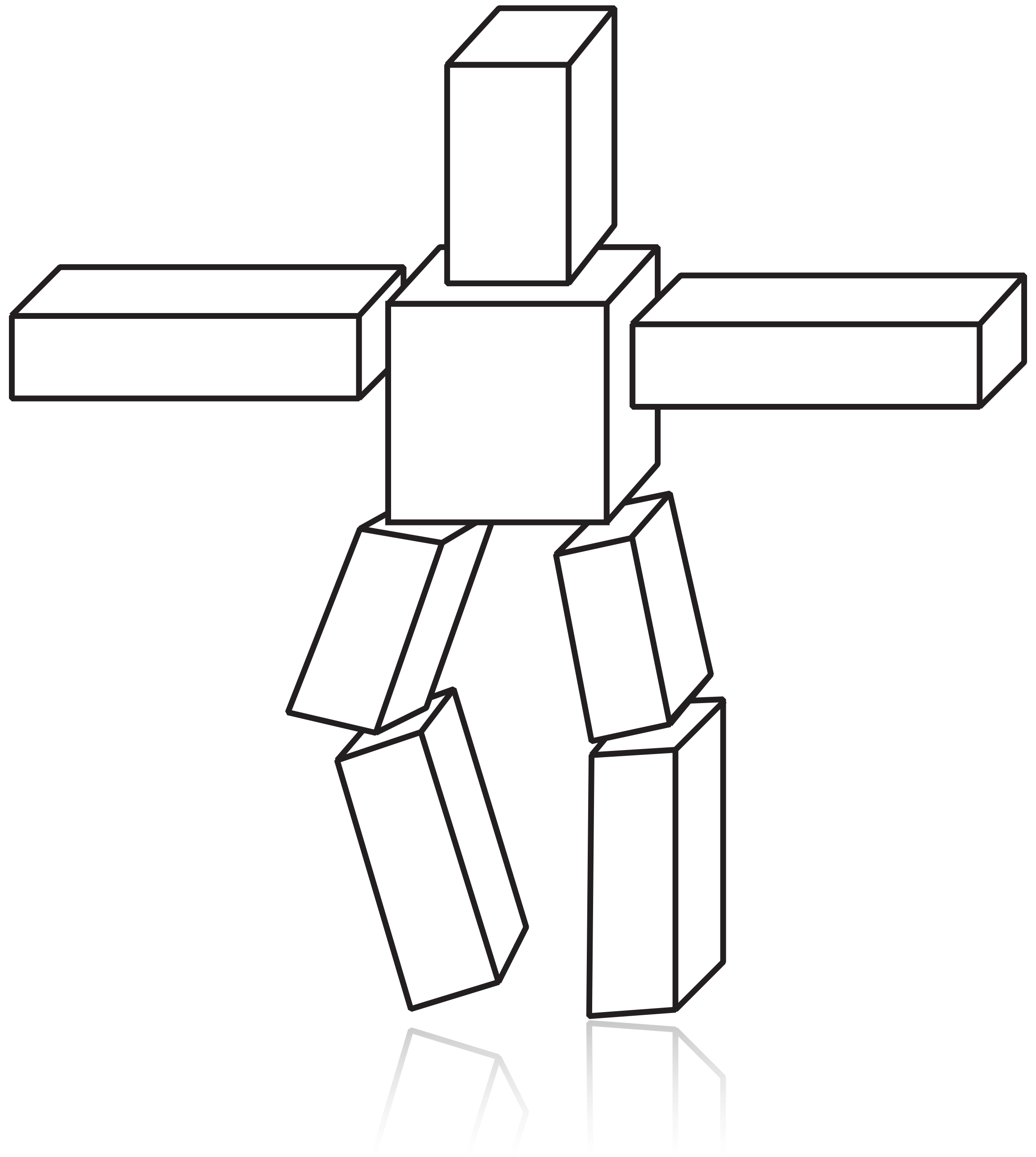
Introduction to Geometry

Computer Graphics
CMU 15-462/15-662

Increasing the complexity of our models

Transformations



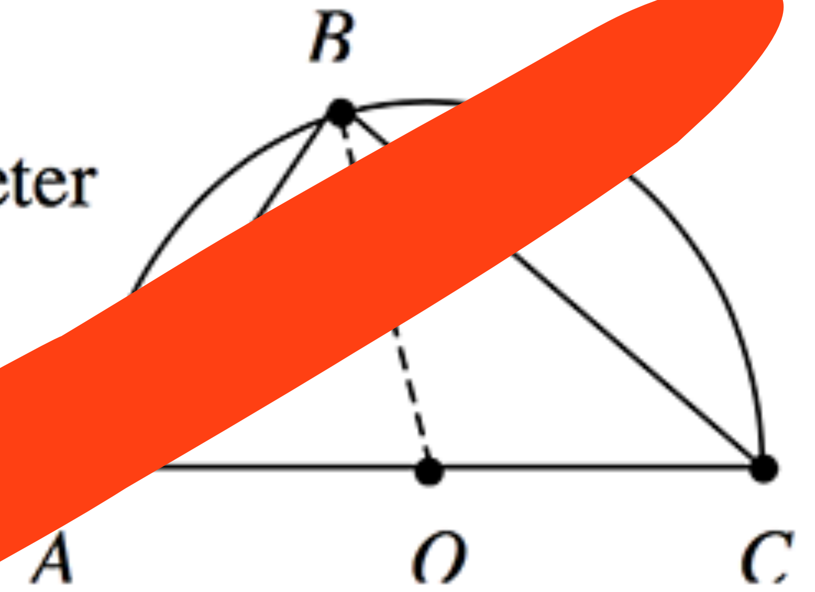
Geometry

Materials, lighting, ...

Q: What is geometry?

A: Geometry is the study of two-column proofs.

TH 9.5. Let ΔABC be inscribed in a semicircle with diameter $\overline{A\ C}$.
Then $\angle ABC$ angle.



Proof:

Statement	
1. Draw radius OB. Then $OB = OC = O$	Given
2. $m\angle OBC = m\angle BCA$ $m\angle OBA = m\angle BAC$	sceles Triangle Theorem
3. $m\angle ABC = m\angle OBA +$	3. Ang ostulate
4. $m\angle ABC + m\angle BC$ $BAC = 180$	4. The sum les of a triangle is 180
5. $m\angle ABC +$ $+ m\angle OBA = 180$	5. Substitution (li
6. 2 m 180	6. Substitution (line 3)
7. $C = 90$	7. Division Property of Equality
8. $\angle ABC$ is a right angle	8. Definition of Right Angle

Ceci n'est pas géométrie.

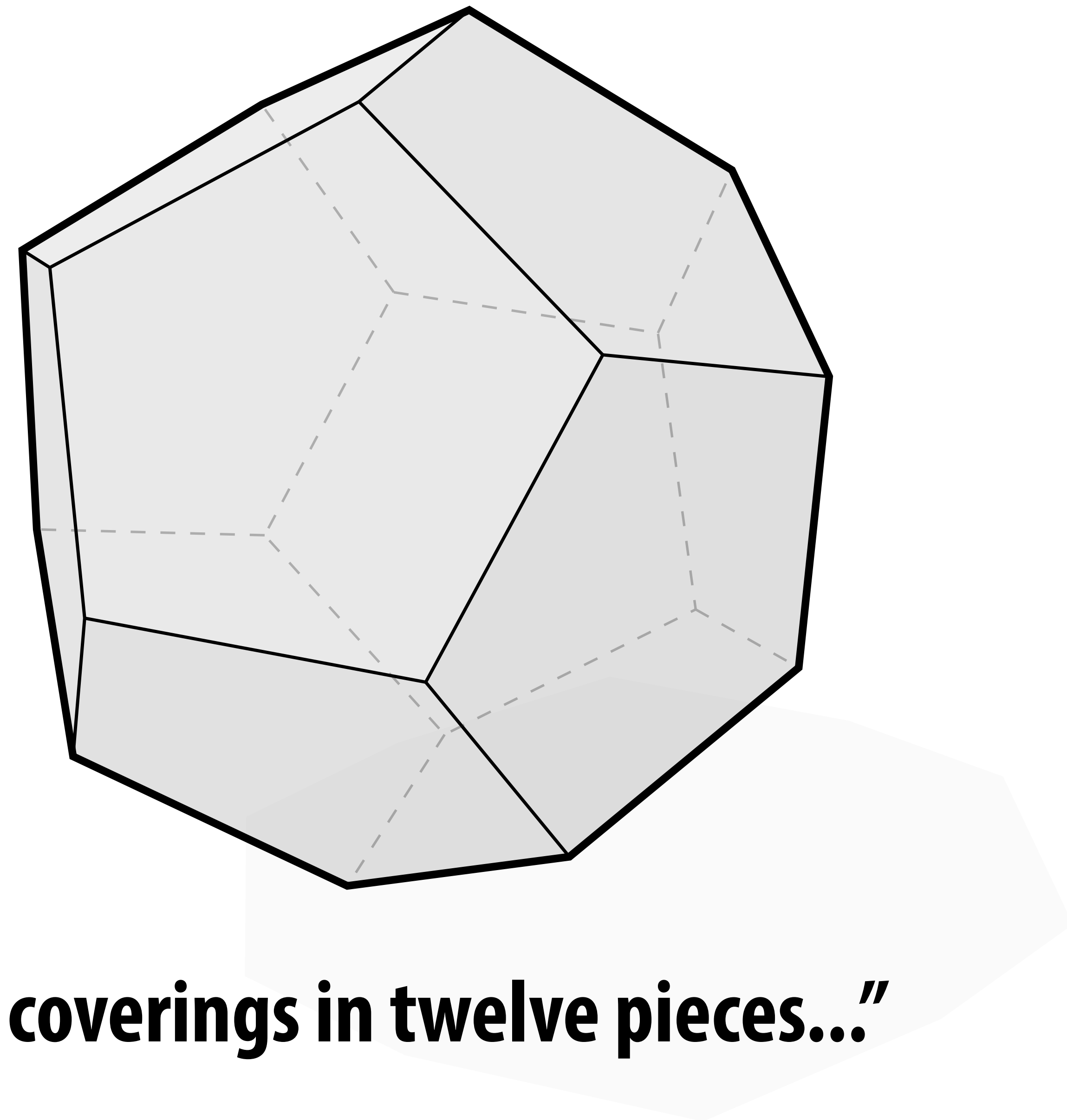
See: Paul Lockhart, "A Mathematician's Lament"

What is geometry?

"Earth" "measure"

ge•om•et•ry /jēˈämətrē/ *n.*

1. The study of shapes, sizes, patterns, and positions.
2. The study of spaces where some quantity (lengths, angles, etc.) can be *measured*.



Plato: "...the earth is in appearance like one of those balls which have leather coverings in twelve pieces..."

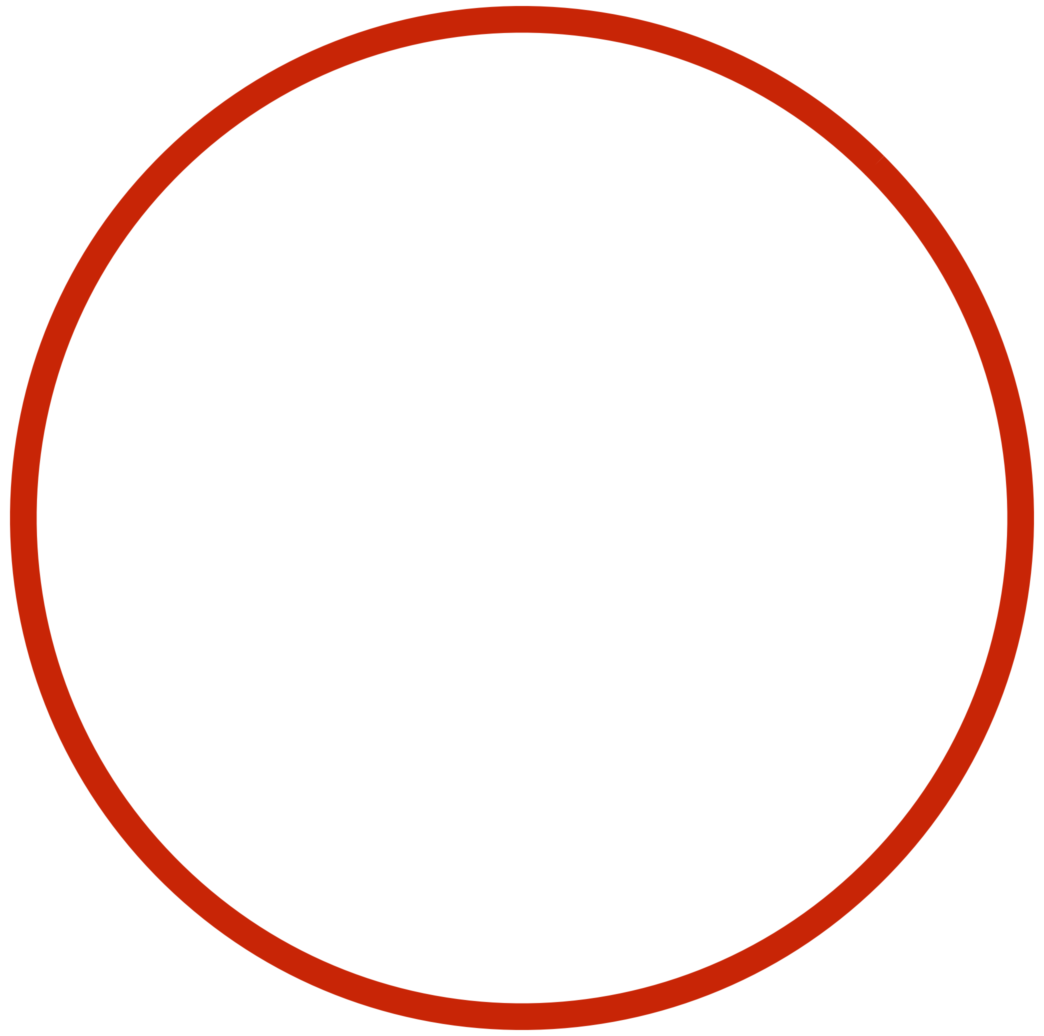
How can we describe geometry?

IMPLICIT

$$x^2 + y^2 = 1$$

LINGUISTIC

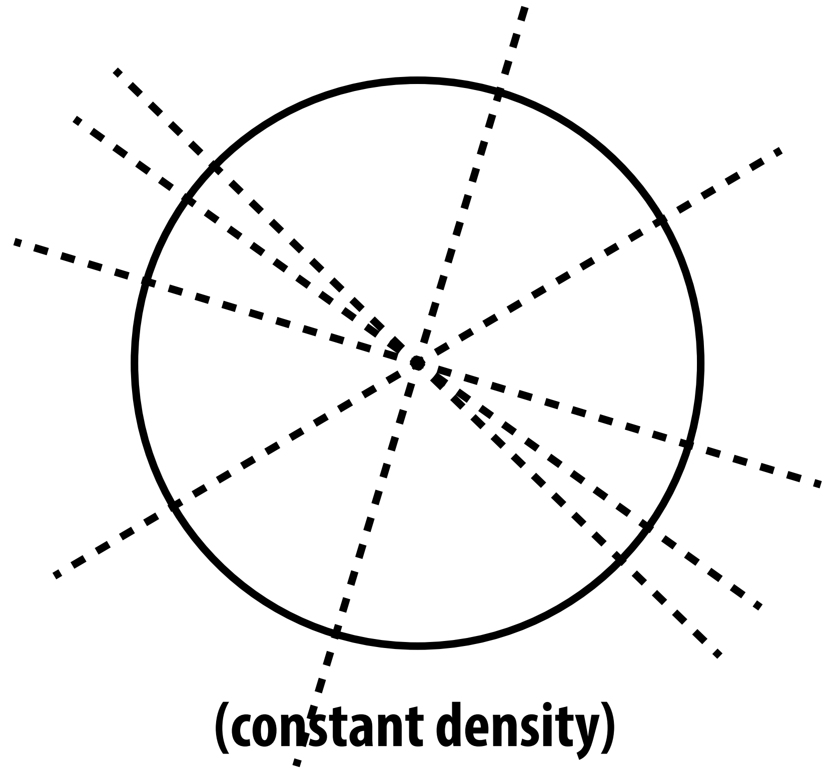
"unit circle"



EXPLICIT

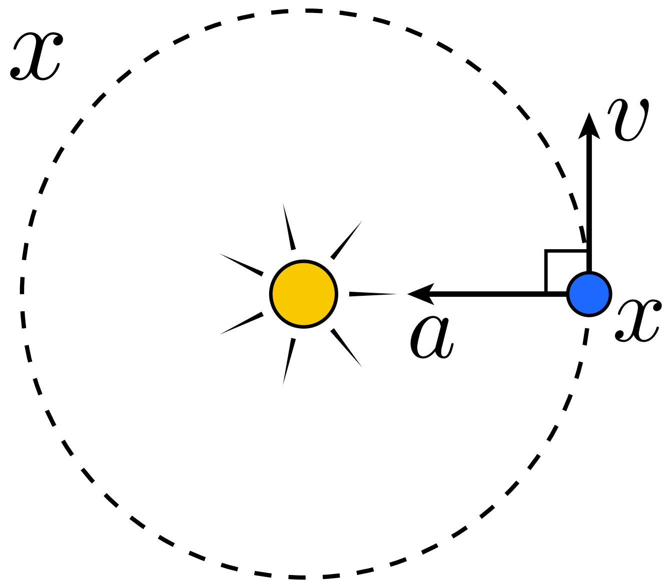
$$(\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y})$$

TOMOGRAPHIC



DYNAMIC

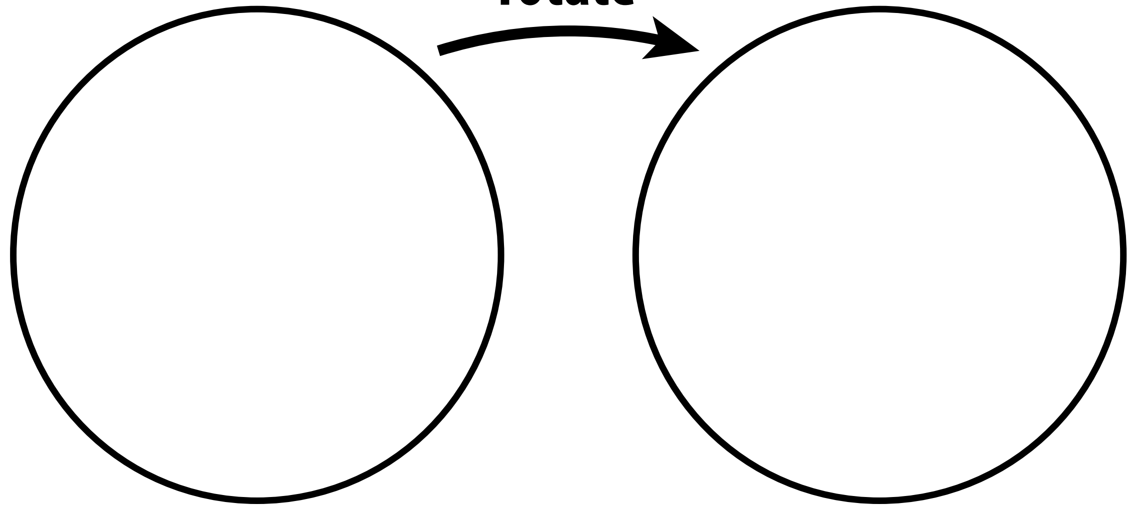
$$\frac{d^2}{dt^2}x = -x$$



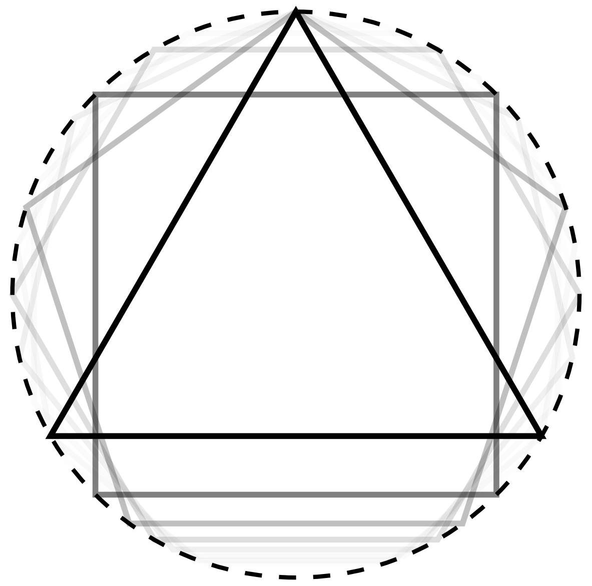
CURVATURE

$$\kappa = 1$$

SYMMETRIC



DISCRETE

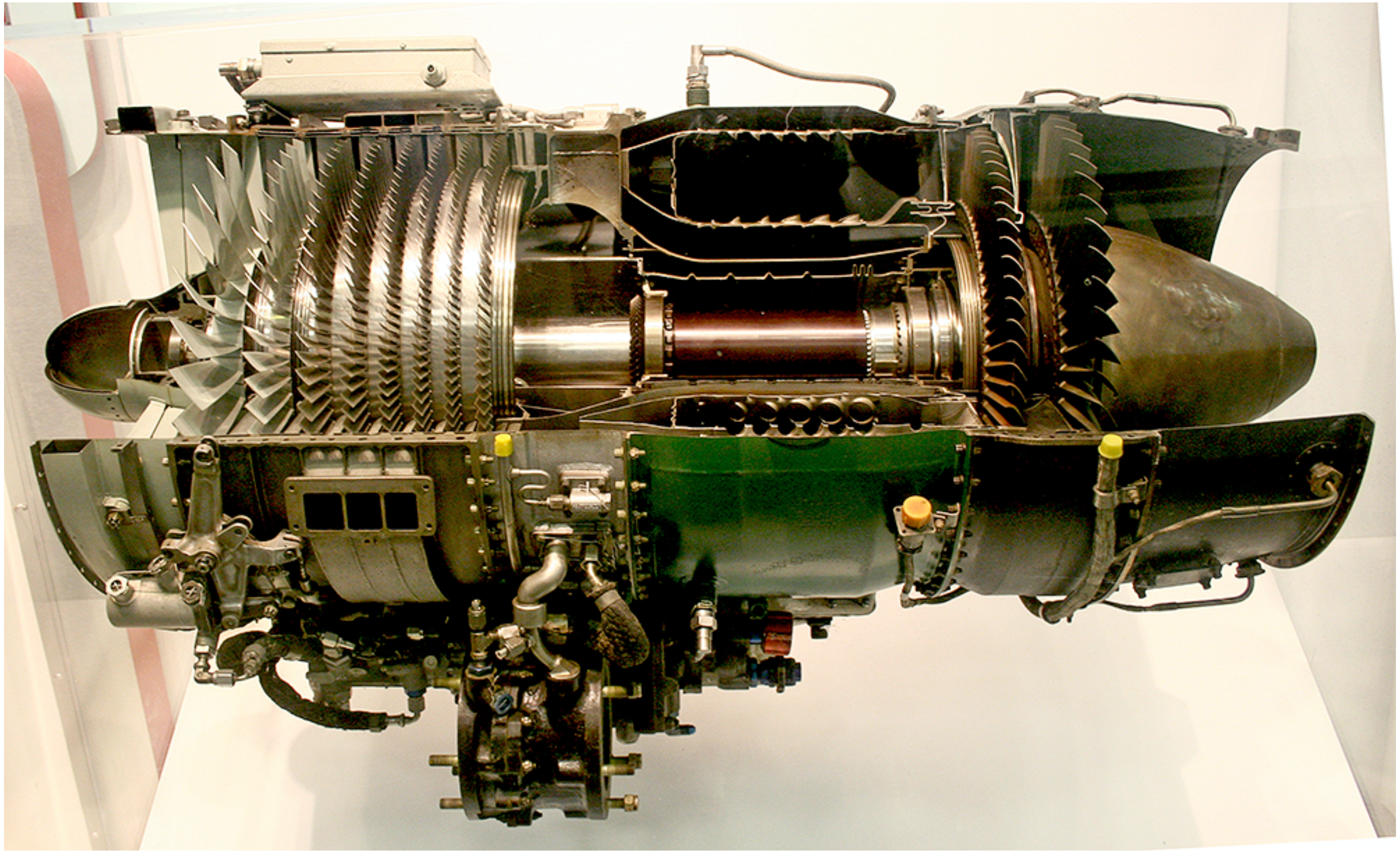


$n \to \infty$

Given all these options, what's the <u>best</u> way to encode geometry on a computer?

Examples of geometry

Examples of geometry



Examples of geometry

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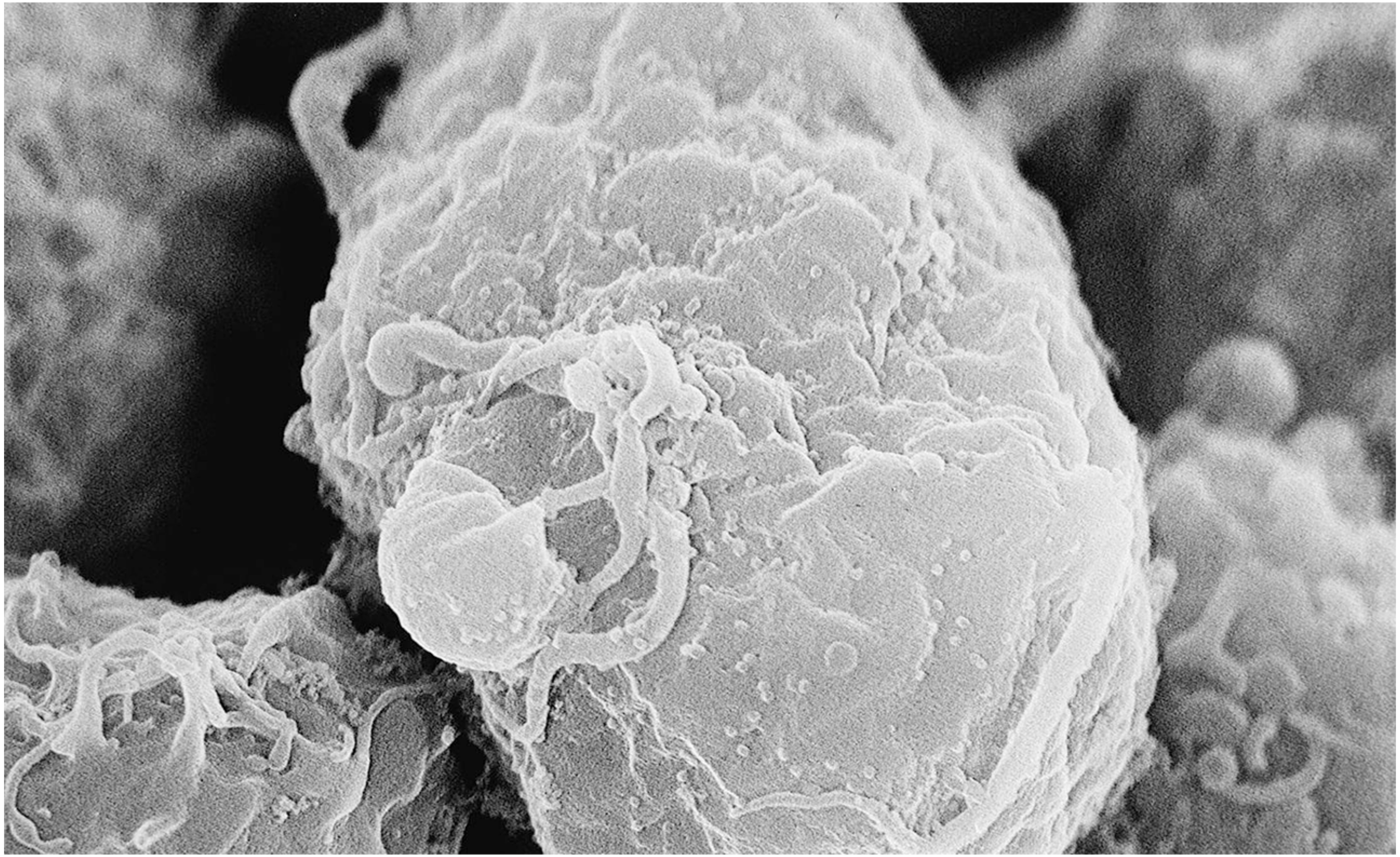
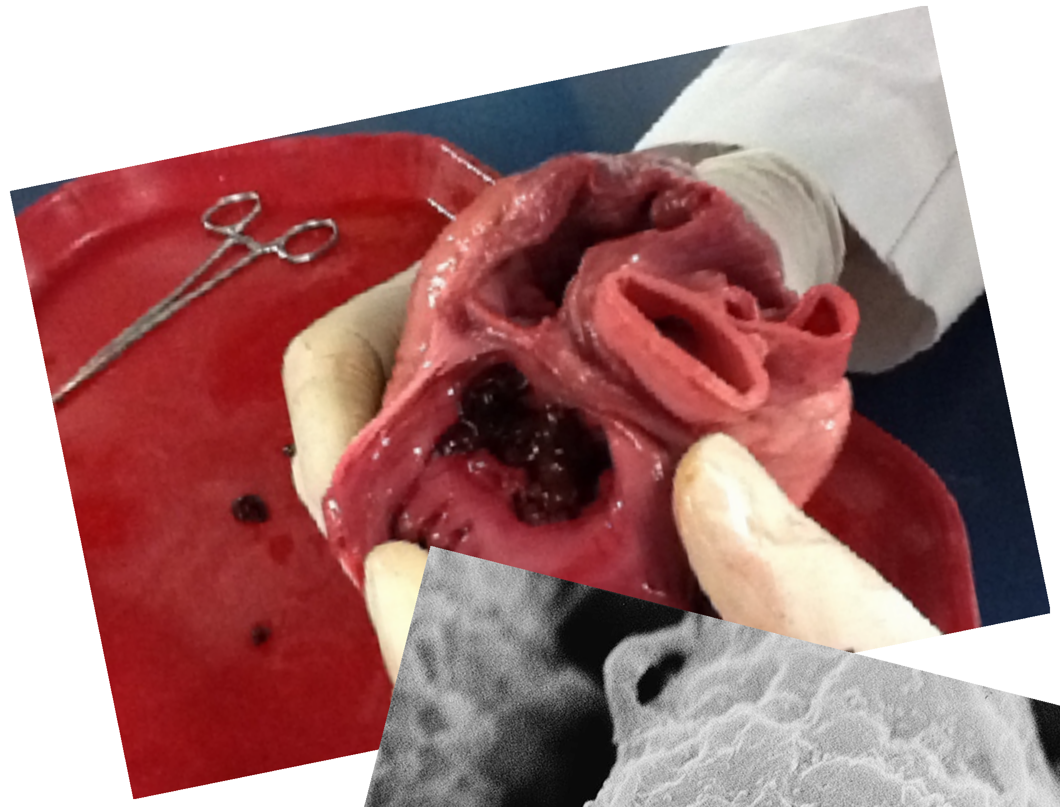


It's a Jungle Out There!






