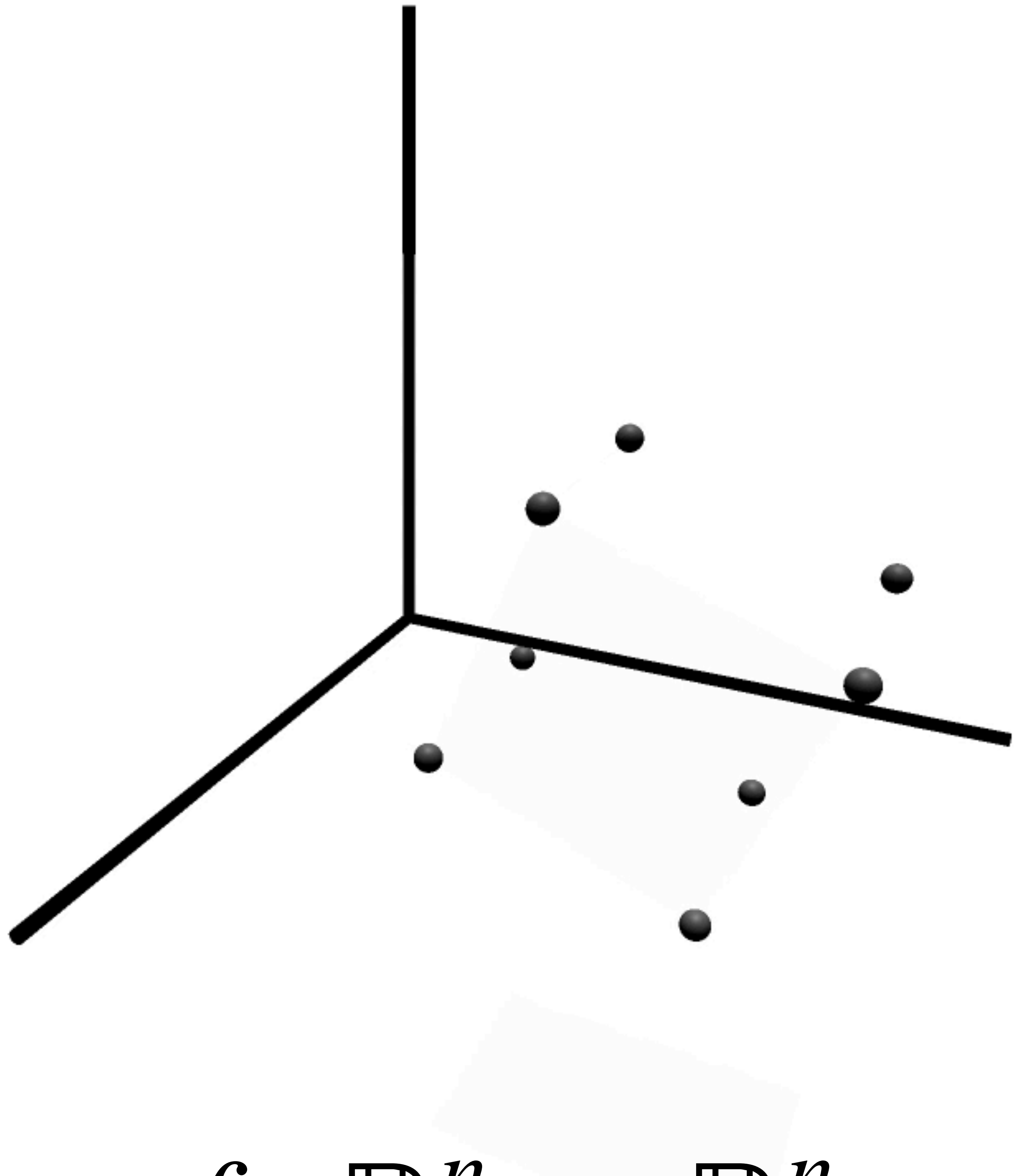
Spatial Transformations

Computer Graphics
CMU 15-462/15-662

Spatial Transformation

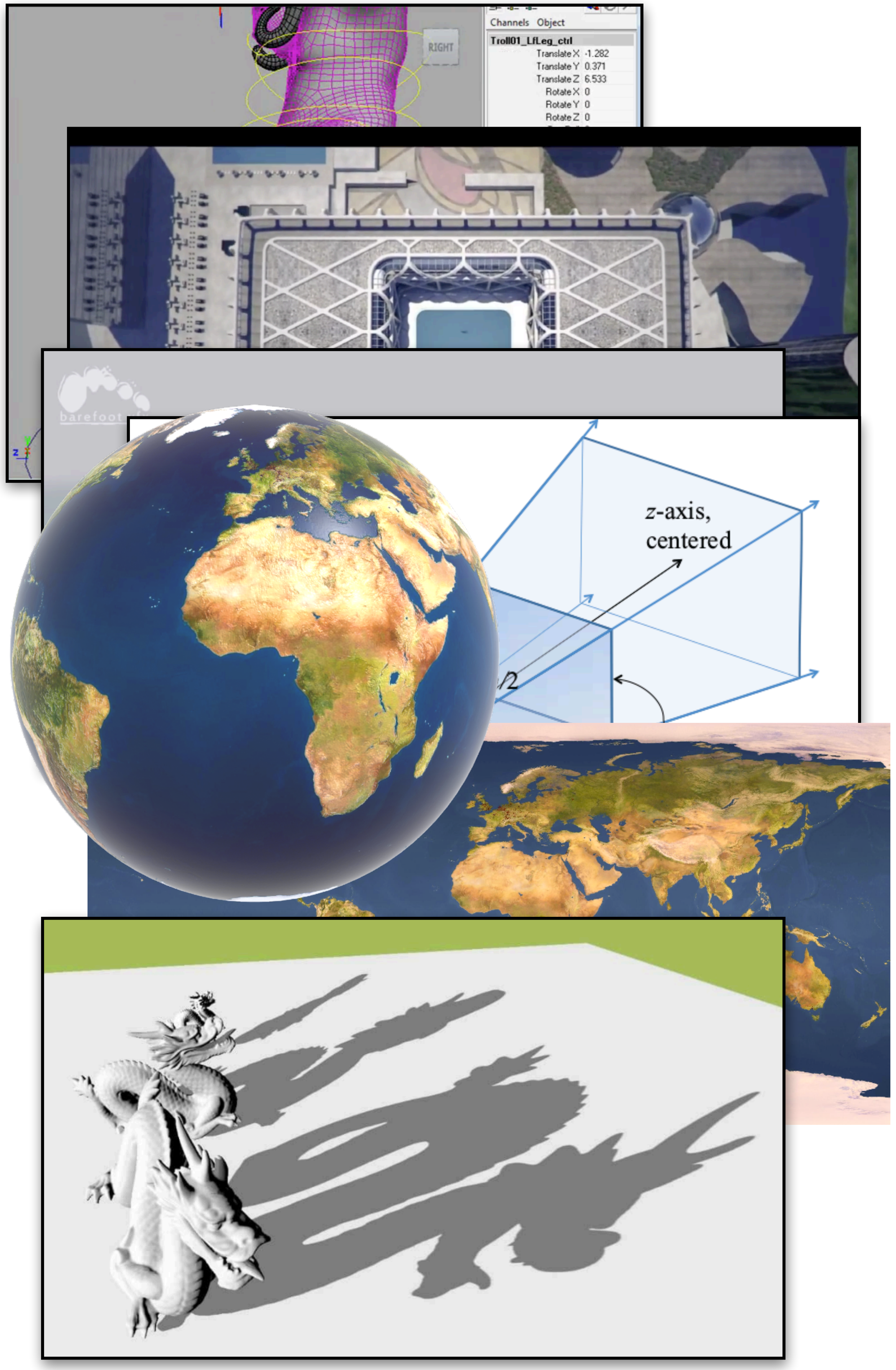
- **Basically any function that assigns each point a new location**
- **Today we'll focus on common transformations of space (rotation, scaling, etc.) encoded by <u>linear</u> maps**



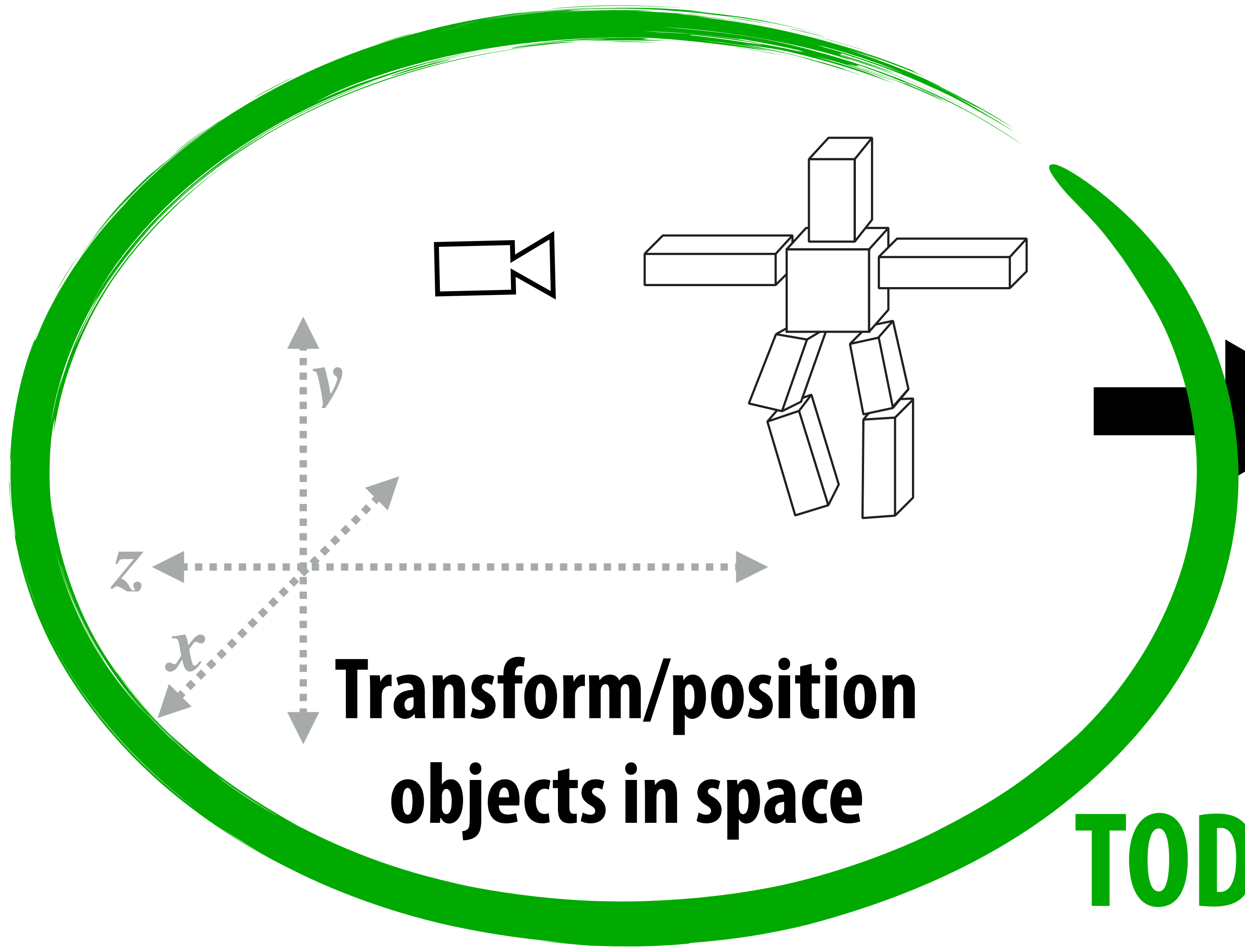
$$f : \mathbb{R}^n \to \mathbb{R}^n$$

Transformations in Computer Graphics

- Where are linear transformations used in computer graphics?
- All over the place!
  - Position/deform objects in space
  - Move the camera
  - Animate objects over time
  - Project 3D objects onto 2D images
  - Map 2D textures onto 3D objects
  - Project shadows of 3D objects onto other 3D objects
  - ...

