otherwise ignore.**

**"low throughput" threshold**

$$X' = \begin{cases} X & \text{if } t > t_{\text{min}} \\ KX/p_{\text{rr}} & \text{otherwise} \end{cases}$$

$$K \sim p(k) \quad \text{where } p(1) = p_{\text{rr}}, \; p(0) = 1 - p_{\text{rr}}$$

**probability of rejecting**

- **Same *expected value* as original estimator:**

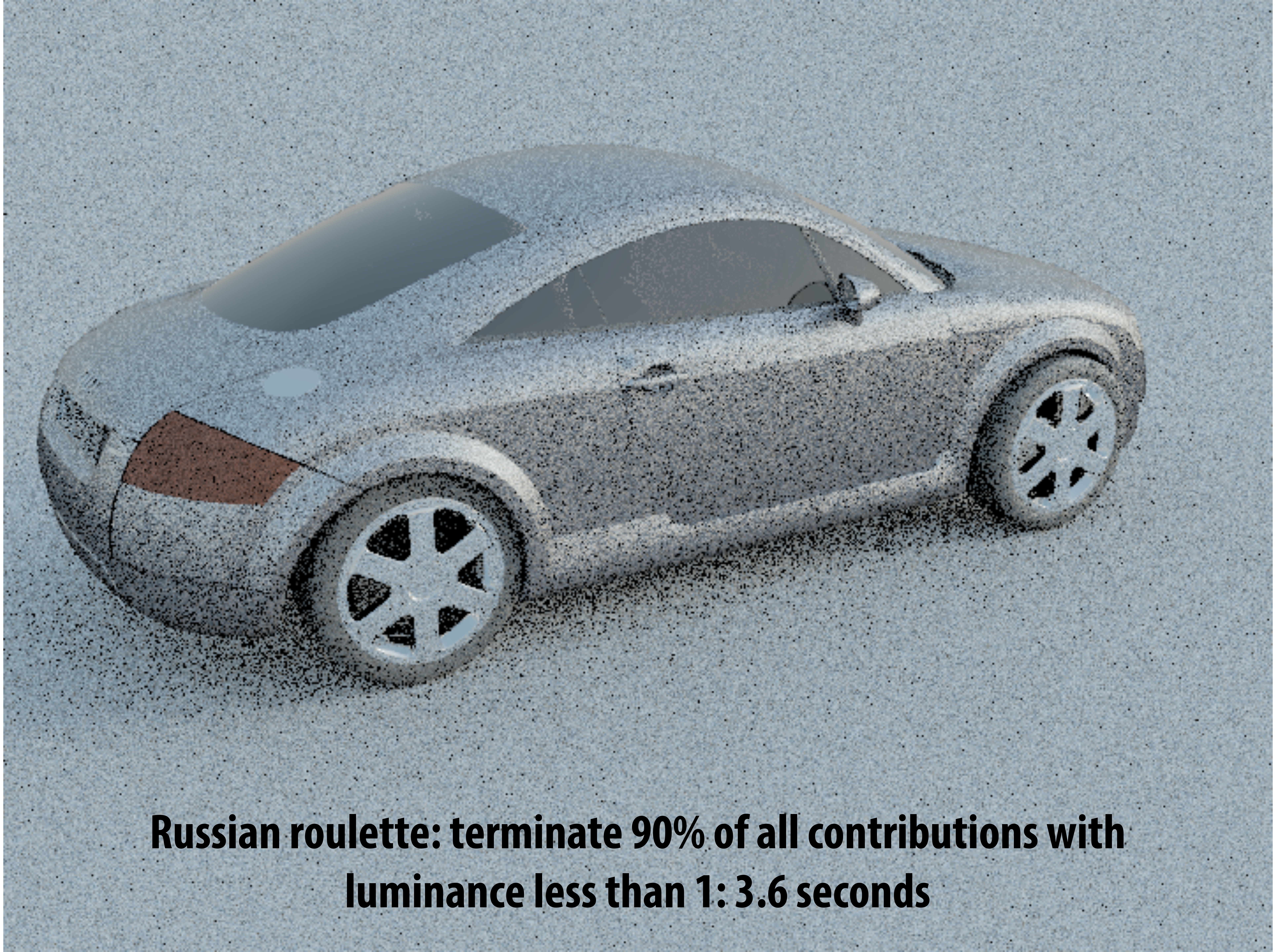
$$E[X'] = p_{\text{rr}} E\left[\frac{X}{p_{\text{rr}}}\right] + (1 - p_{\text{rr}})E[0] = E[X]$$