No one “best” choice—geometry is hard!

“I hate meshes.
I cannot believe how hard this is.
Geometry is hard.”

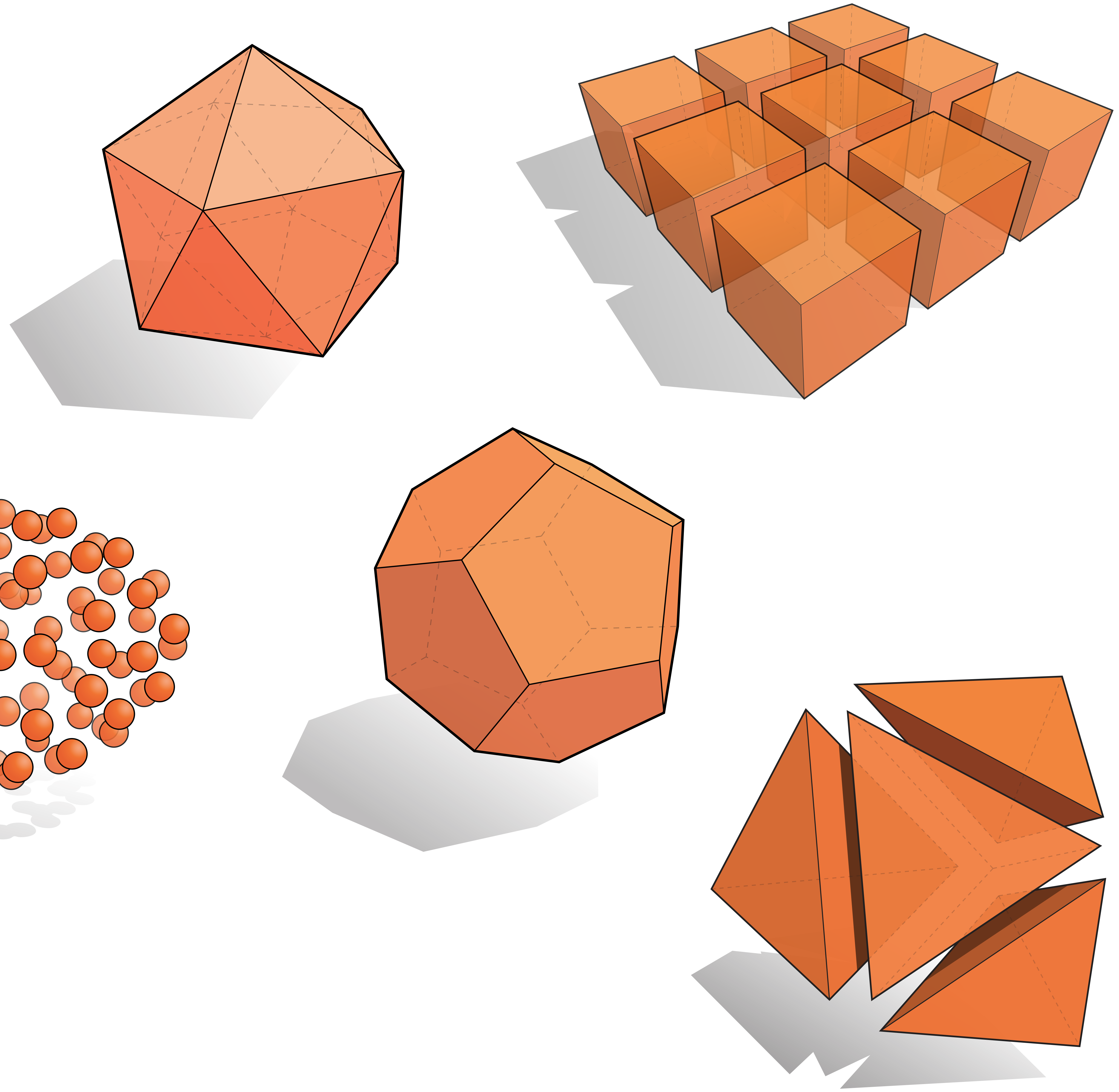
—David Baraff

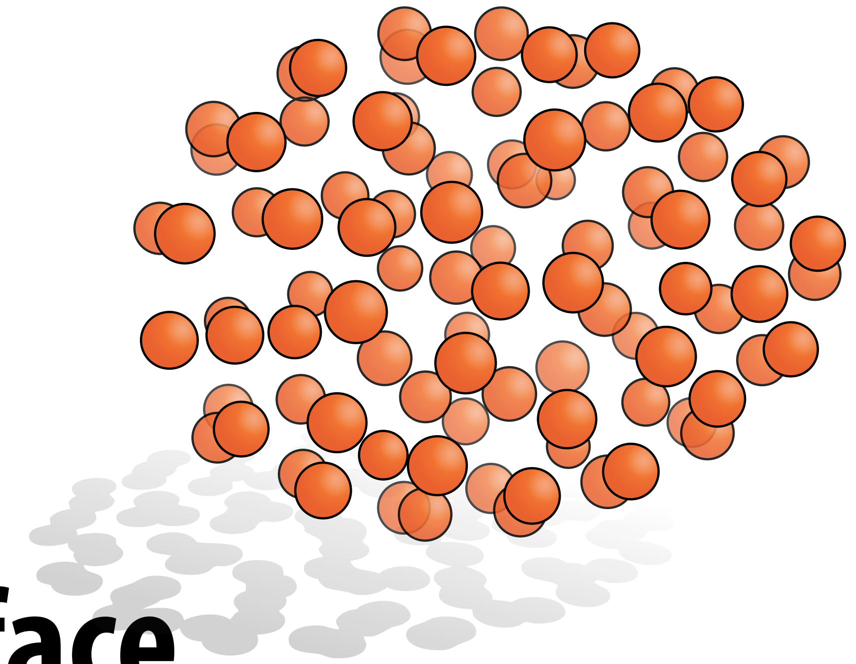
Senior Research Scientist
Pixar Animation Studios

Slide cribbed from Jeff Erickson.

Many ways to digitally encode geometry

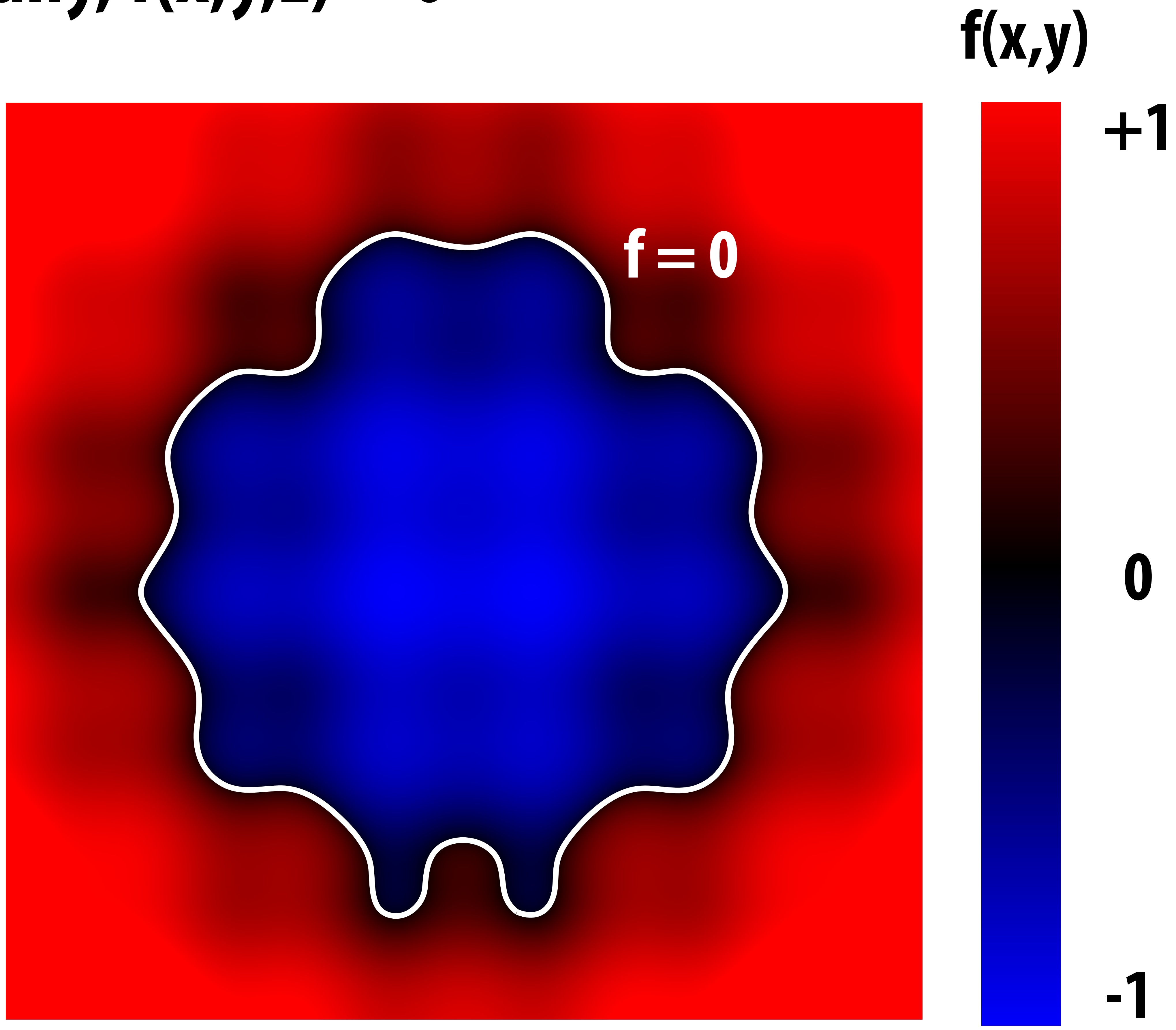
- EXPLICIT
  - point cloud
  - polygon mesh
  - subdivision, NURBS
  - ...
- IMPLICIT
  - level set
  - algebraic surface
  - L-systems
  - ...
- Each choice best suited to a different task/type of geometry





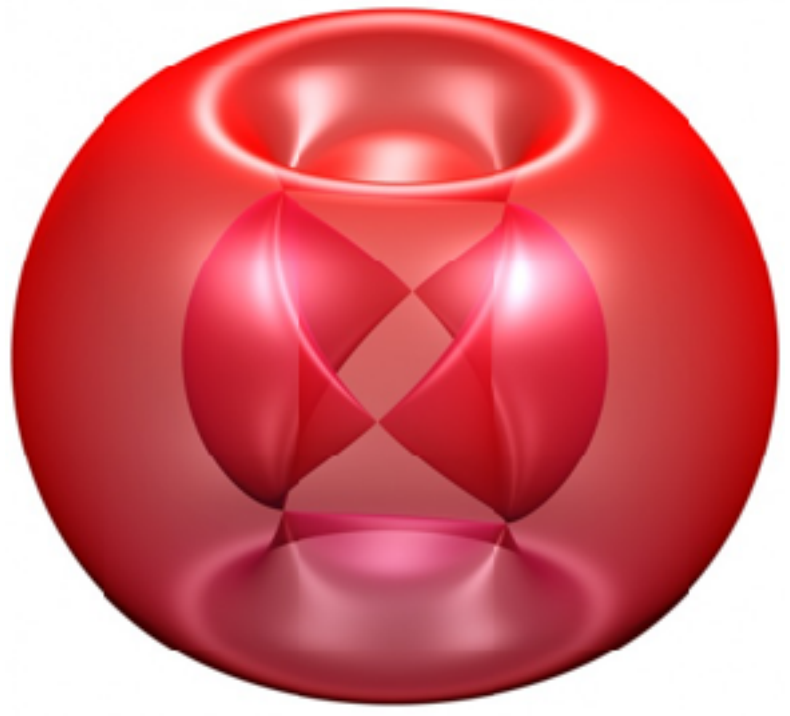
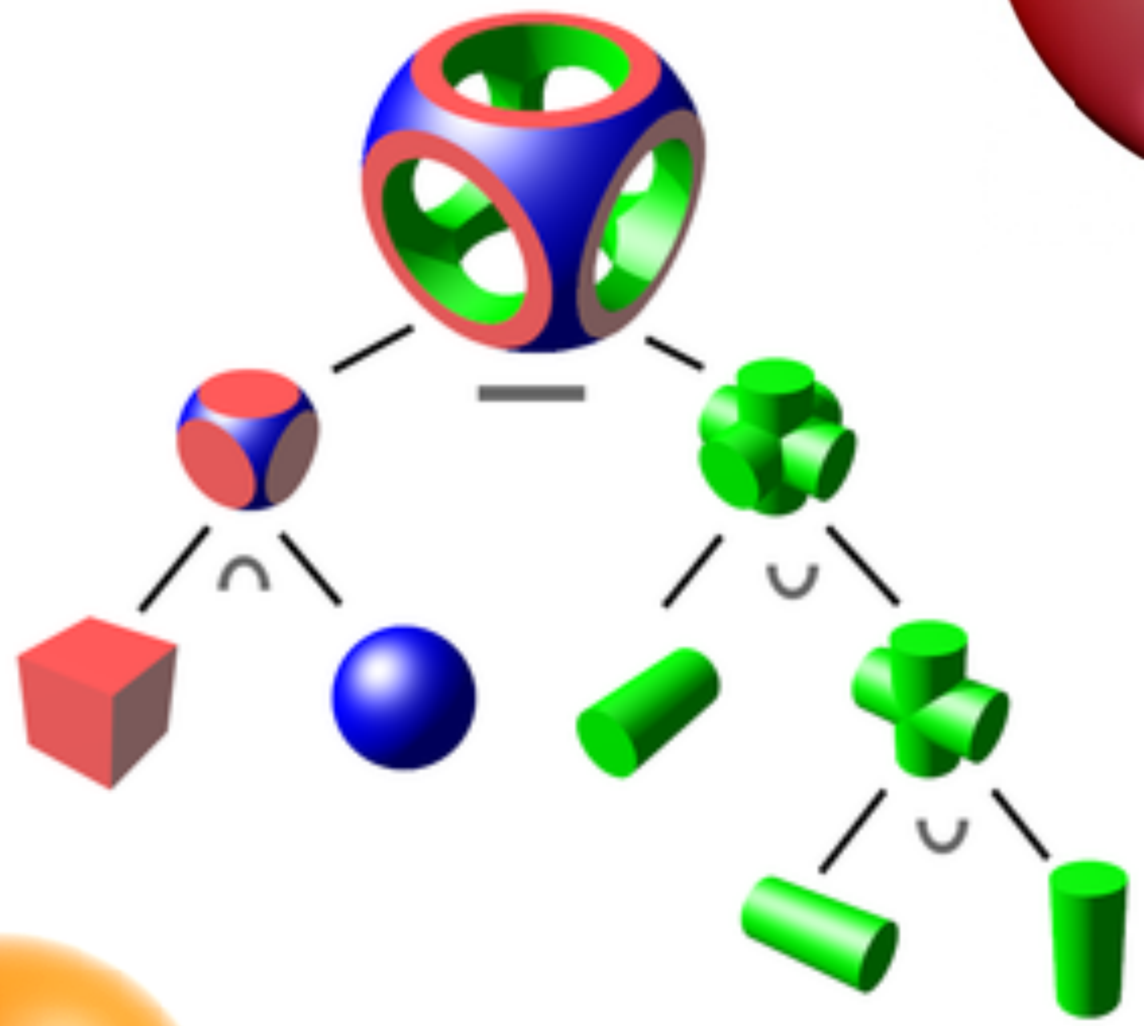
"Implicit" Representations of Geometry

- Points aren't known directly, but satisfy some relationship
- E.g., unit sphere is all points such that $x^2+y^2+z^2=1$
- More generally, $f(x,y,z) = 0$



Many implicit representations in graphics

- algebraic surfaces
- constructive solid geometry
- level set methods
- blobby surfaces
- fractals
- ...


(Will see some of these a bit later.)

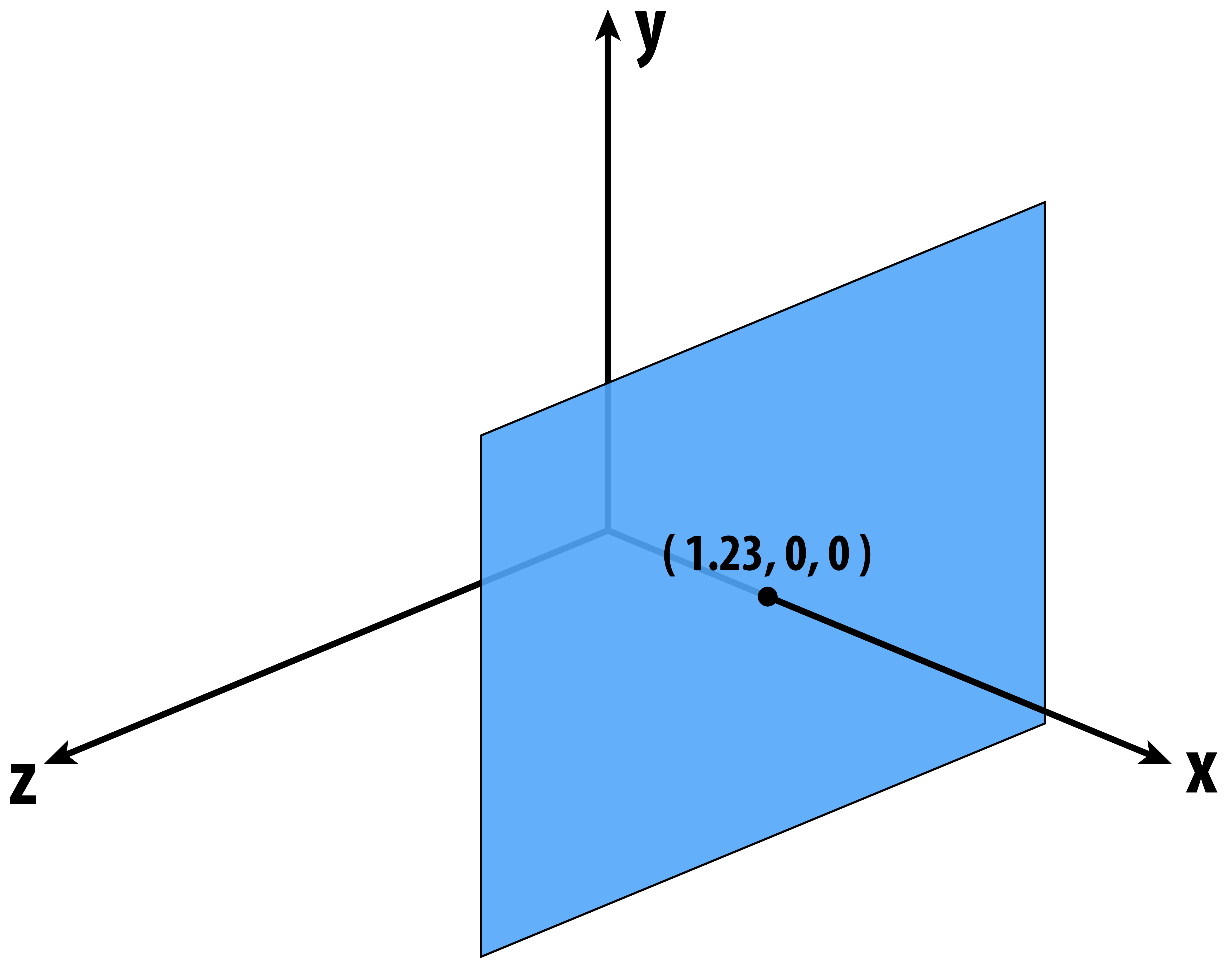
But first, let's play a game:

I'm thinking of an implicit surface $f(x,y,z)=0$.

Find *any* point on it.

Give up?

My function was $f(x,y,z) = x - 1.23$ (a plane):



Observation: implicit surfaces make some tasks hard (like sampling)

Let’s play another game.

I have a new surface $f(x,y,z) = x^2 + y^2 + z^2 - 1$.

I want to see if a point is *inside* it.

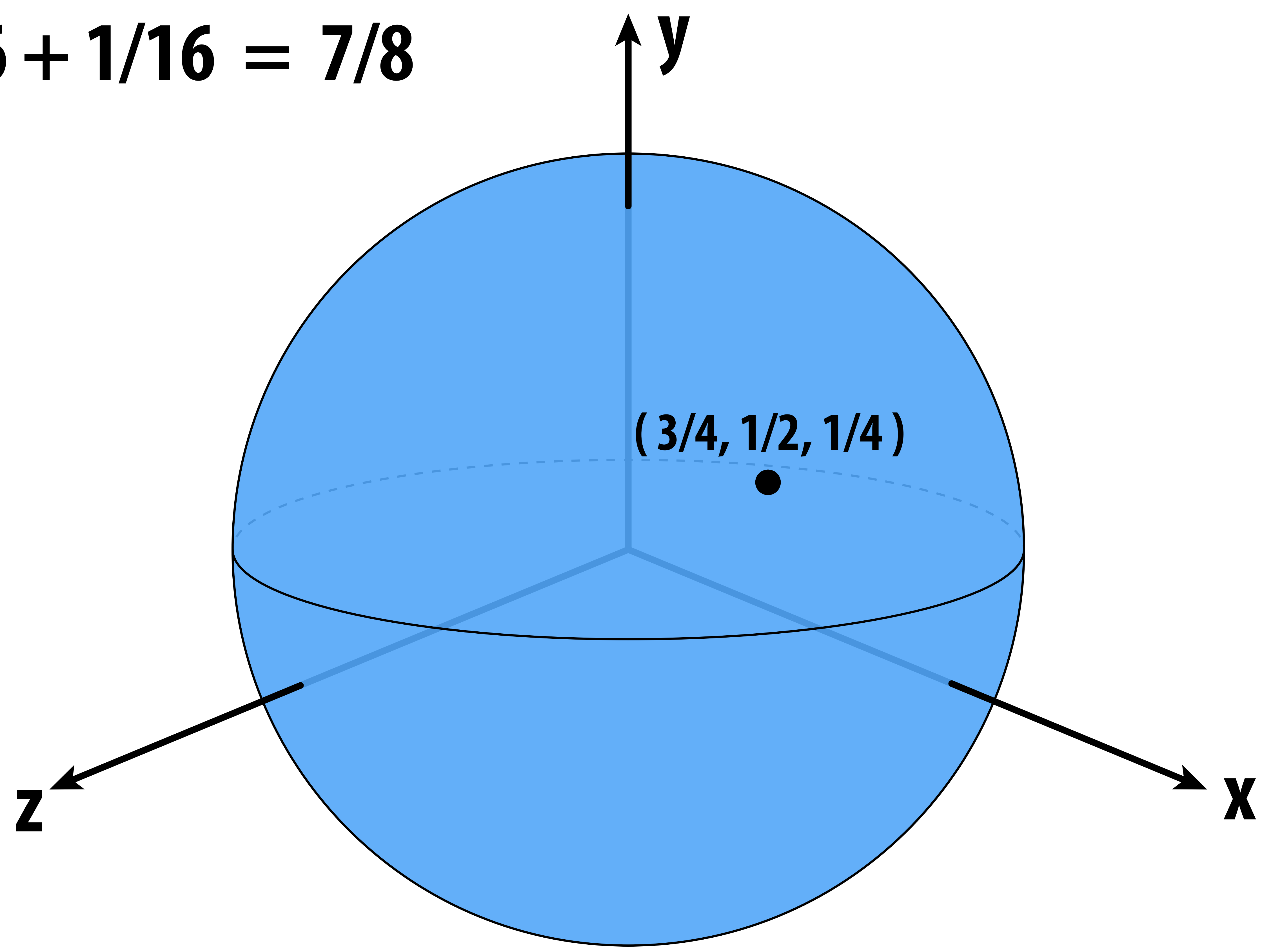
Check if this point is inside the unit sphere

How about the point (3/4, 1/2, 1/4)?

9/16 + 4/16 + 1/16 = 7/8

7/8 < 1

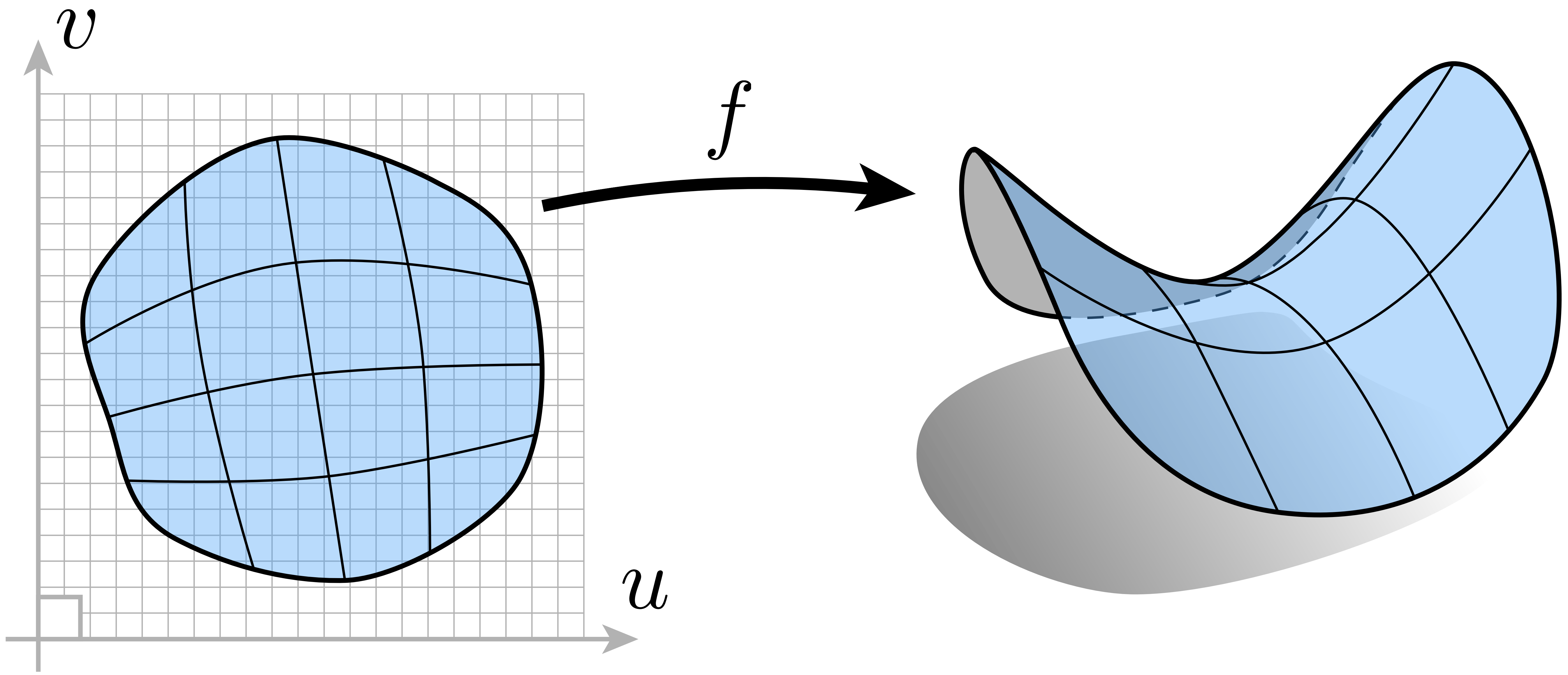
YES.



Implicit surfaces make other tasks easy (like inside/outside tests).

