Numerical Integration
Recall the rendering equation, which models light “bouncing around the scene”:

\[
L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{\mathcal{H}^2} f_r(p, \omega_i \rightarrow \omega_o) L_i(p, \omega_i) \cos \theta \, d\omega_i
\]

How can we possibly evaluate this integral?
Numerical Integration—Overview

- In graphics, many quantities we’re interested in are naturally expressed as integrals (total brightness, total area, …)
- For very, very simple integrals, we can compute the solution analytically
- For everything else, we have to compute a numerical approximation

Basic idea:
- integral is “area under curve”
- sample the function at many points
- integral is approximated as weighted sum

\[
\int_0^1 \frac{1}{3} x^2 \, dx = \left[ x^3 \right]_0^1 = 1
\]
Rendering: what are we integrating?

- Recall this view of the world:

  Want to “sum up”—i.e., integrate!—light from all directions
  (But let’s start a little simpler…)
Review: integral as “area under curve”

\[ \int_{a}^{b} f(x) \, dx \]
Or: average value times size of domain

\[ \int_{a}^{b} f(x) \, dx = (b - a) \text{mean}(f) \]
Review: fundamental theorem of calculus

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]

\[ f(x) = \frac{d}{dx} F(x) \]
Simple case: constant function

\[
\int_a^b C \, dx = (b - a)C
\]
Affine function: \( f(x) = cx + d \)

\[
\int_a^b f(x) \, dx = \frac{1}{2} (f(a) + f(b))(b - a)
\]

Need only one sample of the function (at just the right place...)
More general polynomials?
Gauss Quadrature

- For any polynomial of degree $n$, we can always obtain the exact integral by sampling at a special set of $n$ points and taking a special weighted combination.
Piecewise affine function

For piecewise functions, just sum integral of each piece:

$$\int_{a}^{b} f(x)dx = \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)(f(x_i) + f(x_{i+1}))$$
Key idea so far:
To approximate an integral, we need
(i) quadrature points, and
(ii) weights for each point

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} w_{i} f(x_{i})
\]
Arbitrary function $f(x)$?

\[ f(x) = a + x^1 + x^2 + x^3 = b \]
Trapezoid rule

Approximate integral of $f(x)$ by pretending function is piecewise affine

For equal length segments:

$$ h = \frac{b - a}{n - 1} $$

$$ \int_a^b f(x) \, dx = h \left( \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} (f(x_0) + f(x_n)) \right) $$
Trapezoid rule

Consider cost and accuracy of estimate as $n \to \infty$ (or $h \to 0$)

Work: $O(n)$

Error can be shown to be: $O(h^2) = O\left(\frac{1}{n^2}\right)$

(for $f(x)$ with continuous second derivative)
What about a 2D function?

How should we approximate the area underneath?
Integration in 2D

Consider integrating $f(x, y)$ using the trapezoidal rule (apply rule twice: when integrating in $x$ and in $y$)

$$
\int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x, y) \, dx \, dy = \int_{a_y}^{b_y} \left( O(h^2) + \sum_{i=0}^{n} A_i f(x_i, y) \right) \, dy
$$

First application of rule

$$
= O(h^2) + \sum_{i=0}^{n} A_i \int_{a_y}^{b_y} f(x_i, y) \, dy
$$

Second application

$$
= O(h^2) + \sum_{i=0}^{n} A_i \left( O(h^2) + \sum_{j=0}^{n} A_j f(x_i, y_j) \right)
$$

$$
= O(h^2) + \sum_{i=0}^{n} \sum_{j=0}^{n} A_i A_j f(x_i, y_j)
$$

Errors add, so error still: $O(h^2)$

But work is now: $O(n^2)$

$(n \times n$ set of measurements$)$

Must perform much more work in 2D to get same error bound on integral!

In K-D, let $N = n^k$

Error goes as: $O\left(\frac{1}{N^{2/k}}\right)$
Curse of Dimensionality

- How much does it cost to apply the trapezoid rule as we go up in dimension?
  - 1D: $O(n)$
  - 2D: $O(n^2)$
  - ...  
  - kD: $O(n^k)$

- For many problems in graphics (like rendering), $k$ is very, very big (e.g., tens or hundreds or thousands)

- Applying trapezoid rule does not scale!