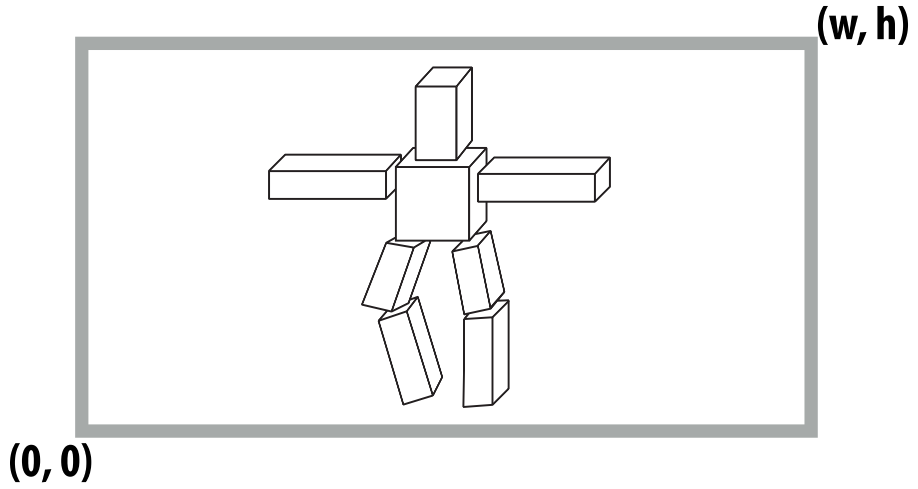


The Rasterization Pipeline

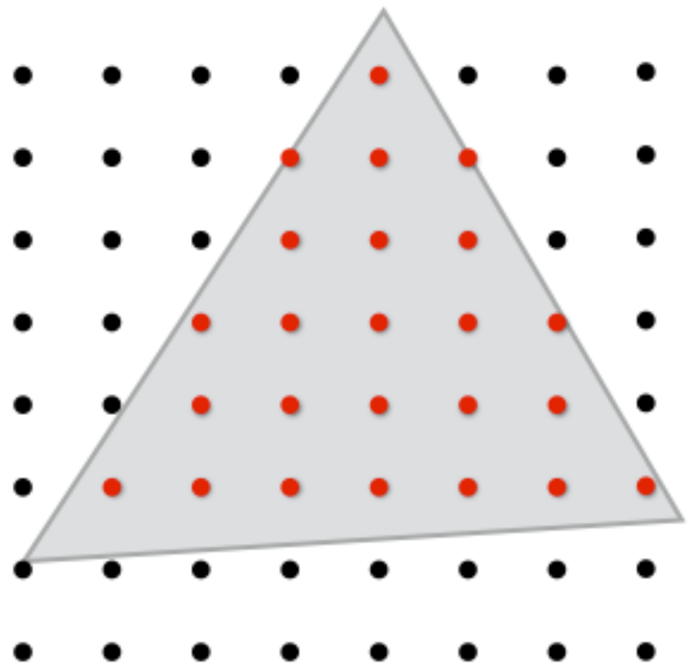


Transform/position objects in space

TODAY



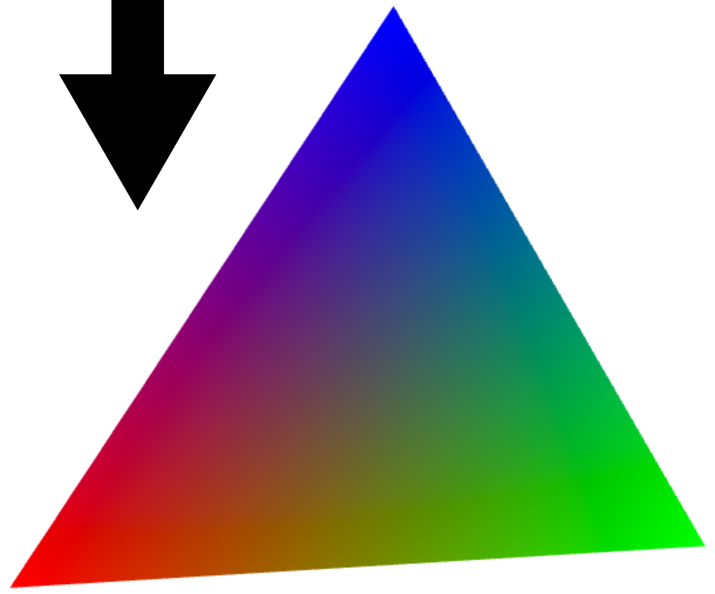
Project objects onto the screen



Sample triangle coverage

Combine samples into final image (depth, alpha, …)