"Explicit" Representations of Geometry

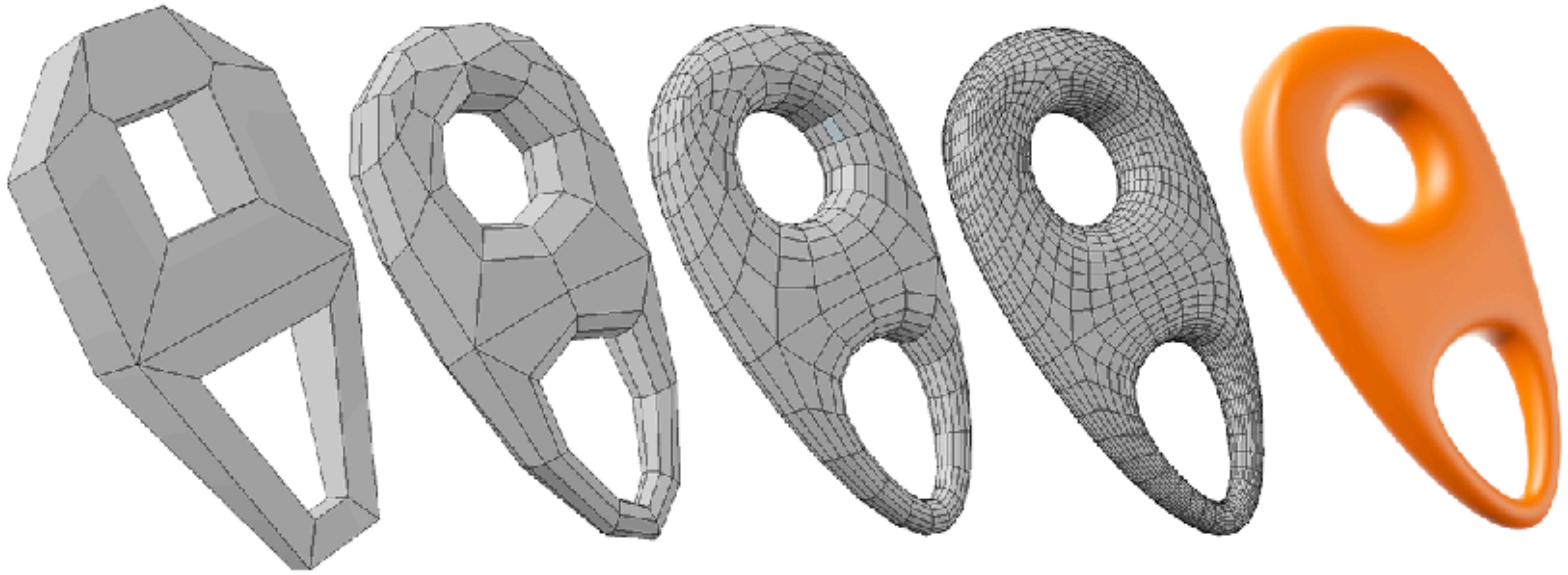
- **All points are given directly**
- **E.g., points on sphere are** $(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)),$
  $\text{for } 0 \leq u < 2\pi \text{ and } 0 \leq v \leq \pi$
- **More generally:** $f : \mathbb{R}^2 \to \mathbb{R}^3; (u, v) \mapsto (x, y, z)$

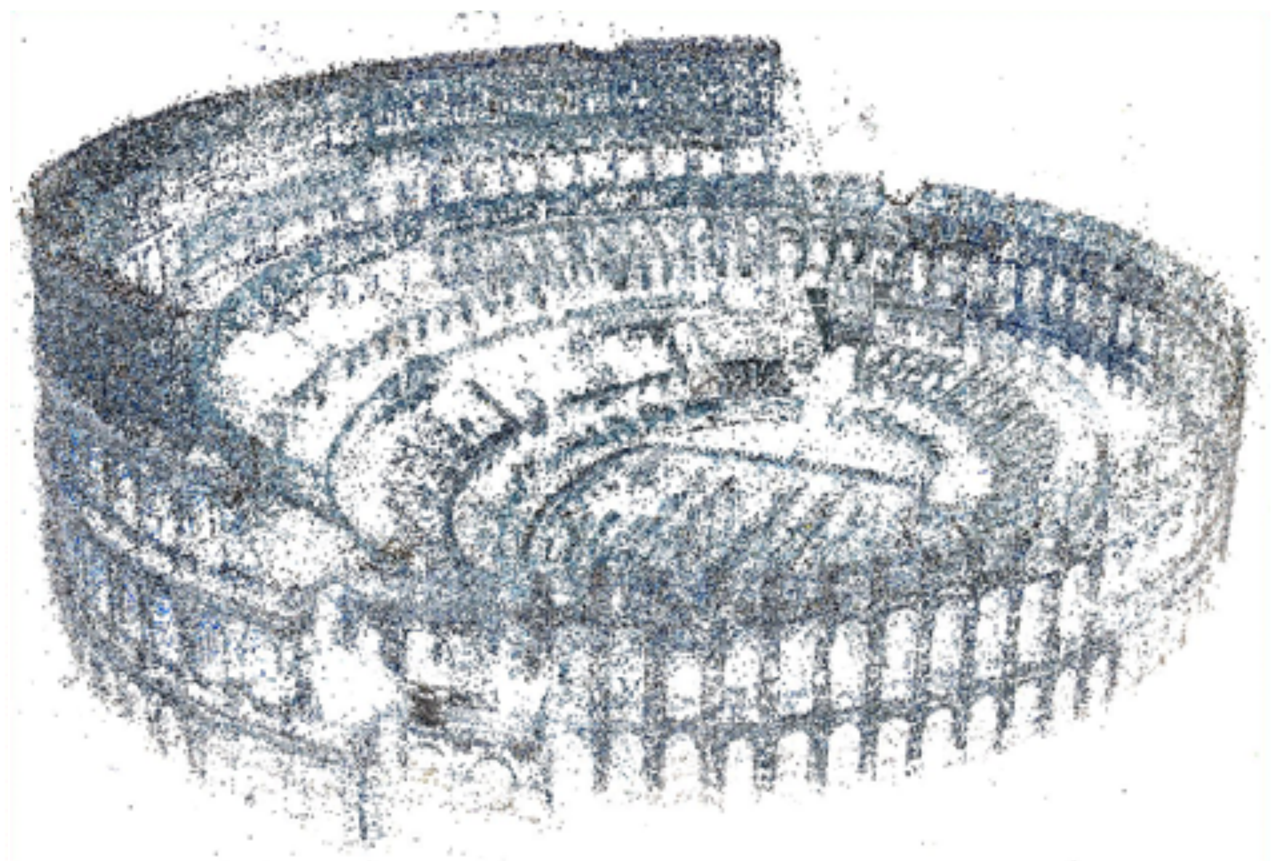


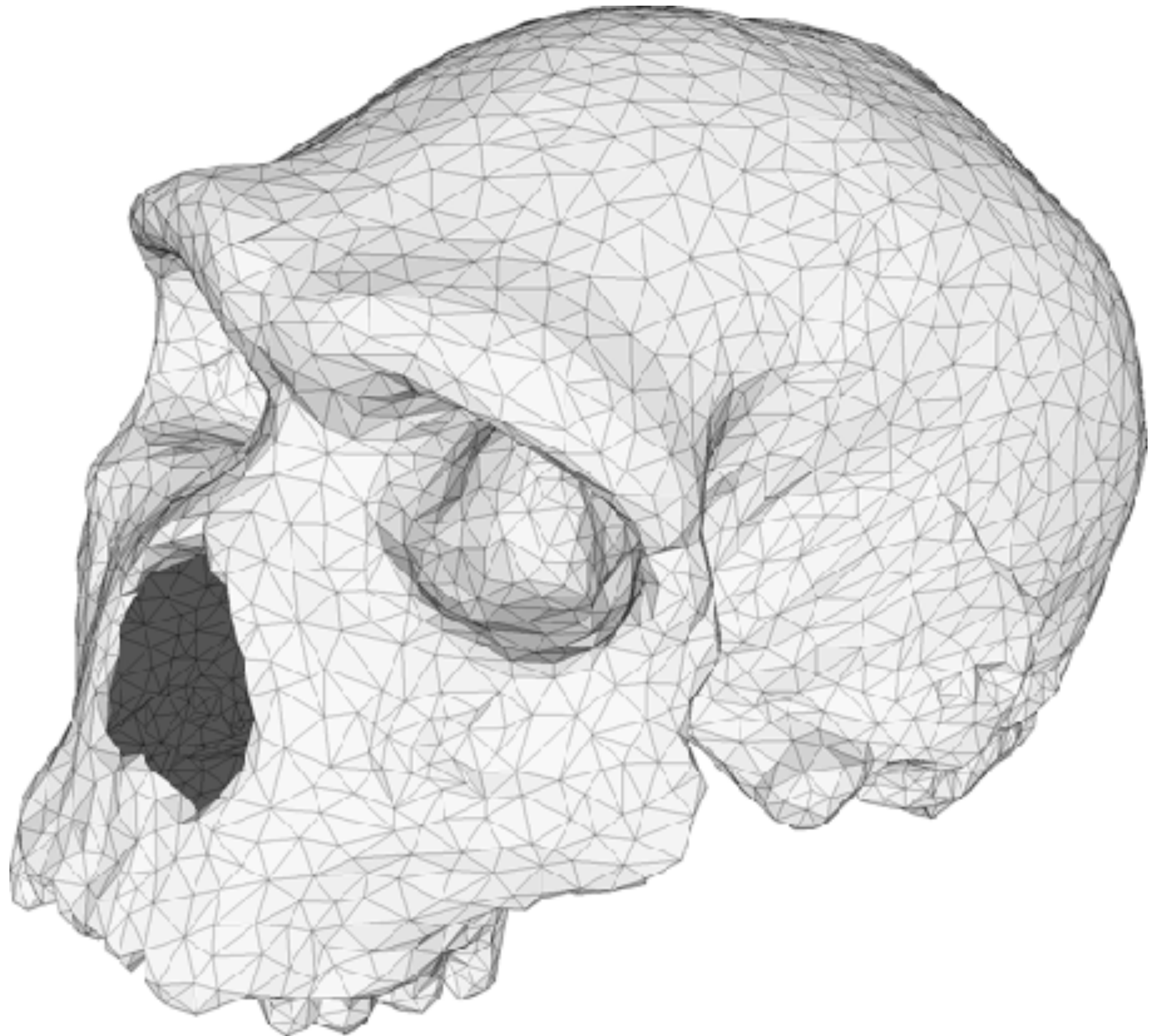
- **(Might have a bunch of these maps, e.g., one per triangle!)**

Many explicit representations in graphics

- triangle meshes
- polygon meshes
- subdivision surfaces
- NURBS
- point clouds
- ...







(Will see some of these a bit later.)

But first, let's play a game:

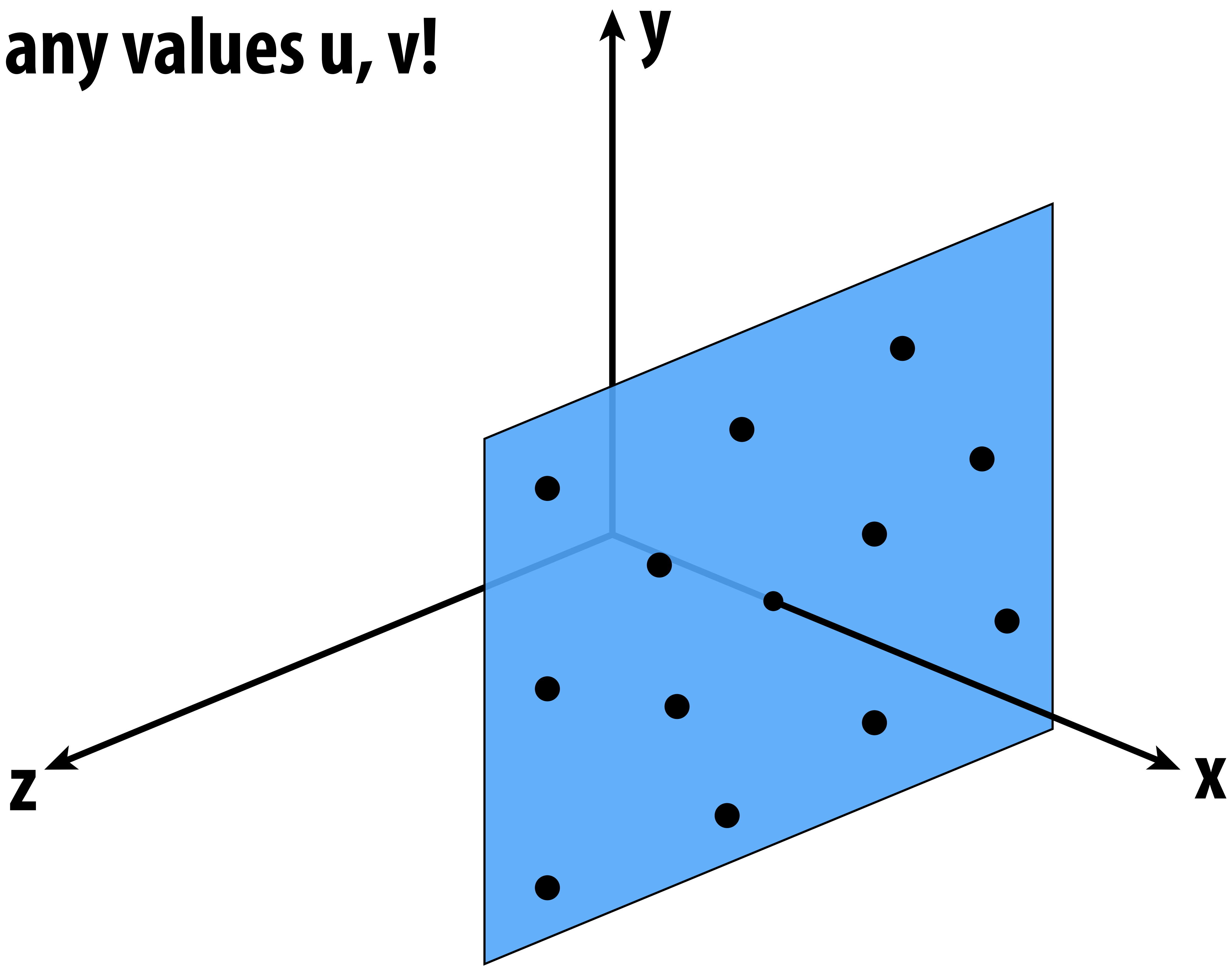
I'll give you an explicit surface.

You give me some points on it.

Sampling an explicit surface

My surface is $f(u, v) = (1.23, u, v)$.

Just plug in any values u, v!



Explicit surfaces make some tasks easy (like sampling).

Let's play another game.

I have a new surface $f(u,v)$.

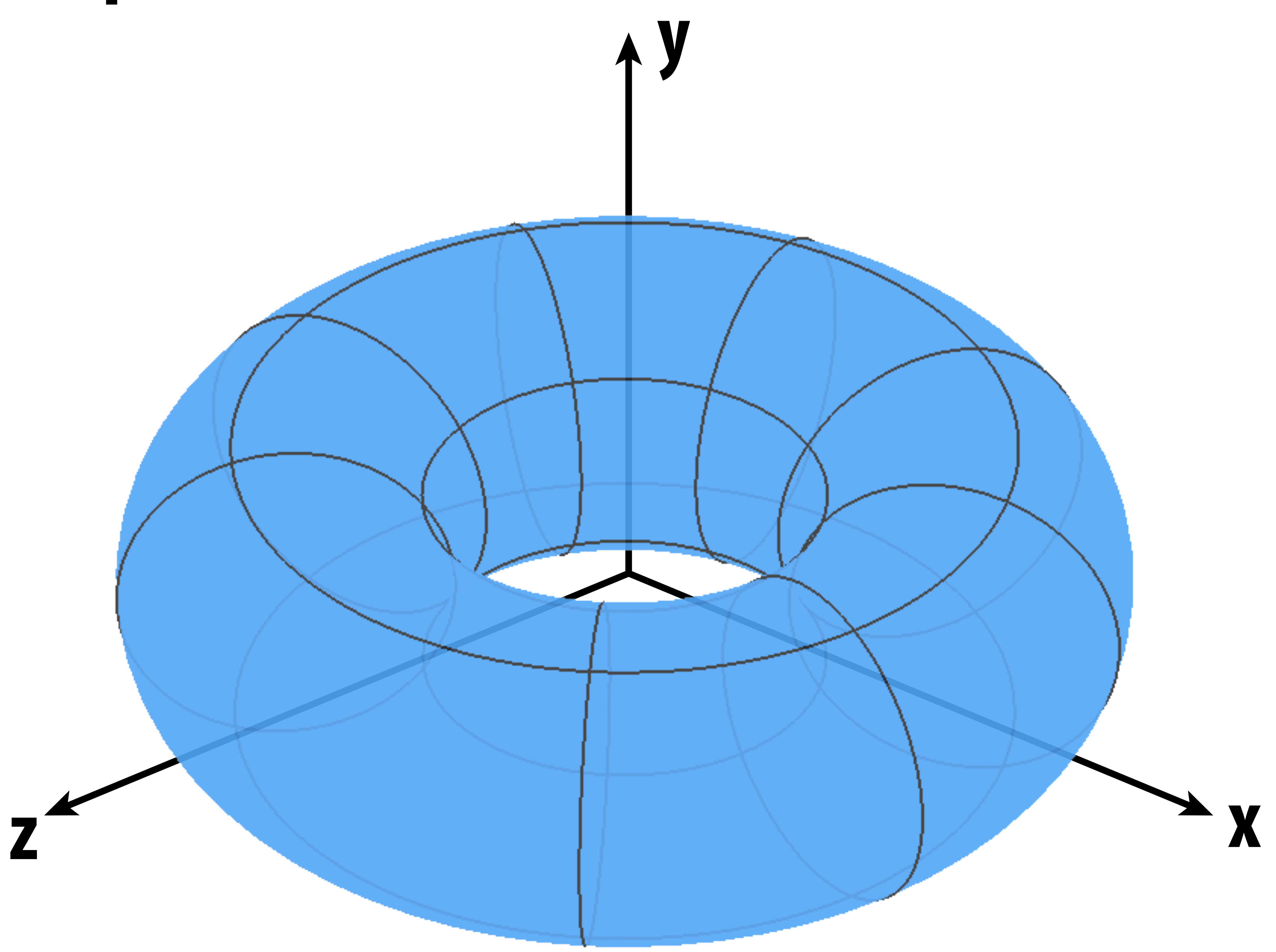
I want to see if a point is *inside* it.

Check if this point is inside the torus

My surface is f(u,v) = ((2+cos u)cos v, (2+cos u)sin v, sin u)

How about the point (1.96, -0.39, 0.9)?

...NO!



Explicit surfaces make other tasks hard (like inside/outside tests).

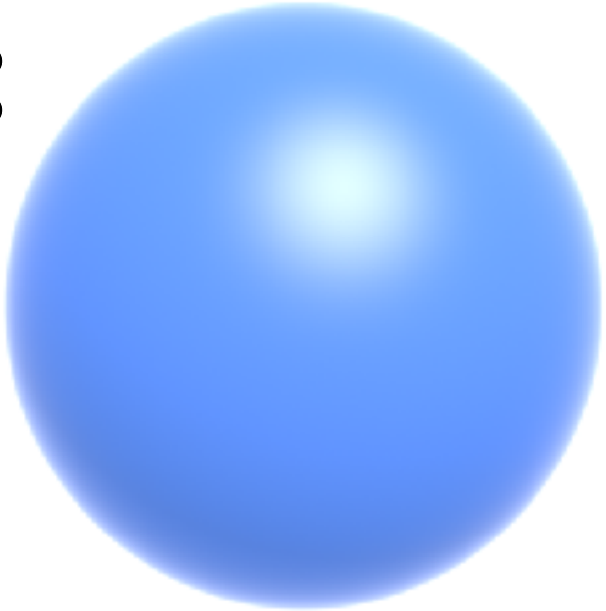
CONCLUSION:
Some representations work better than others—depends on the task!

Different representations will also be better suited to different types of geometry.

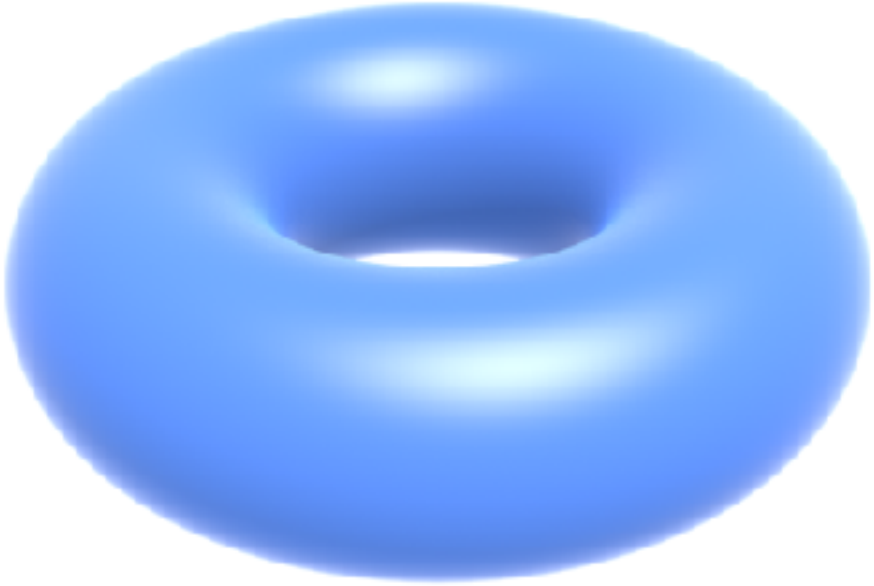
Let’s take a look at some common representations used in computer graphics.

Algebraic Surfaces (Implicit)

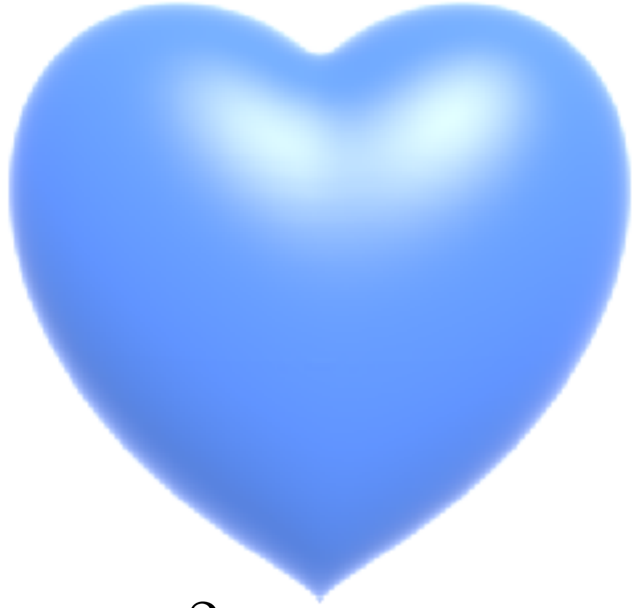
- Surface is zero set of a polynomial in x, y, z
- Examples:



$$x^2+y^2+z^2=1$$



$$(R-\sqrt{x^2+y^2})^2+z^2=r^2$$



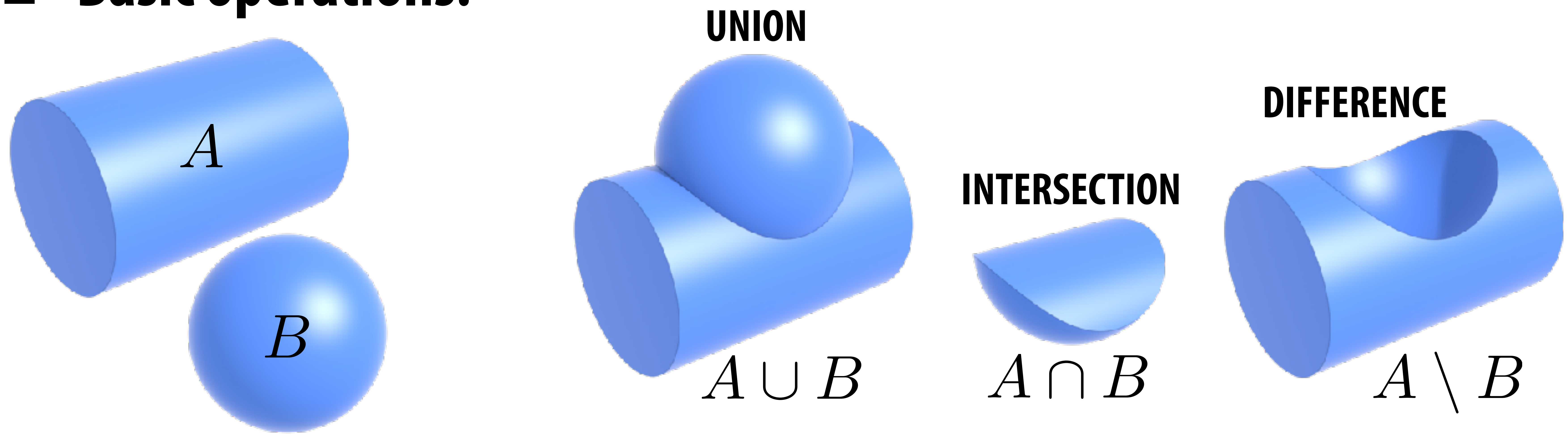
$$(x^2+\frac{9y^2}{4}+z^2-1)^3 = x^2z^3+\frac{9y^2z^3}{80}$$

- What about more complicated shapes?

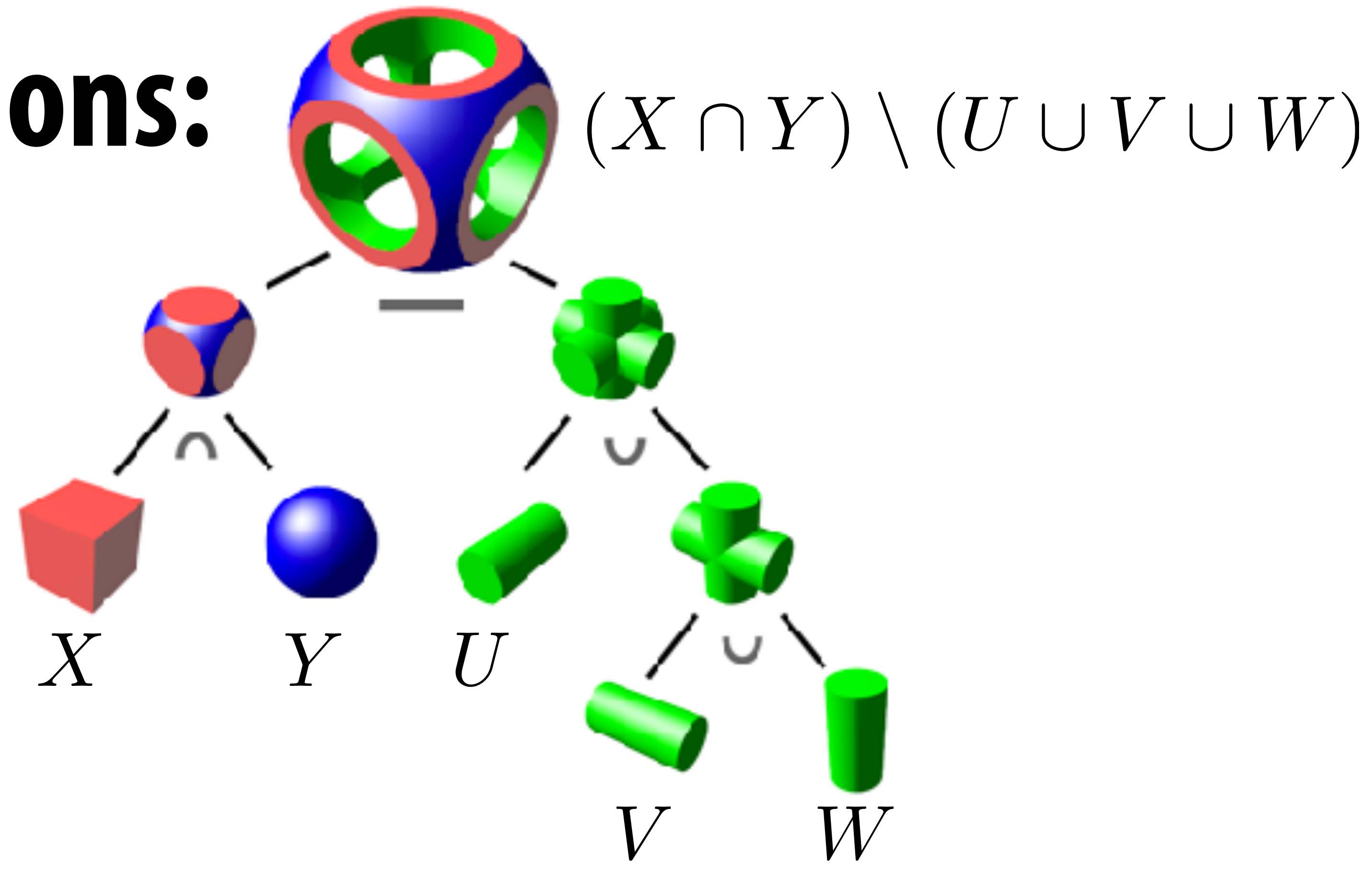
- Very hard to come up with polynomials!

