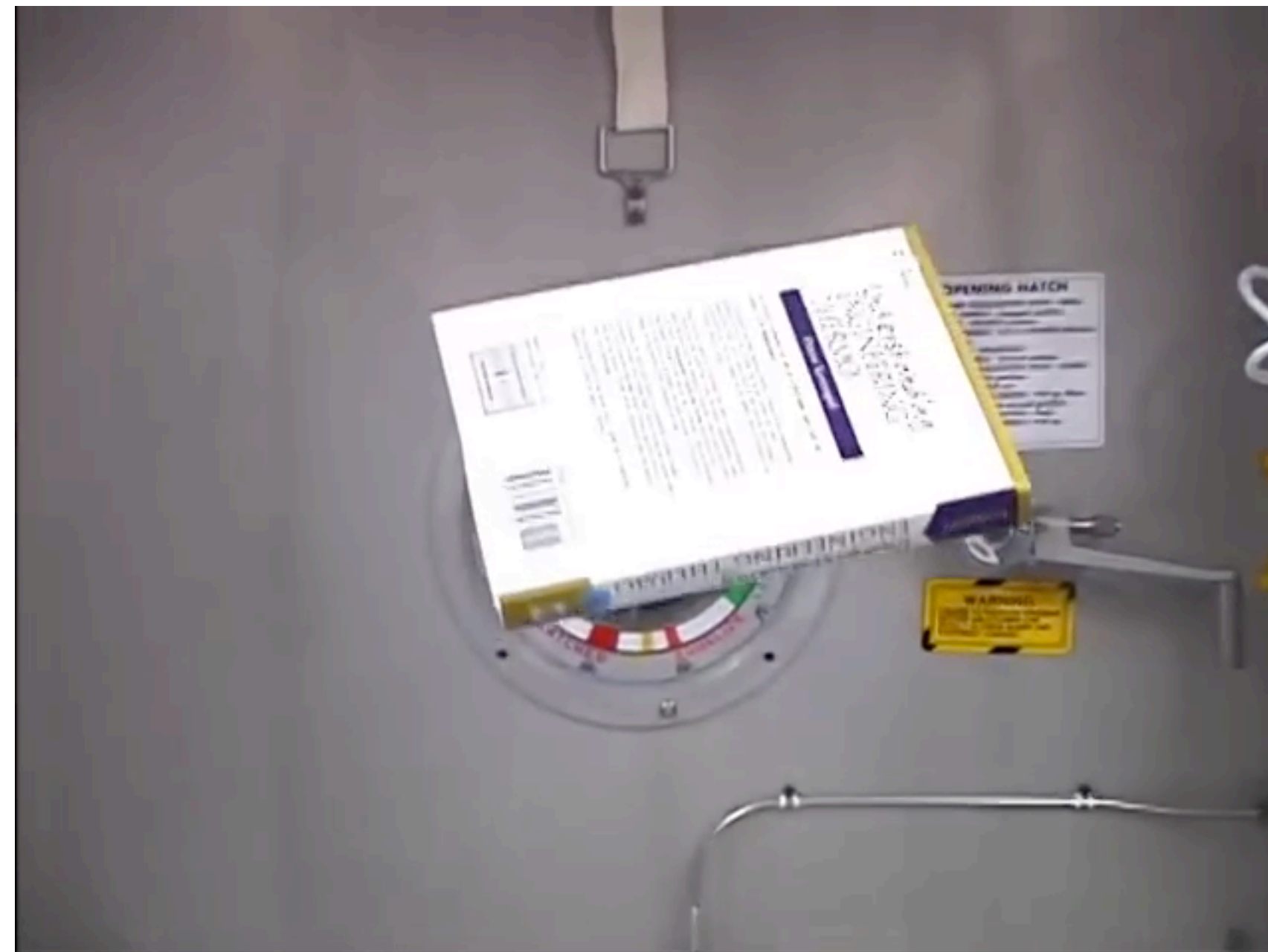


3D Rotations and Complex Representations

Computer Graphics
CMU 15-462/15-662

Rotations in 3D

- **What is a rotation, intuitively?**
- ***How do you know a rotation when you see it?***
 - **length/distance is preserved (no stretching/shearing)**
 - **orientation is preserved (e.g., text remains readable)**
 - **origin is preserved (otherwise it's a rotation + translation)**



3D Rotations—Degrees of Freedom

- How many numbers do we need to specify a rotation in 3D?
- For instance, we could use rotations around X, Y, Z . But do we *need* all three?
- Well, to rotate Pittsburgh to another city (say, São Paulo), we have to specify two numbers: latitude & longitude:
- Do we really need both latitude and longitude? Or will one suffice?
- Is that the *only* rotation from Pittsburgh to São Paulo? (How many more numbers do we need?)

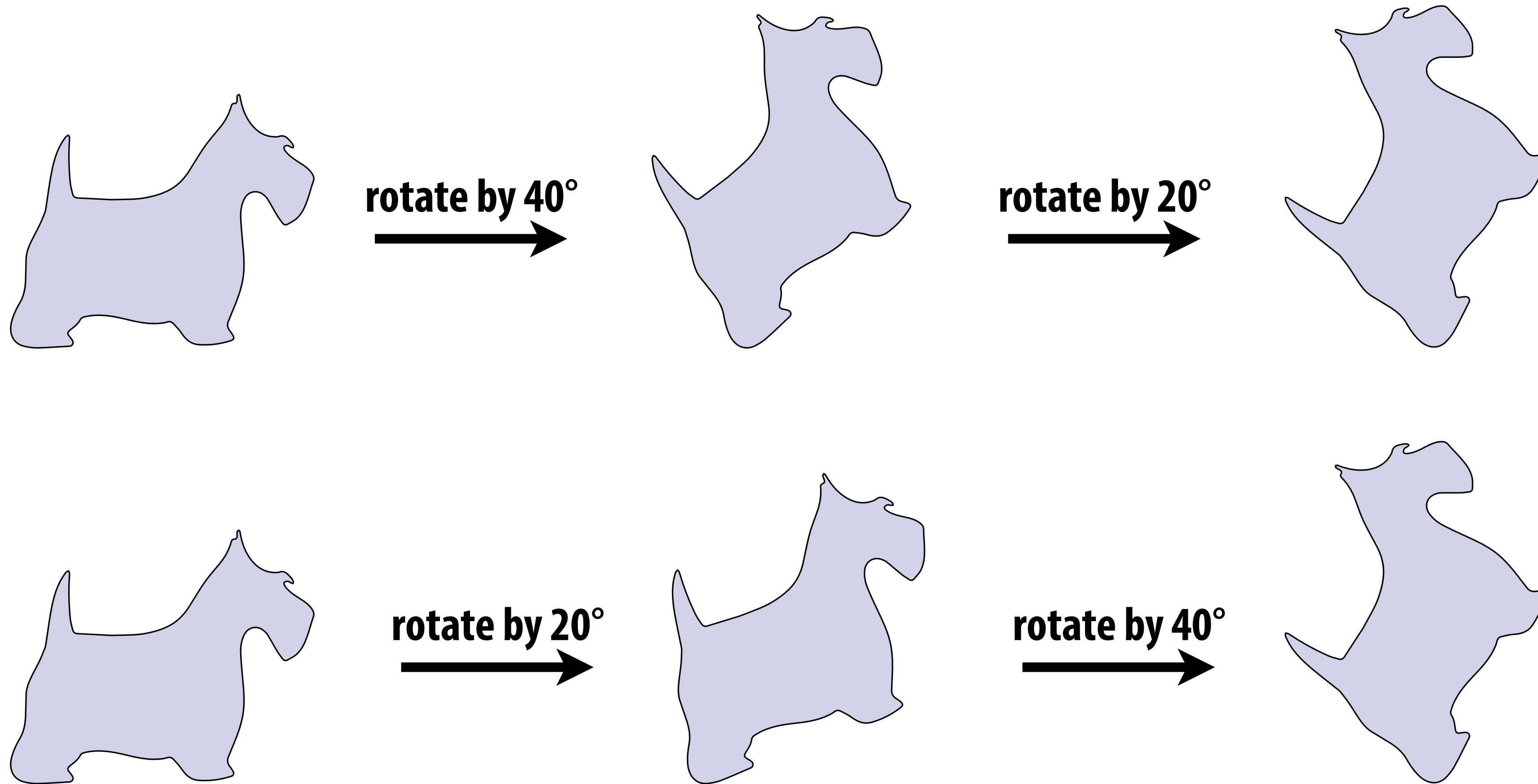


NO: We can keep São Paulo fixed as we rotate the globe.

Hence, we MUST have three degrees of freedom.

Commutativity of Rotations—2D

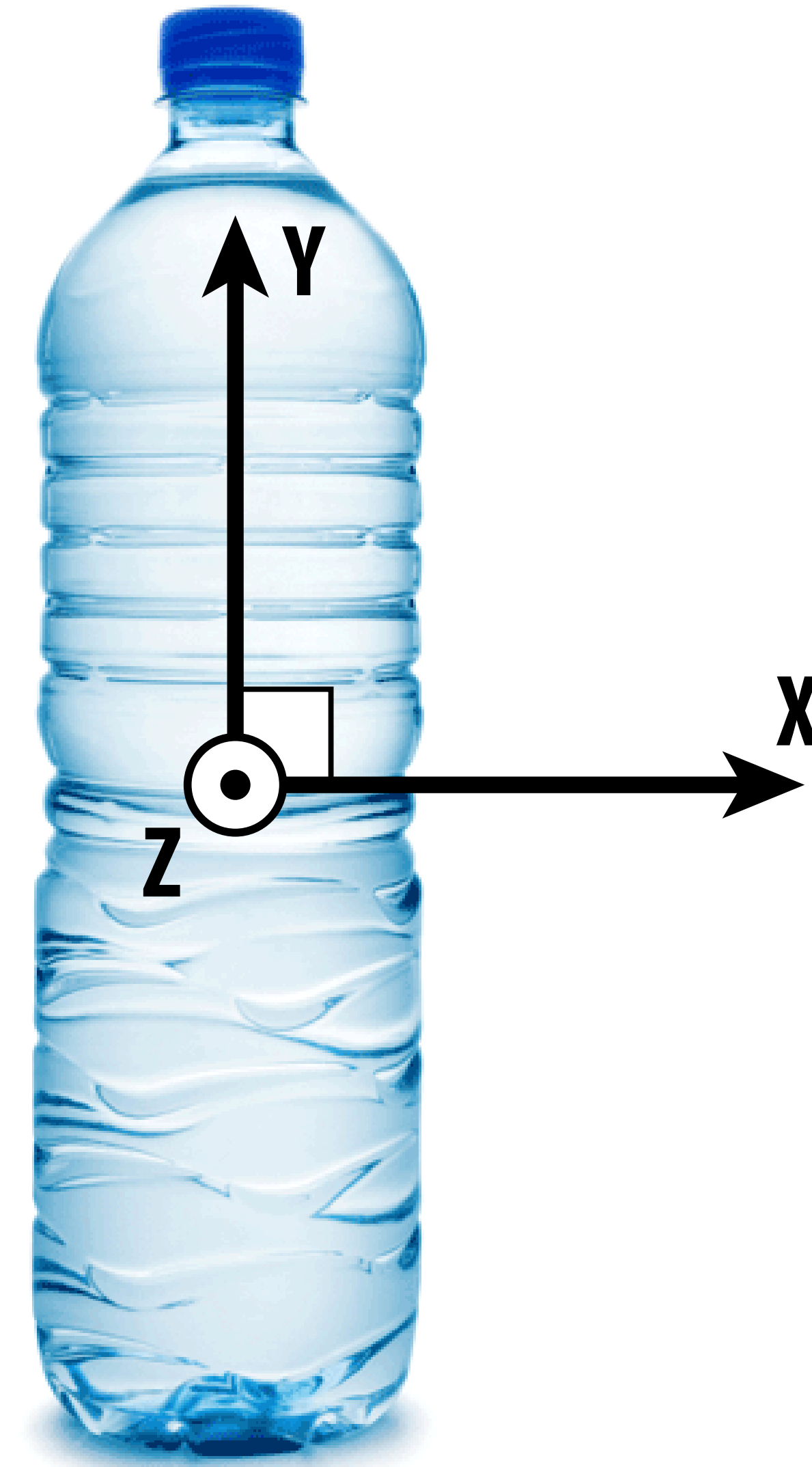
- In 2D, order of rotations doesn't matter:



Same result! ("2D rotations commute")

Commutativity of Rotations—3D

- What about in 3D?
- Try it at home—grab a water bottle!
 - Rotate 90° around Y, then 90° around Z, then 90° around X
 - Rotate 90° around Z, then 90° around Y, then 90° around X
 - (Was there any difference?)



CONCLUSION: bad things can happen if we're not careful about the order in which we apply rotations!