- Need a fundamentally different approach...
Monte Carlo Integration

Credit: many of these slides were created by Matt Pharr and Pat Hanrahan
Monte Carlo Integration

- Estimate value of integral using random sampling of function
  - Value of estimate depends on random samples used
  - But algorithm gives the correct value of integral “on average”

- Only requires function to be evaluated at random points on its domain
  - Applicable to functions with discontinuities, functions that are impossible to integrate directly

- Error of estimate is independent of the dimensionality of the integrand
  - Depends on the number of random samples used: $O(n^{1/2})$
Review: random variables

$X$ random variable. Represents a distribution of potential values

$X \sim p(x)$ probability density function (PDF). Describes relative probability of a random process choosing value $x$

Uniform PDF: all values over a domain are equally likely

e.g., for an unbiased die
$X$ takes on values 1,2,3,4,5,6

$p(1) = p(2) = p(3) = p(4) = p(5) = p(6)$
Discrete probability distributions

\( n \) discrete values \( x_i \)

With probability \( p_i \)

Requirements of a PDF:

\[
p_i \geq 0
\]

\[
\sum_{i=1}^{n} p_i = 1
\]

Six-sided die example: \( p_i = \frac{1}{6} \)

Think: \( p_i \) is the probability that a random measurement of \( X \) will yield the value \( x_i \)

\( X \) takes on the value \( x_i \) with probability \( p_i \)
Cumulative distribution function (CDF)
(For a discrete probability distribution)

Cumulative PDF: \( P_j = \sum_{i=1}^{j} p_i \)

where:

\[ 0 \leq P_i \leq 1 \]
\[ P_n = 1 \]
How do we generate samples of a discrete random variable (with a known PDF?)
Sampling from discrete probability distributions

To randomly select an event, select $x_i$ if

$$P_{i-1} < \xi \leq P_i$$

Uniform random variable $\in [0, 1)$

Note: “inversion method” is $O(\log N)$ work per sample. But $O(1)$ is possible! (More on the “Alias Method” later.)
Continuous probability distributions

**PDF** $p(x)$

$p(x) \geq 0$

**CDF** $P(x)$

$P(x) = \int_0^x p(x) \, dx$

$P(x) = \Pr(X < x)$

$P(1) = 1$

$\Pr(a \leq X \leq b) = \int_a^b p(x) \, dx$

$= P(b) - P(a)$

**Uniform distribution**

(for random variable $X$ defined on $[0,1]$ domain)
Sampling continuous random variables using the inversion method

Cumulative probability distribution function

\[ P(x) = \Pr(X < x) \]

Construction of samples:
Solve for \( x = P^{-1}(\xi) \)

Must know the formula for:
1. The integral of \( p(x) \)
2. The inverse function \( P^{-1}(x) \)
Example—Sampling Quadratic Distribution

- As a toy example, consider the simple probability distribution \( p(x) := 3(1-x)^2 \) over the interval \([0,1]\)
- How do we pick random samples distributed according to \( p(x) \)?
- First, integrate probability distribution \( p(x) \) to get cumulative distribution \( P(x) \)
- Invert \( P(x) \) by solving \( y = P(x) \) for \( x \)
- Finally, plug uniformly distributed random values \( y \) in \([0,1]\) into this expression
How do we uniformly sample the unit circle?

I.e., choose any point $P=(px, py)$ in circle with equal probability.
Uniformly sampling unit circle: first try

- $\theta =$ uniform random angle between 0 and $2\pi$
- $r =$ uniform random radius between 0 and 1
- Return point: $(r \cos \theta, r \sin \theta)$

This algorithm does not produce the desired uniform sampling of the area of a circle. Why?
Because sampling is not uniform in area!