Constructive Solid Geometry (Implicit)

- Build more complicated shapes via Boolean operations
- Basic operations:

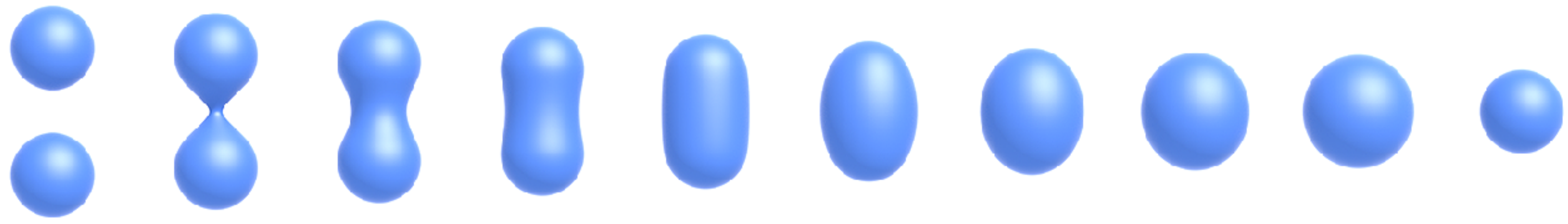


- Then chain together expressions:

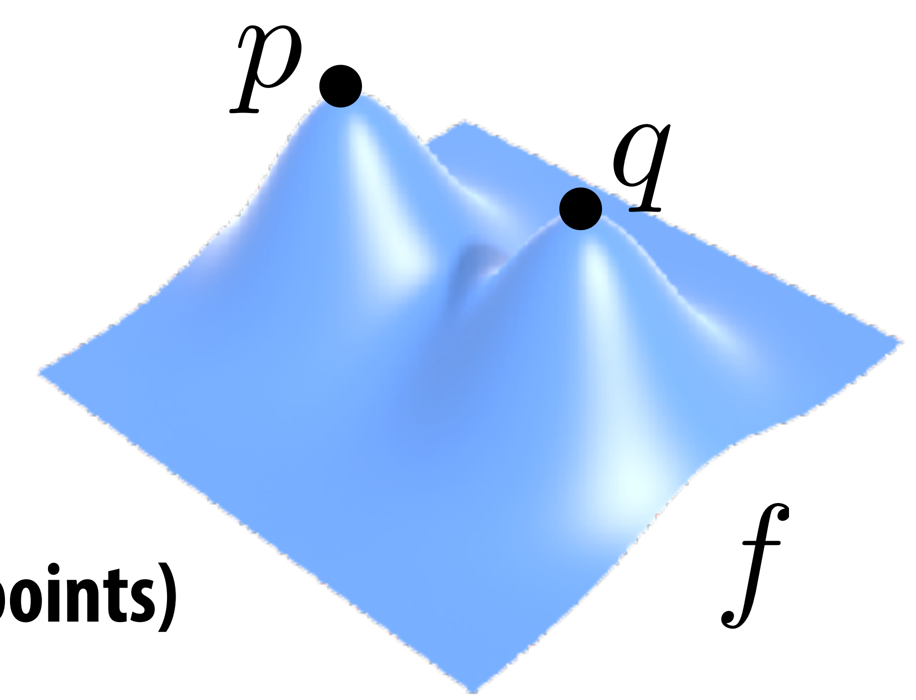


Blobby Surfaces (Implicit)

- **Instead of Booleans, gradually blend surfaces together:**

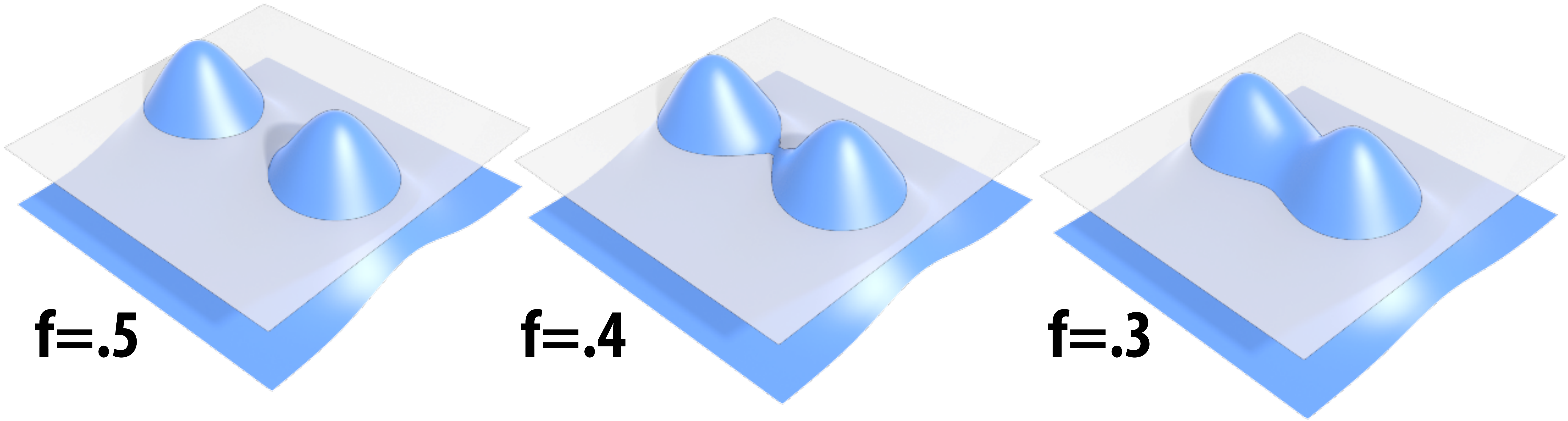


- **Easier to understand in 2D:**



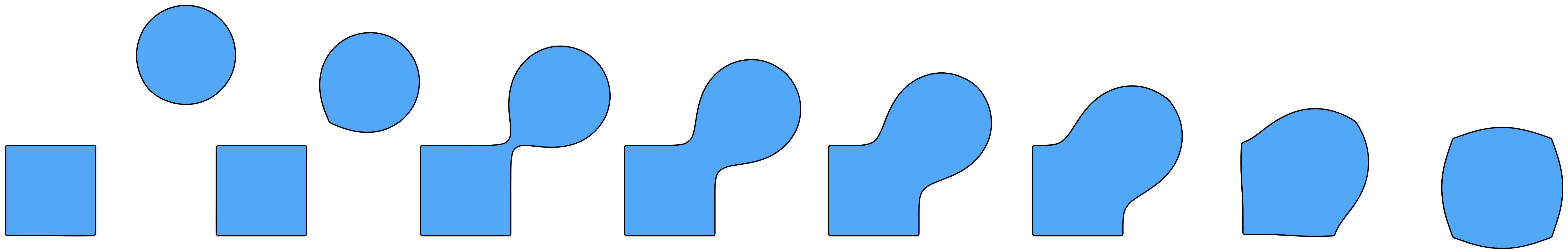
$$\phi_p(x) := e^{-|x-p|^2}$$ (Gaussian centered at p)

$$f := \phi_p + \phi_q$$ (Sum of Gaussians centered at different points)



Blending Distance Functions (Implicit)

- A *distance function* gives distance to closest point on object
- Can blend any two distance functions d_1, d_2:

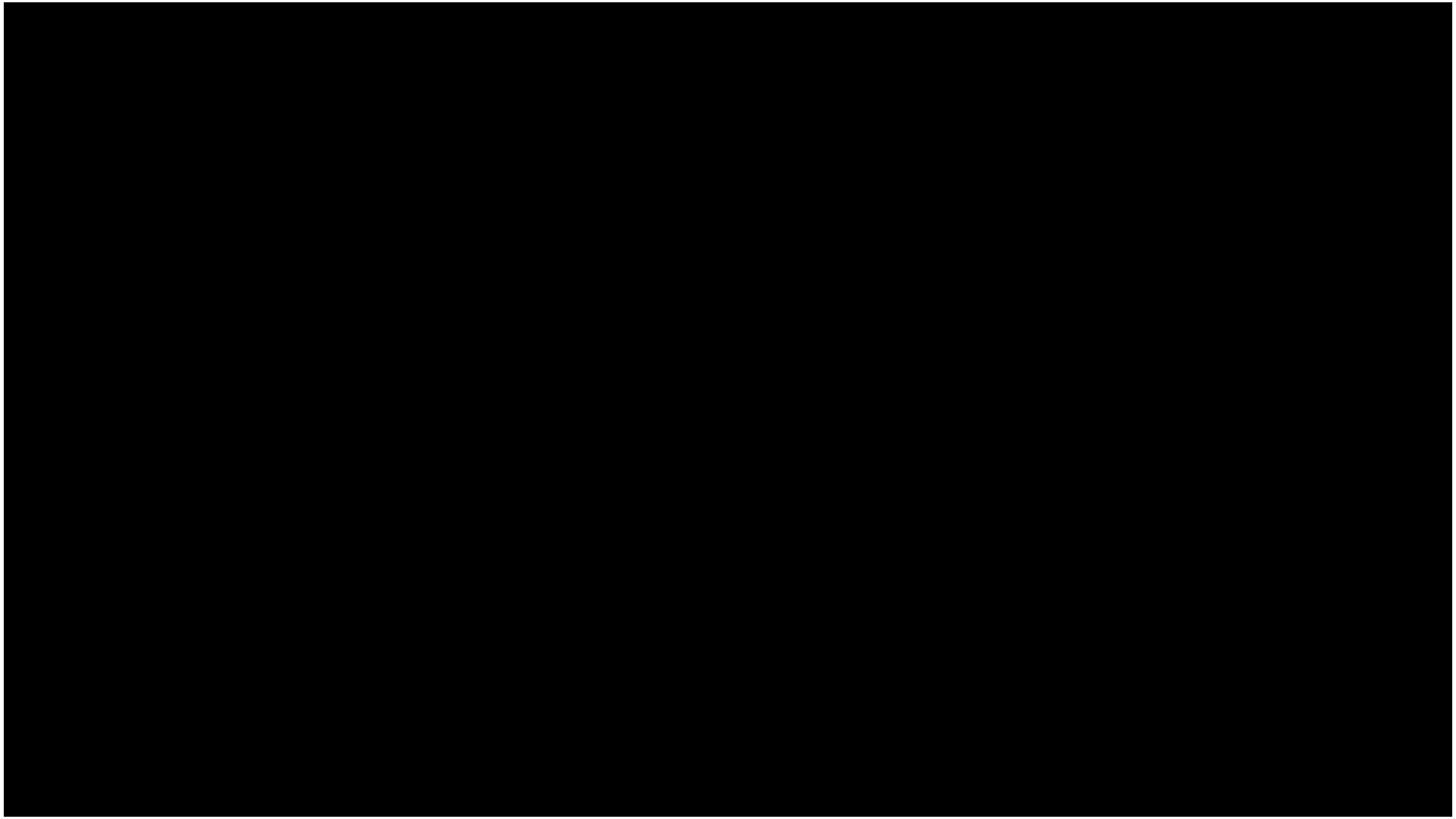


- Similar strategy to points, though many possibilities. E.g.,

$$f(x) := e^{d_1(x)^2} + e^{d_2(x)^2} - \frac{1}{2}$$

- Appearance depends on how we combine functions
- Q: How do we implement a Boolean union of $d_1(x), d_2(x)$?
- A: Just take the minimum: $f(x) = \min(d_1(x), d_2(x))$

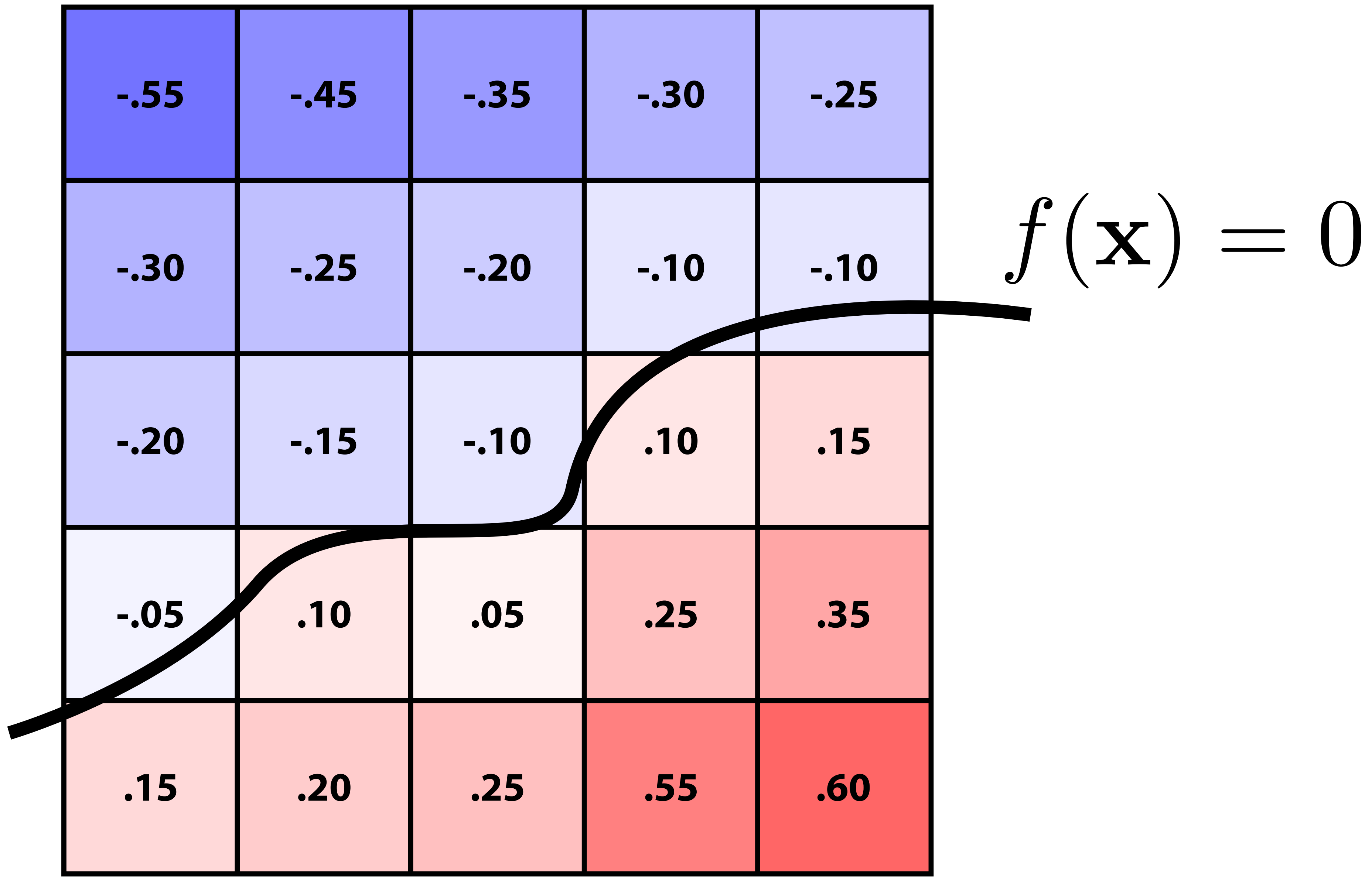
Scene of pure distance functions (not easy!)



see http://iquilezles.org/

Level Set Methods (Implicit)

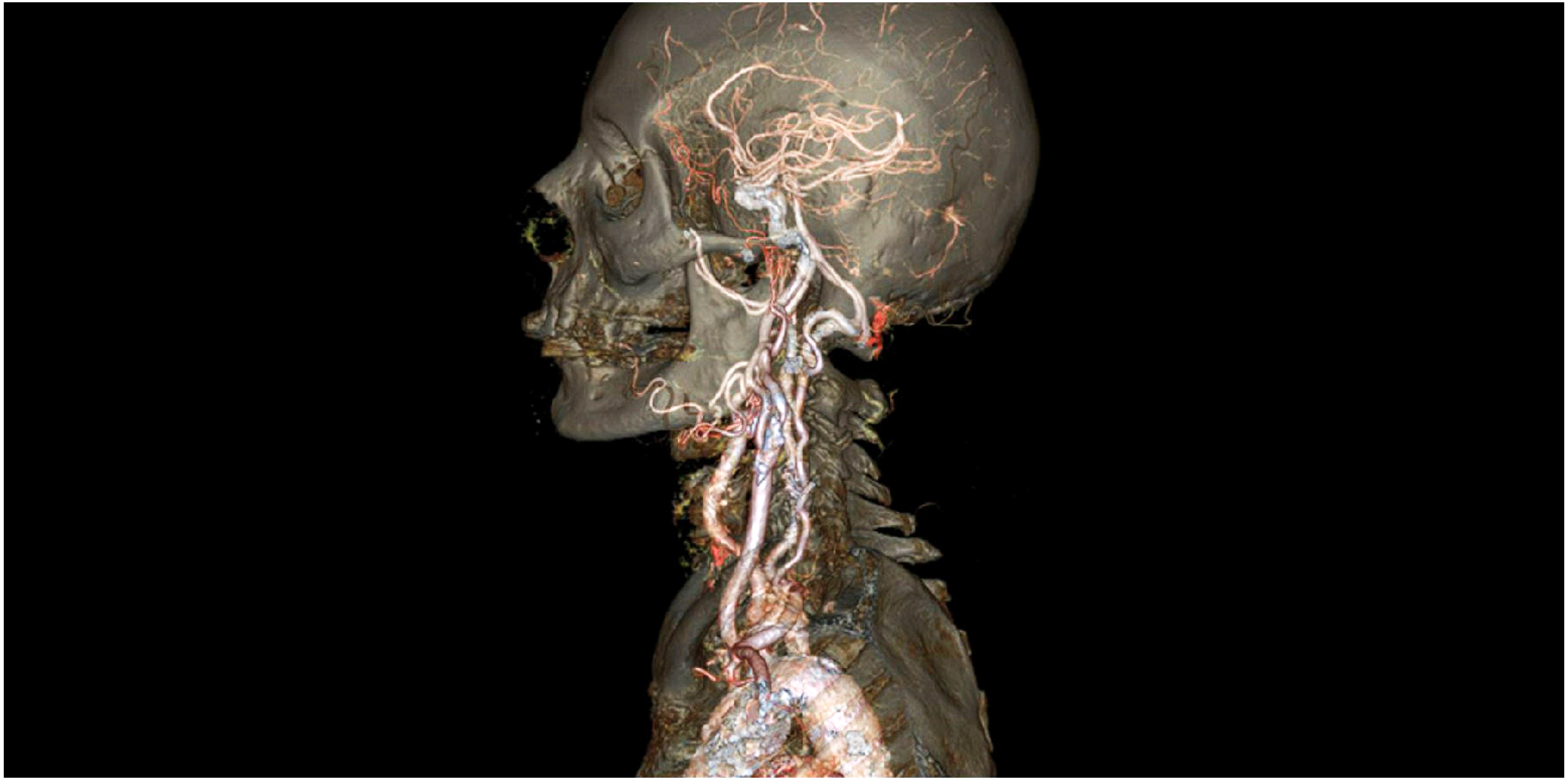
- Implicit surfaces have some nice features (e.g., merging/splitting)
- But, hard to describe complex shapes in closed form
- Alternative: store a grid of values approximating function



- Surface is found where *interpolated* values equal zero
- Provides much more explicit control over shape (like a texture)
- Unlike closed-form expressions, run into problems of aliasing!

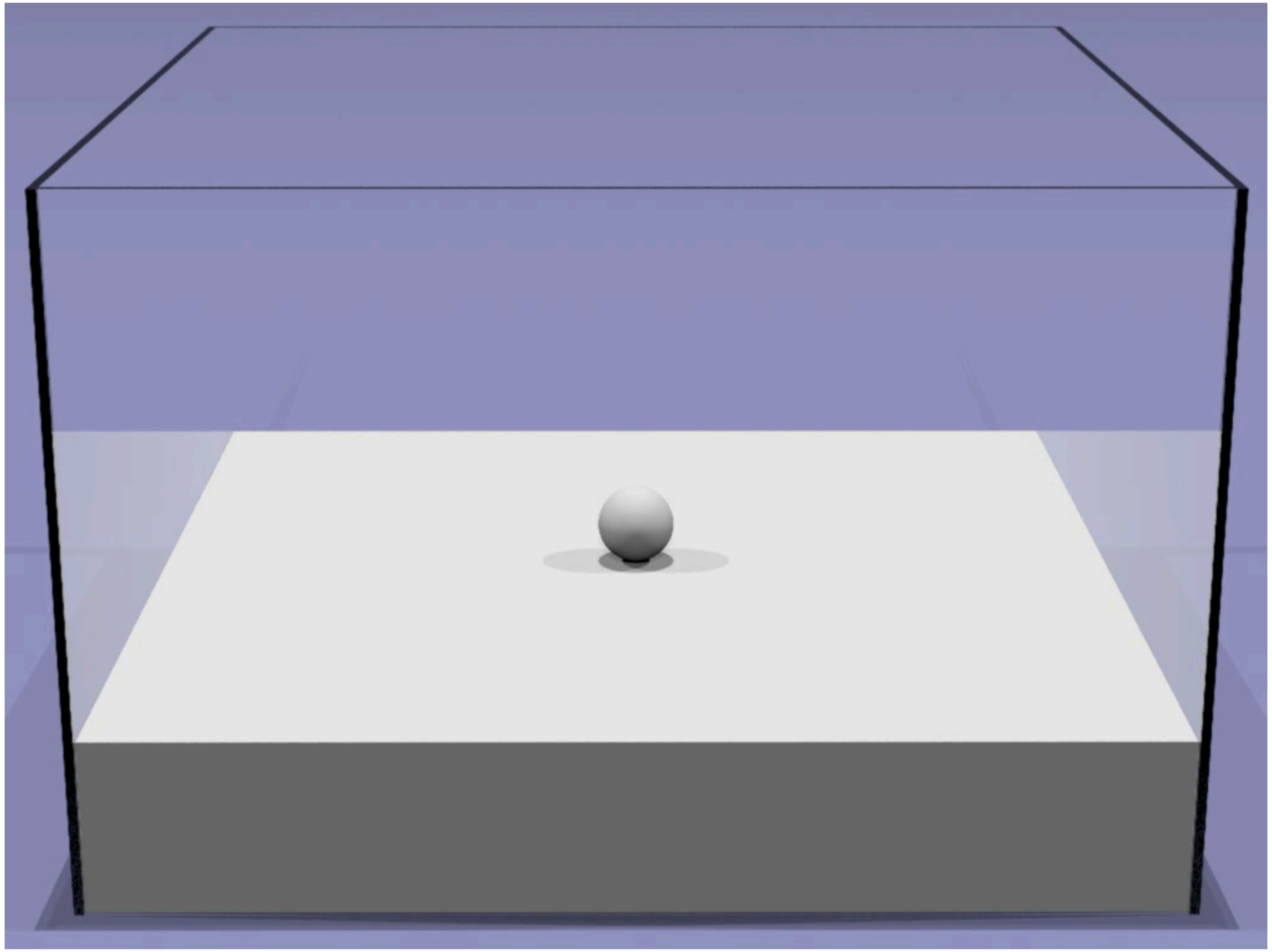
Level Sets from Medical Data (CT, MRI, etc.)

- **Level sets encode, e.g., constant tissue density**



Level Sets in Physical Simulation

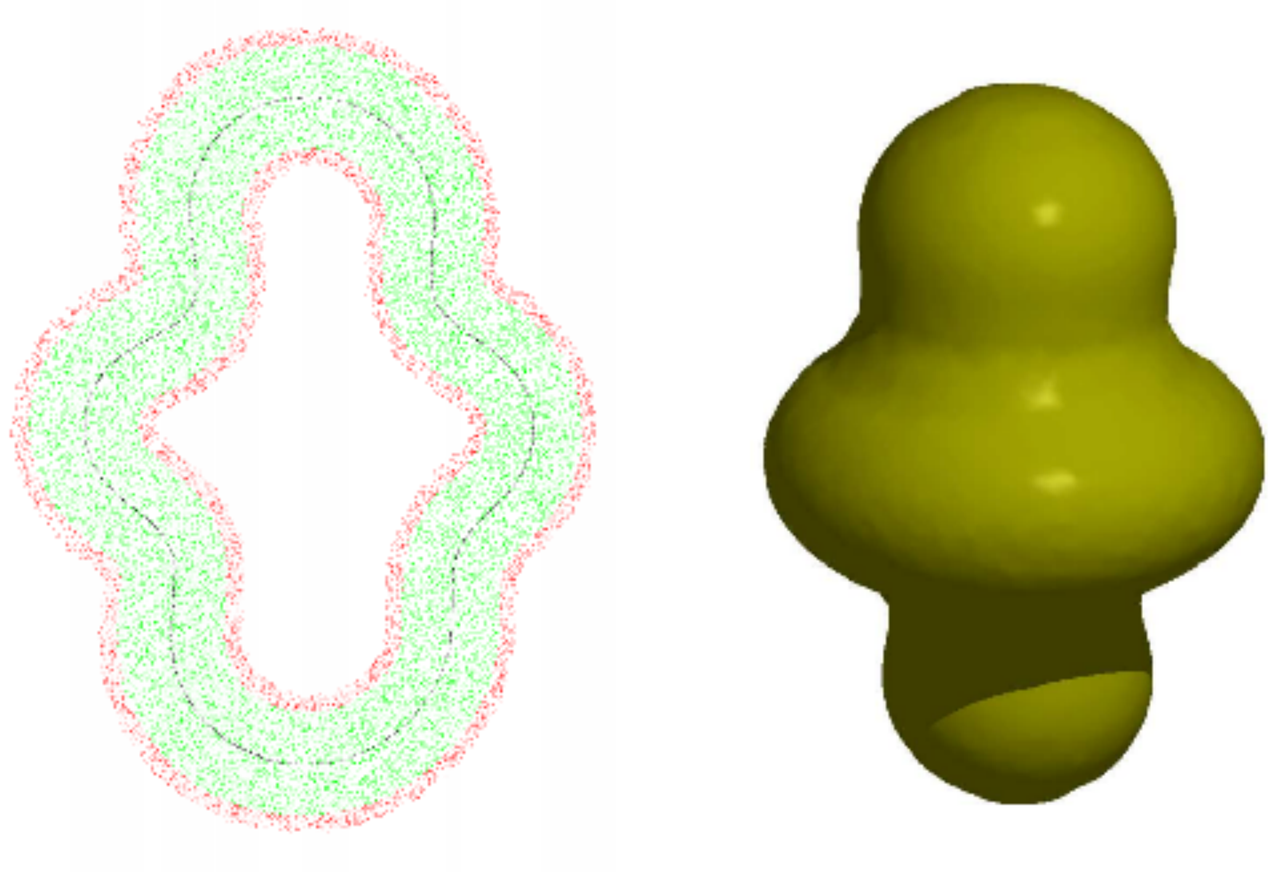
Level set encodes distance to air-liquid boundary:

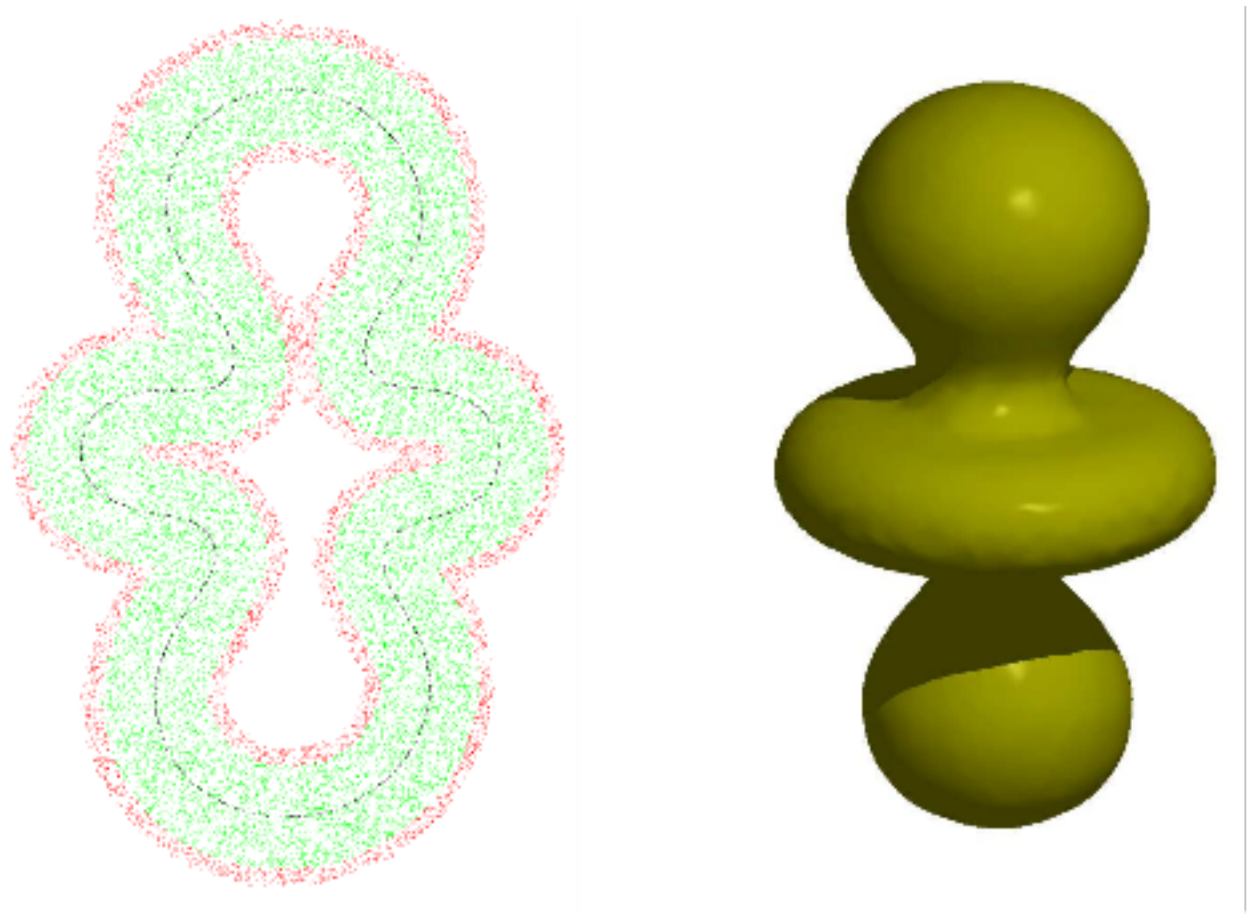


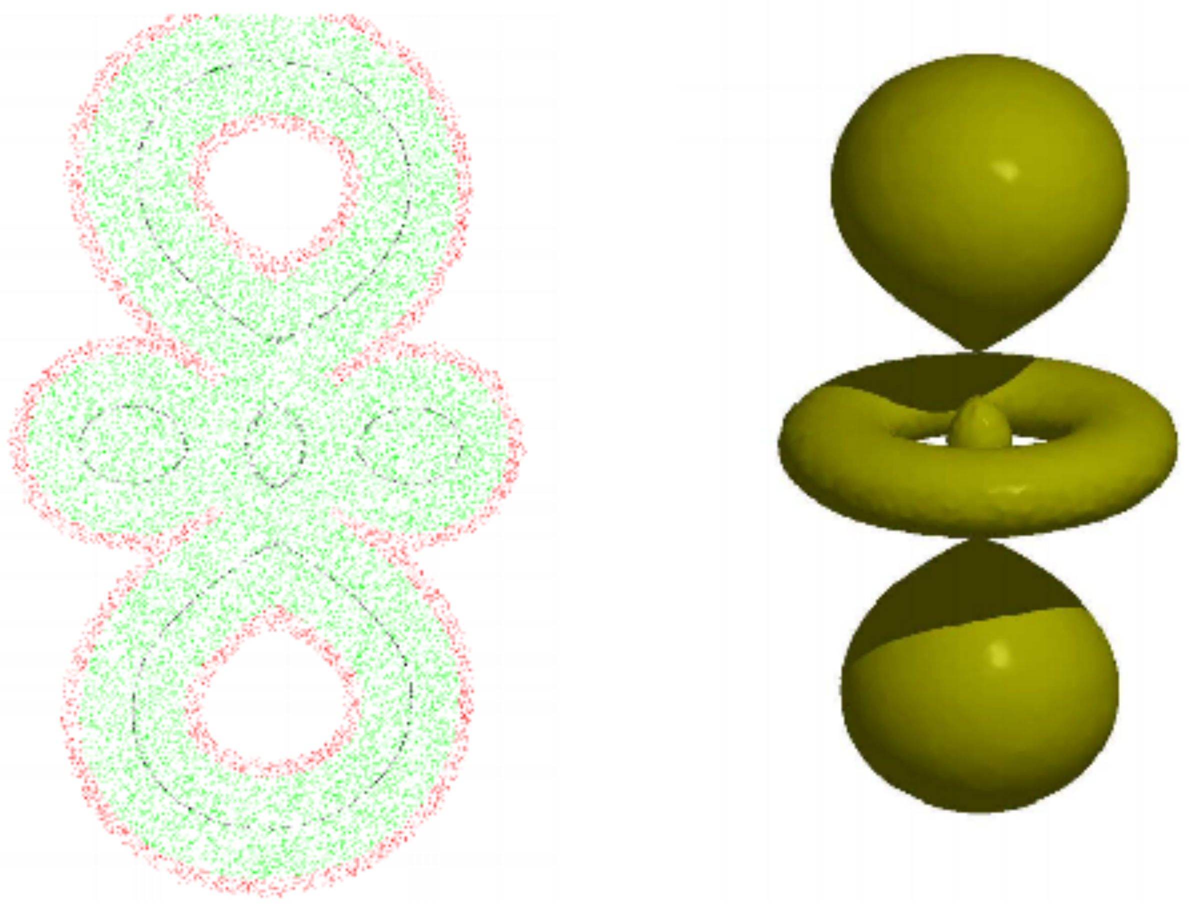
see http://physbam.stanford.edu

Level Set Storage

- Drawback: storage for 2D surface is now $O(n^3)$
- Can reduce cost by storing only a narrow band around surface:

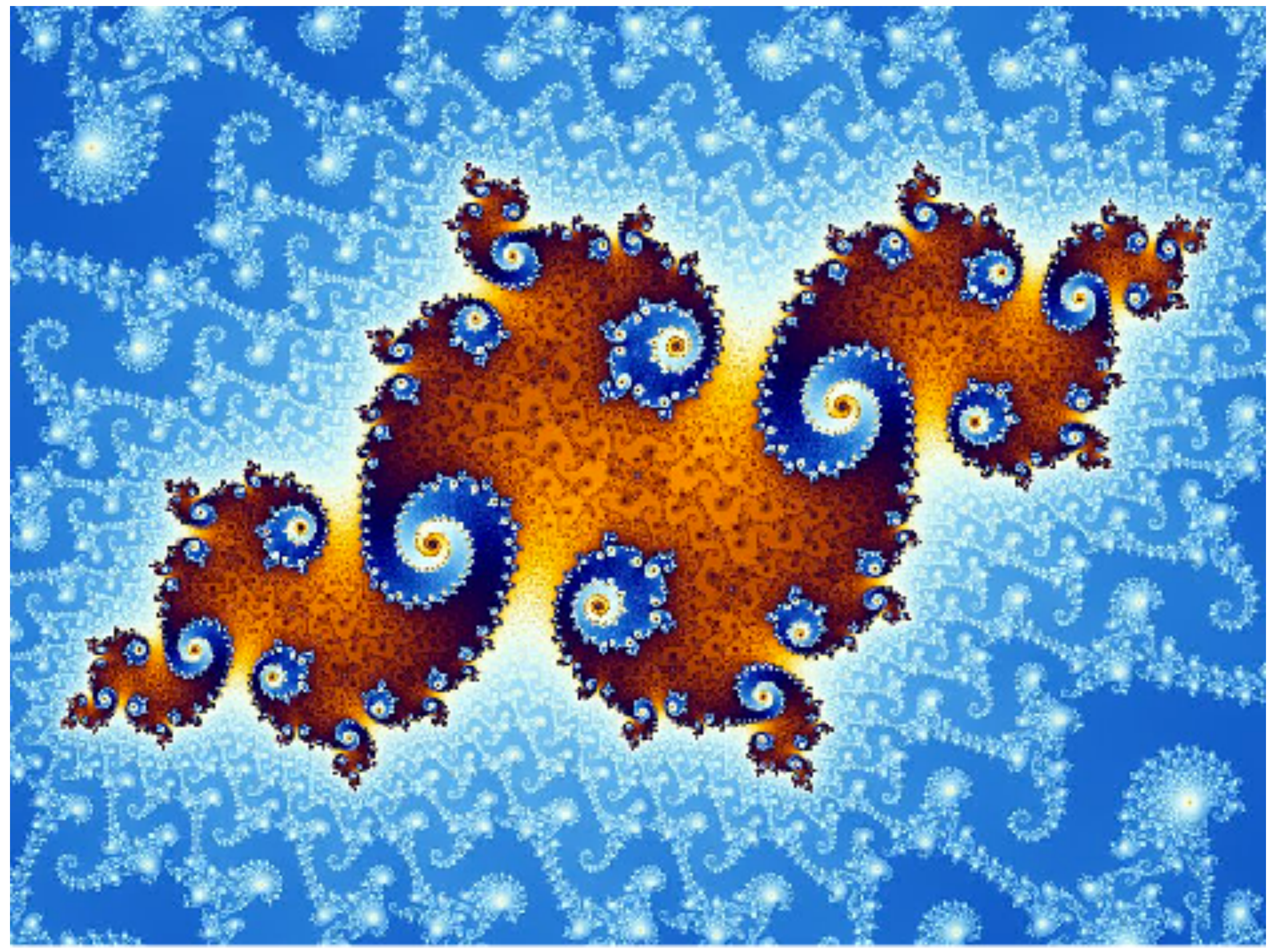






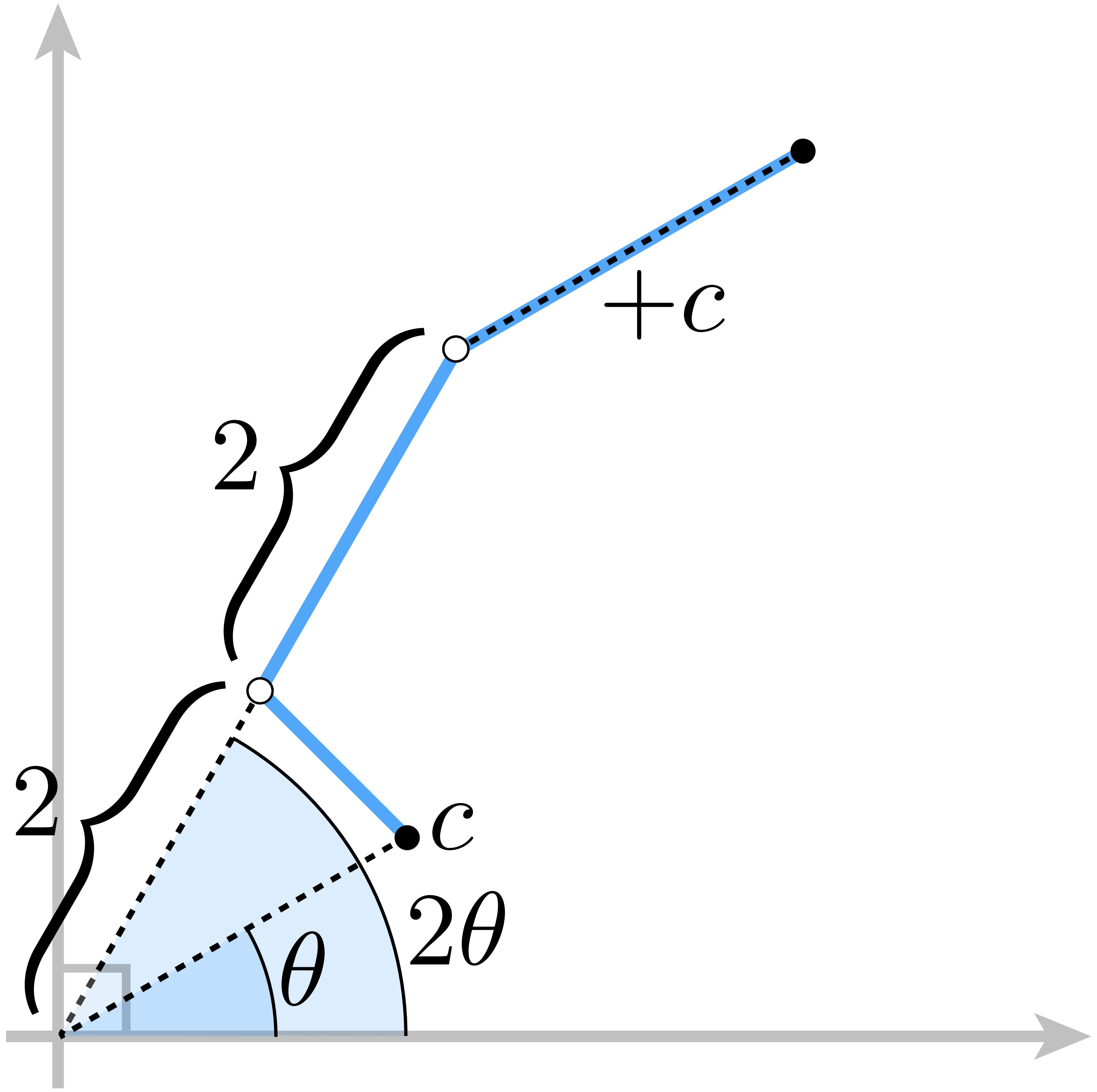
Fractals (Implicit)

- No precise definition; exhibit self-similarity, detail at all scales
- New "language" for describing natural phenomena
- Hard to control shape!



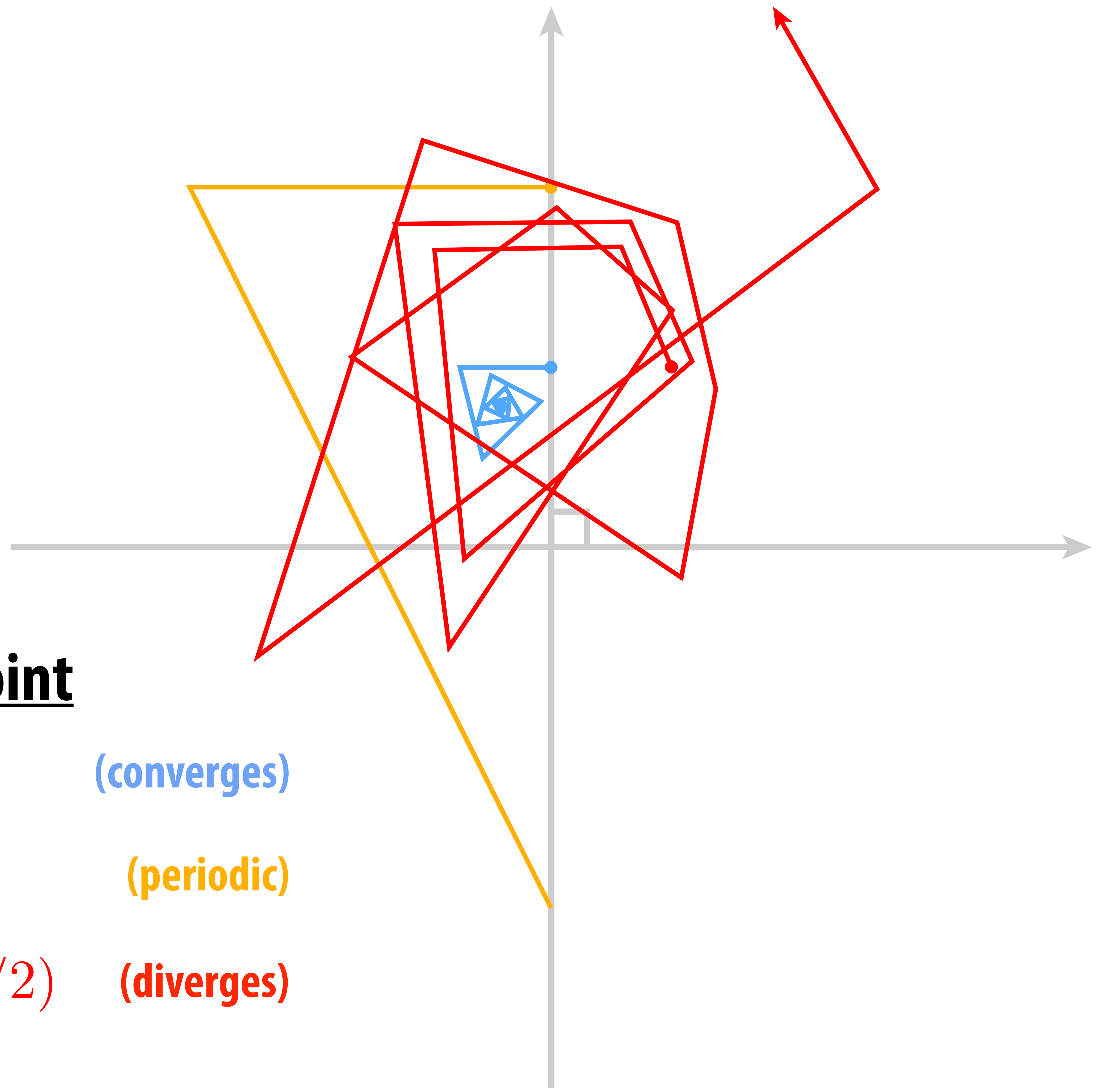
Mandelbrot Set - Definition

- For each point c in the plane:
  - double the angle
  - square the magnitude
  - add the original point c
  - repeat
- Complex version:
  - *Replace* z *with* $z^2 + c$
  - repeat



If magnitude remains bounded (never goes to ∞), it's in the Mandelbrot set.

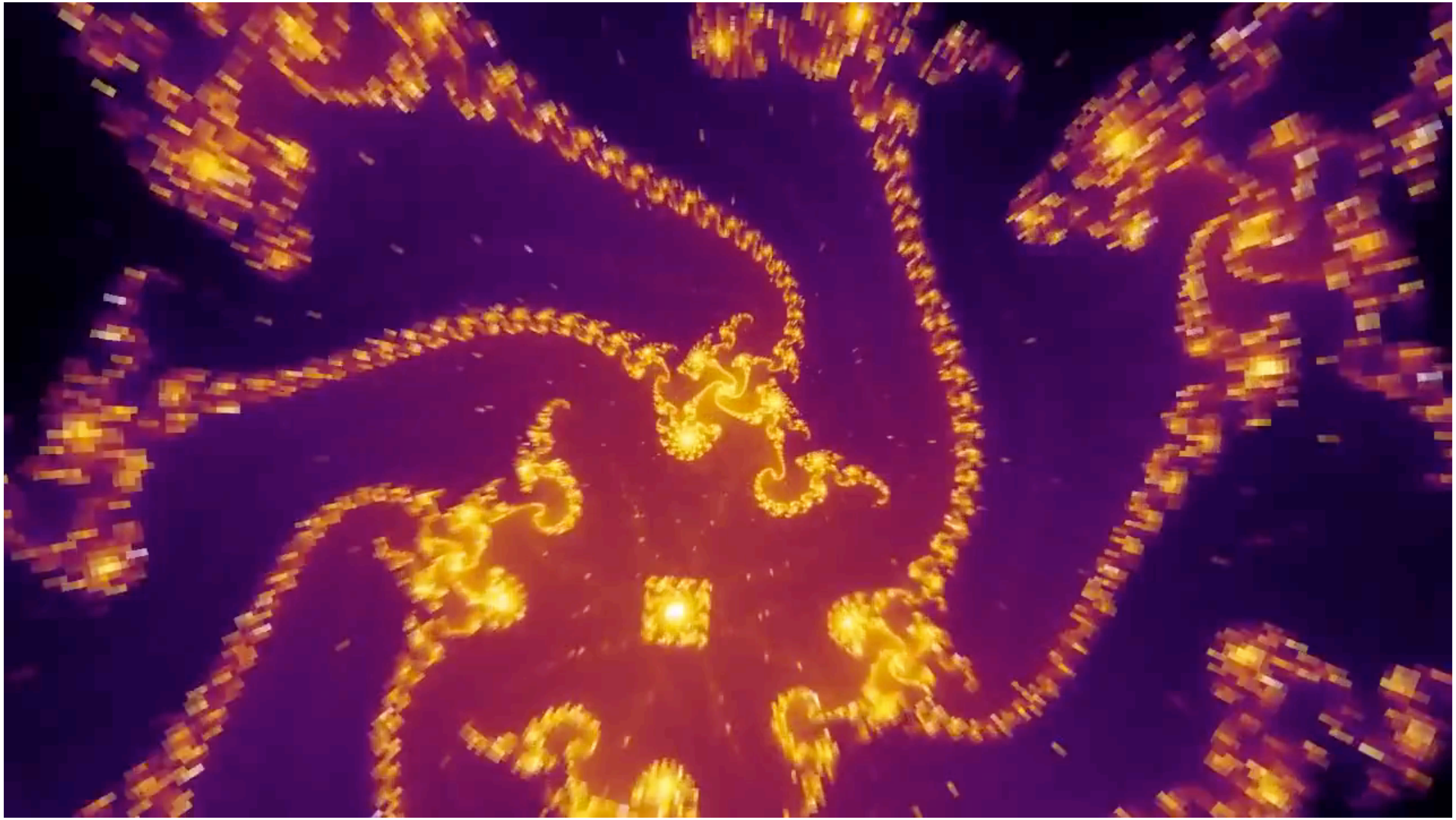
Mandelbrot Set - Examples



starting point

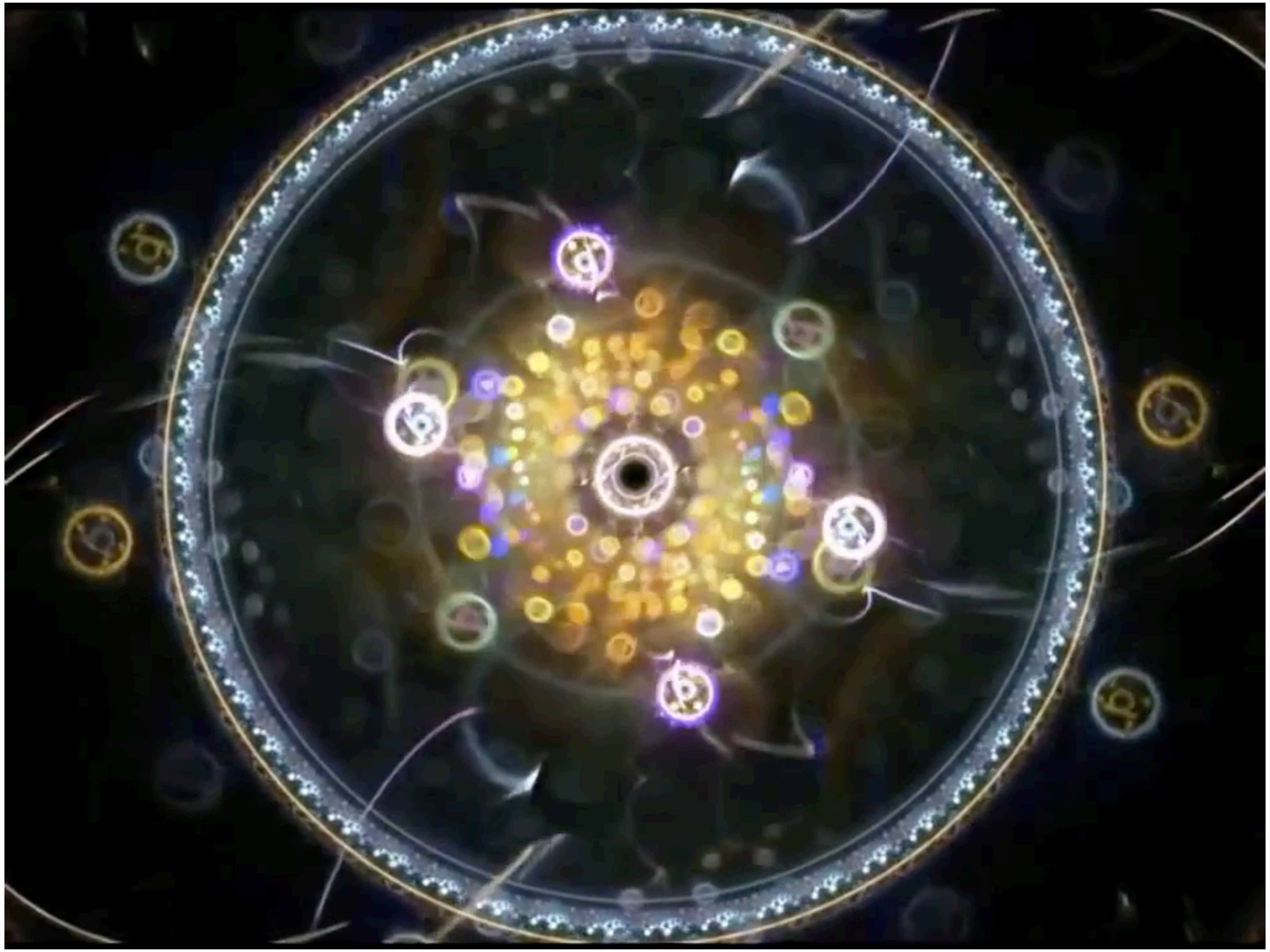
- $(0, 1/2)$ **(converges)**
- $(0, 1)$ **(periodic)**
- $(1/3, 1/2)$ **(diverges)**

Mandelbrot Set - Zooming In



(Colored according to how quickly each point diverges/converges.)

Iterated Function Systems



Scott Draves (CMU alumn) - see http://electricsheep.org

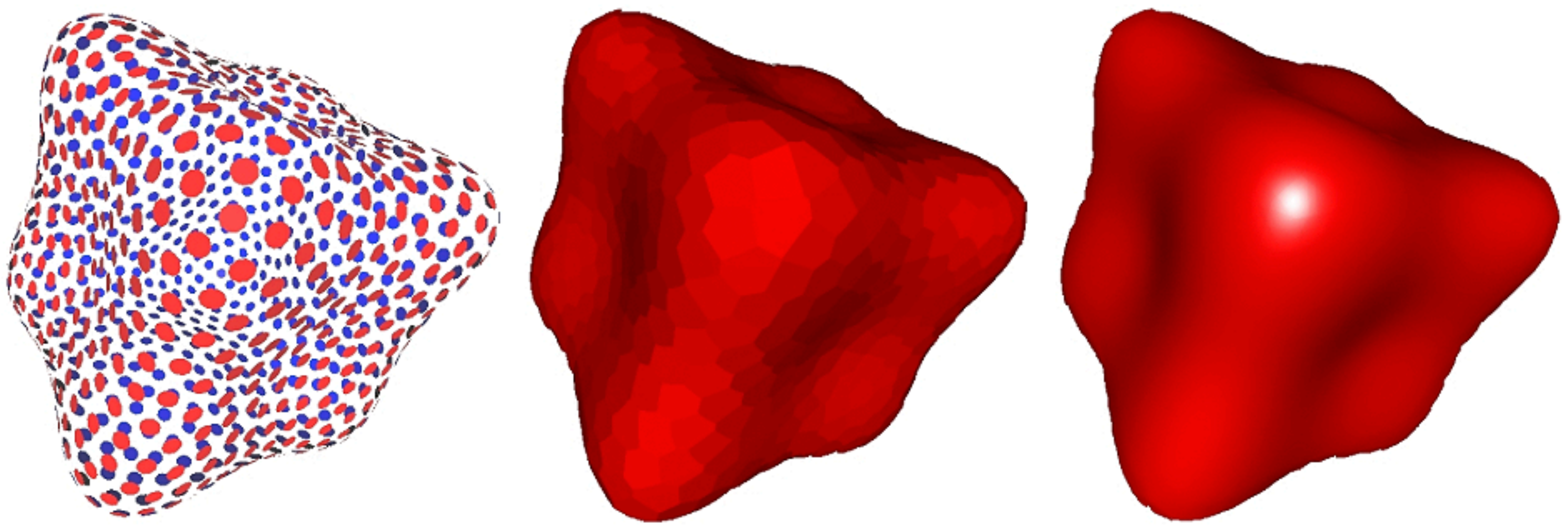
Implicit Representations - Pros & Cons

- **Pros:**
  - description can be very compact (e.g., a polynomial)
  - easy to determine if a point is in our shape (just plug it in!)
  - other queries may also be easy (e.g., distance to surface)
  - for simple shapes, exact description/no sampling error
  - easy to handle changes in topology (e.g., fluid)
- **Cons:**
  - expensive to find all points in the shape (e.g., for drawing)
  - *very difficult to model complex shapes*

What about explicit representations?

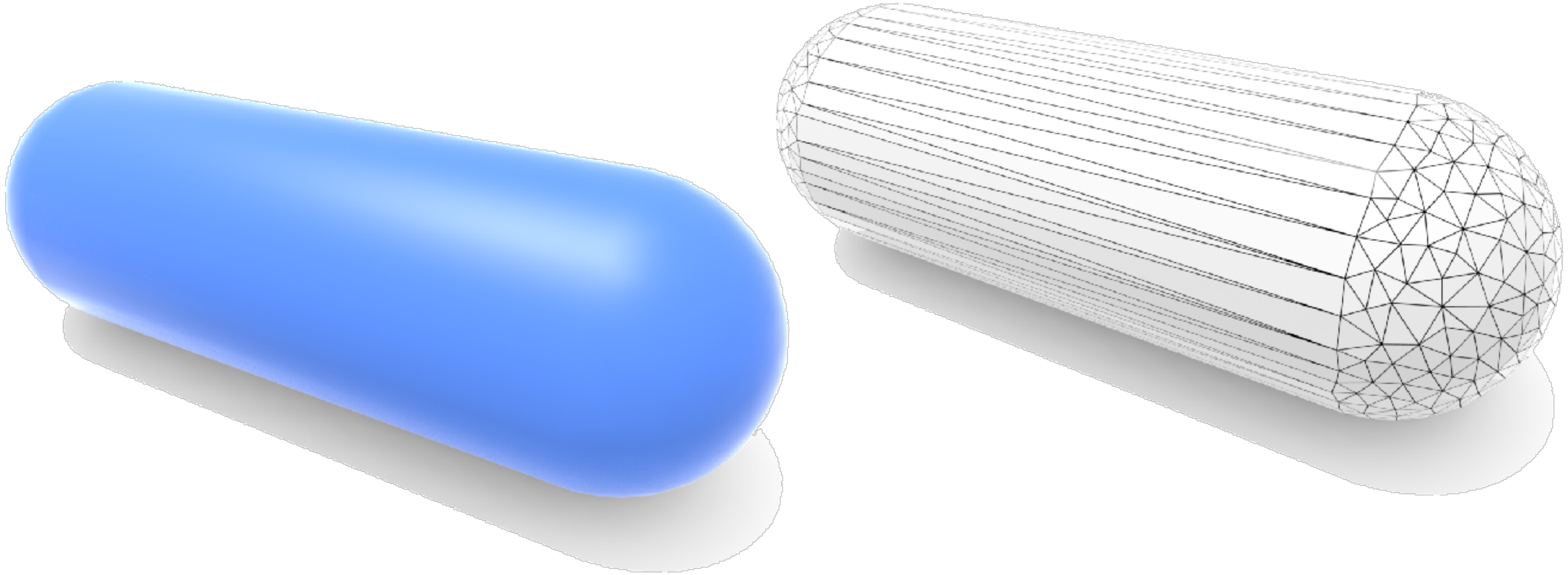
Point Cloud (Explicit)

- Easiest representation: list of points (x,y,z)
- Often augmented with *normals*
- Easily represent any kind of geometry
- Easy to draw dense cloud (>>1 point/pixel)
- Hard to interpolate undersampled regions
- Hard to do processing / simulation / …



Polygon Mesh (Explicit)

- Store vertices *and* polygons (most often triangles or quads)
- Easier to do processing/simulation, adaptive sampling
- More complicated data structures
- Irregular neighborhoods



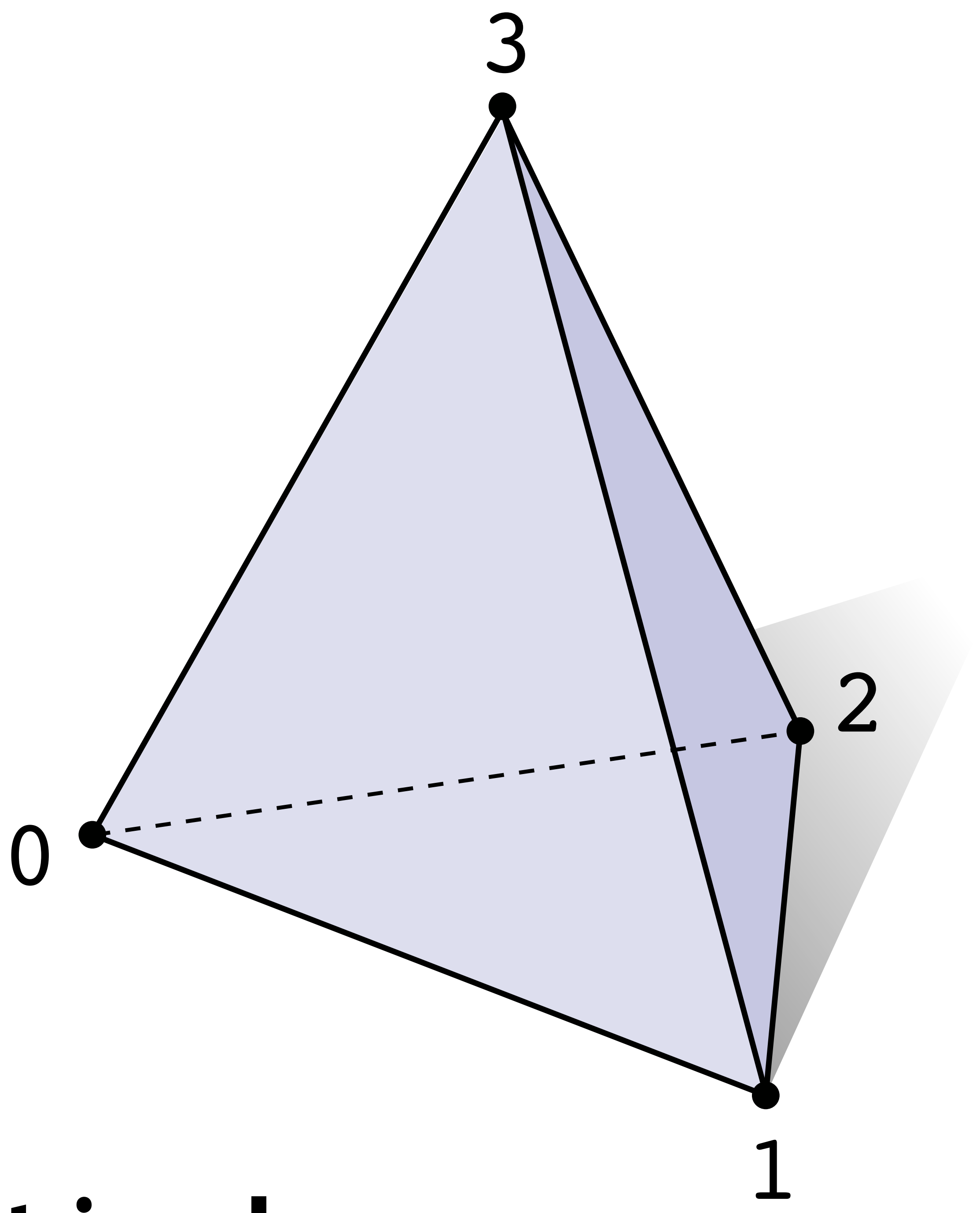
(Much more about polygon meshes in upcoming lectures!)