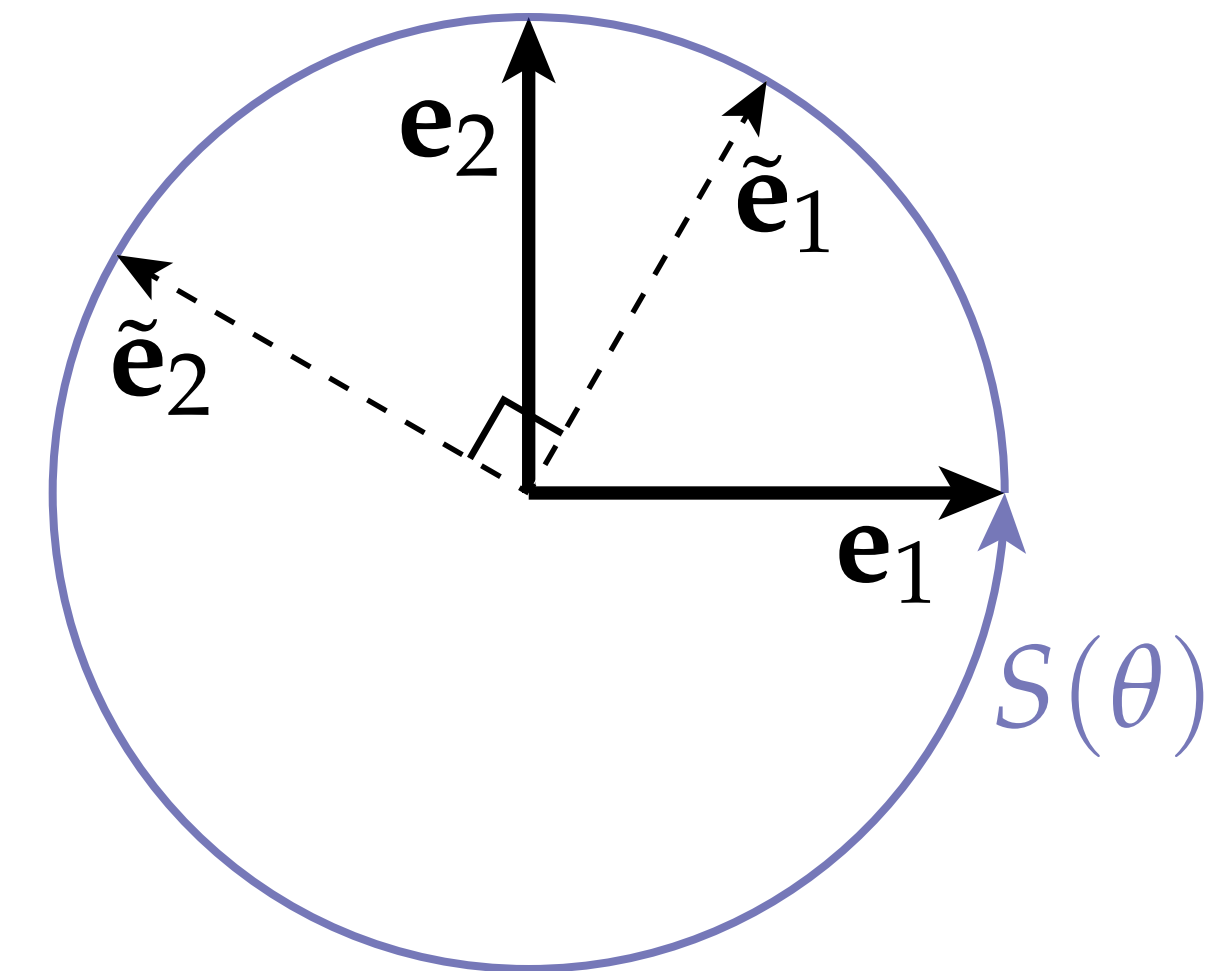
Representing Rotations—2D

- **First things first: how do we get a rotation matrix in 2D? (Don't just regurgitate the formula!)**
- **Suppose I have a function $S(\theta)$ that for a given angle θ gives me the point (x,y) around a circle (CCW).**

- **Right now, I do not care how this function is expressed!***

- **What's e_1 rotated by θ ? $\tilde{e}_1 = S(\theta)$**
- **What's e_2 rotated by θ ? $\tilde{e}_2 = S(\theta + \pi/2)$**
- **How about $\mathbf{u} := a\mathbf{e}_1 + b\mathbf{e}_2$?**

$$\mathbf{u} := aS(\theta) + bS(\theta + \pi/2)$$



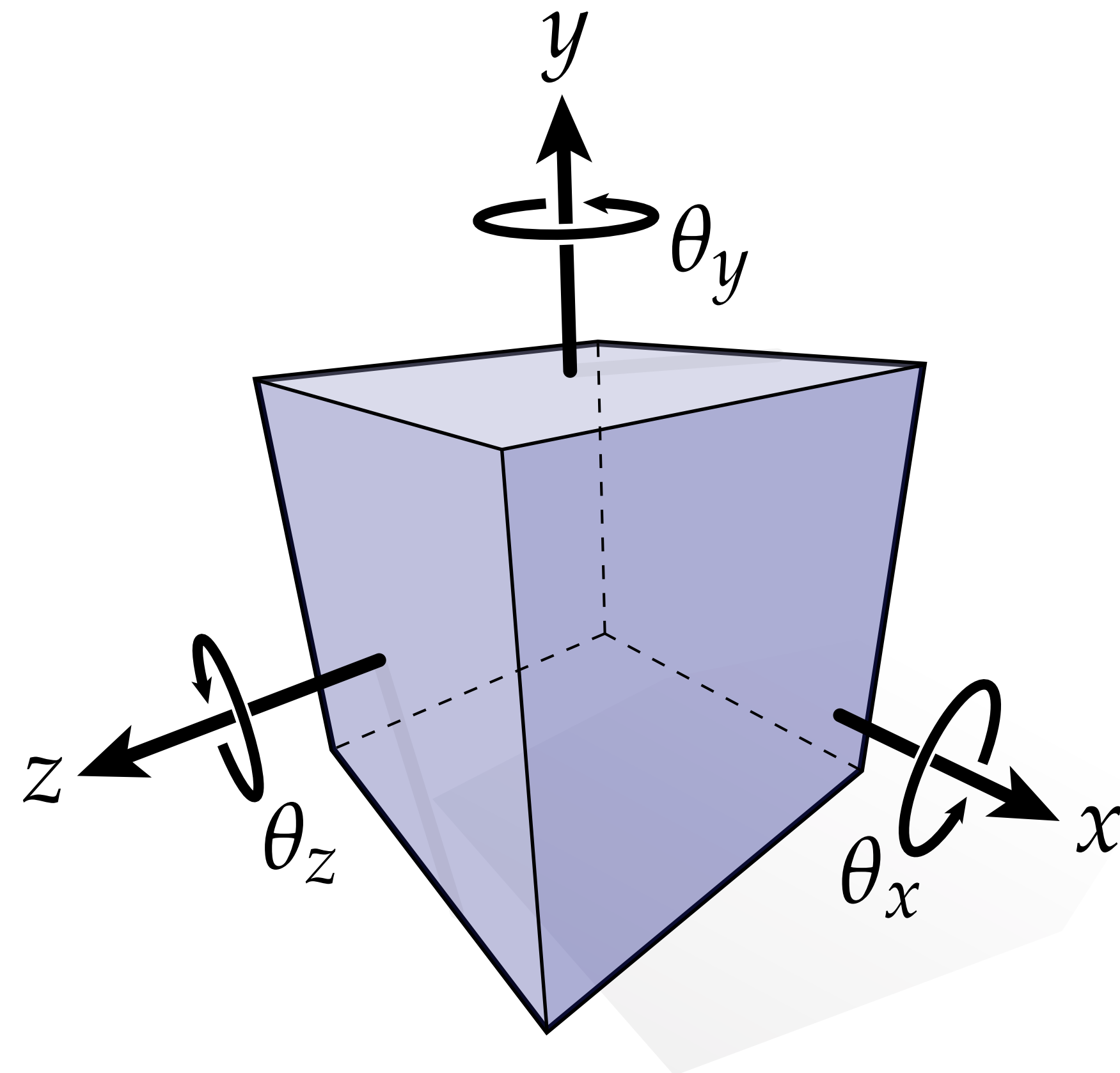
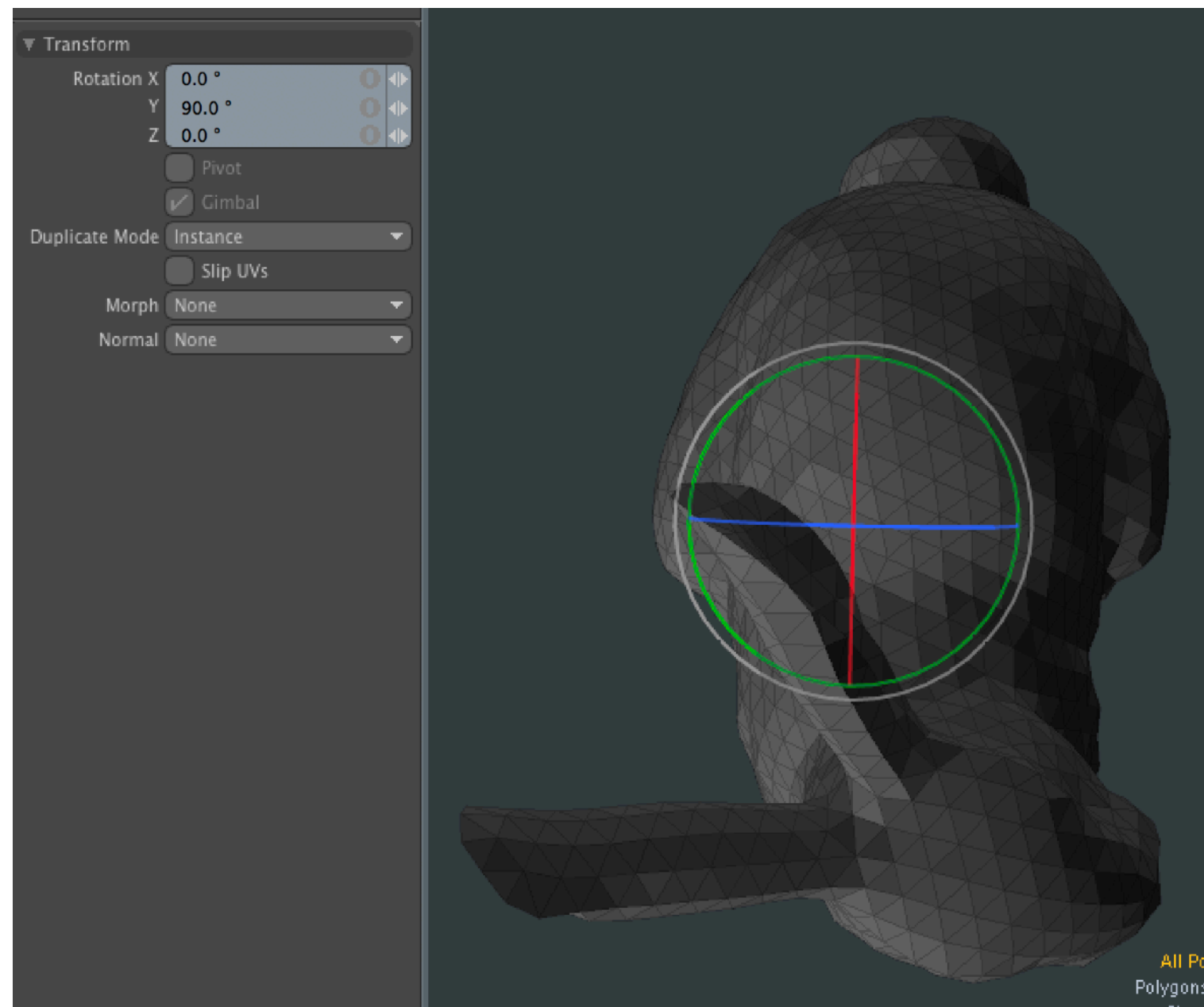
What then must the matrix look like?

$$\begin{bmatrix} S(\theta) & S(\theta + \pi/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \cos(\theta + \pi/2) \\ \sin(\theta) & \sin(\theta + \pi/2) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

***I.e., I don't yet care about sines and cosines and so forth.**

Representing Rotations in 3D—Euler Angles

- How do we express rotations in 3D?
- One idea: we know how to do 2D rotations.
- Why not simply apply rotations around the three axes? (X,Y,Z)
- Scheme is called *Euler angles*
- “Gimbal Lock”



Gimbal Lock

- When using Euler angles $\theta_x, \theta_y, \theta_z$, may reach a configuration where there is *no way to rotate around one of the three axes!*

- Recall rotation matrices around three axes:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x \\ 0 & \sin \theta_x & \cos \theta_x \end{bmatrix} \quad R_y = \begin{bmatrix} \cos \theta_y & 0 & \sin \theta_y \\ 0 & 1 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y \end{bmatrix} \quad R_z = \begin{bmatrix} \cos \theta_z & -\sin \theta_z & 0 \\ \sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Product of these matrices represents rotation by Euler angles:

$$R_x R_y R_z = \begin{bmatrix} \cos \theta_y \cos \theta_z & -\cos \theta_y \sin \theta_z & \sin \theta_y \\ \cos \theta_z \sin \theta_x \sin \theta_y + \cos \theta_x \sin \theta_z & \cos \theta_x \cos \theta_z - \sin \theta_x \sin \theta_y \sin \theta_z & -\cos \theta_y \sin \theta_x \\ -\cos \theta_x \cos \theta_z \sin \theta_y + \sin \theta_x \sin \theta_z & \cos \theta_z \sin \theta_x + \cos \theta_x \sin \theta_y \sin \theta_z & \cos \theta_x \cos \theta_y \end{bmatrix}$$

- Consider special case $\theta_y = \pi/2$ (so, $\cos \theta_y = 0, \sin \theta_y = 1$):

$$\implies \begin{bmatrix} 0 & 0 & 1 \\ \cos \theta_z \sin \theta_x + \cos \theta_x \sin \theta_z & \cos \theta_x \cos \theta_z - \sin \theta_x \sin \theta_z & 0 \\ -\cos \theta_x \cos \theta_z + \sin \theta_x \sin \theta_z & \cos \theta_z \sin \theta_x + \cos \theta_x \sin \theta_z & 0 \end{bmatrix}$$