Points farther from center of circle are less likely to be chosen

So how should we pick samples? Well, think about how we integrate over a disk in polar coordinates…

$$\theta = 2\pi \xi_1 \quad r = \xi_2$$
Sampling a circle (via inversion in 2D)

\[ A = \int_{0}^{2\pi} \int_{0}^{1} r \, dr \, d\theta = \int_{0}^{1} r \, dr \int_{0}^{2\pi} d\theta = \left( \frac{r^2}{2} \right) \left| \frac{1}{\theta} \right|_{0}^{2\pi} = \pi \]

\[ p(r, \theta) \, dr \, d\theta = \frac{1}{\pi} r \, dr \, d\theta \rightarrow p(r, \theta) = \frac{r}{\pi} \]

\[ p(r, \theta) = p(r)p(\theta) \quad \text{\( r, \theta \) independent} \]

\[ p(\theta) = \frac{1}{2\pi} \]

\[ P(\theta) = \frac{1}{2\pi} \theta \quad \theta = 2\pi \xi_1 \]

\[ p(r) = 2r \]

\[ P(r) = r^2 \quad r = \sqrt{\xi_2} \]

so that we integrate to 1 instead of area
Uniform area sampling of a circle

**WRONG**
probability is uniform; samples are not!

\[
\theta = 2\pi \xi_1 \\
\xi_2
\]

**RIGHT**
probability is nonuniform; samples are uniform

\[
\theta = 2\pi \xi_1 \\
r = \sqrt{\xi_2}
\]
Uniform sampling via rejection sampling

Completely different idea: pick uniform samples in square (easy)
Then toss out any samples not in square (easy)

Efficiency of technique: area of circle / area of square
Efficiency of Rejection Sampling

- If the region we care about covers only a very small fraction of the region we’re sampling, rejection is probably a bad idea:

Smarter in this case to “warp” our random variables to follow the spiral.
Expected Value

- What value does a random variable $Y$ take, on average?
- E.g., consider a fair coin where heads = 1, tails = 0
- Equal probability of heads & is tails (1/2 for both)
- Expected value is then $(1/2)*1 + (1/2)*0 = 1/2$

Properties of expectation:

\[
E \left[ \sum_i Y_i \right] = \sum_i E[Y_i]
\]

\[
E[aY] = aE[Y]
\]

(Can you show these are true?)
Putting it all together: Monte Carlo Integration

- **Definite integral**
  What we seek to estimate
  \[ \int_a^b f(x) \, dx \]

- **Random variables**
  \( X_i \) is the value of a random sample drawn from the distribution \( p(x) \)
  \( Y_i \) is also a random variable.

- **Expectation of \( f \)**
  For a *continuous* random variable
  \[ E[Y_i] = E[f(X_i)] = \int_a^b f(x) \, p(x) \, dx \]

- **Estimator**
  Monte Carlo estimate of \( \int_a^b f(x) \, dx \)
  \[ F_N = \frac{b - a}{N} \sum_{i=1}^{N} Y_i \]

Assuming samples \( X_i \) drawn from uniform pdf. I will provide estimator for arbitrary PDFs next lecture.
Basic unbiased Monte Carlo estimator

Let’s compute expected value of our numerical estimate of an integral:

\[
E[F_N] = E \left[ \frac{b - a}{N} \sum_{i=1}^{N} Y_i \right] \\
= \frac{b - a}{N} \sum_{i=1}^{N} E[Y_i] = \frac{b - a}{N} \sum_{i=1}^{N} E[f(X_i)] \\
= \frac{b - a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) p(x) \, dx \\
= \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \, dx \\
= \int_{a}^{b} f(x) \, dx
\]

Since expected value is equal to the exact value of the integral we wanted, say this estimator is “unbiased”
Next Time: Monte Carlo Ray Tracing
Fun Stuff: The Alias Method

You can generate samples from *any* discrete probability distribution in $O(1)$ time!

How? Pre-process to arrange the probability mass to perfectly cover a rectangle:

Hey, is this pre-processing even *possible* in general?

(I did only show you one example...)

At most two outcomes per column

Why is this $O(1)$ to sample?