Sample texture maps / evaluate shaders

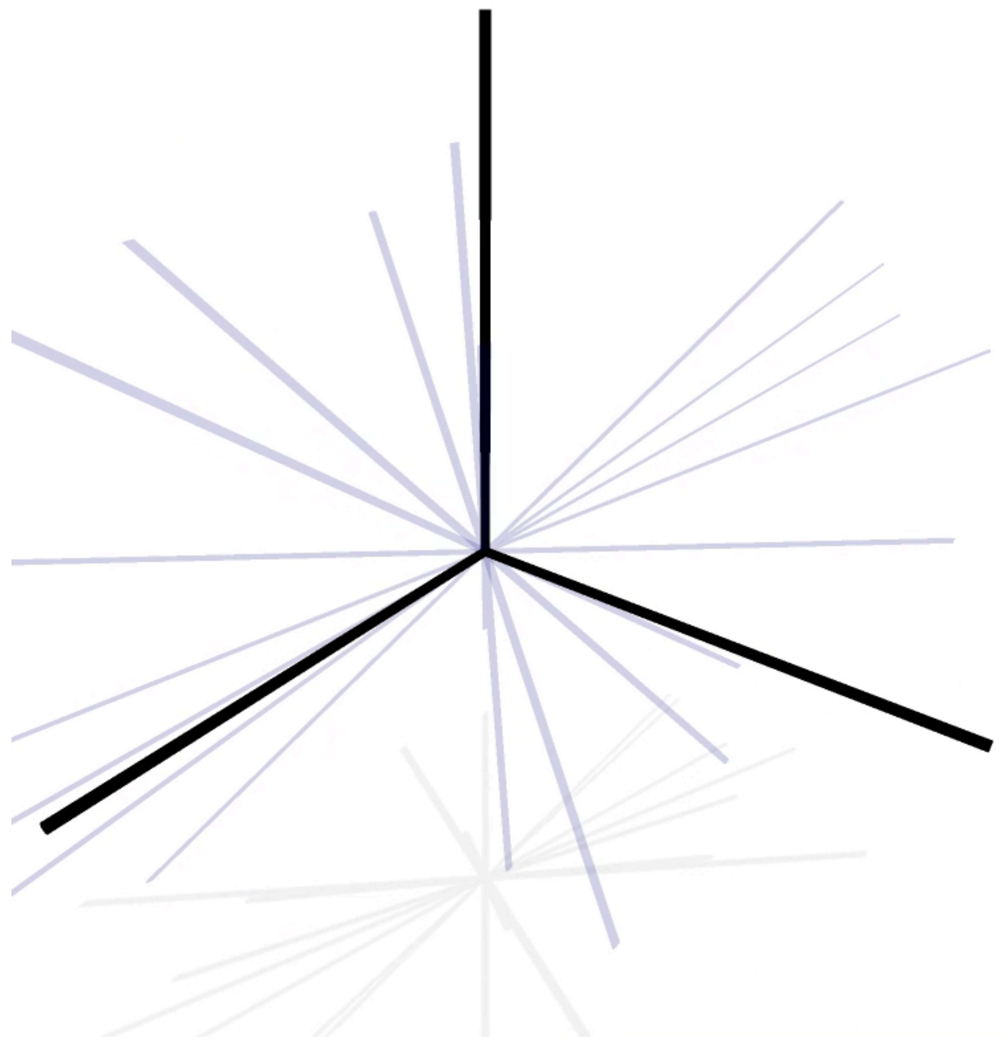


Interpolate triangle attributes at covered samples

Review: Linear Maps

Q: What does it mean for a map $f : \mathbb{R}^n \to \mathbb{R}^n$ to be <u>linear</u>?

Geometrically: it maps <u>lines</u> to <u>lines</u>, and preserves the origin



Algebraically: preserves vector space operations (addition & scaling)

$$\begin{tikzcd}
\mathbf{x}, \mathbf{y} \arrow[r, "\text{add first}"] \arrow[d, "\text{apply f first}"] & \mathbf{x} + \mathbf{y} \arrow[d, "\text{then apply f}"] \\
f(\mathbf{x}), f(\mathbf{y}) \arrow[r, "\text{then add}"] & f(\mathbf{x}) + f(\mathbf{y}) = f(\mathbf{x} + \mathbf{y})
\end{tikzcd}$$

Why do we care about *linear* transformations?

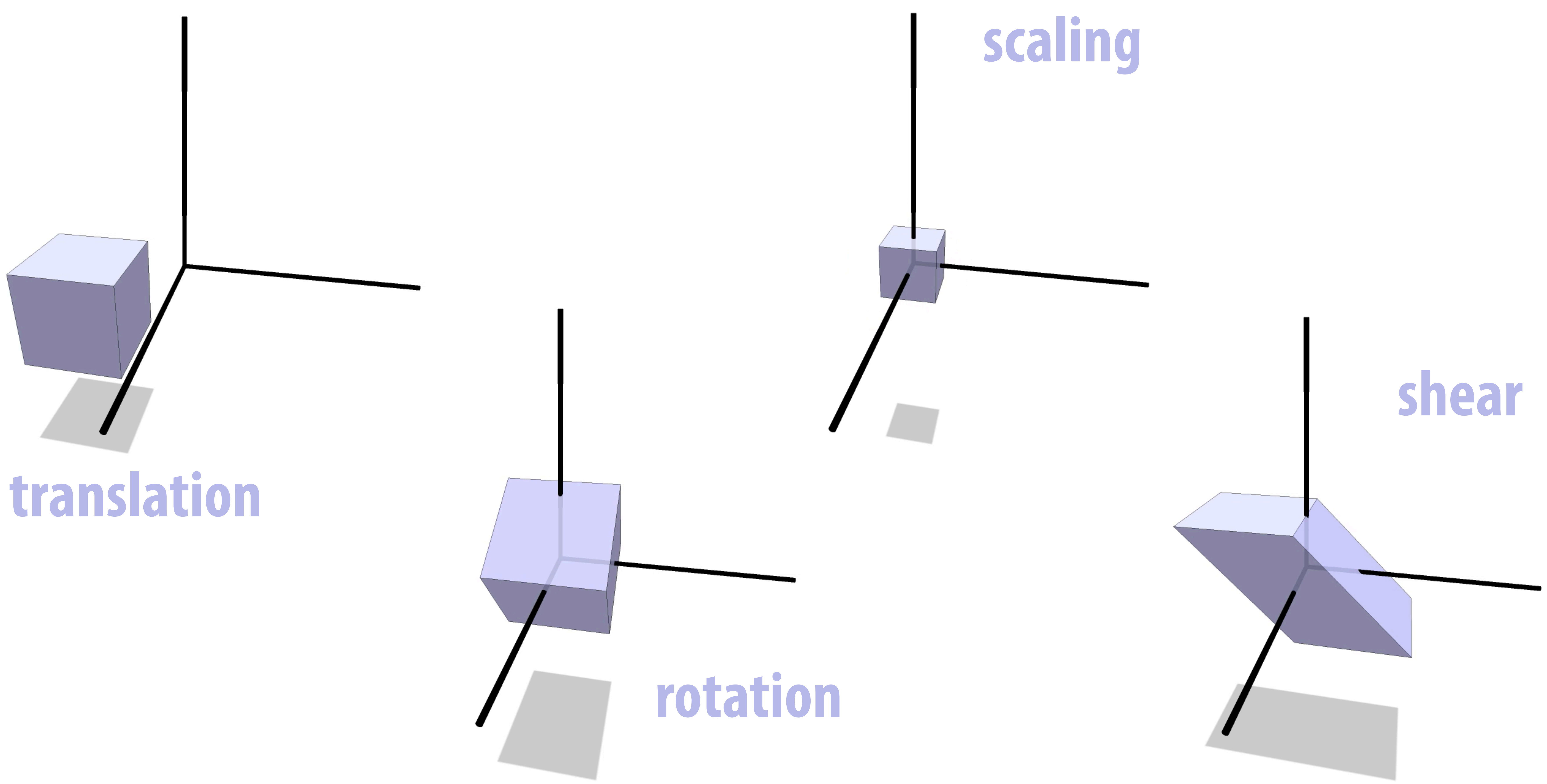
- Cheap to apply
- Usually pretty easy to solve for (linear systems)
- Composition of linear transformations is linear
  - product of <u>many</u> matrices is a <u>single</u> matrix
  - gives uniform representation of transformations
  - simplifies graphics algorithms, systems (e.g., GPUs & APIs)