Gimbal Lock, continued

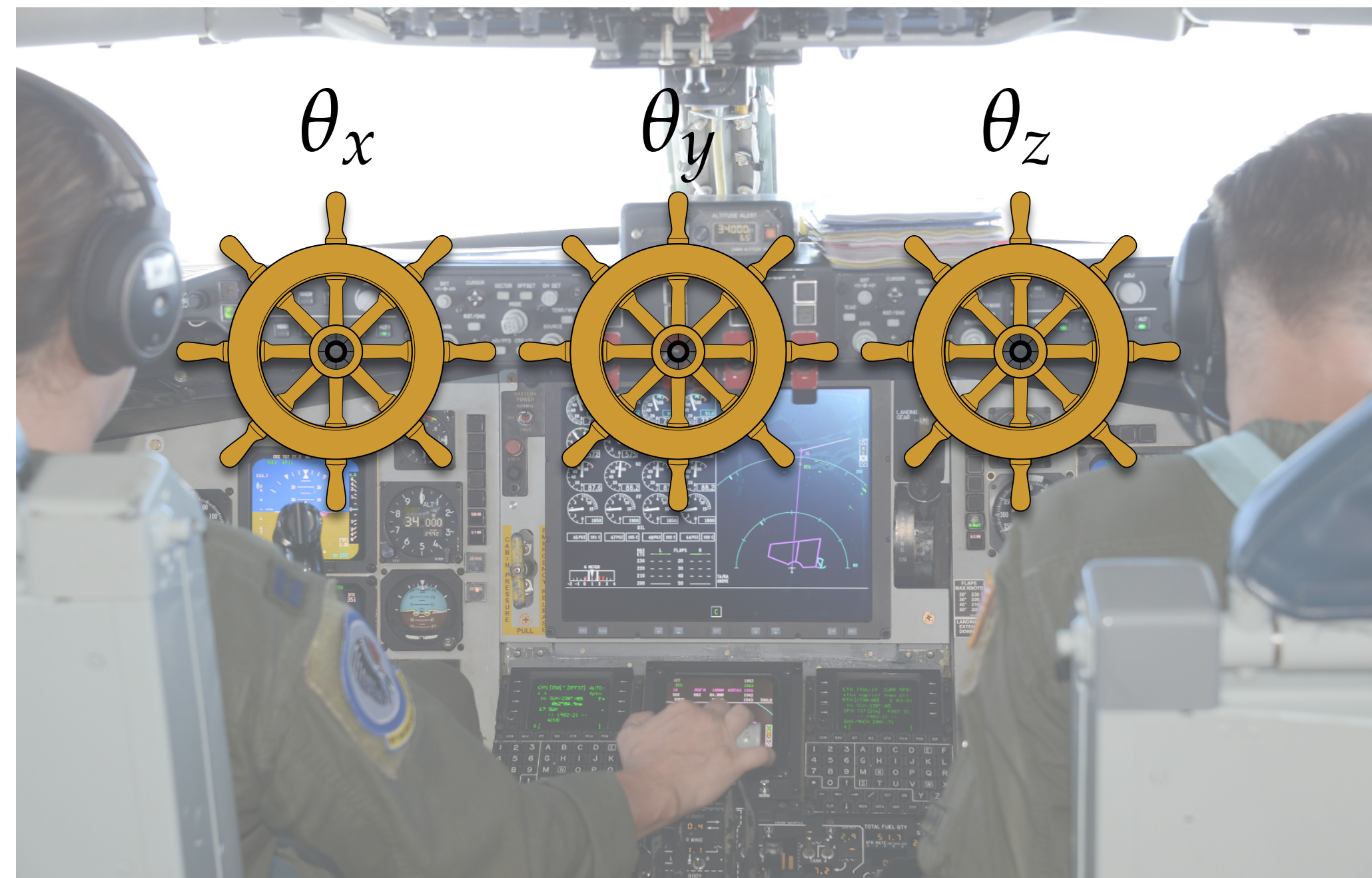
- Simplifying matrix from previous slide, we get

no matter how we adjust θ_x, θ_z ,
can only rotate in one plane!

$$\begin{bmatrix} 0 & 0 & 1 \\ \sin(\theta_x + \theta_z) & \cos(\theta_x + \theta_z) & 0 \\ -\cos(\theta_x + \theta_z) & \sin(\theta_x + \theta_z) & 0 \end{bmatrix}$$

Q: What does this matrix do?

- We are now “locked” into a single axis of rotation
- Not a great design for airplane controls!



Rotation from Axis/Angle

- Alternatively, there is a general expression for a matrix that performs a rotation around a given axis u by a given angle θ :

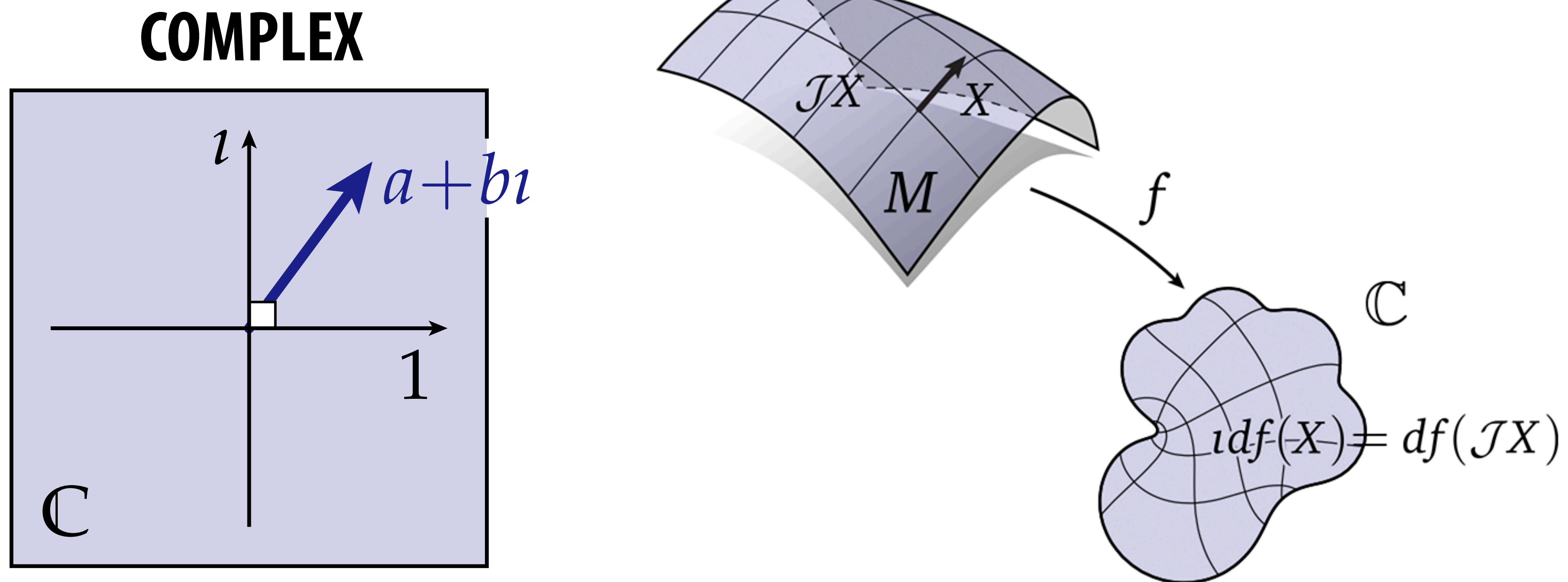
$$\begin{bmatrix} \cos \theta + u_x^2 (1 - \cos \theta) & u_x u_y (1 - \cos \theta) - u_z \sin \theta & u_x u_z (1 - \cos \theta) + u_y \sin \theta \\ u_y u_x (1 - \cos \theta) + u_z \sin \theta & \cos \theta + u_y^2 (1 - \cos \theta) & u_y u_z (1 - \cos \theta) - u_x \sin \theta \\ u_z u_x (1 - \cos \theta) - u_y \sin \theta & u_z u_y (1 - \cos \theta) + u_x \sin \theta & \cos \theta + u_z^2 (1 - \cos \theta) \end{bmatrix}$$

Just memorize this matrix! :-)

...we'll see a much easier way, later on.

Complex Analysis—Motivation

- Natural way to encode geometric transformations in 2D
- Simplifies code / notation / debugging / *thinking*
- *Moderate* reduction in computational cost/bandwidth/storage
- Fluency with complex analysis can lead into deeper/novel solutions to problems...



Truly: no good reason to use 2D vectors instead of complex numbers...

DON'T: Think of these numbers as “complex.”

DO: Imagine we're simply defining additional operations (like dot and cross).

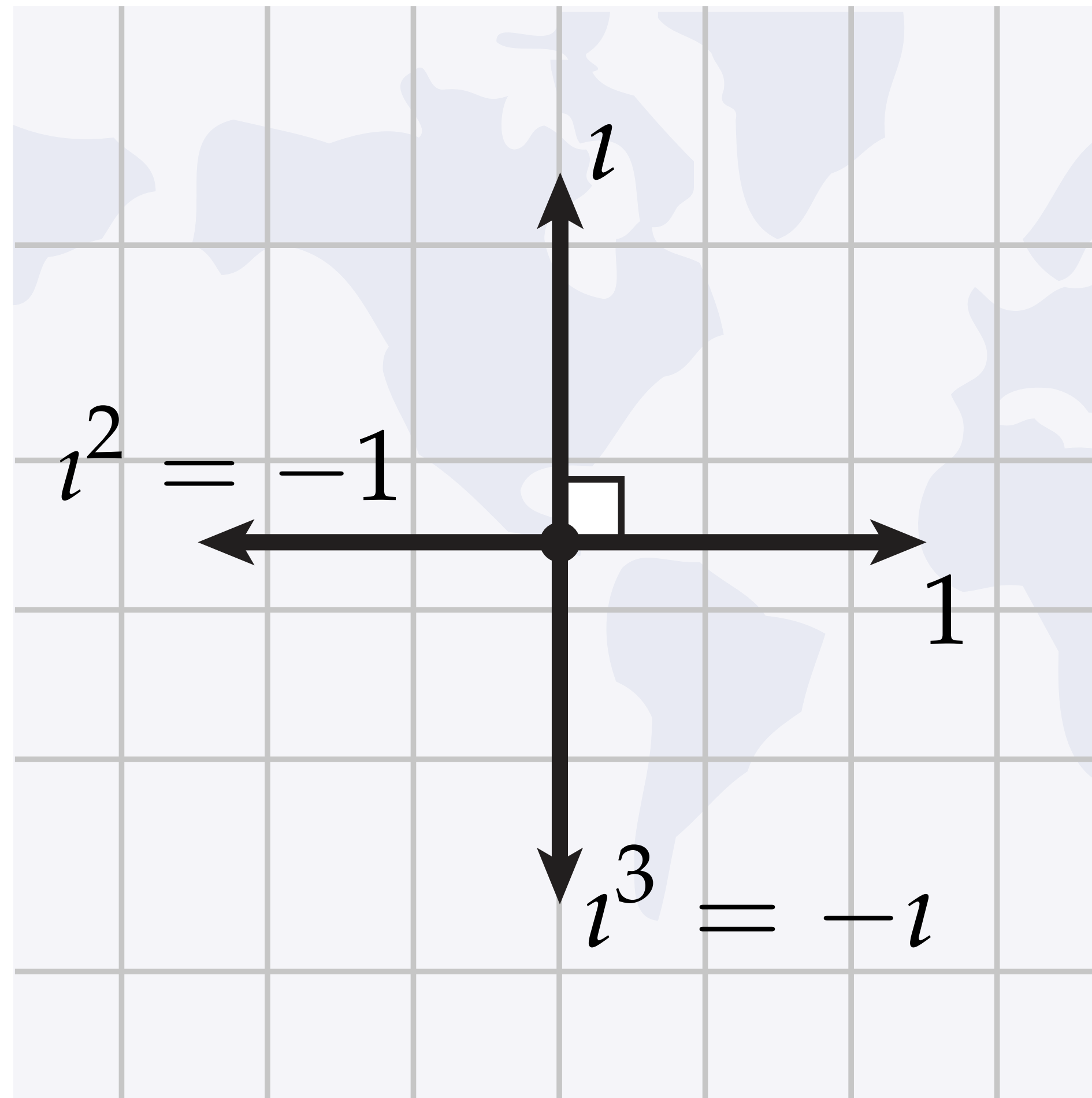
Imaginary Unit

$$i := \sqrt{-1}$$

nonsense!

More importantly: obscures geometric meaning.