Triangle Mesh (Explicit)

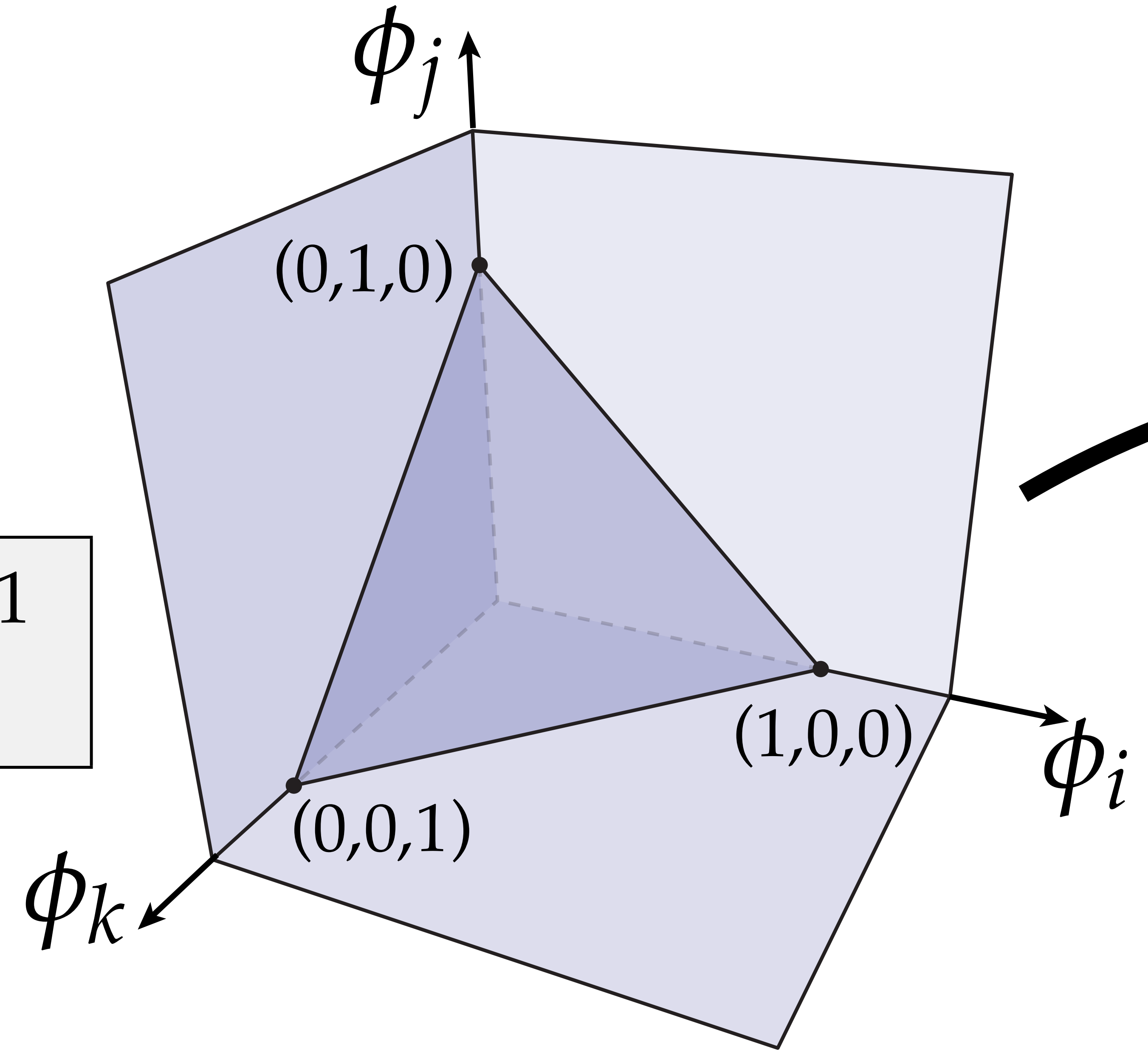
- **Store vertices as triples of coordinates (x,y,z)**
- **Store triangles as triples of indices (i,j,k)**
- **E.g., tetrahedron:**

VERTICES	x	y	z
0:	−1	−1	−1
1:	1	−1	1
2:	1	1	−1
3:	−1	1	1

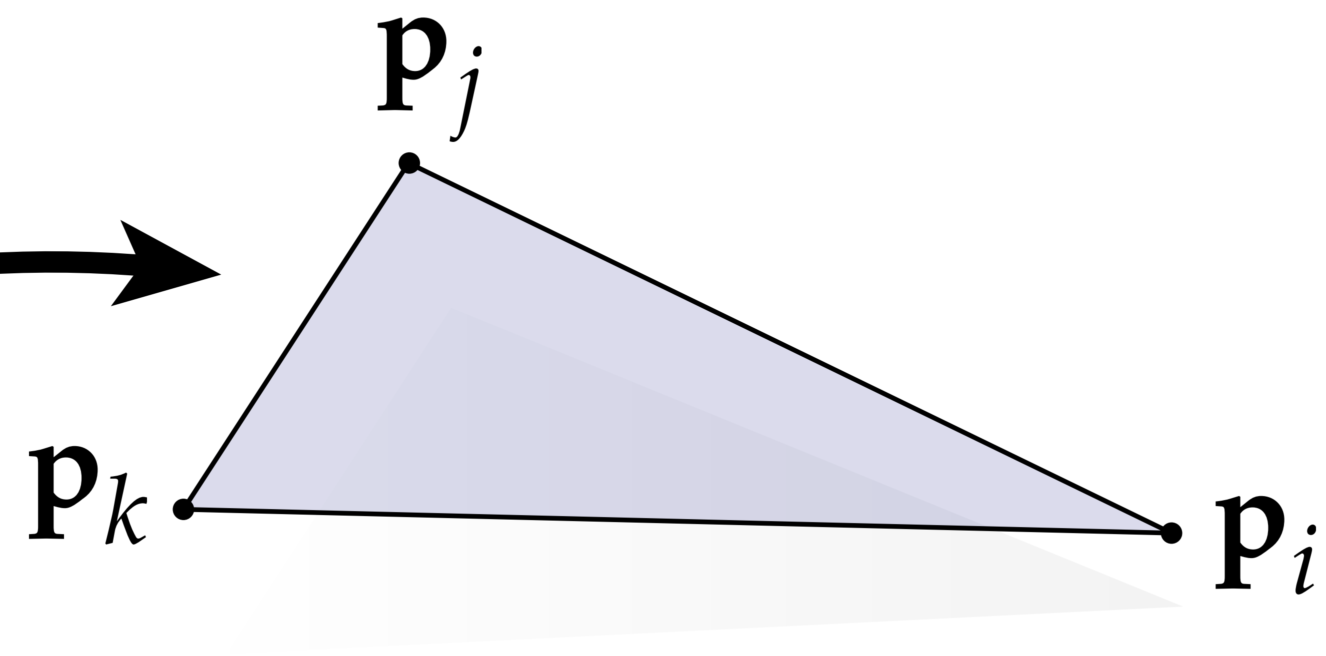
TRIANGLES i	j	k
0	2	1
0	3	2
3	0	1
3	1	2



- **Use barycentric interpolation to define points inside triangles:**



$$\phi_i + \phi_j + \phi_k = 1$$
$$\phi_i, \phi_j, \phi_k > 0$$



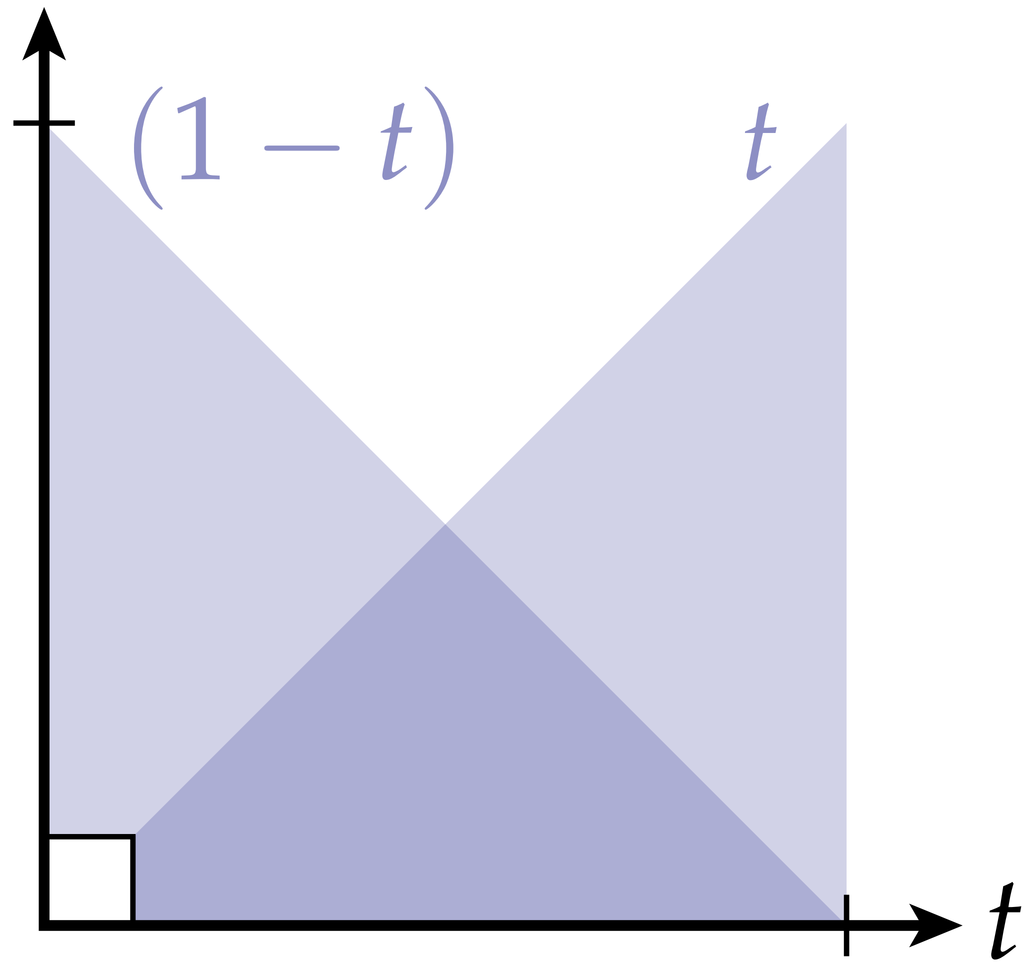
$$p = \phi_i \mathbf{p}_i + \phi_j \mathbf{p}_j + \phi_k \mathbf{p}_k$$

Recall: Linear Interpolation (1D)

- **Interpolate values using *linear interpolation*; in 1D:**

$$\hat{f}(t) = (1 - t)f_i + t f_j$$

- **Can think of this as a linear combination of two functions:**



- **Why limit ourselves to <u>linear</u> basis functions?**
- **Can we get more interesting geometry with other bases?**

Bernstein Basis

- Linear interpolation essentially uses 1st-order polynomials
- Provide more flexibility by using higher-order polynomials
- Instead of usual basis (1, x, x^2, x^3, ...), use Bernstein basis:

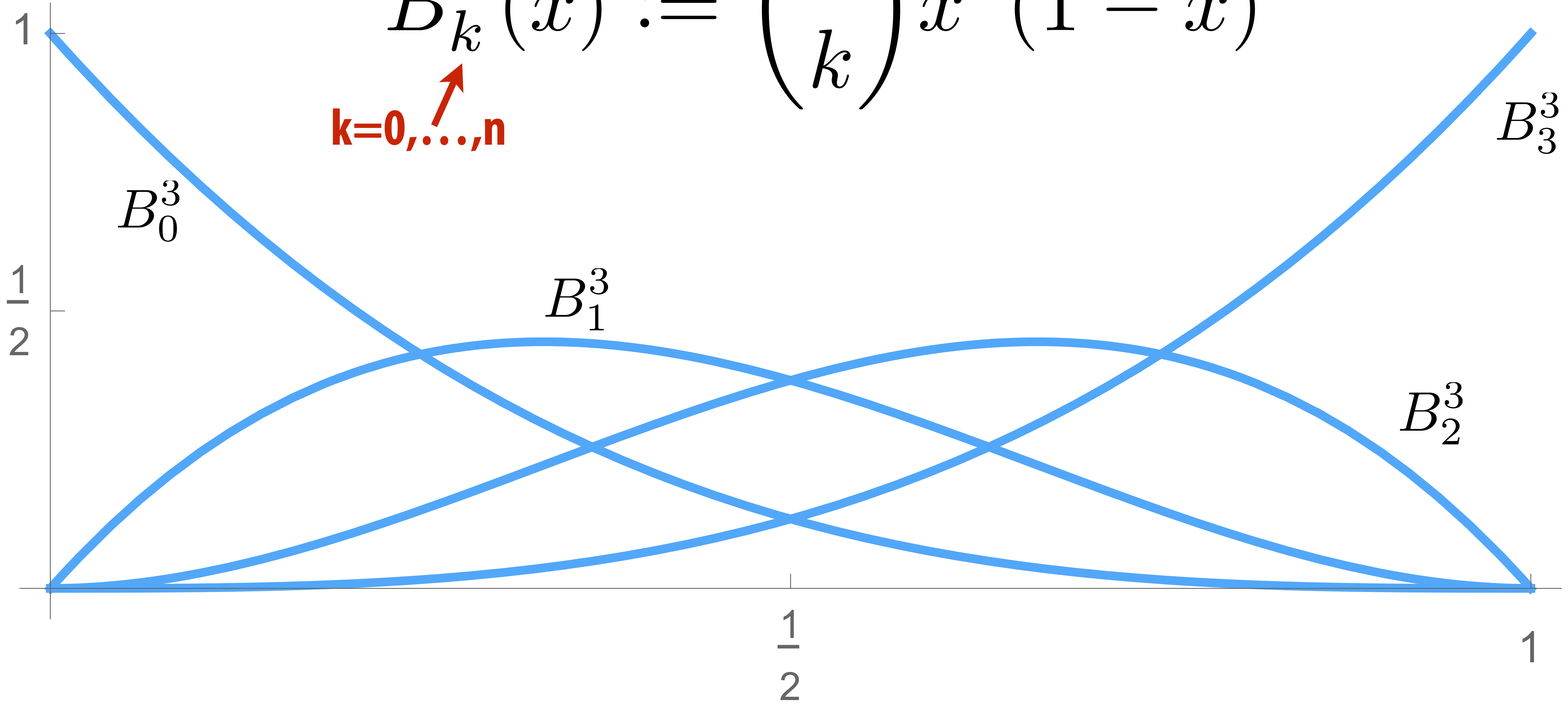
degree

$0 \leq x \leq 1$

"n choose k"

$$B_k^n(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

$k=0,\ldots,n$



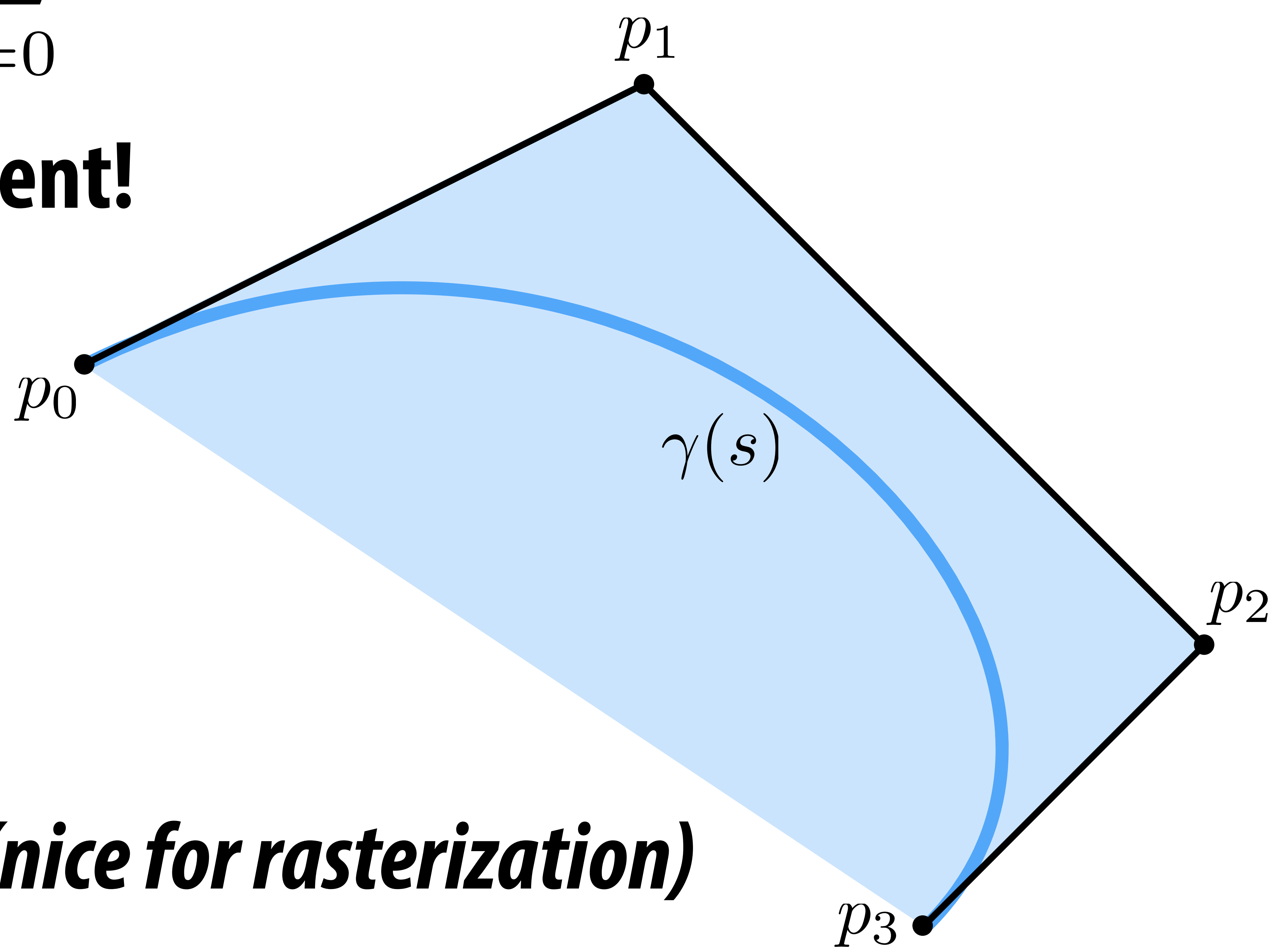
Bézier Curves (Explicit)

- A Bézier curve is a curve expressed in the Bernstein basis:

$$\gamma(s) := \sum_{k=0}^{n} B_{n,k}(s)\,p_k$$

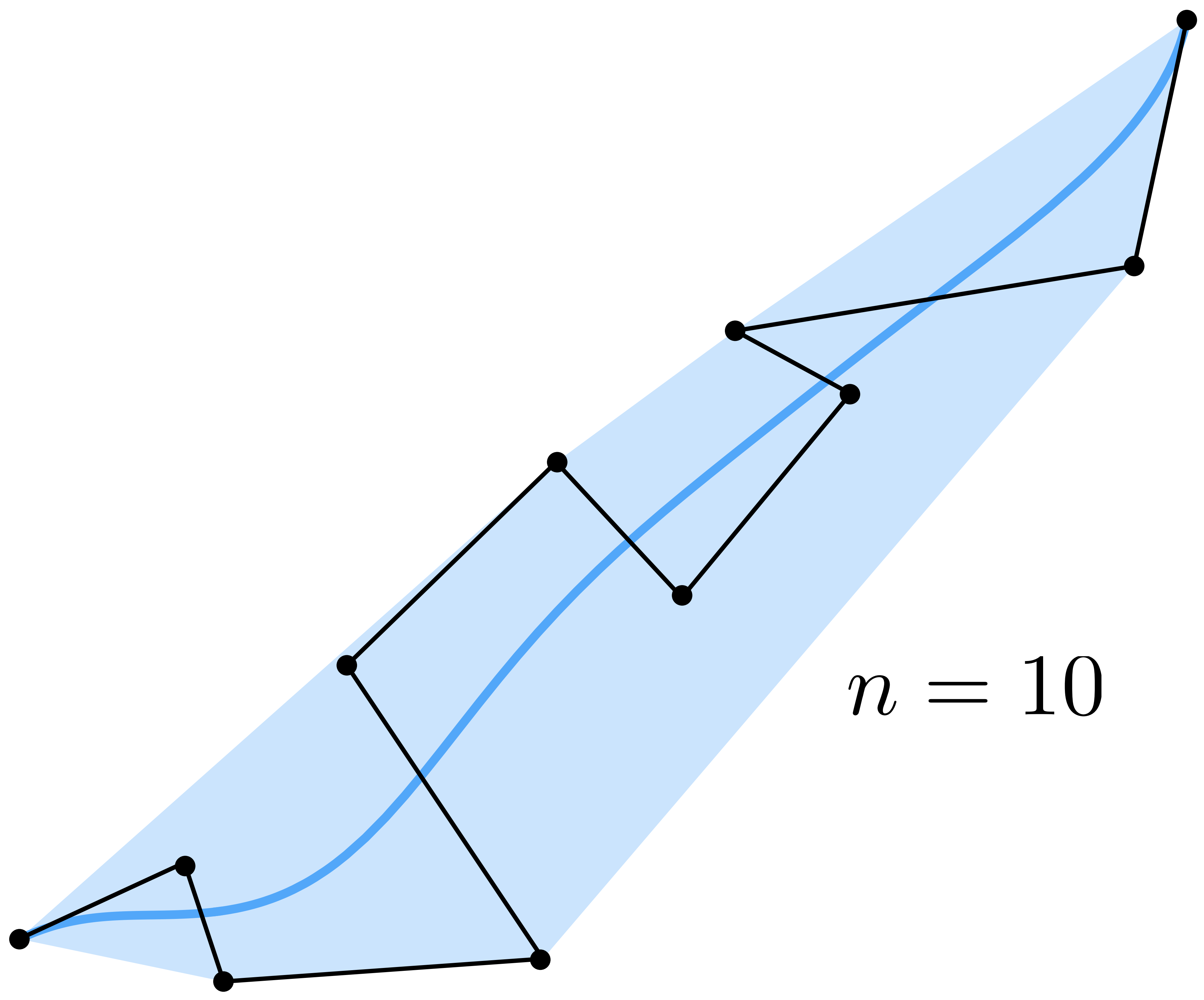
control points

- For n=1, just get a line segment!
- For n=3, get "cubic Bézier":
- Important features:
  1. interpolates endpoints
  2. tangent to end segments
  3. contained in convex hull *(nice for rasterization)*



Just keep going…?

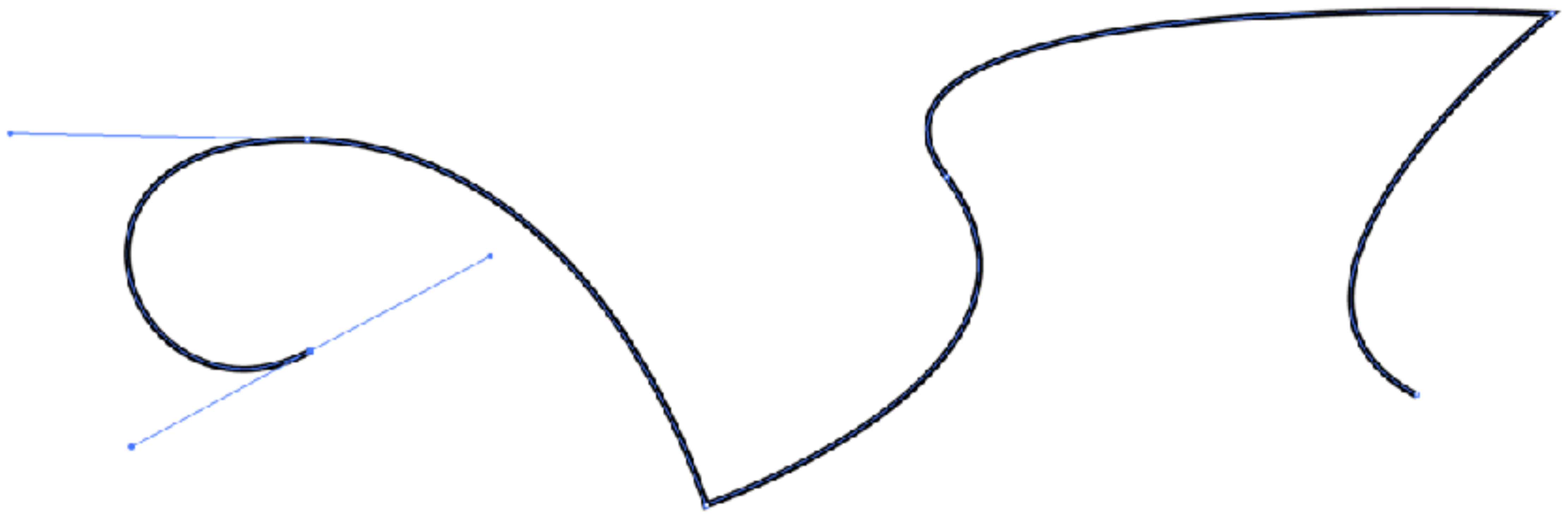
- What if we want an even more interesting curve?
- High-degree Bernstein polynomials don't interpolate well:

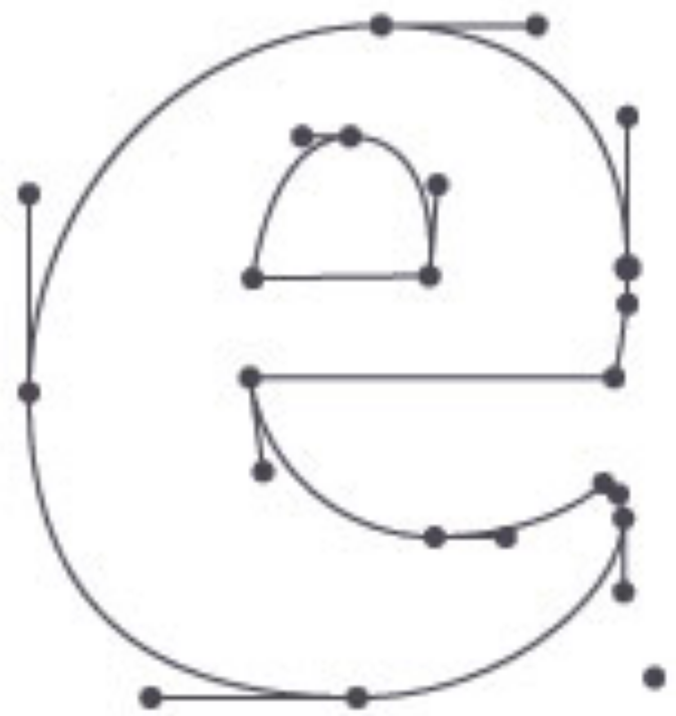


Very hard to control!