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\textit{rotation}} \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\textit{scale}} \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{\textit{rotation}} \cdots = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}}_{\textit{composite transformation}}$$

What kinds of linear transformations can we compose?

Types of Transformations

What would you call each of these types of transformations?



Q: How did you know that? (Hint: you did <u>not</u> inspect a formula!)

Invariants of Transformation

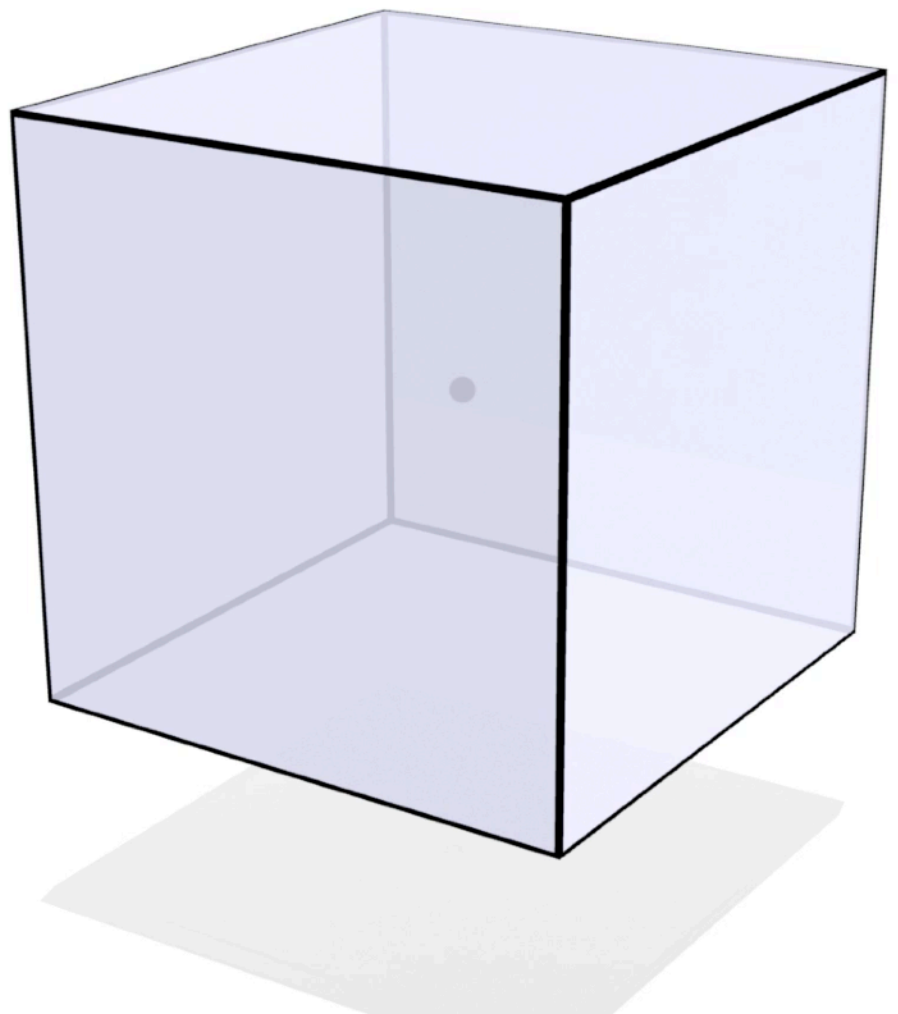
A transformation is determined by the invariants it preserves

transformation	invariants	algebraic description
linear	*straight lines / origin*	$f(a\mathbf{x}+\mathbf{y}) = af(\mathbf{x}) + f(\mathbf{y})$, $f(0) = 0$
translation	*differences between pairs of points*	$f(\mathbf{x}-\mathbf{y}) = \mathbf{x}-\mathbf{y}$
scaling	*lines through the origin / direction of vectors*	$f(\mathbf{x})/\lvert f(\mathbf{x})\rvert = \mathbf{x}/\lvert\mathbf{x}\rvert$
rotation	*origin / distances between points / orientation*	$\lvert f(\mathbf{x})-f(\mathbf{y})\rvert = \lvert\mathbf{x}-\mathbf{y}\rvert$, $\det(f) > 0$
…	…	…