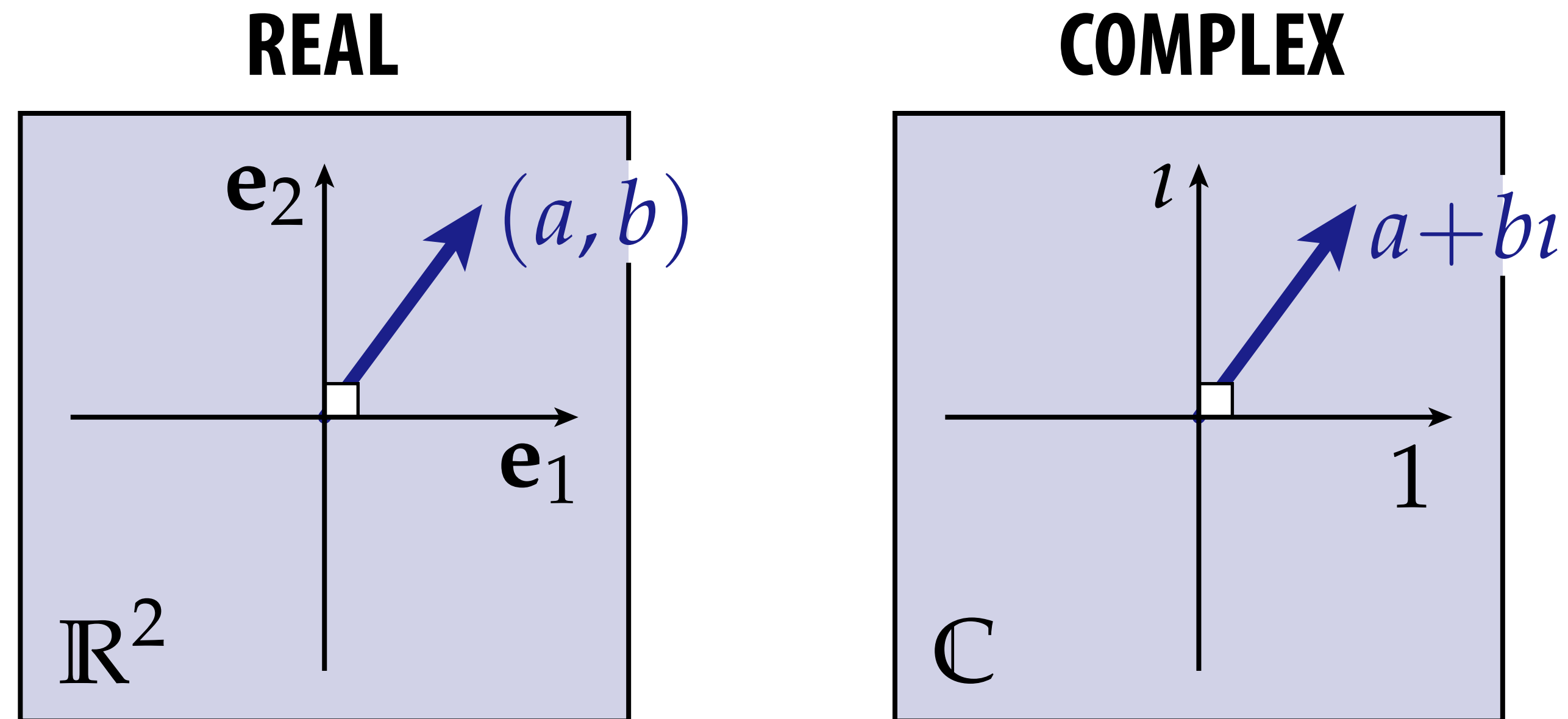
Imaginary Unit—Geometric Description



**Imaginary unit is just a quarter-turn
in the counter-clockwise direction.**

Complex Numbers

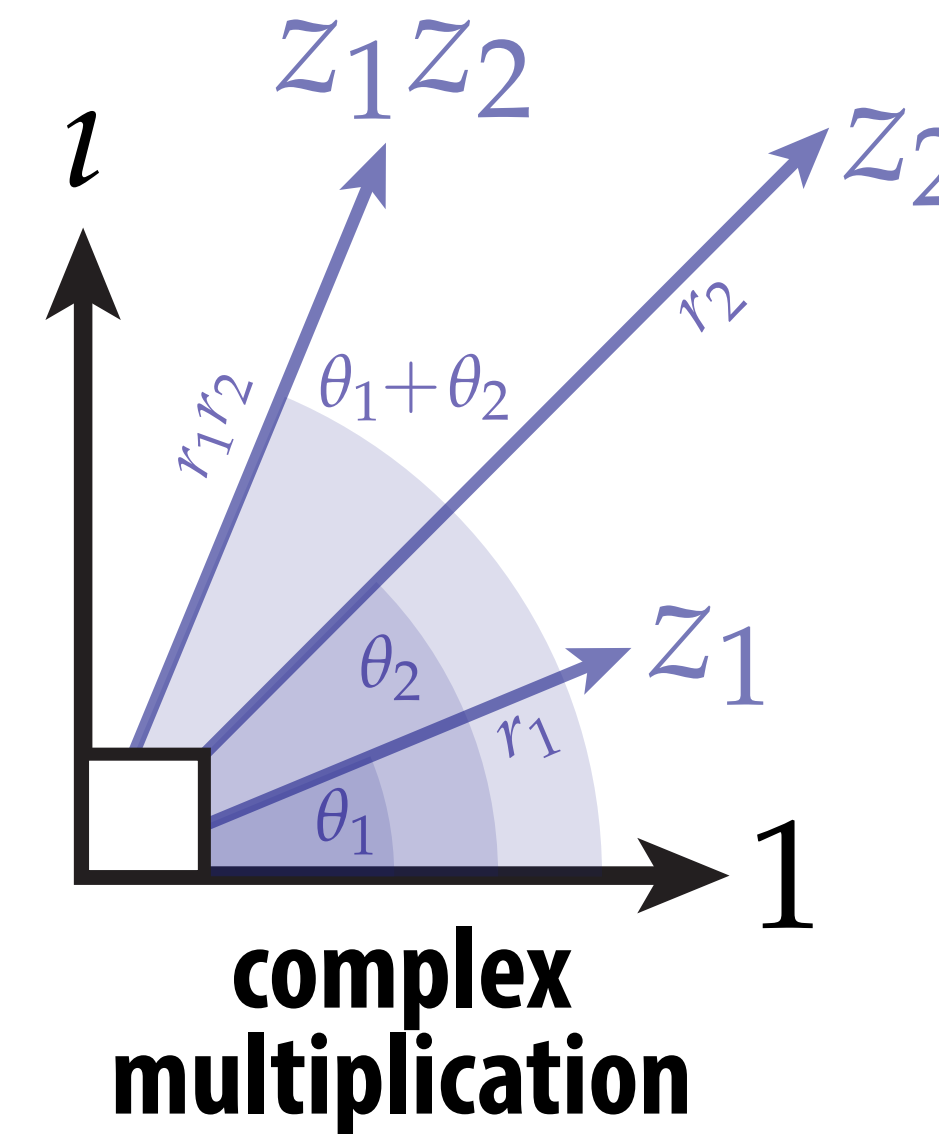
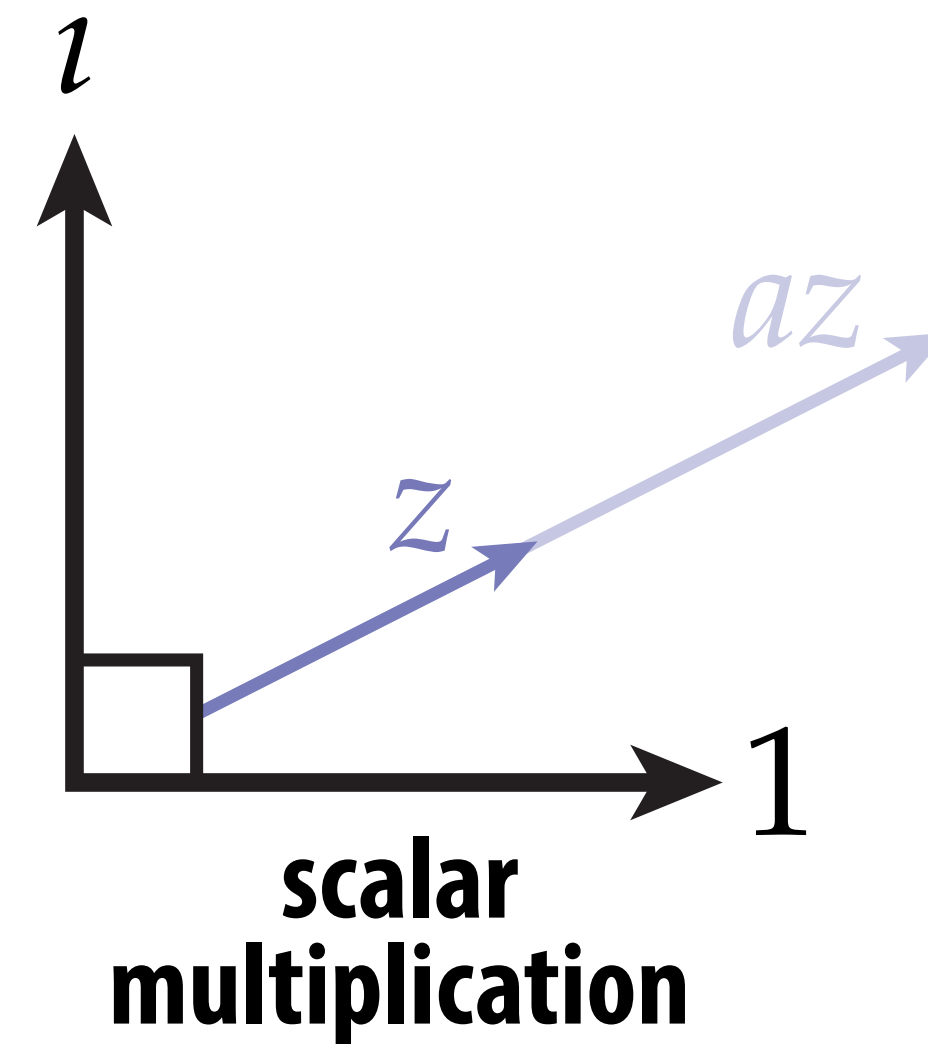
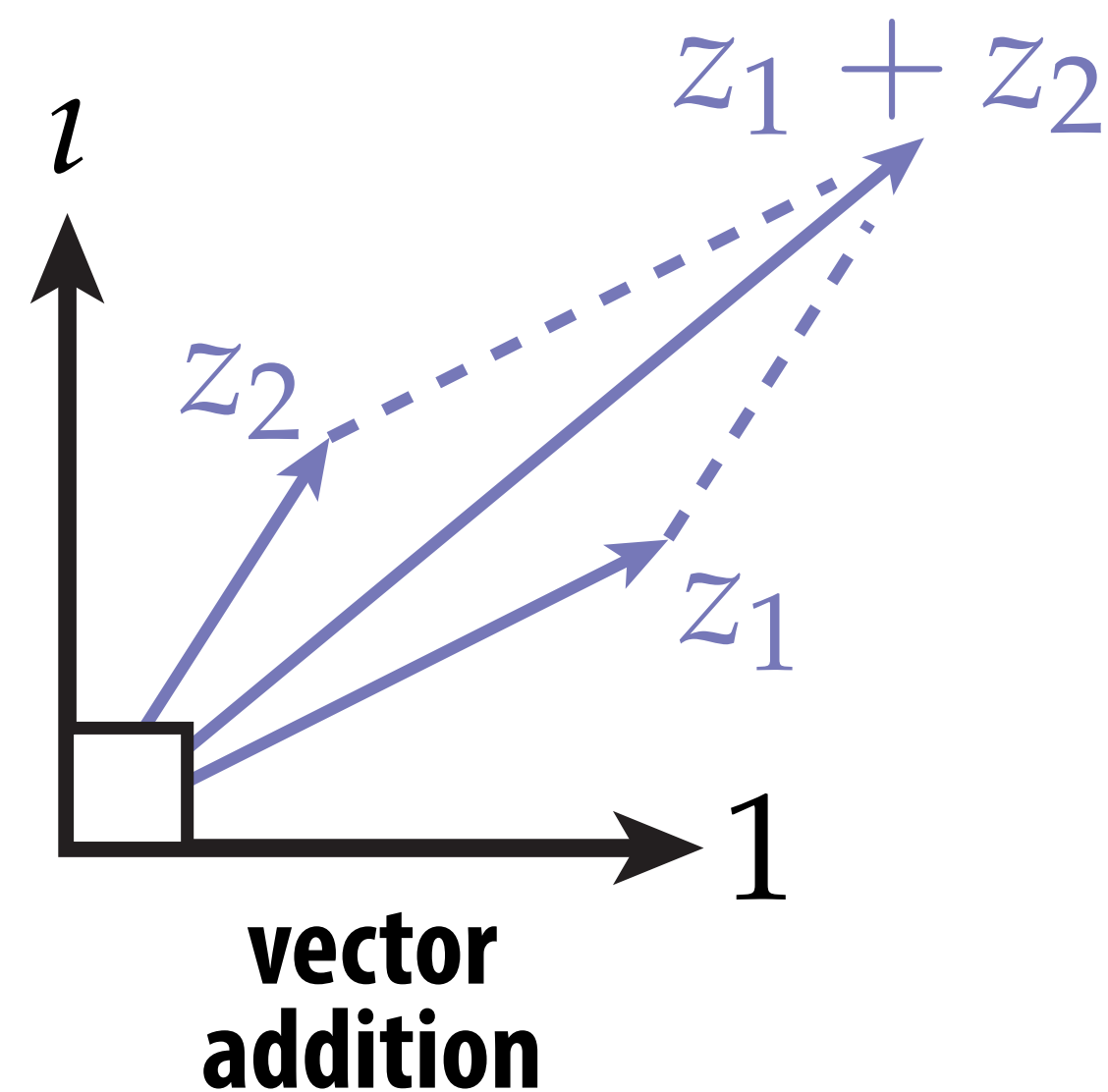
- Complex numbers are then just 2-vectors
- Instead of e_1, e_2 , use “1” and “ i ” to denote the two bases
- Otherwise, behaves exactly like a real 2-dimensional space



- ...except that we're also going to get a very useful new notion of the *product* between two vectors.

Complex Arithmetic

- Same operations as before, plus one more:



- Complex multiplication:
 - angles *add*
 - magnitudes *multiply*

“POLAR FORM”*:

$$z_1 := (r_1, \theta_1)$$

$$z_2 := (r_2, \theta_2)$$

$$z_1 z_2 = (r_1 r_2, \theta_1 + \theta_2)$$

have to be more careful here!



*Not quite how it really works, but basic idea is right.