Piecewise Bézier Curves (Explicit)

- **Alternative idea: piece together many Bézier curves**
- **Widely-used technique (Illustrator, fonts, SVG, etc.)**





- **Formally, piecewise Bézier curve:**

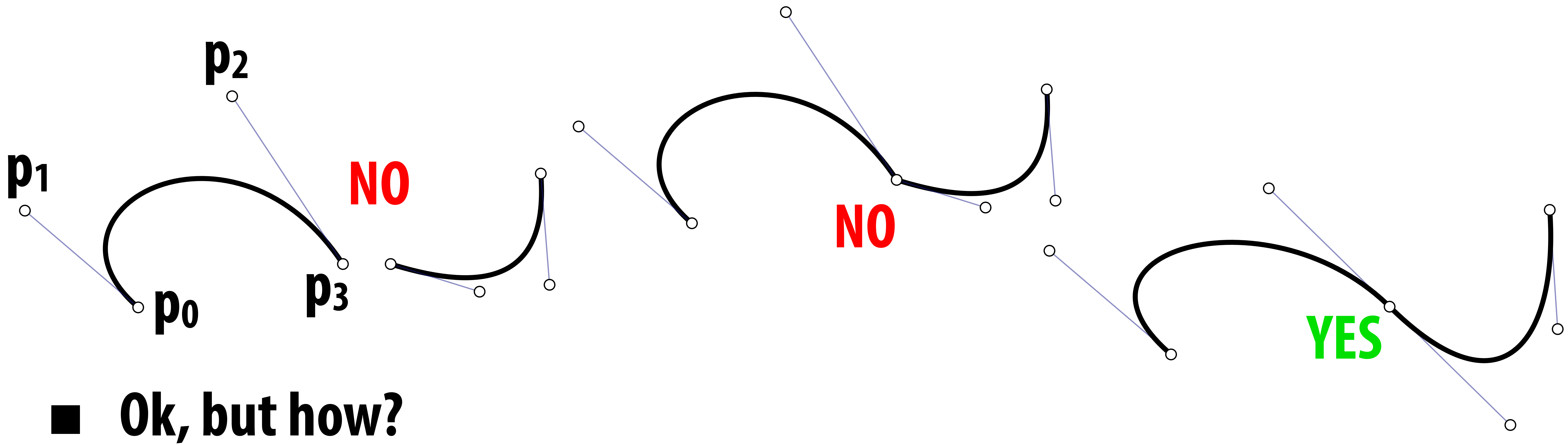
piecewise Bézier

$$\gamma(u) := \gamma_i\left(\frac{u - u_i}{u_{i+1} - u_i}\right), \qquad u_i \leq u < u_{i+1}$$

single Bézier

Bézier Curves — tangent continuity

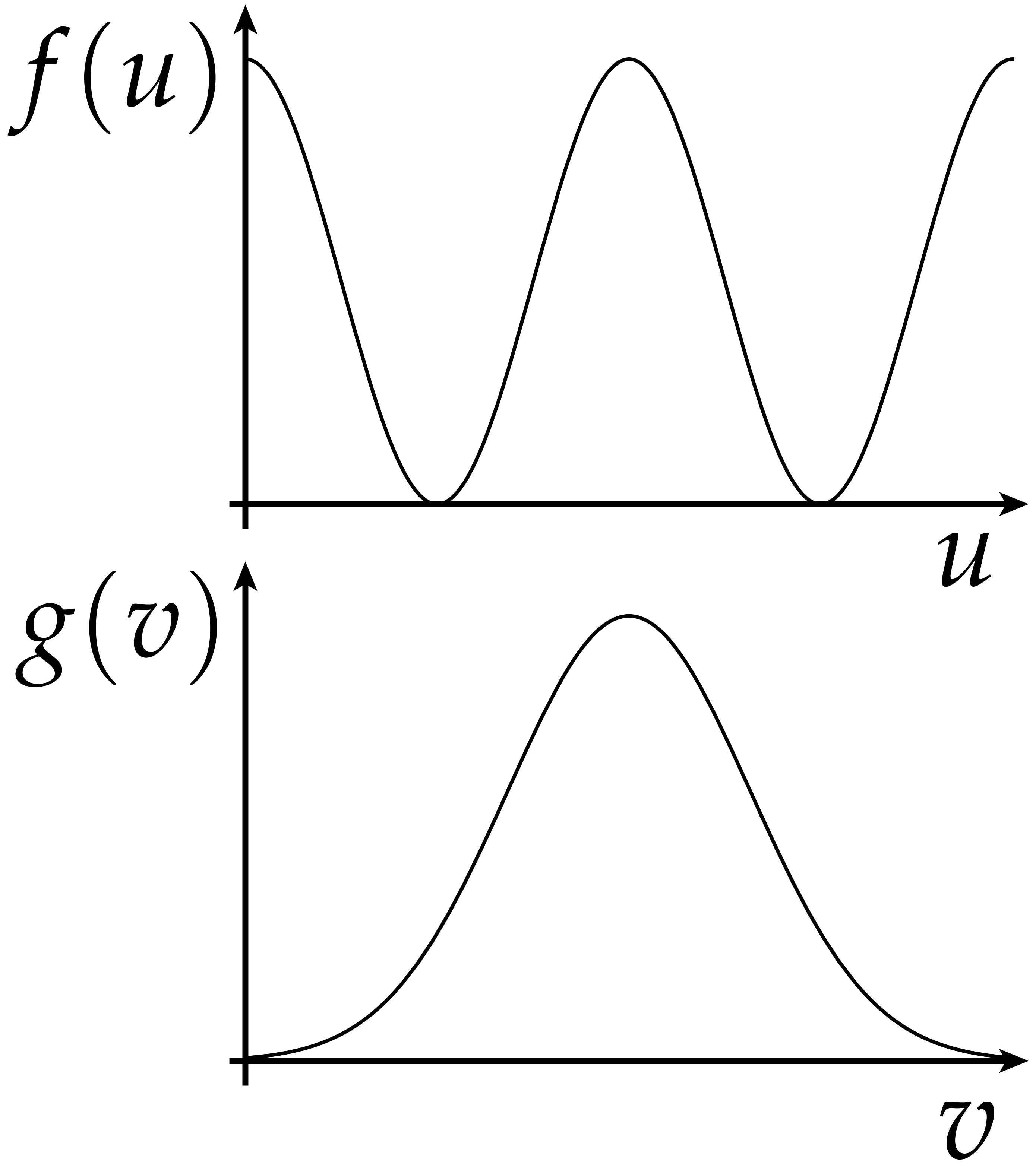
- To get "seamless" curves, need *points* <u>and</u> *tangents* to line up:

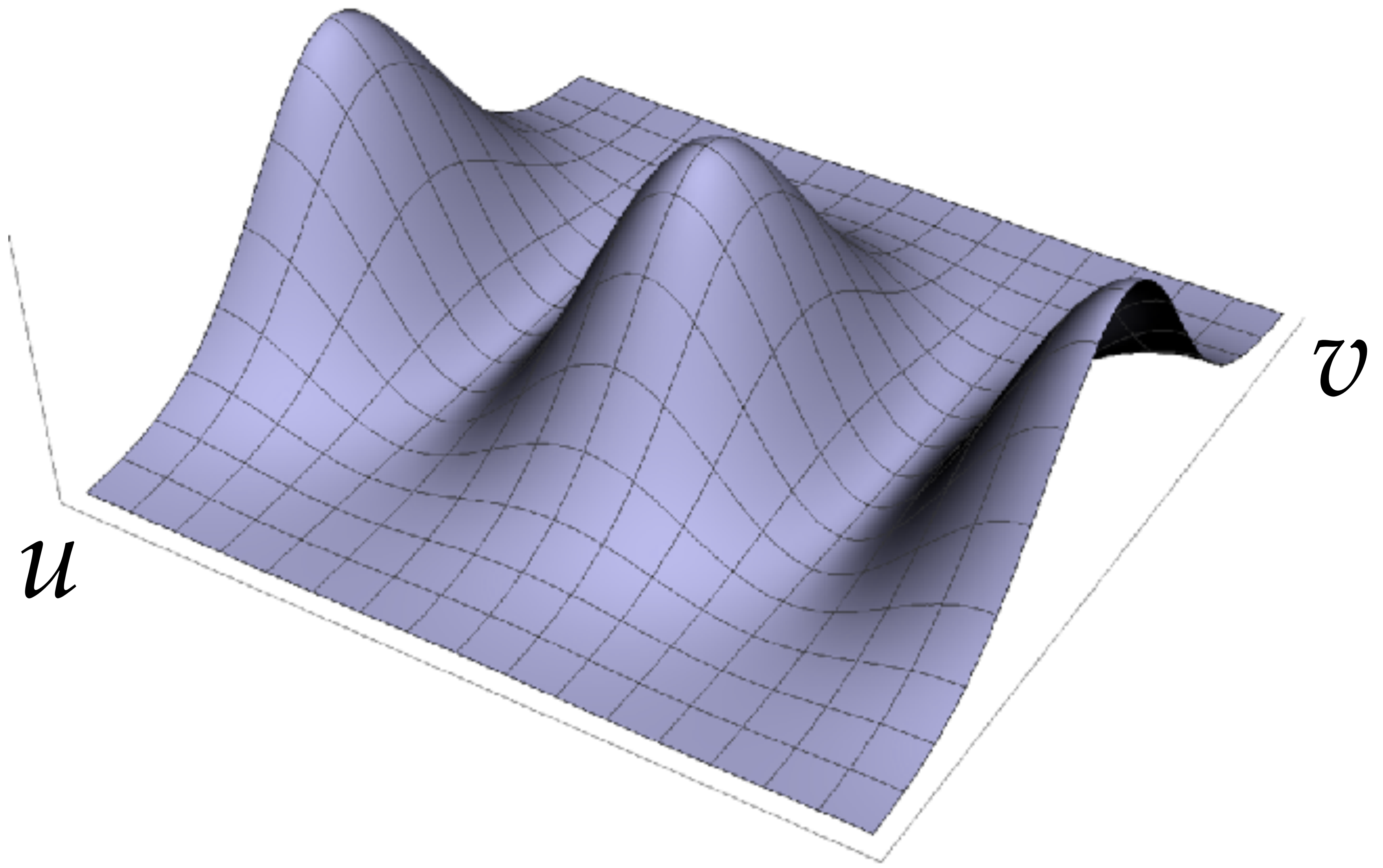


- Ok, but how?
- Each curve is cubic: $u^3p_0 + 3u^2(1\text{-}u)p_1 + 3u(1\text{-}u)^2p_2 + (1\text{-}u)^3p_3$
- Want endpoints of each segment to meet
- Want tangents at endpoints to meet
- Q: Could you do this with *quadratic* Bézier? *Linear* Bézier?

Tensor Product

- **Can use a pair of curves to get a surface**
- **Value at any point (u,v) given by product of a curve f at u and a curve g at v (sometimes called the *"tensor product"*):**

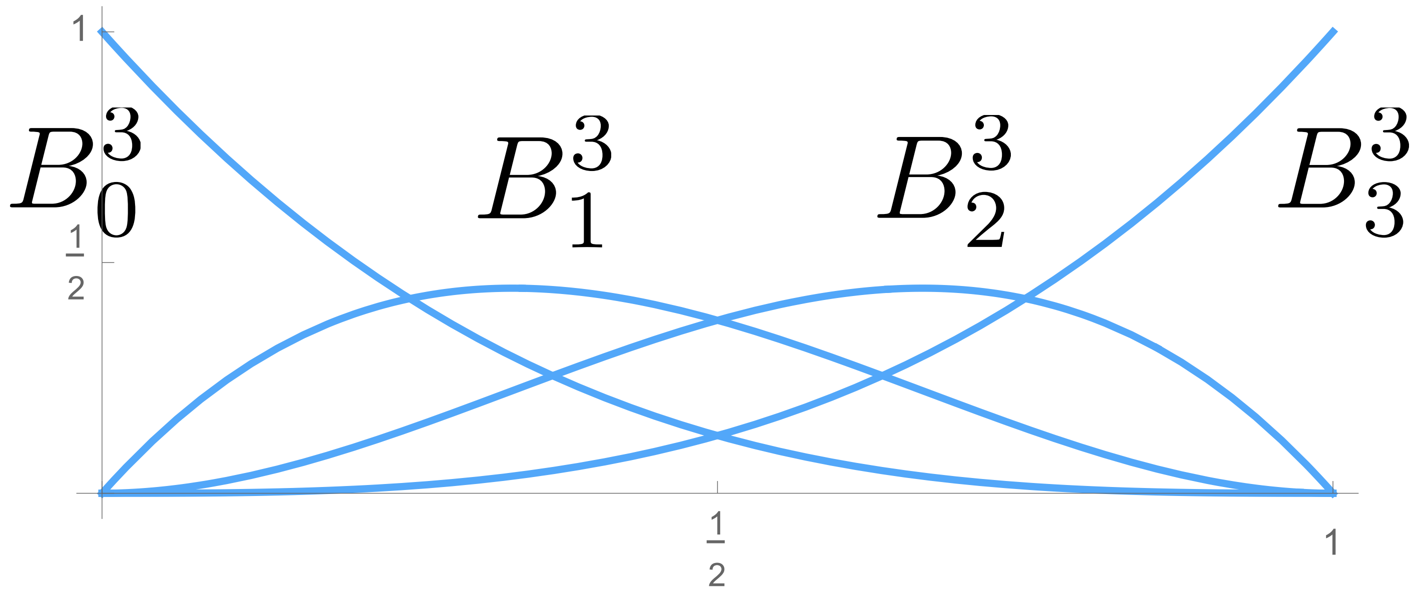


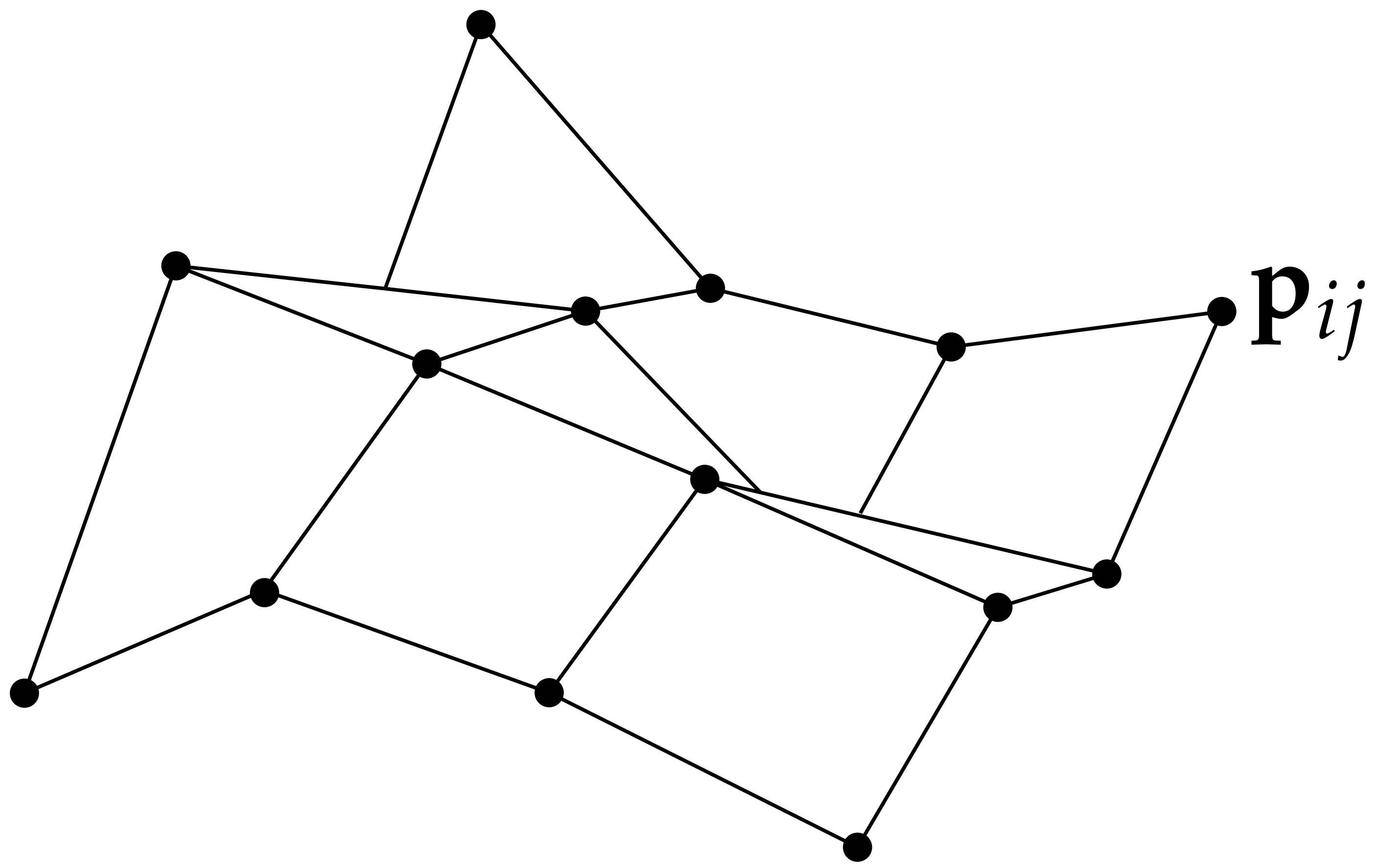


$$(f \otimes g)(u,v) := f(u)g(v)$$

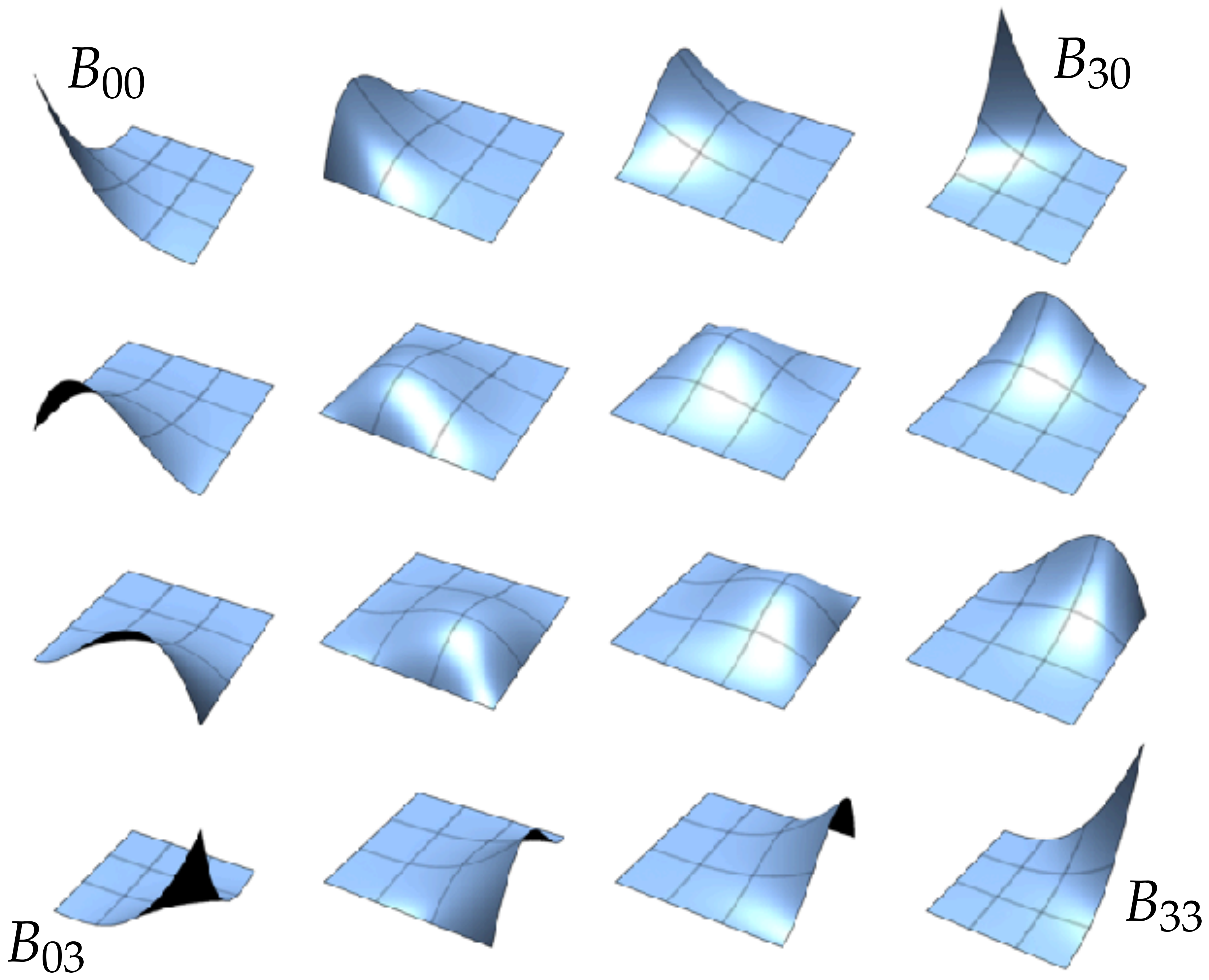
Bézier Patches

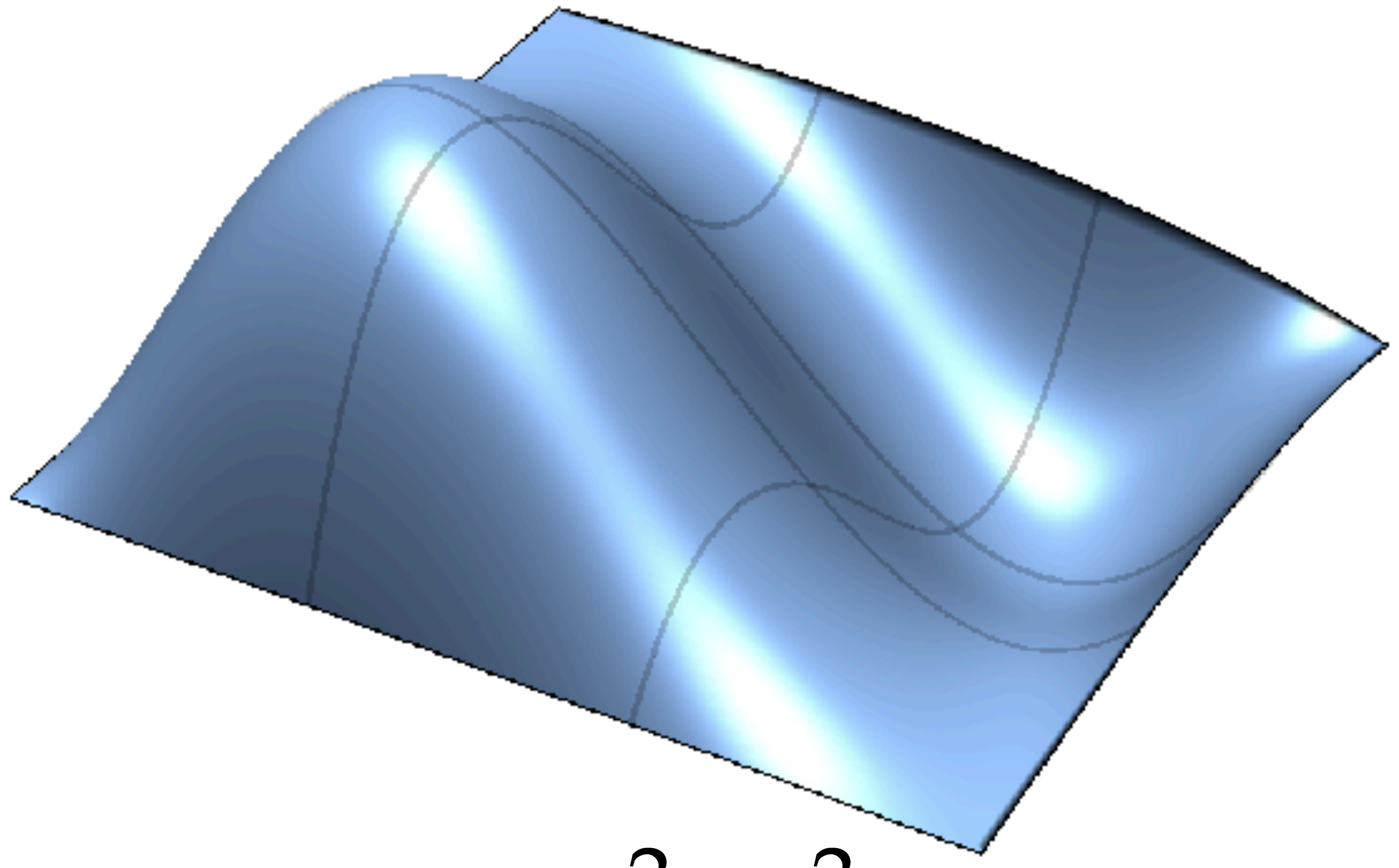
- ***Bézier patch*** **is sum of (tensor) products of Bernstein bases**





$$B^3_{i,j}(u,v) := B^3_i(u)B^3_j(v)$$

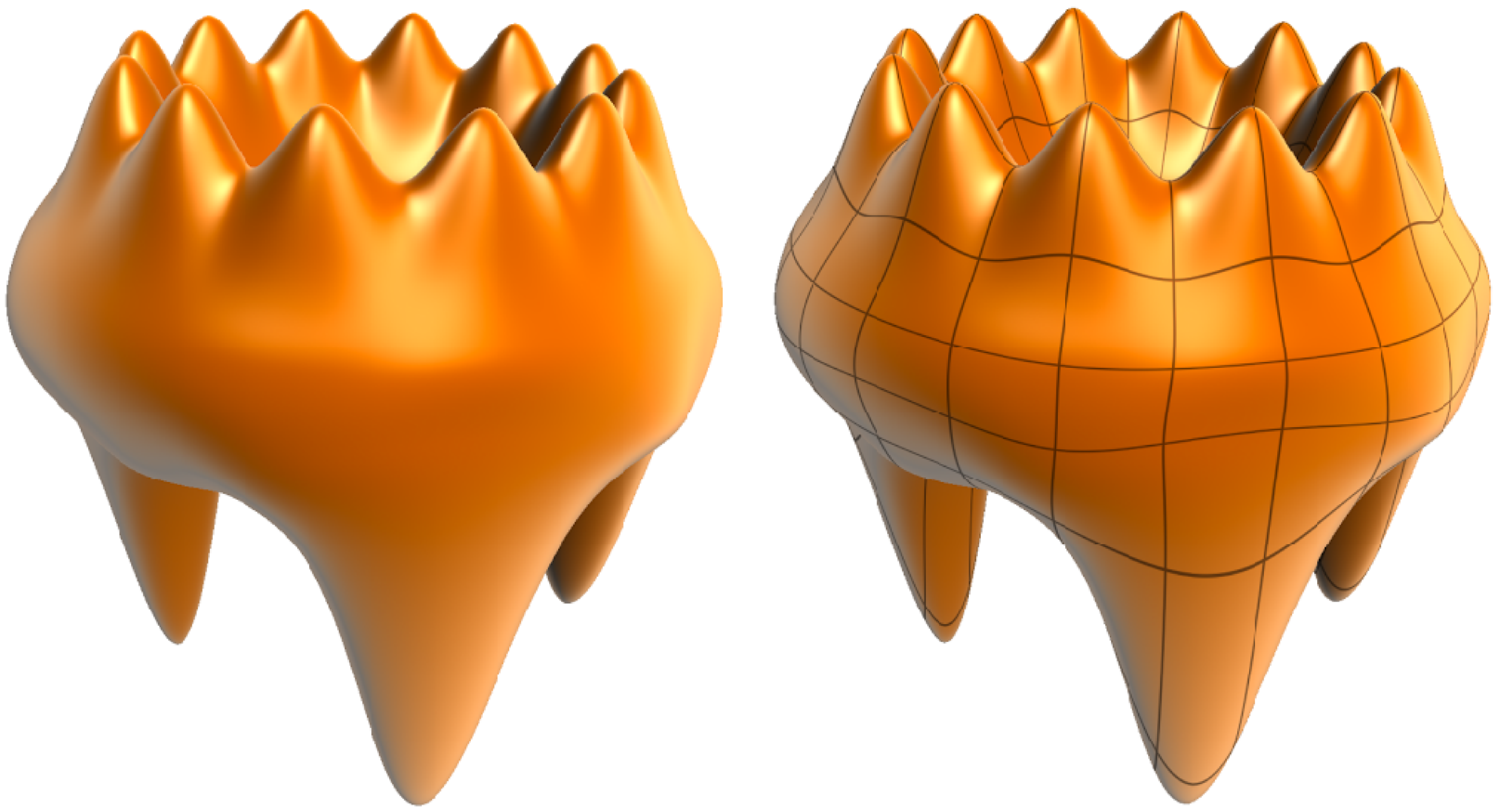




$$S(u,v) := \sum_{i=0}^{3}\sum_{j=0}^{3} B^3_{i,j}(u,v)\,\mathbf{p}_{ij}$$

Bézier Surface

- Just as we connected Bézier *curves*, can connect Bézier *patches* to get a surface:



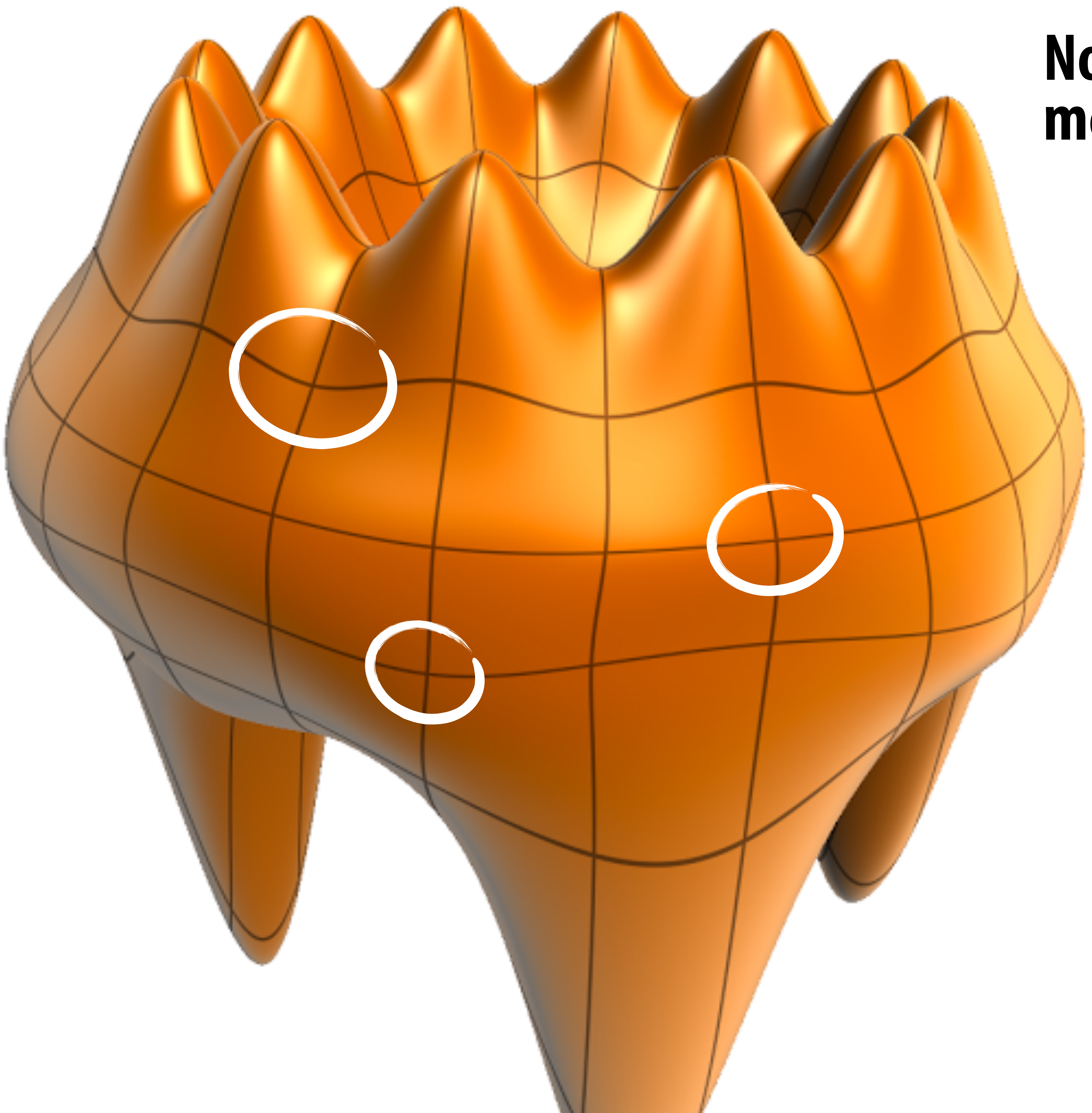
- *Very* easy to draw: just dice each patch into regular (u,v) grid!

Q: Can we always get tangent continuity?

(Think: how many constraints? How many degrees of freedom?)

Notice anything fishy about the last picture?

Bézier Patches are Too Simple

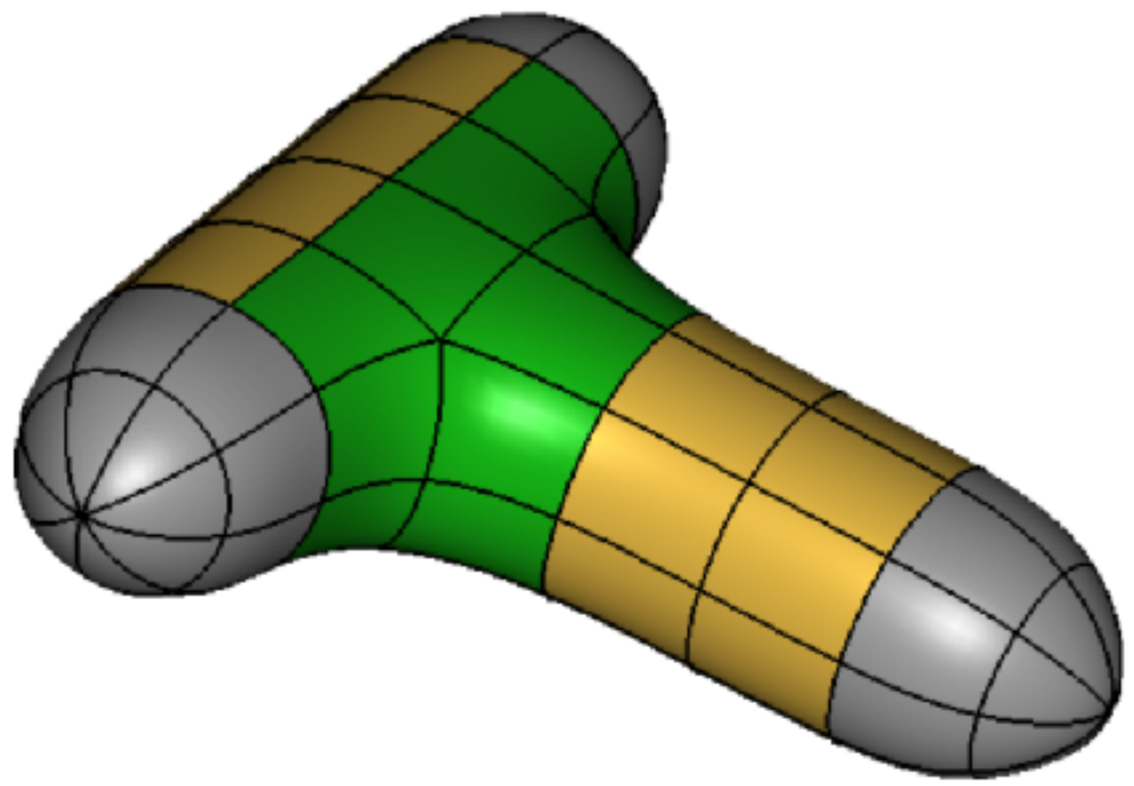


Notice that exactly four patches meet around *every* vertex!