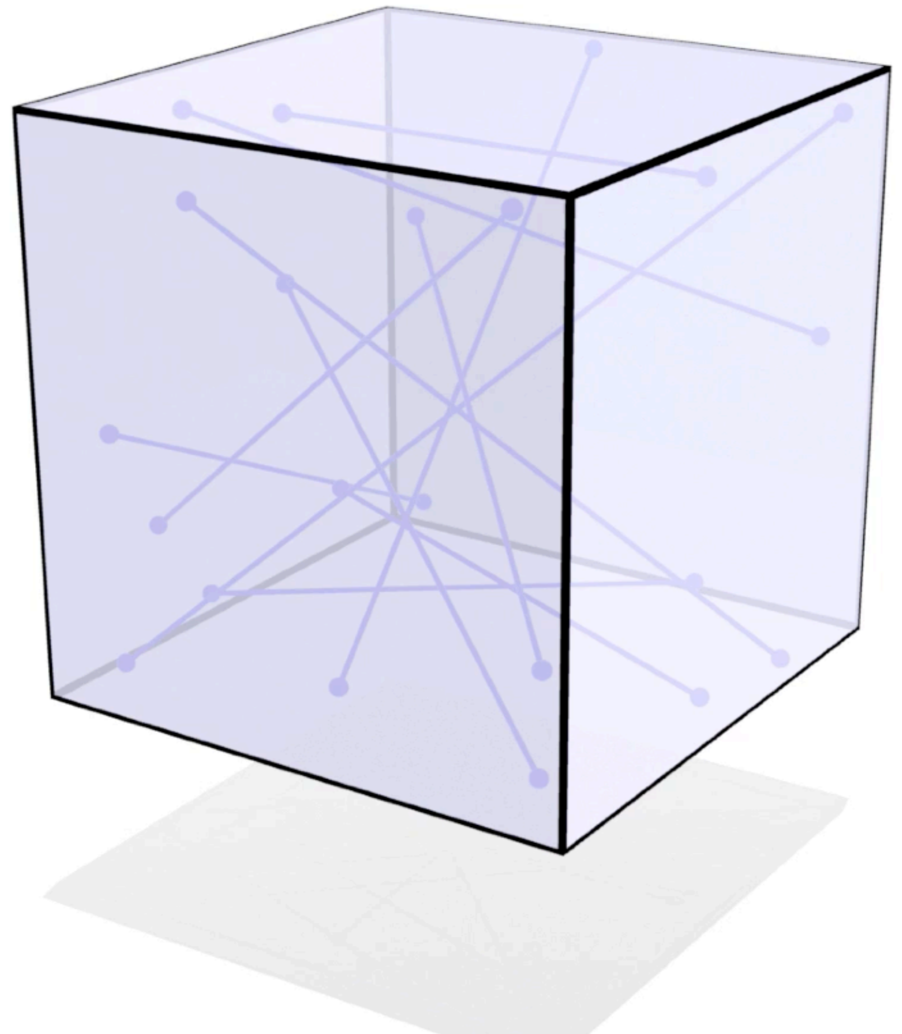
(Essentially how your brain "knows" what kind of transformation you're looking at…)

Rotation

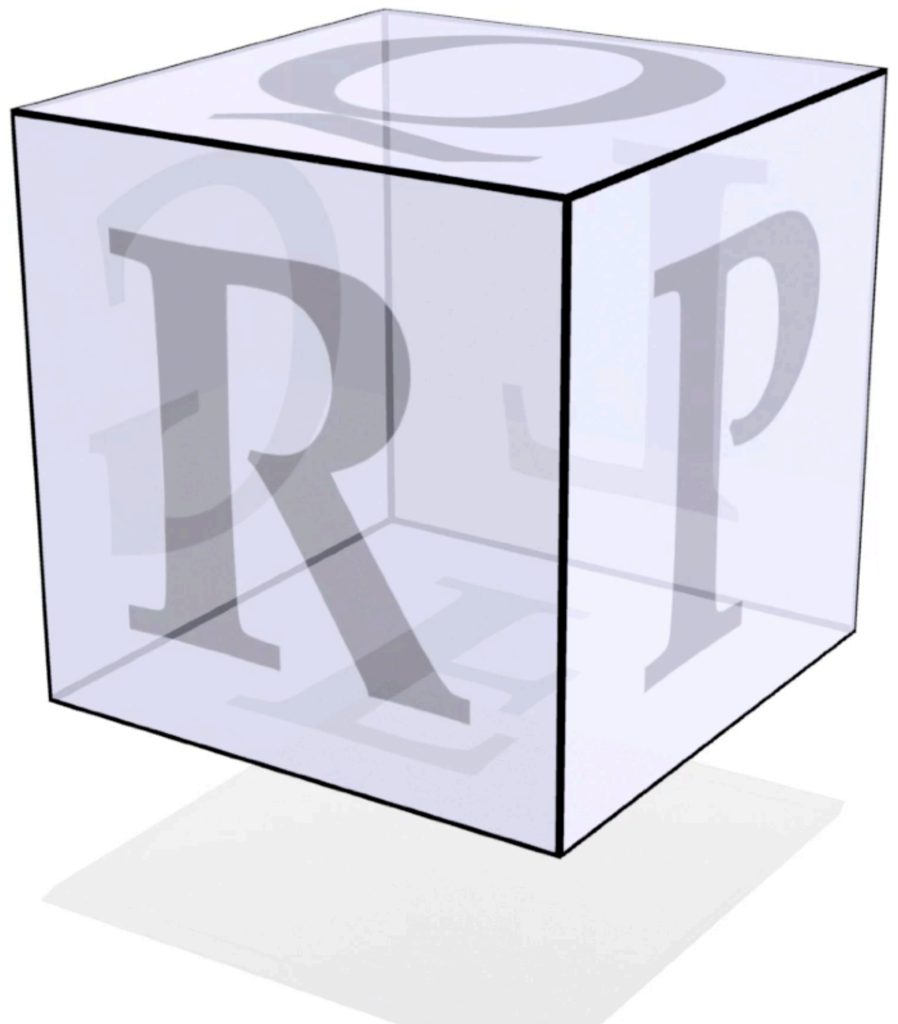
Rotations defined by three basic properties:



keeps origin fixed



preserves distances



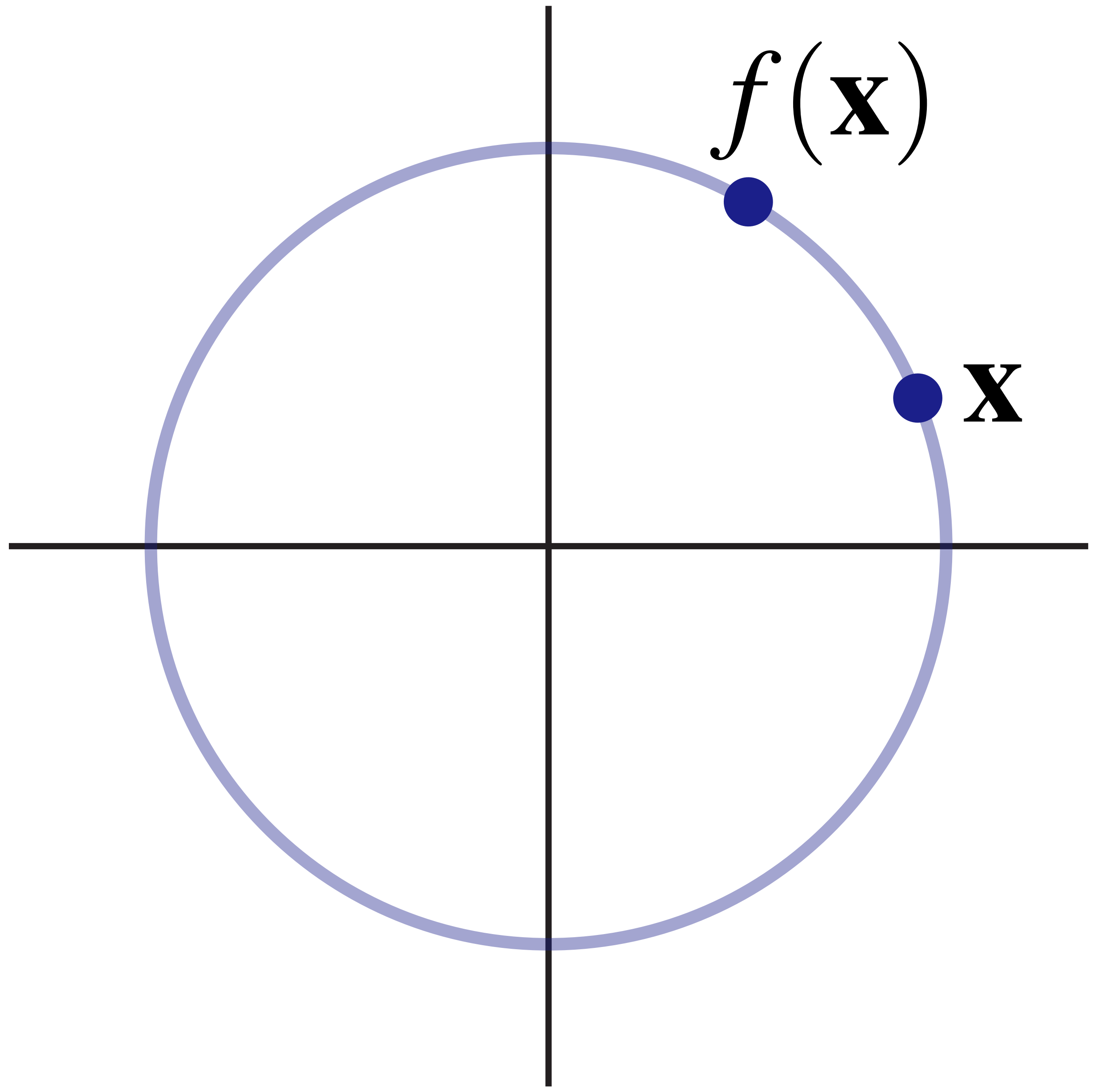
preserves orientation

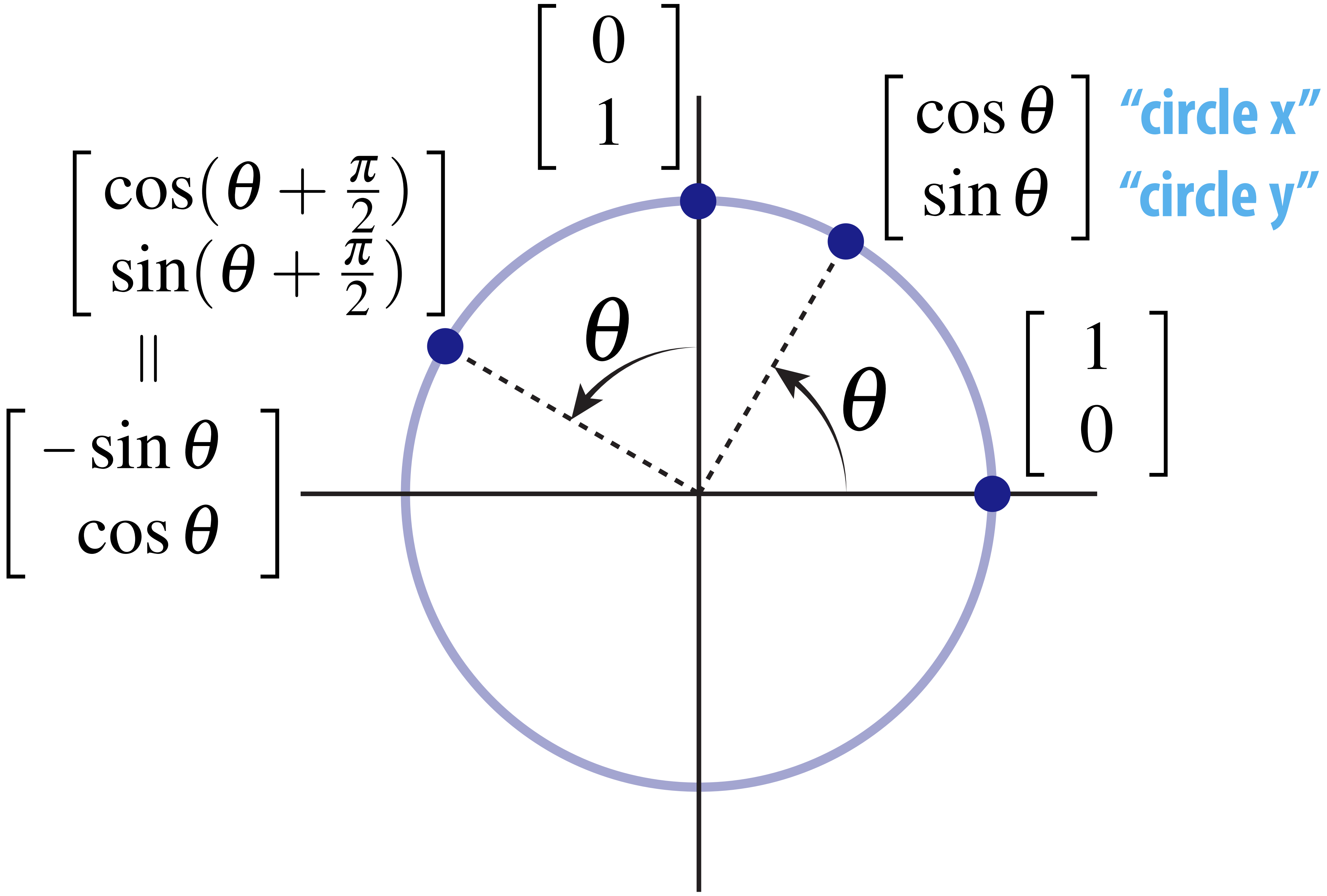
First two properties together imply that rotations are <u>linear</u>.