Complex Product—Rectangular Form

- Complex product in “rectangular” coordinates (1, i):

$$z_1 = (a + bi)$$

$$z_2 = (c + di)$$

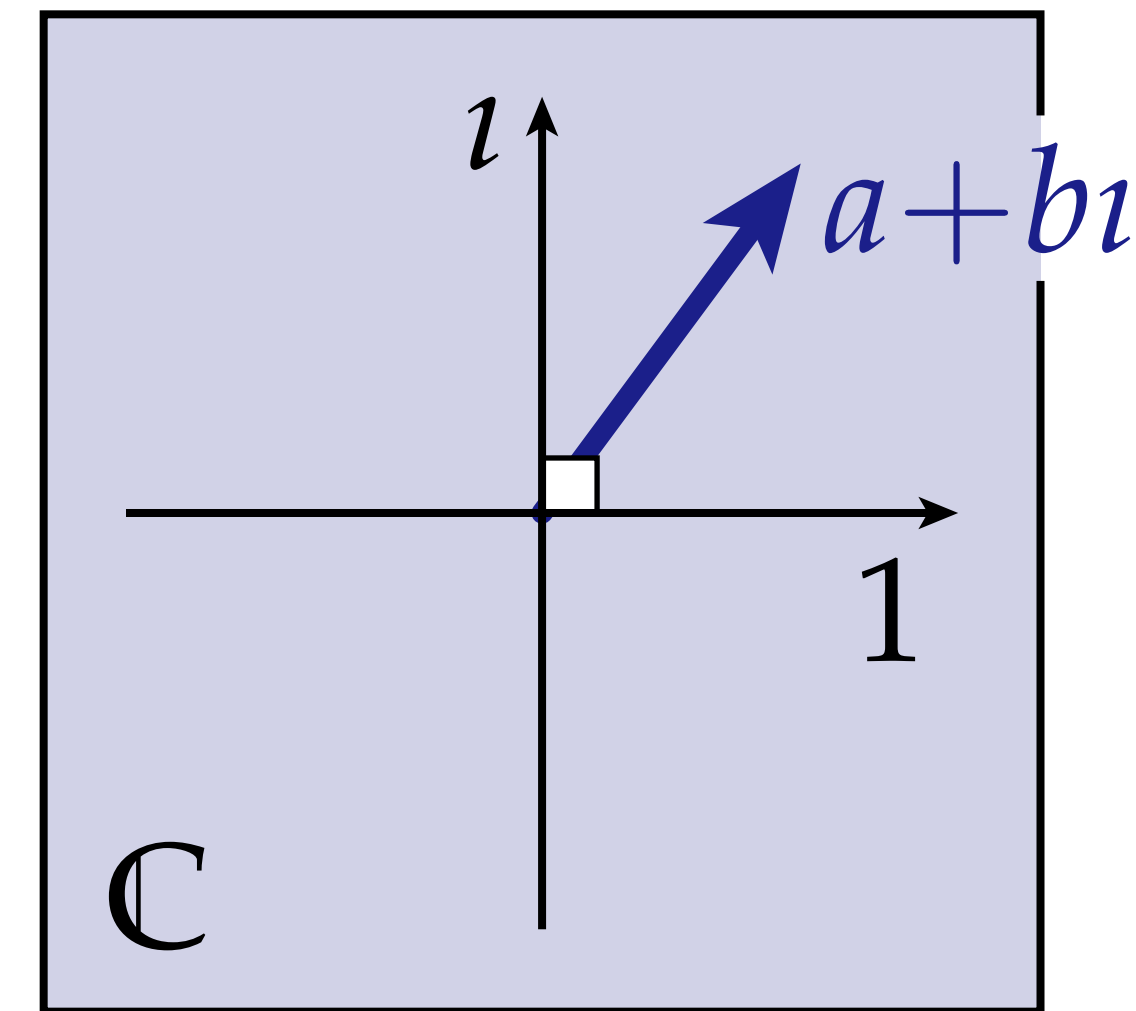
$$z_1 z_2 = ac + adi + bci + bd \overset{\text{two quarter turns—}}{\text{same as -1}} i^2 =$$

$$\boxed{(ac - bd) + (ad + bc)i.}$$

↑
“real part”
 $\text{Re}(z_1 z_2)$

↑
“imaginary part”
 $\text{Im}(z_1 z_2)$

- We used a lot of “rules” here. Can you justify them geometrically?
- Does this product agree with our geometric description (last slide)?



Complex Product—Polar Form

- Perhaps most beautiful identity in math:

$$e^{i\pi} + 1 = 0$$

- Specialization of *Euler's formula*:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

- Can use to “implement” complex product:

$$z_1 = ae^{i\theta}, \quad z_2 = be^{i\phi}$$

$$z_1 z_2 = abe^{i(\theta+\phi)}$$

(as with real exponentiation, exponents *add*)



Leonhard Euler
(1707–1783)

Q: How does this operation differ from our earlier, “fake” polar multiplication?

2D Rotations: Matrices vs. Complex

- Suppose we want to rotate a vector \mathbf{u} by an angle θ , then by an angle ϕ .

REAL / RECTANGULAR	COMPLEX / POLAR
$\mathbf{u} = (x, y)$	$u = re^{i\alpha}$
$\mathbf{A} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	$a = e^{i\theta}$
$\mathbf{B} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$	$b = e^{i\phi}$
$\mathbf{A}\mathbf{u} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}$	$abu = re^{i(\alpha+\theta+\phi)}$
$\mathbf{B}\mathbf{A}\mathbf{u} = \begin{bmatrix} (x \cos \theta - y \sin \theta) \cos \phi - (x \sin \theta + y \cos \theta) \sin \phi \\ (x \cos \theta - y \sin \theta) \sin \phi + (x \sin \theta + y \cos \theta) \cos \phi \end{bmatrix}$	
$= \dots \text{some trigonometry} \dots =$	
$\mathbf{B}\mathbf{A}\mathbf{u} = \begin{bmatrix} x \cos(\theta + \phi) - y \sin(\theta + \phi) \\ x \sin(\theta + \phi) + y \cos(\theta + \phi) \end{bmatrix}$	

Pervasive theme in graphics:

**Sure, there are often many
“equivalent” representations.**

**...But why not choose the one
that makes life easiest*?**

***Or most efficient, or most accurate...**

Quaternions

- TLDR: Kind of like complex numbers but for 3D rotations
- **Weird situation:** can't do 3D rotations w/ only 3 components!



William Rowan Hamilton
(1805-1865)



(Not Hamilton)

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
 $i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Quaternions in Coordinates

- Hamilton's insight: in order to do 3D rotations in a way that mimics complex numbers for 2D, actually need **FOUR** coords.
- One real, *three* imaginary:

"H" is for Hamilton! $\mathbb{H} := \text{span}(\{1, i, j, k\})$
 $q = a + bi + cj + dk \in \mathbb{H}$

- Quaternion product determined by

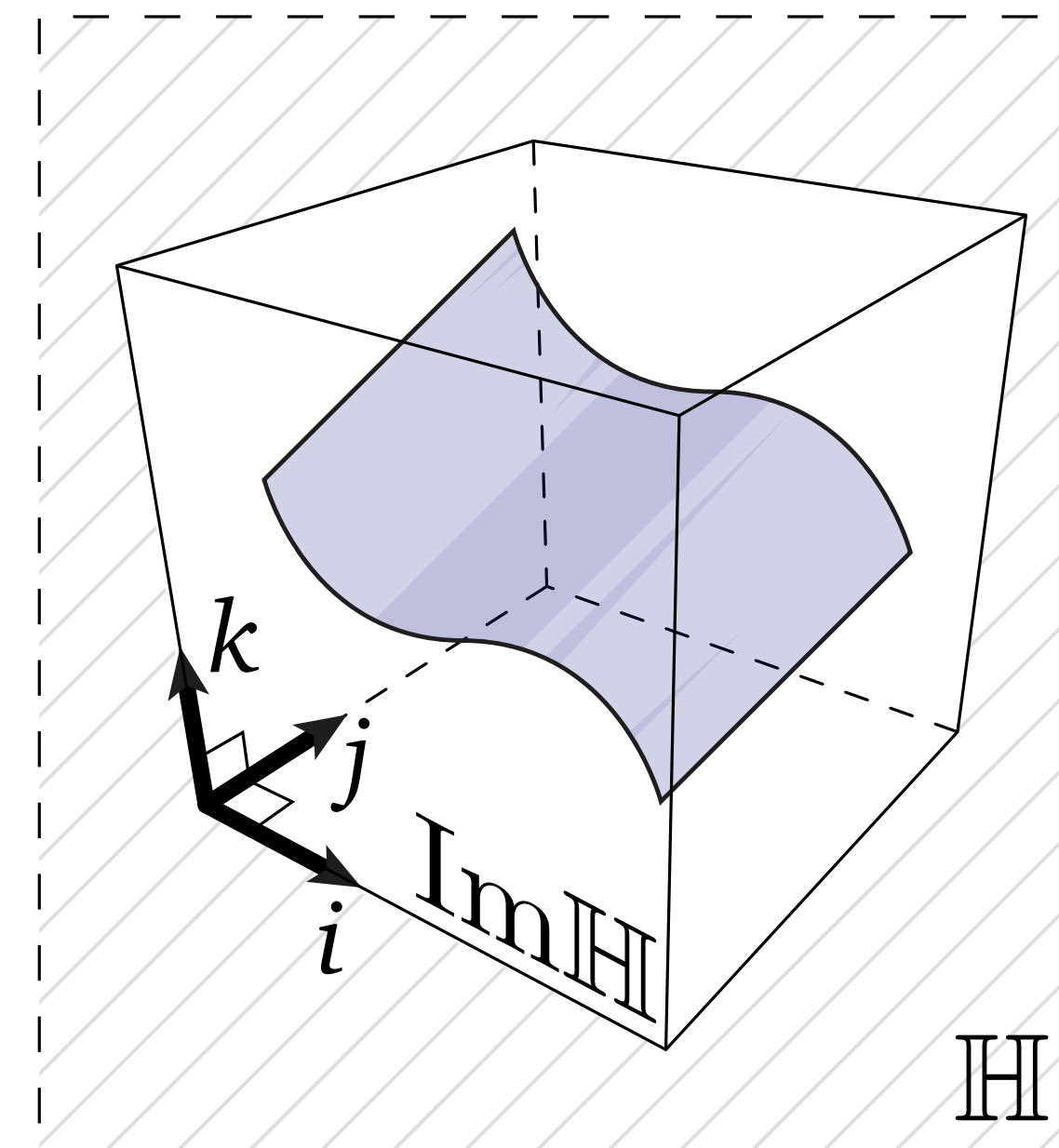
$$i^2 = j^2 = k^2 = ijk = -1$$

together w/ "natural" rules (distributivity, associativity, etc.)

- **WARNING:** product no longer commutes!

$$\text{For } q, p \in \mathbb{H}, \quad qp \neq pq$$

(Why might it make sense that it doesn't commute?)



Quaternion Product in Components

- Given two quaternions

$$q = a_1 + b_1i + c_1j + d_1k$$

$$p = a_2 + b_2i + c_2j + d_2k$$

- Can express their product as

$$\begin{aligned} qp = & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ & + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j \\ & + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k \end{aligned}$$

...fortunately there is a (much) nicer expression.

Quaternions—Scalar + Vector Form

- If we have *four* components, how do we talk about pts in 3D?
- Natural idea: we have three imaginary parts—why not use these to encode 3D vectors?

$$(x, y, z) \mapsto 0 + xi + yj + zk$$

- Alternatively, can think of a quaternion as a pair

$$\left(\underbrace{\text{scalar}}_{\mathbb{R}}, \underbrace{\text{vector}}_{\mathbb{R}^3} \right) \in \mathbb{H}$$

- Quaternion product then has simple(r) form:

$$(a, \mathbf{u})(b, \mathbf{v}) = (ab - \mathbf{u} \cdot \mathbf{v}, a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v})$$

- For vectors in \mathbb{R}^3 , gets even simpler:

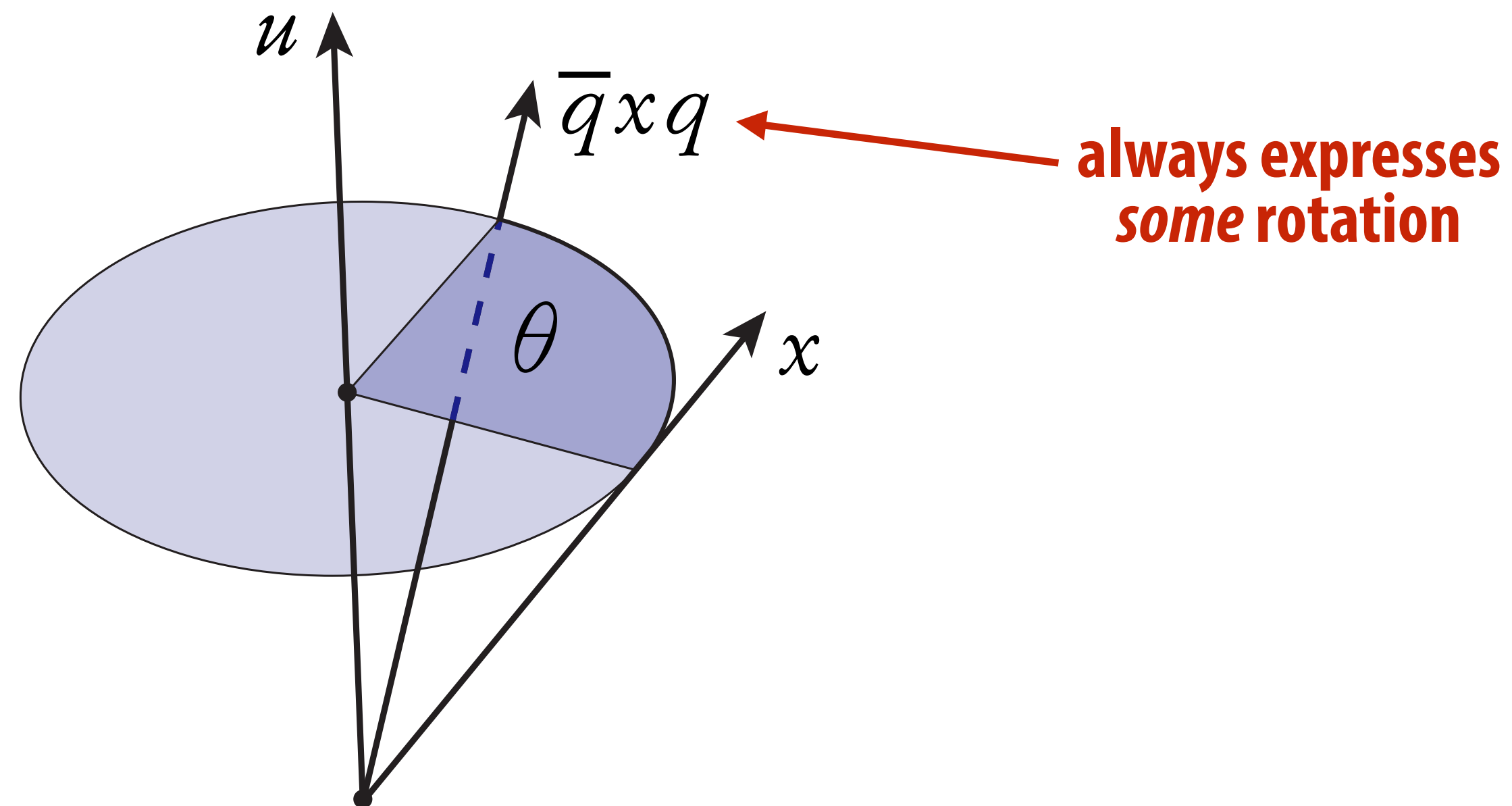
$$\mathbf{u}\mathbf{v} = \mathbf{u} \times \mathbf{v} - \mathbf{u} \cdot \mathbf{v}$$