No Russian roulette: 6.4 seconds

**Russian roulette: terminate 50% of all contributions with luminance less than 0.25: 5.1 seconds**

Russian roulette: terminate 50% of all contributions with luminance less than 0.5: 4.9 seconds

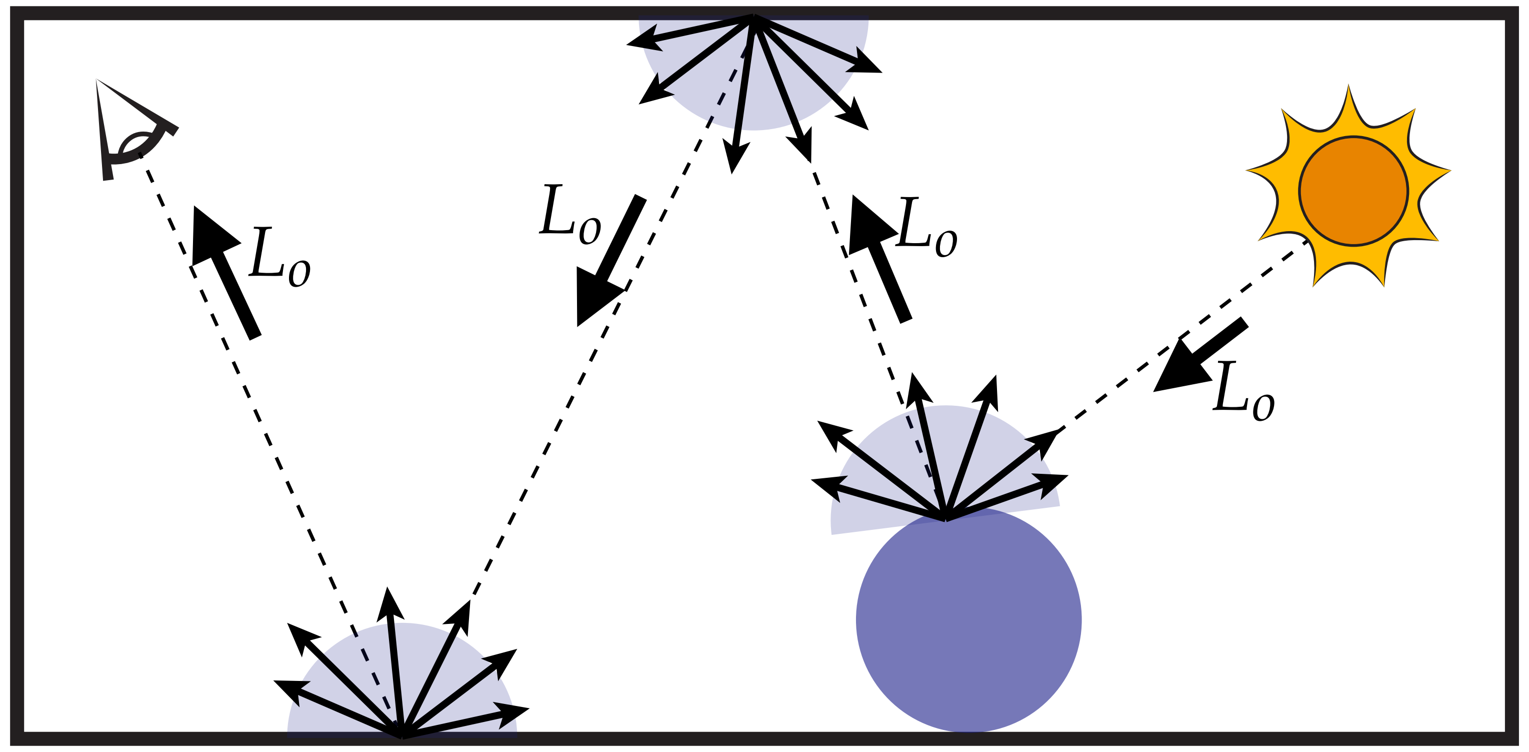
**Russian roulette: terminate 90% of all contributions with luminance less than 0.125: 4.8 seconds**



**Russian roulette: terminate 90% of all contributions with luminance less than 1: 3.6 seconds**

# Monte Carlo Rendering—Summary

- Light hitting a point (e.g., pixel) described by *rendering equation*
  - Expressed as recursive *integral*
  - Can use Monte Carlo to estimate this integral
  - Need to be intelligent about how to sample!




$$L_o(\mathbf{p},\omega_o) = L_e(\mathbf{p},\omega_o) + \int_{\mathcal{H}^2} f_r(\mathbf{p},\omega_i \to \omega_o) L_i(\mathbf{p},\omega_i) \cos\theta \, d\omega_i$$

# Next time:

- **Variance reduction—how do we get the most out of our samples?**