In practice, far too constrained.

To make interesting shapes (with good continuity), we need patches that allow more interesting connectivity...

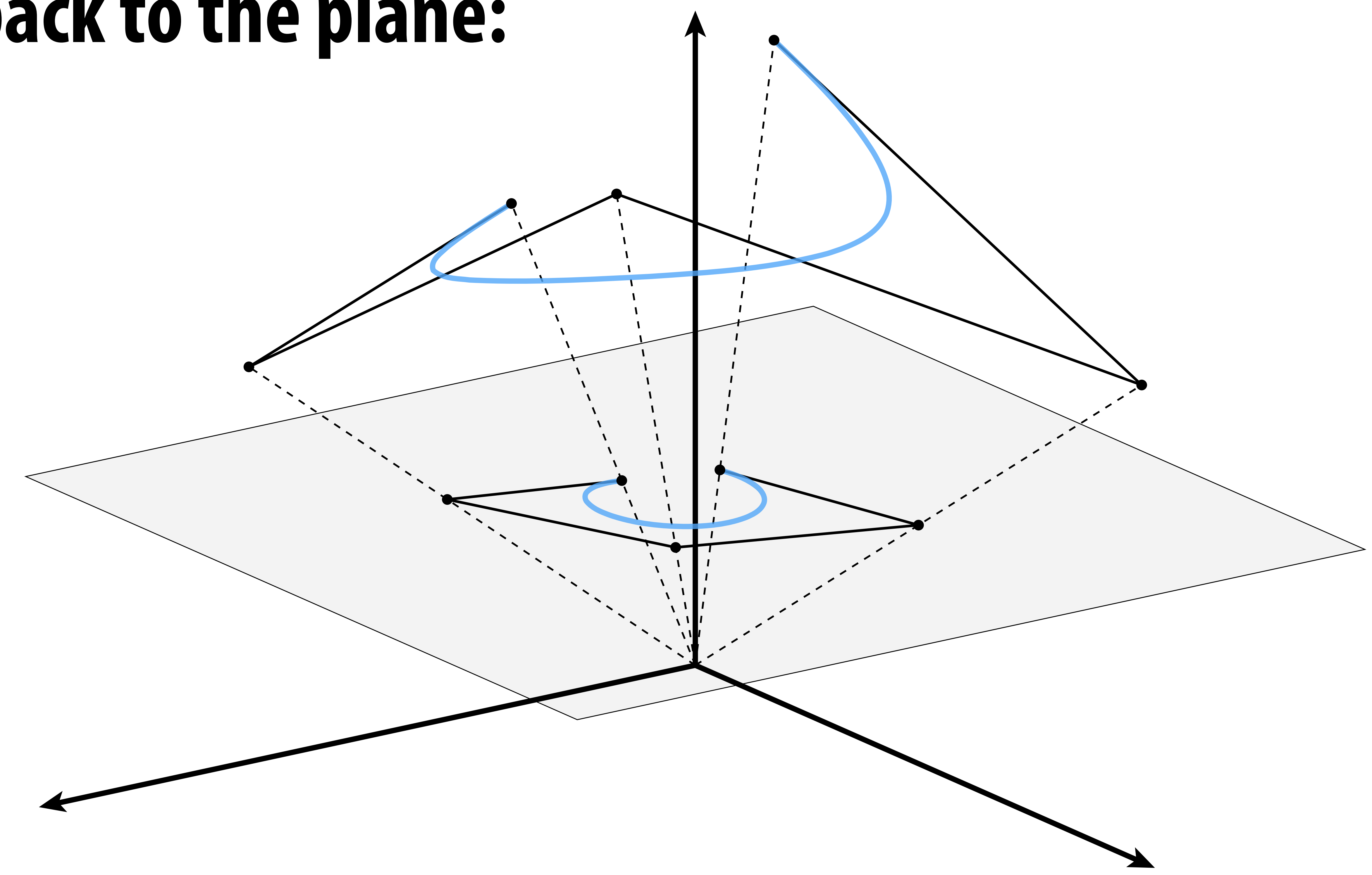
Spline patch schemes



- There are *many* alternatives!
- NURBS, Gregory, Pm, polar…
- Tradeoffs:
  - degrees of freedom
  - continuity
  - difficulty of editing
  - cost of evaluation
  - generality
  - …
- As usual: *pick the right tool for the job!*

Rational B-Splines (Explicit)

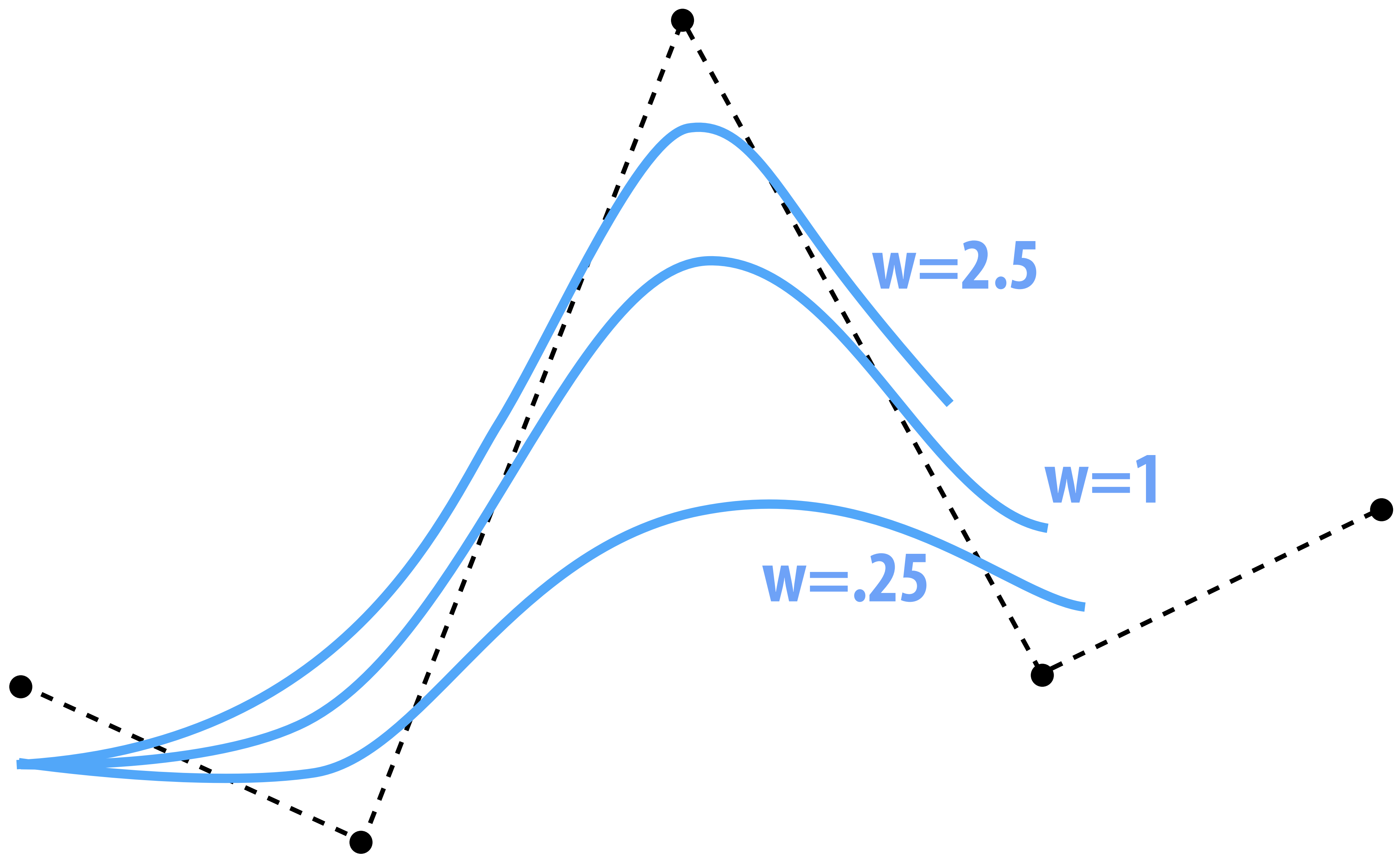
- **Bézier can't exactly represent *conics*—not even the circle!**
- **Solution: interpolate in homogeneous coordinates, then project back to the plane:**



Result is called a *rational* B-spline.

NURBS (Explicit)

- (N)on-(U)niform (R)ational (B)-(S)pline
  - knots at arbitrary locations (non-uniform)
  - expressed in homogeneous coordinates (rational)
  - piecewise polynomial curve (B-Spline)
- Homogeneous coordinate w controls “strength” of a vertex:

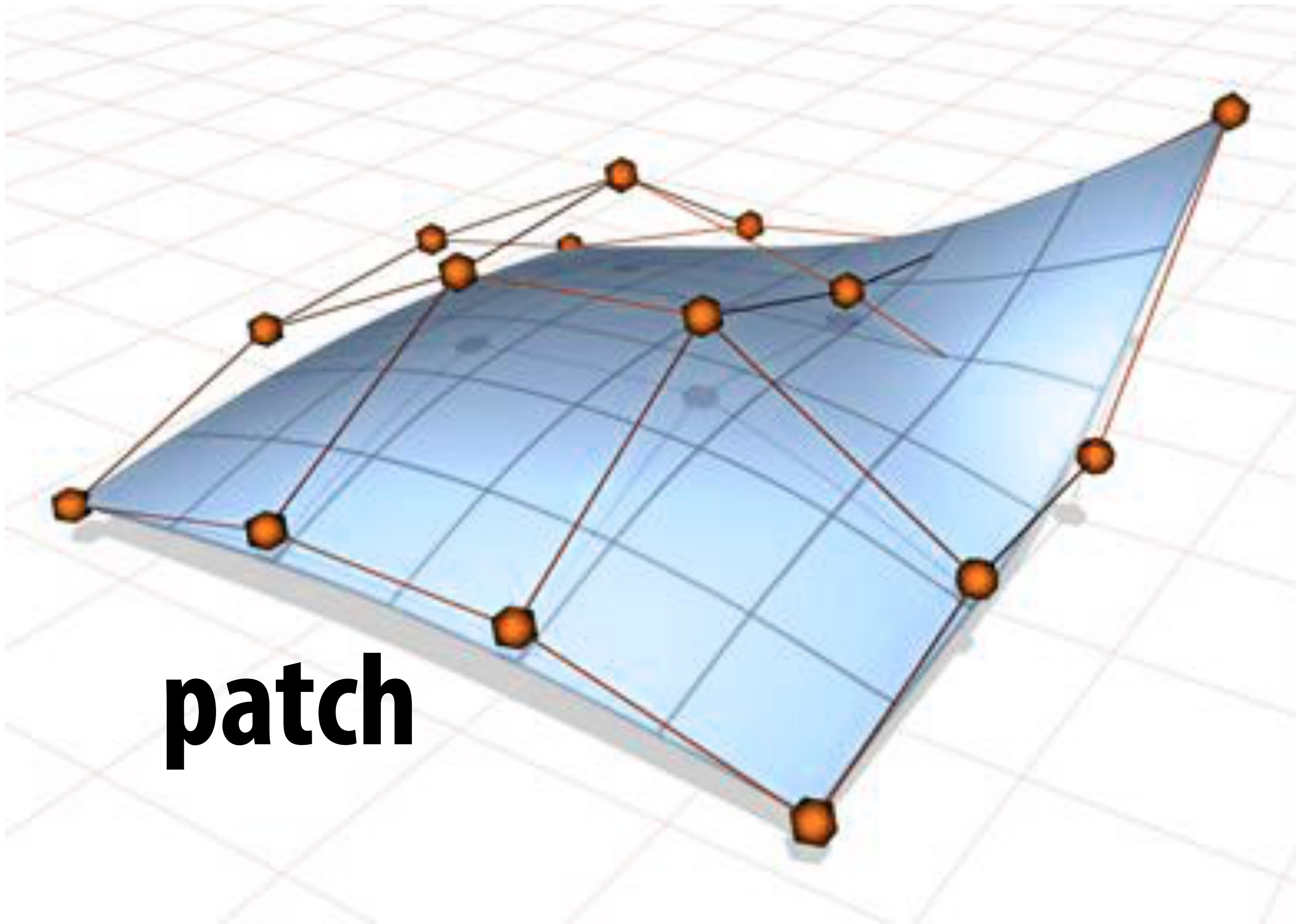


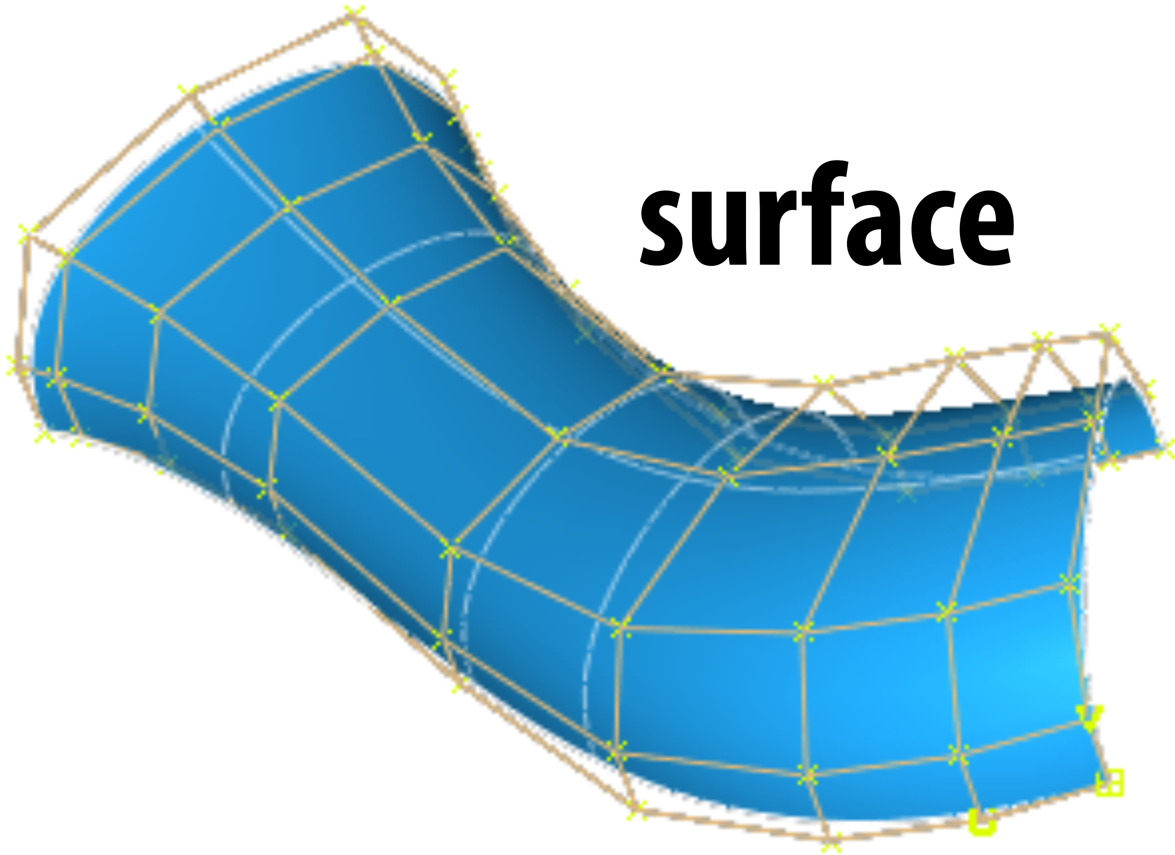
NURBS Surface (Explicit)

- **How do we go from curves to surfaces?**
- **Use *tensor product* of NURBS curves to get a patch:**

$$S(u,v) := N_i(u)N_j(v)p_{ij}$$

- **Multiple NURBS patches form a surface**

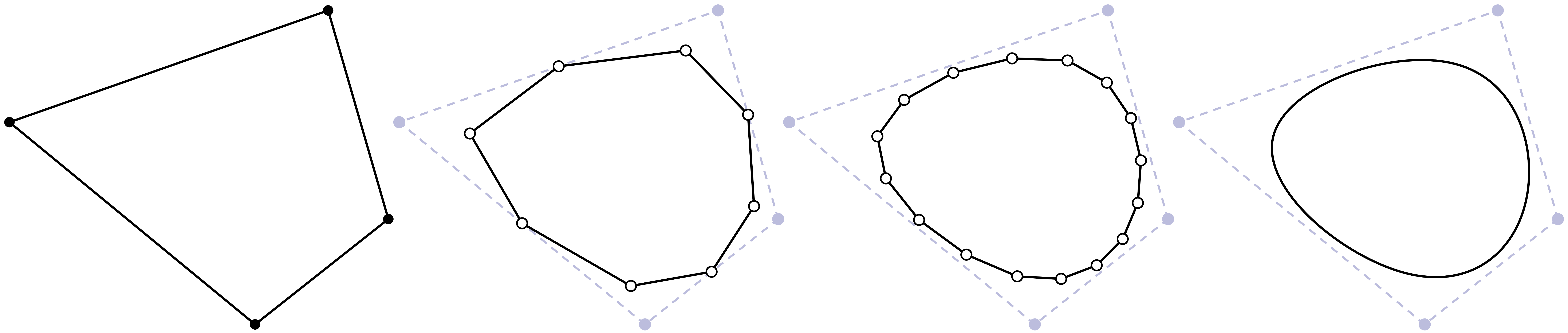




- **Pros: easy to evaluate, exact conics, high degree of continuity**
- **Cons: Hard to piece together patches / hard to edit (many DOFs)**

Subdivision

- Alternative starting point for curves/surfaces: *subdivision*
- Start with "control curve"
- Repeatedly split, take weighted average to get new positions
- For careful choice of averaging rule, approaches nice limit curve
  - *Often exact same curve as well-known spline schemes!*



Q: Is subdivision an explicit or implicit representation?

Subdivision—Example

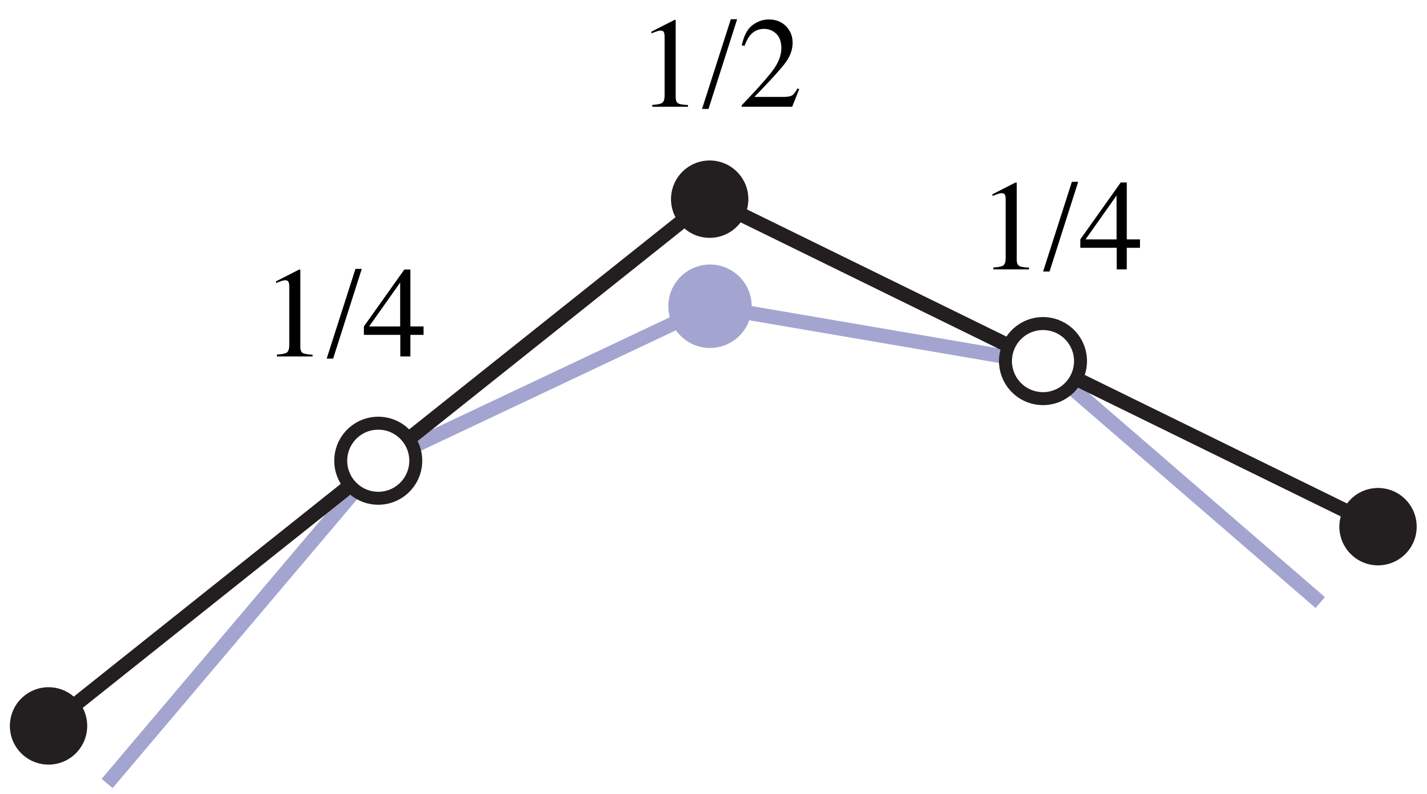
- **One possible scheme: *Lane-Riesenfeld***
  - **insert midpoint of each edge**
  - **use row k of Pascal's triangle (normalized to 1) as weights for neighbors**
  - **e.g., $k = 2$, get weights** $(1/4, 1/2, 1/4)$
  - **limit is B-spline of degree $k + 1$**

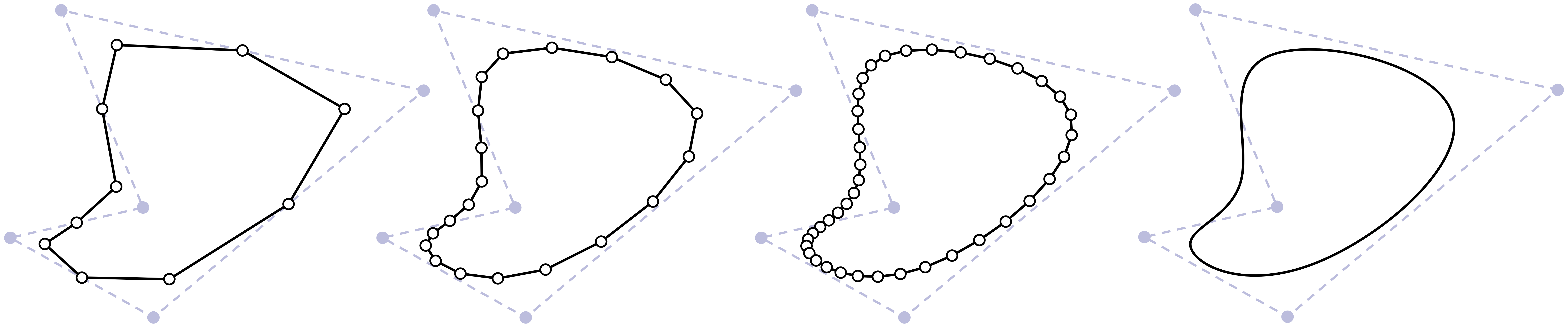
$k = 0:$ 1

$k = 1:$ 1 1

$k = 2:$ 1 2 1

$k = 3:$ 1 3 3 1

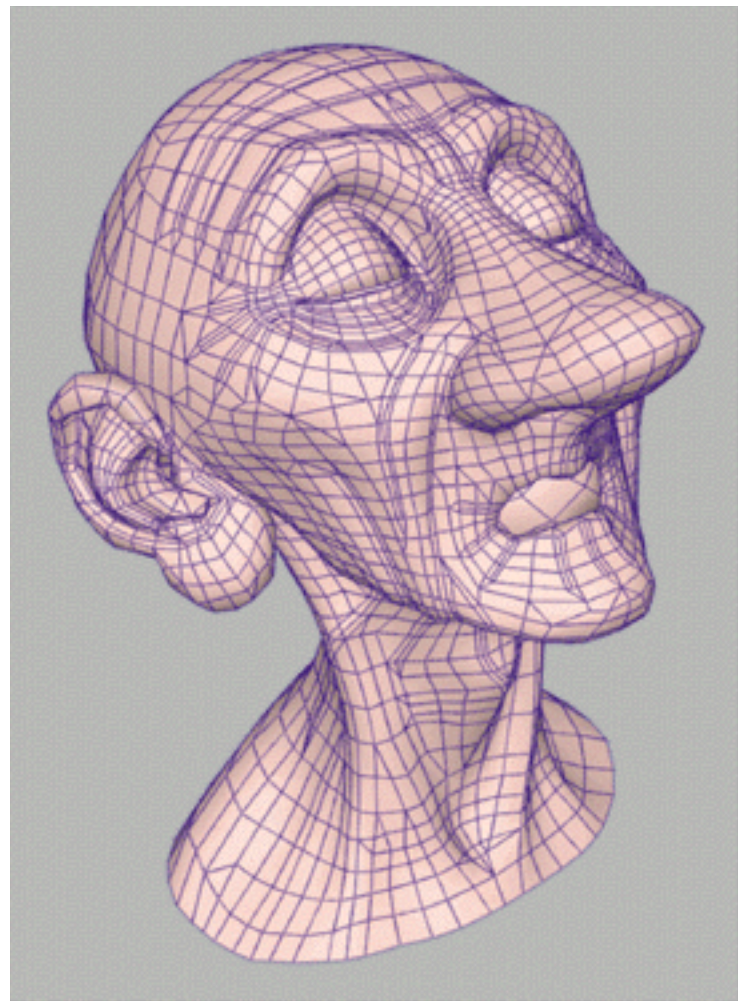


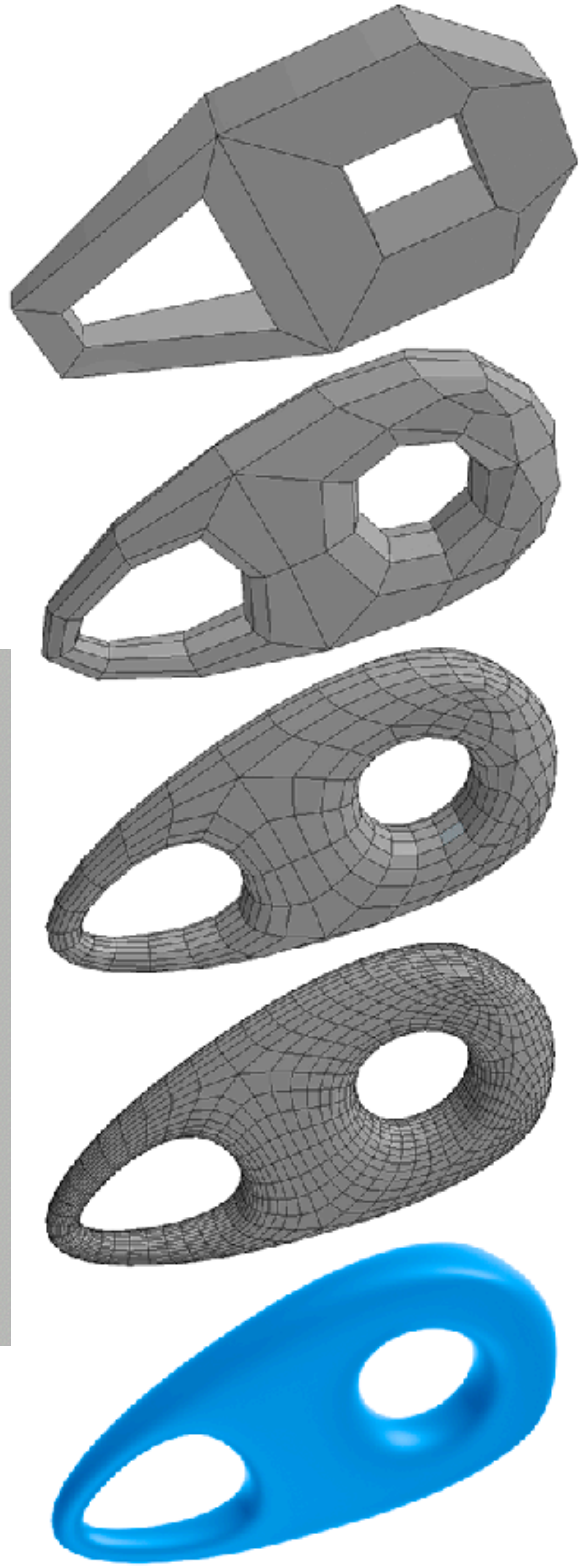


iteration 1 **iteration 2** **iteration 3** **limit curve**

Subdivision Surfaces (Explicit)

- Start with coarse polygon mesh ("control cage")
- Subdivide each element
- Update vertices via local averaging
- Many possible rules:
  - *Catmull-Clark* (quads)
  - *Loop* (triangles)
  - ...
- Common issues:
  - interpolating or approximating?
  - continuity at vertices?
- Easier than splines for modeling; harder to evaluate pointwise
- Widely used in practice (2019 Academy Awards!)





Subdivision in Action (Pixar's "Geri's Game")

see: de Rose et al, "Subdivision Surfaces in Character Animation"

Next time: Curves, Surfaces, & Meshes