3D Transformations via Quaternions

- Main use for quaternions in graphics? *Rotations.*
- Consider vector x (“pure imaginary”) and *unit* quaternion q :

$$x \in \text{Im}(\mathbb{H})$$

$$q \in \mathbb{H}, \quad |q|^2 = 1$$



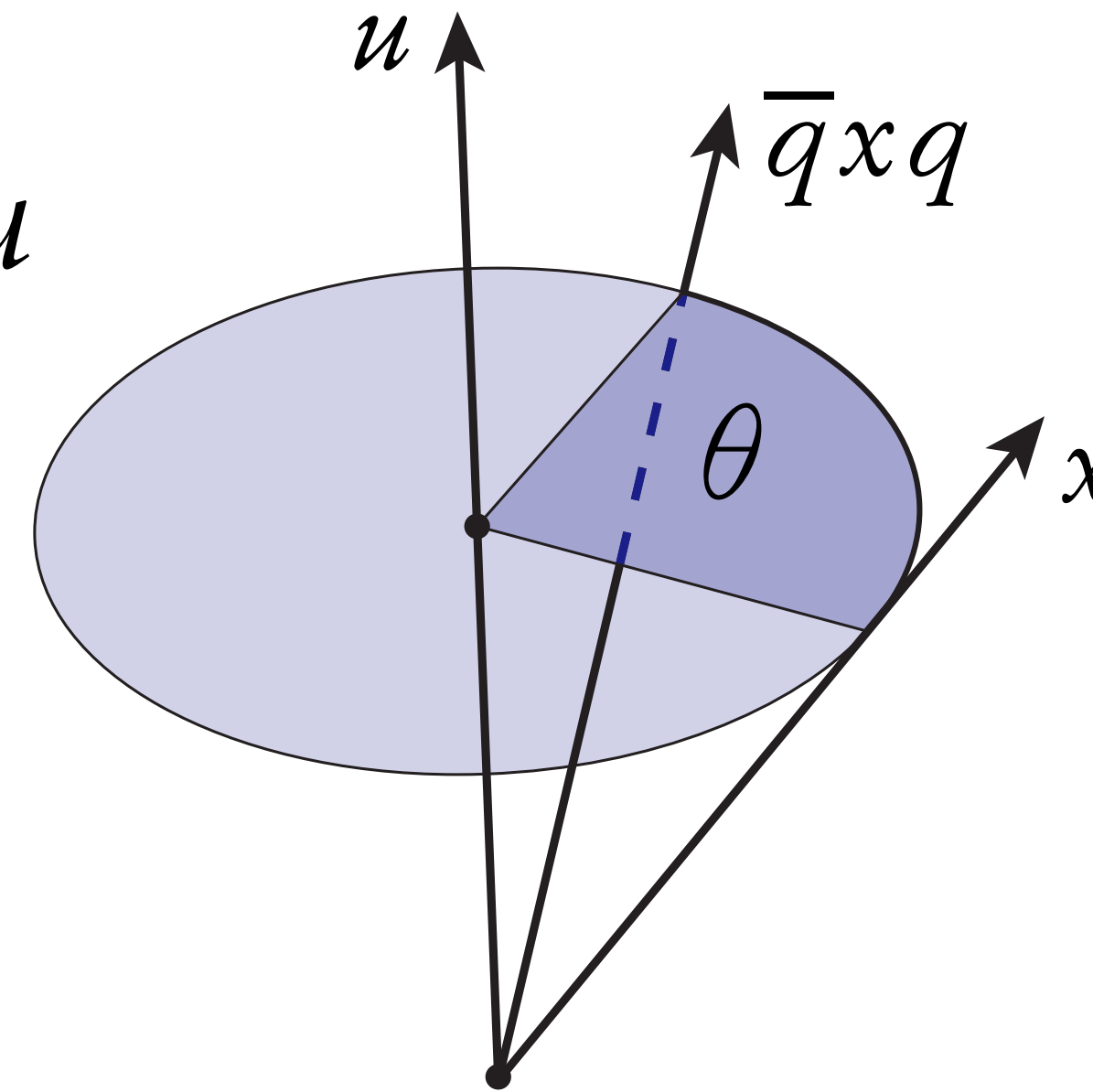
Rotation from Axis/Angle, Revisited

- Given axis u , angle θ , quaternion q representing rotation is

$$q = \cos(\theta/2) + \sin(\theta/2)u$$

angle

axis



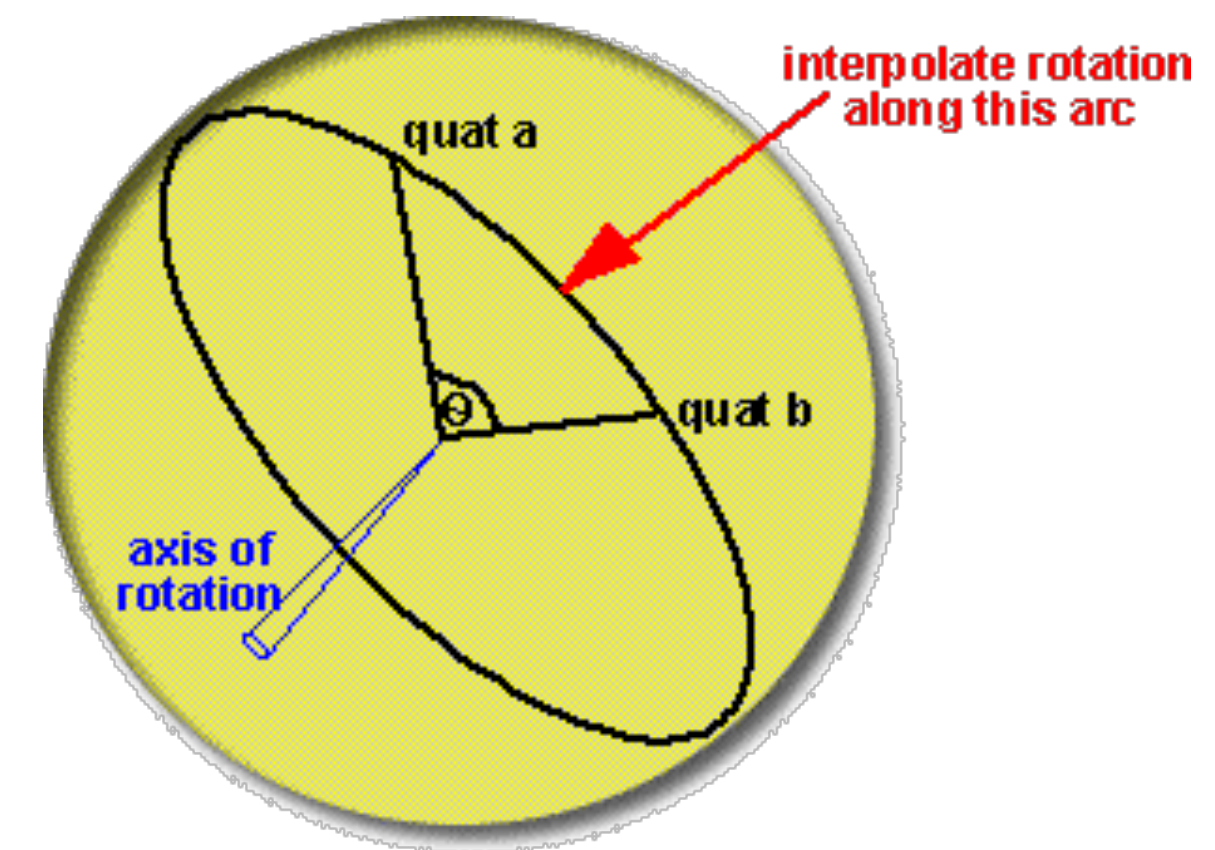
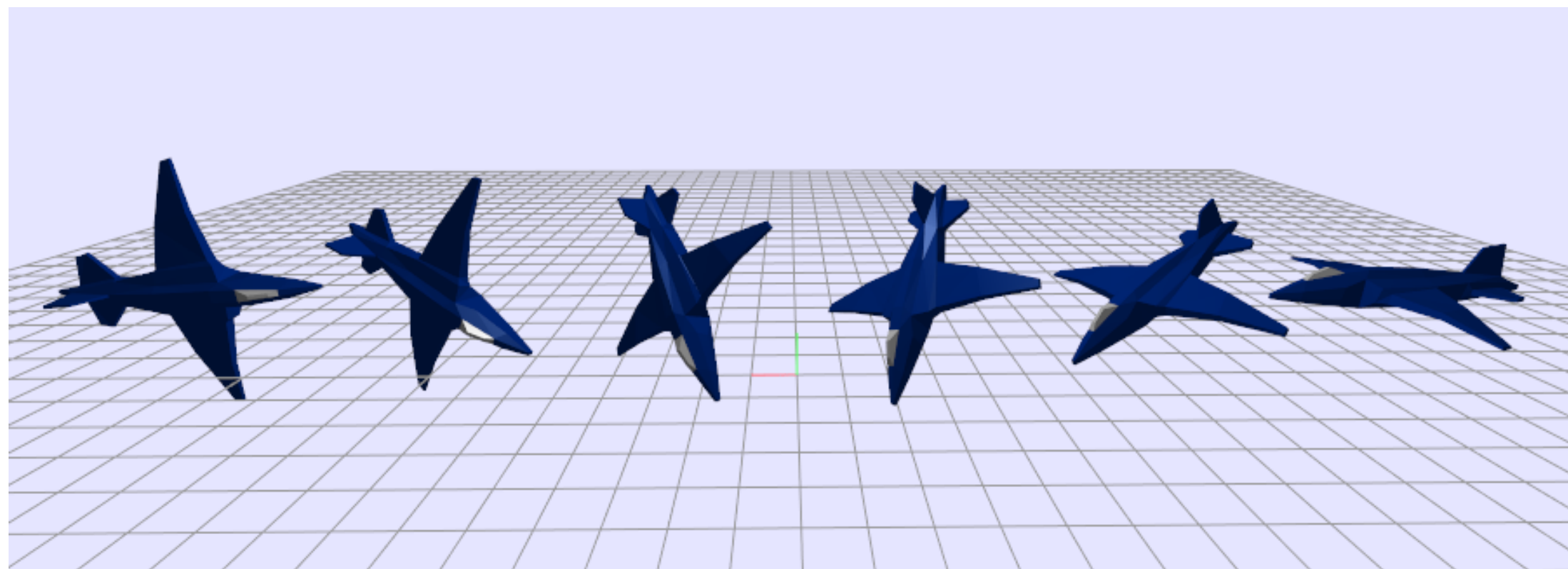
- Much easier to remember (and manipulate) than matrix!

$$\begin{bmatrix} \cos \theta + u_x^2 (1 - \cos \theta) & u_x u_y (1 - \cos \theta) - u_z \sin \theta & u_x u_z (1 - \cos \theta) + u_y \sin \theta \\ u_y u_x (1 - \cos \theta) + u_z \sin \theta & \cos \theta + u_y^2 (1 - \cos \theta) & u_y u_z (1 - \cos \theta) - u_x \sin \theta \\ u_z u_x (1 - \cos \theta) - u_y \sin \theta & u_z u_y (1 - \cos \theta) + u_x \sin \theta & \cos \theta + u_z^2 (1 - \cos \theta) \end{bmatrix}$$

Interpolating Rotations

- Suppose we want to smoothly interpolate between two rotations (e.g., orientations of an airplane)
- Interpolating Euler angles can yield strange-looking paths, non-uniform rotation speed, ...
- Simple solution* w/ quaternions: "SLERP" (spherical linear interpolation):

$$\text{Slerp}(q_0, q_1, t) = q_0(q_0^{-1}q_1)^t, \quad t \in [0, 1]$$

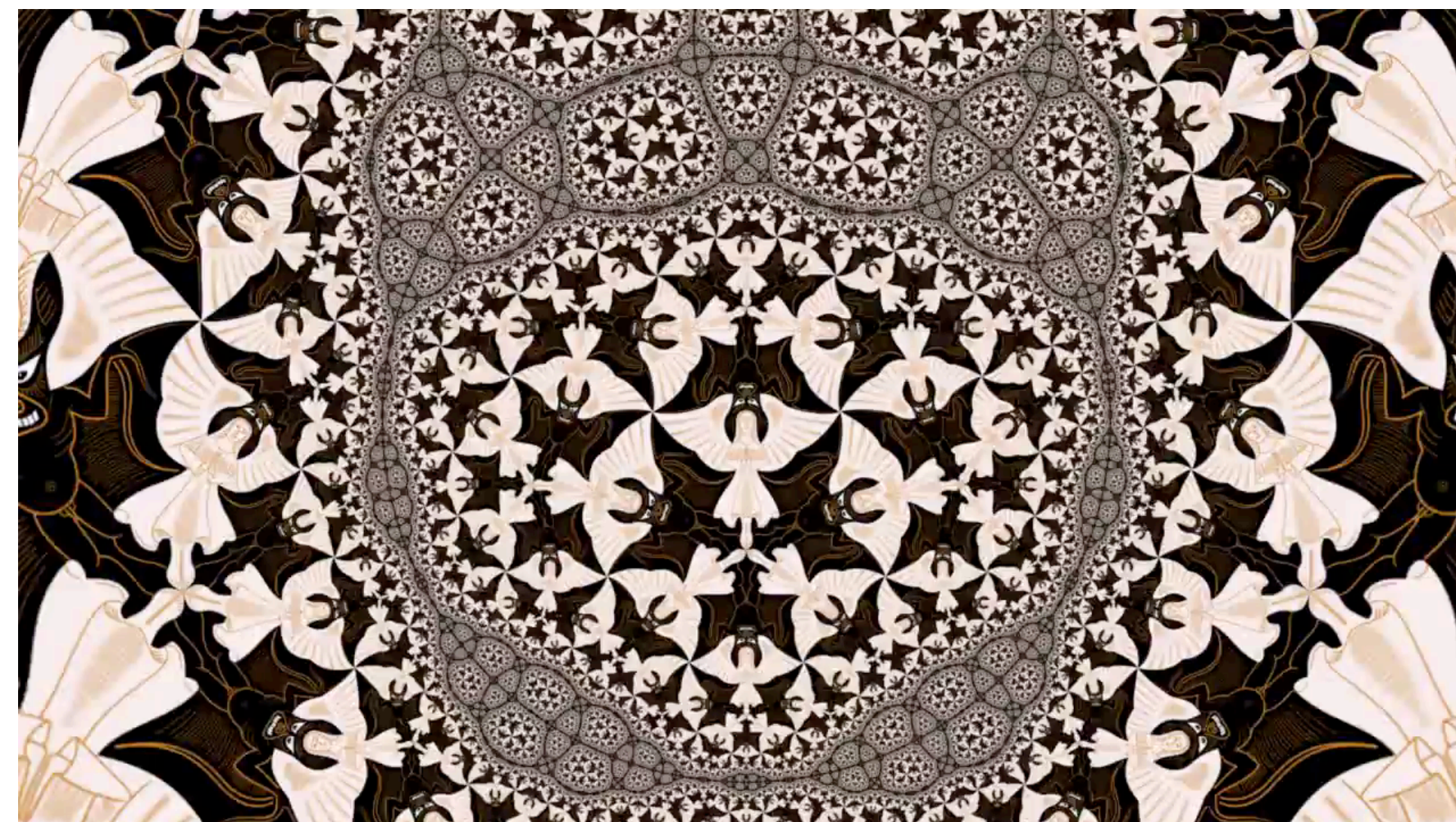
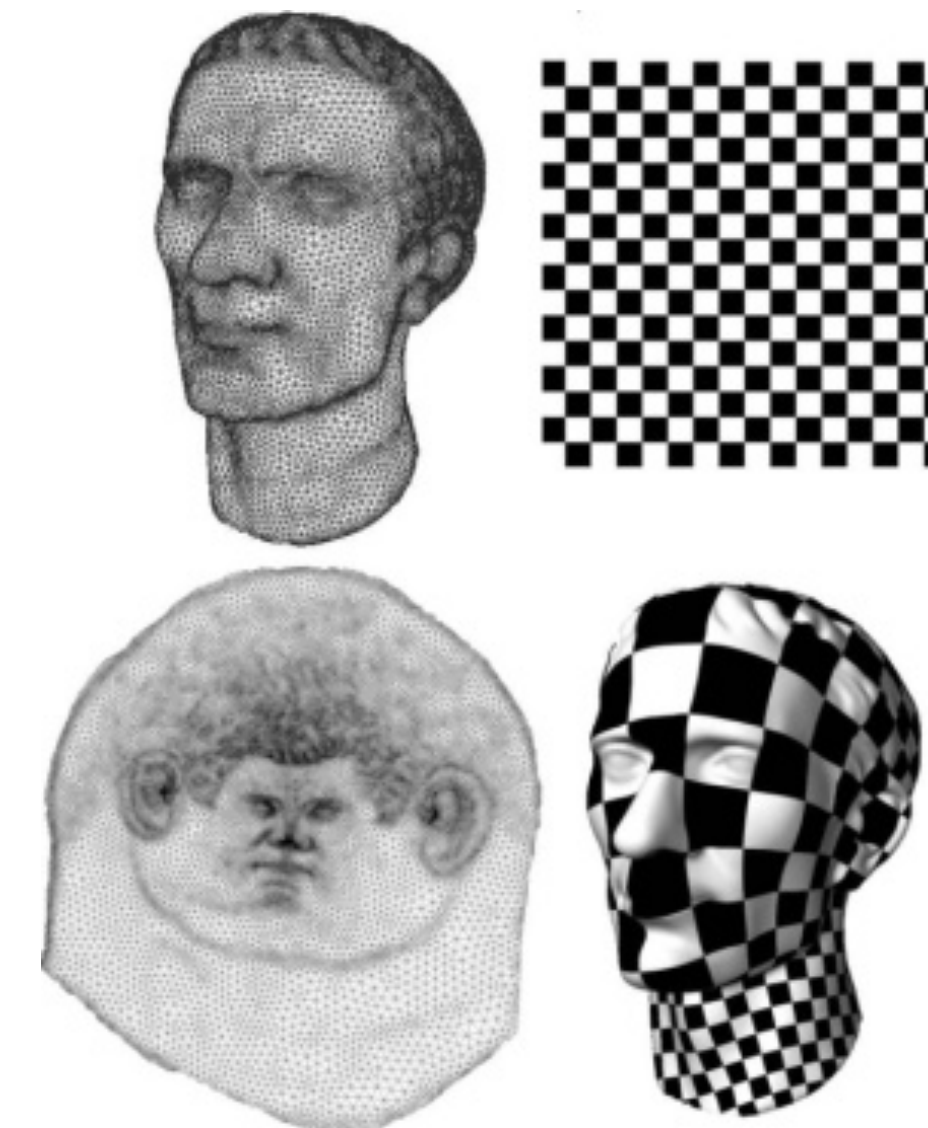
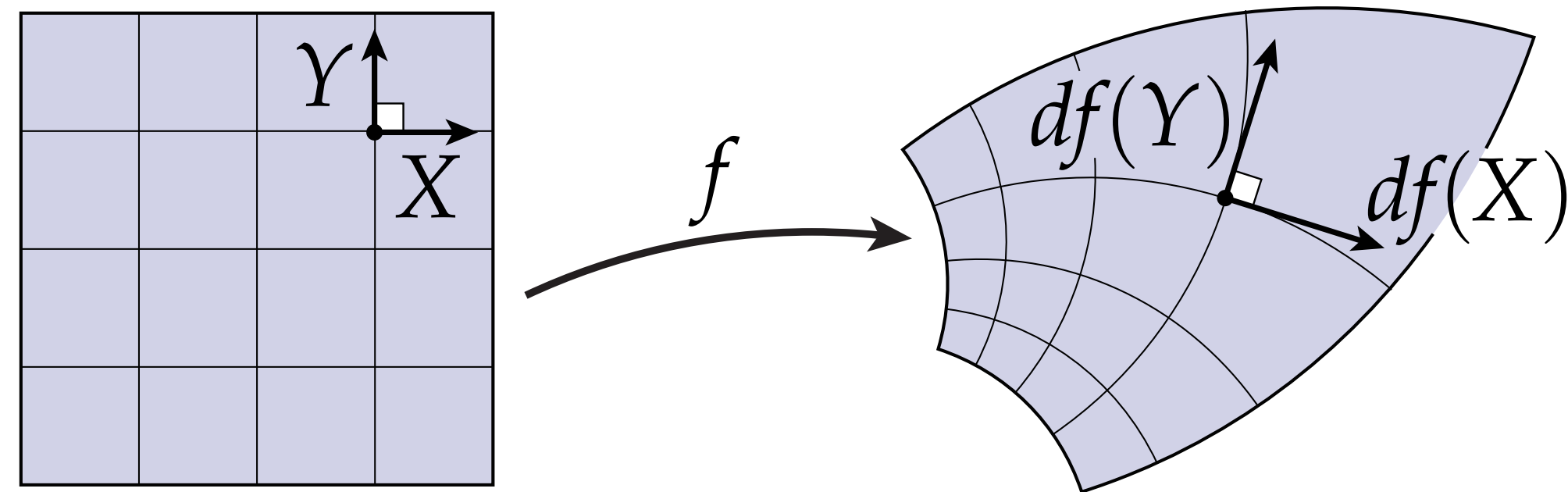


*Shoemake 1985, "Animating Rotation with Quaternion Curves"

**Where else are (hyper-)complex numbers
useful in computer graphics?**

Generating Coordinates for Texture Maps

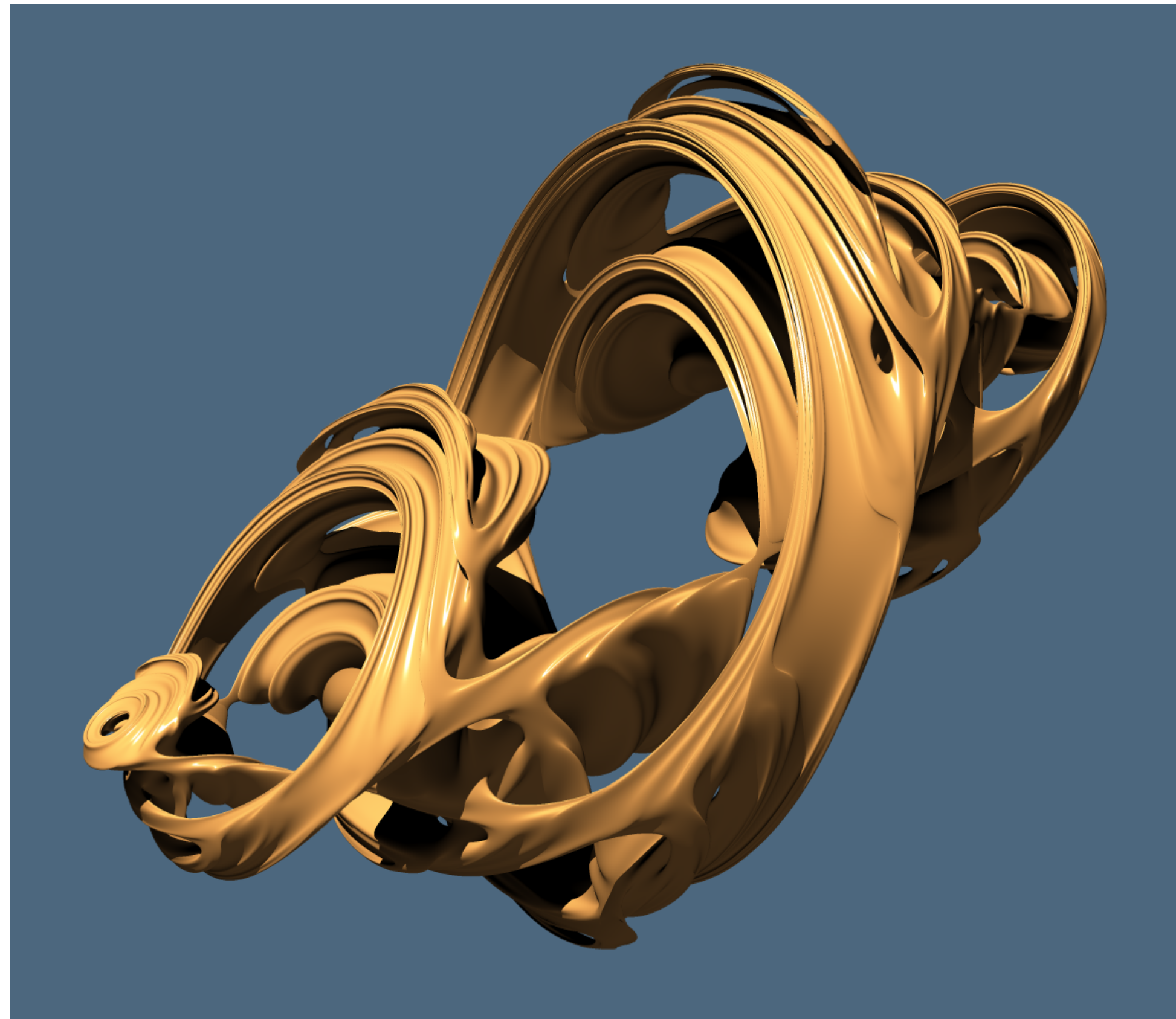
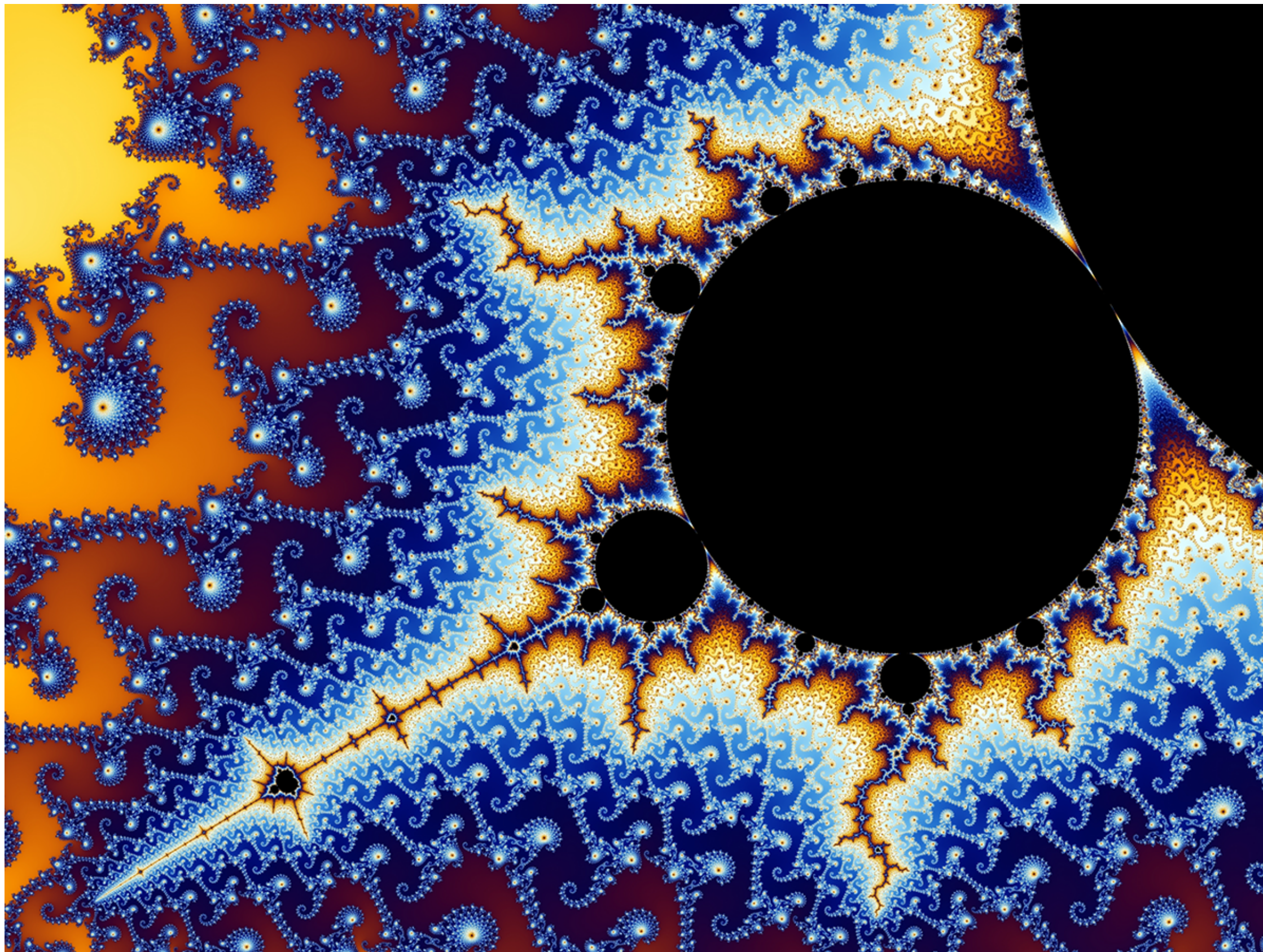
Complex numbers are natural language for angle-preserving (“conformal”) maps



Preserving angles in texture well-tuned to human perception...

Useless-But-Beautiful Example: Fractals

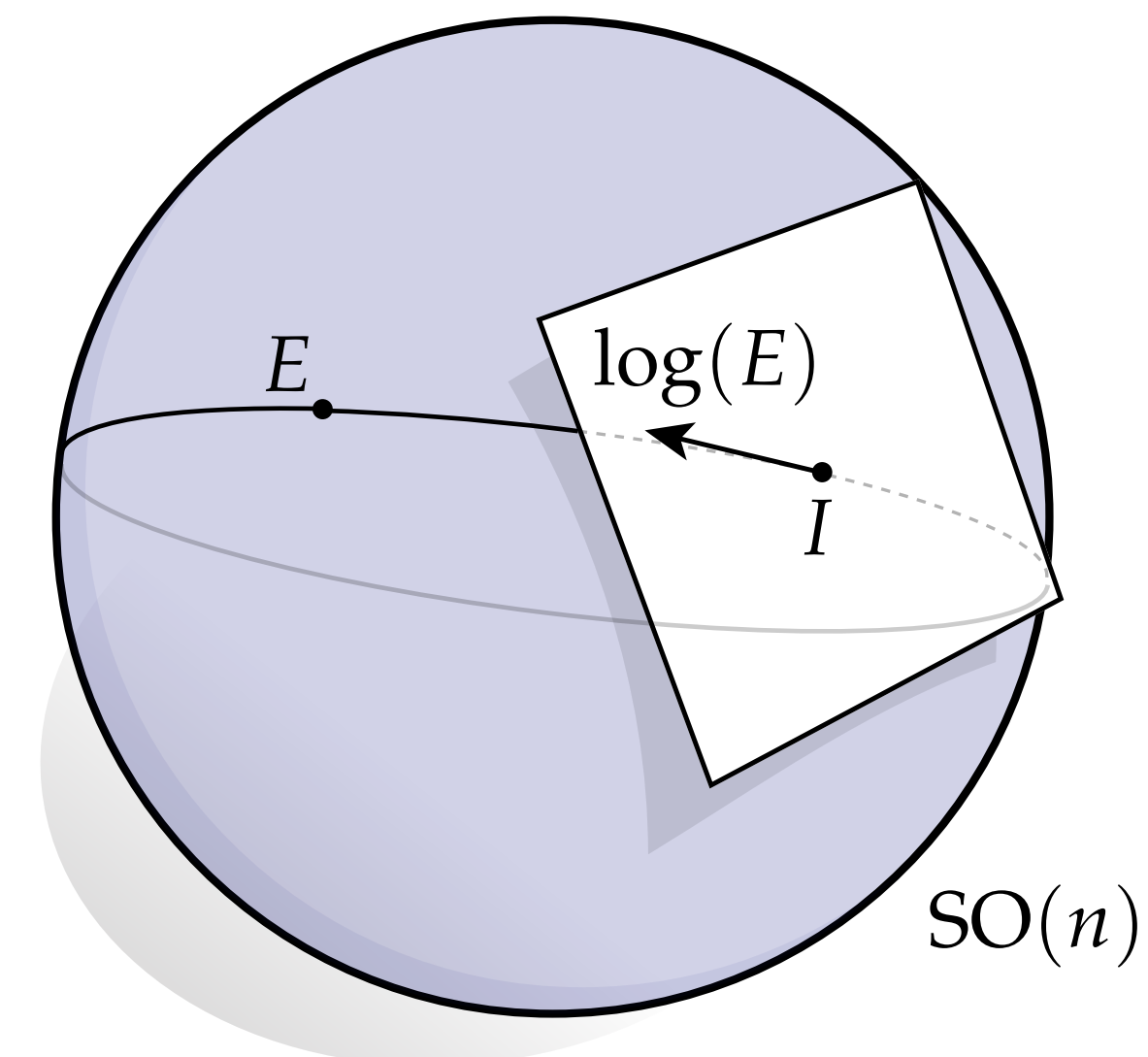
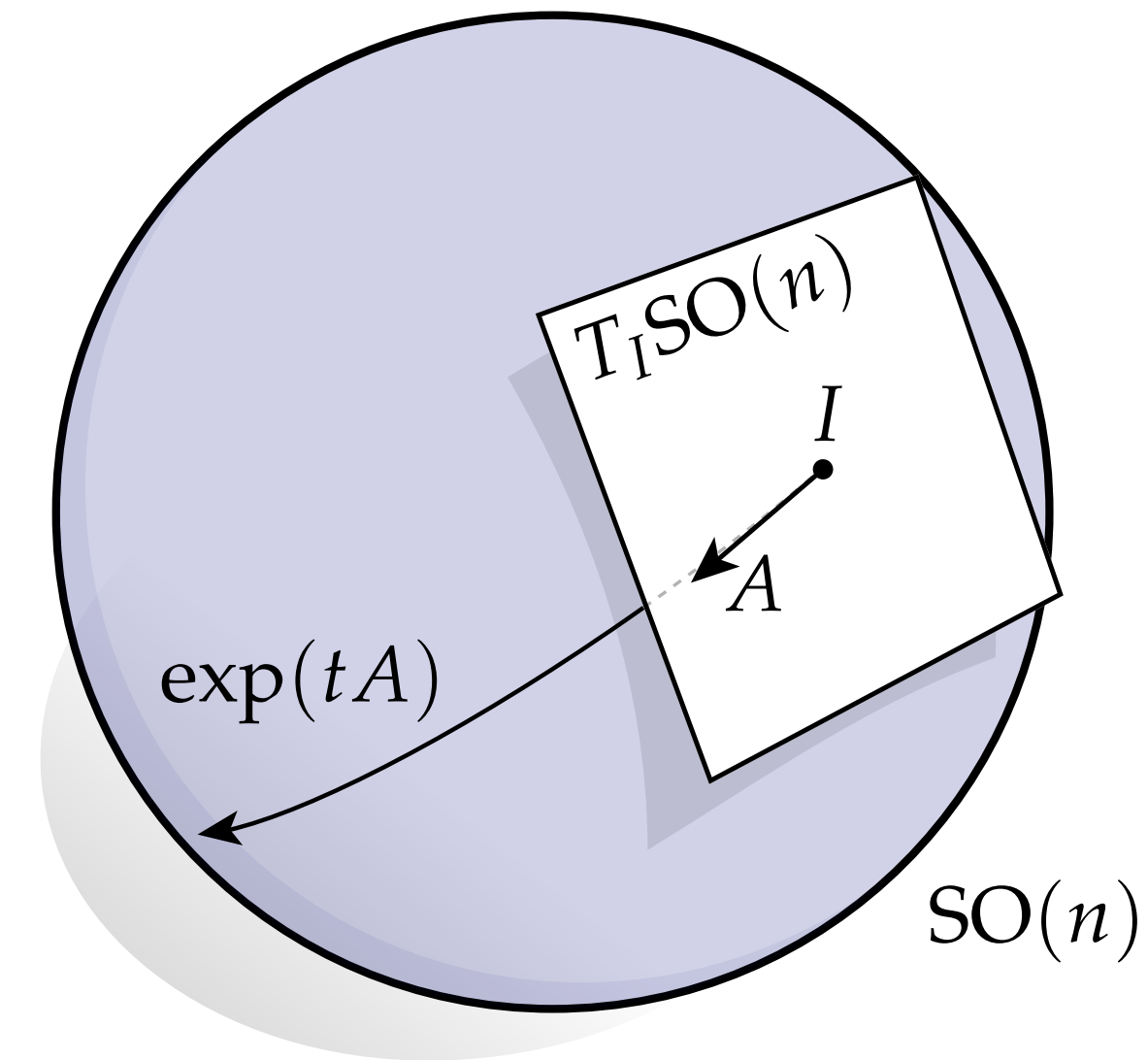
- Defined in terms of iteration on (hyper)complex numbers:



(Will see exactly how this works later in class.)

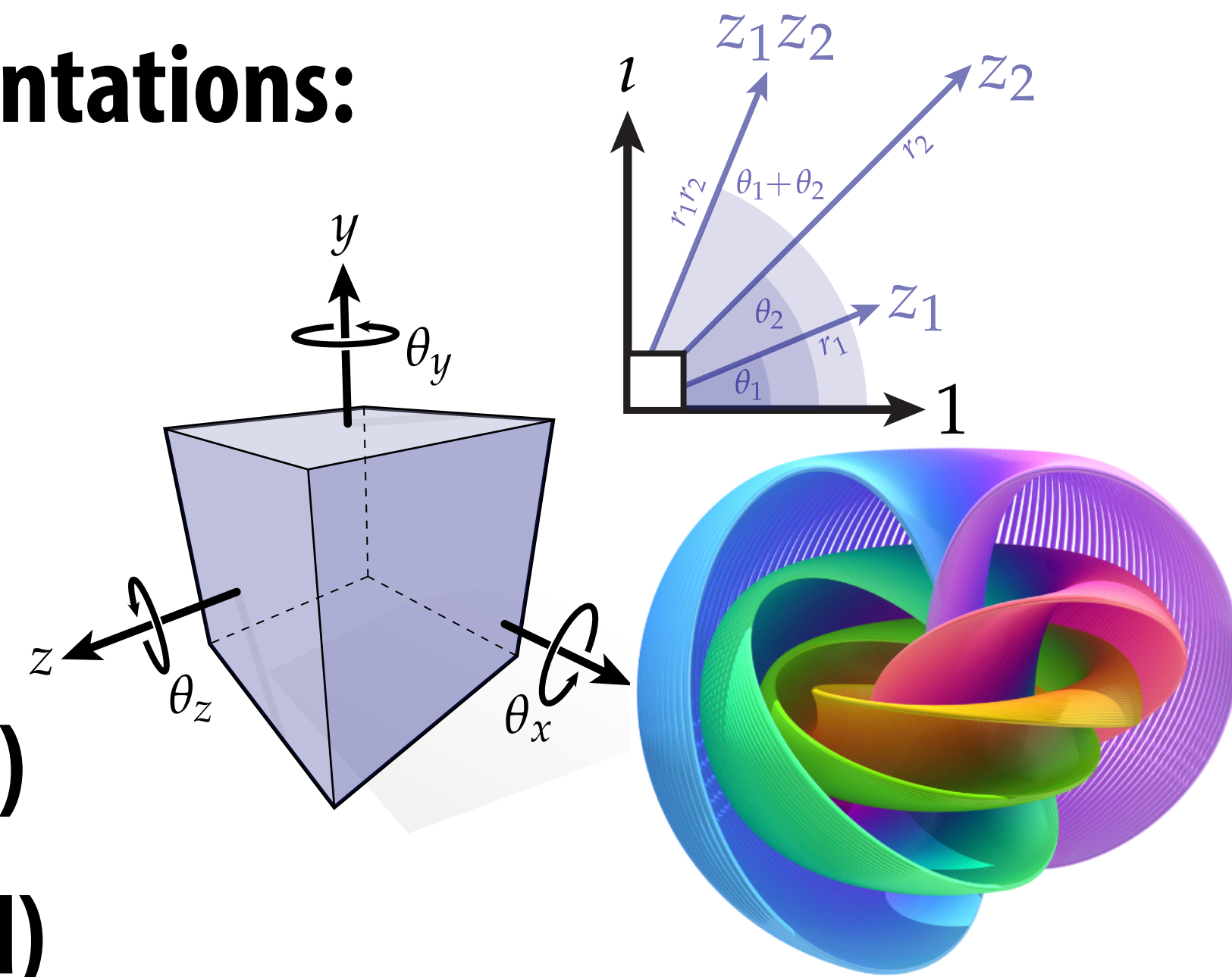
Not Covered: Lie algebras/Lie Groups

- Another super nice/useful perspective on rotations is via “Lie groups” and “Lie algebras”
- More than we have time to cover!
- Many benefits similar to quaternions (easy axis/angle representation, no gimbal lock, ...)
- Nice for encoding angles bigger than 2π
- Also *very* useful for taking averages of rotations
- (Very) short story:
 - exponential map takes you from axis/angle to rotation matrix
 - logarithmic map takes you from rotation matrix to axis/angle



Rotations and Complex Representations—Summary

- Rotations are surprisingly complicated in 3D!
- Today, looked at how complex representations help understand/work with rotations in 3D (& 2D)
- In general, many possible representations:
 - Euler angles
 - axis-angle
 - quaternions
 - Lie group/algebra (not covered)
 - geometric algebra (not covered)
- There's no "right" or "best" way—the more you know, the more you'll be able to do!



Next time: Perspective & Texture Mapping