Will have a *lot* more to say about rotations next lecture…

2D Rotations—Matrix Representation

Rotations preserve distances and the origin—hence, a 2D rotation by an angle θ maps each point $\mathbf{x}$ to a point $f_\theta(\mathbf{x})$ on the circle of radius $|\mathbf{x}|$:

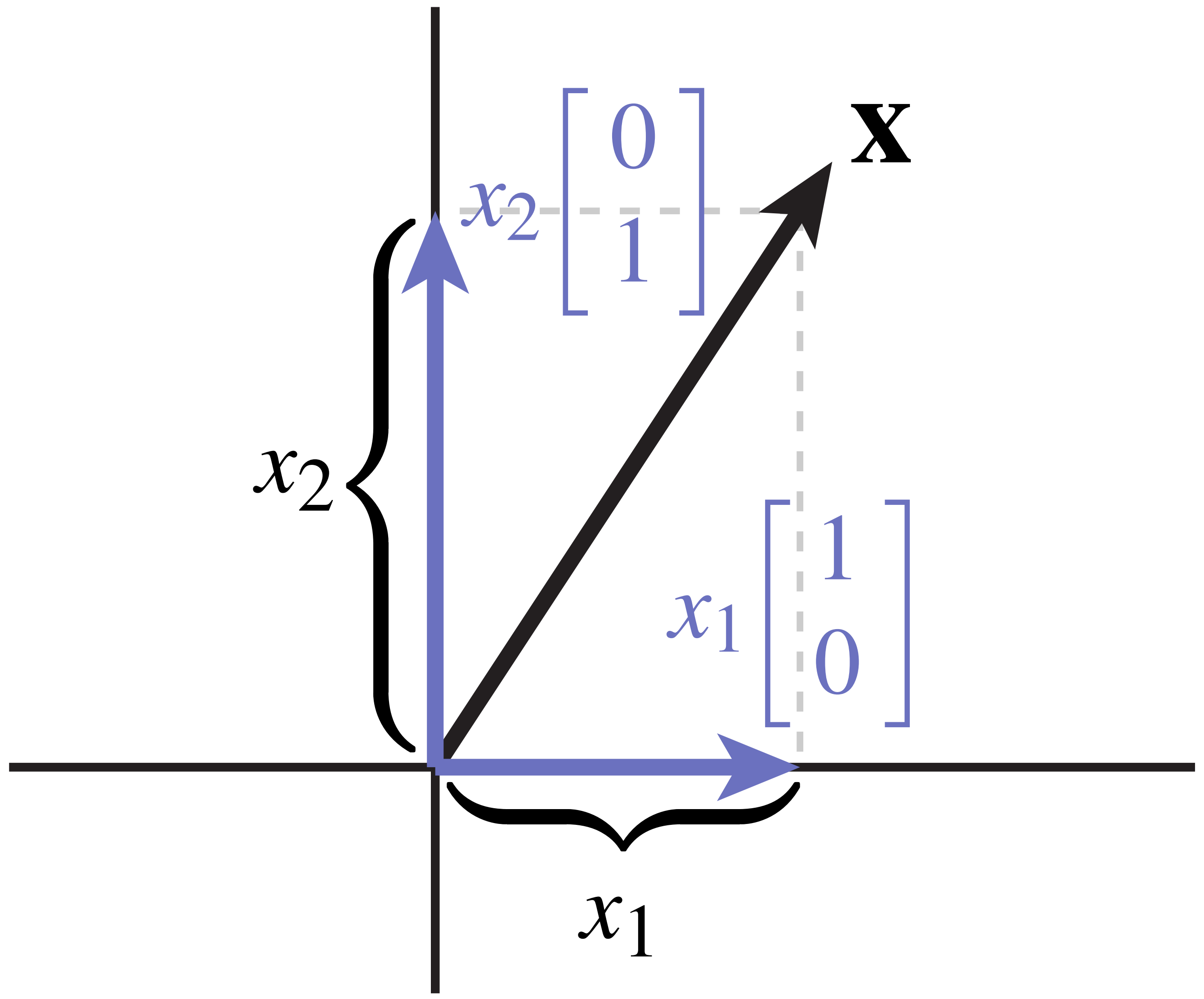


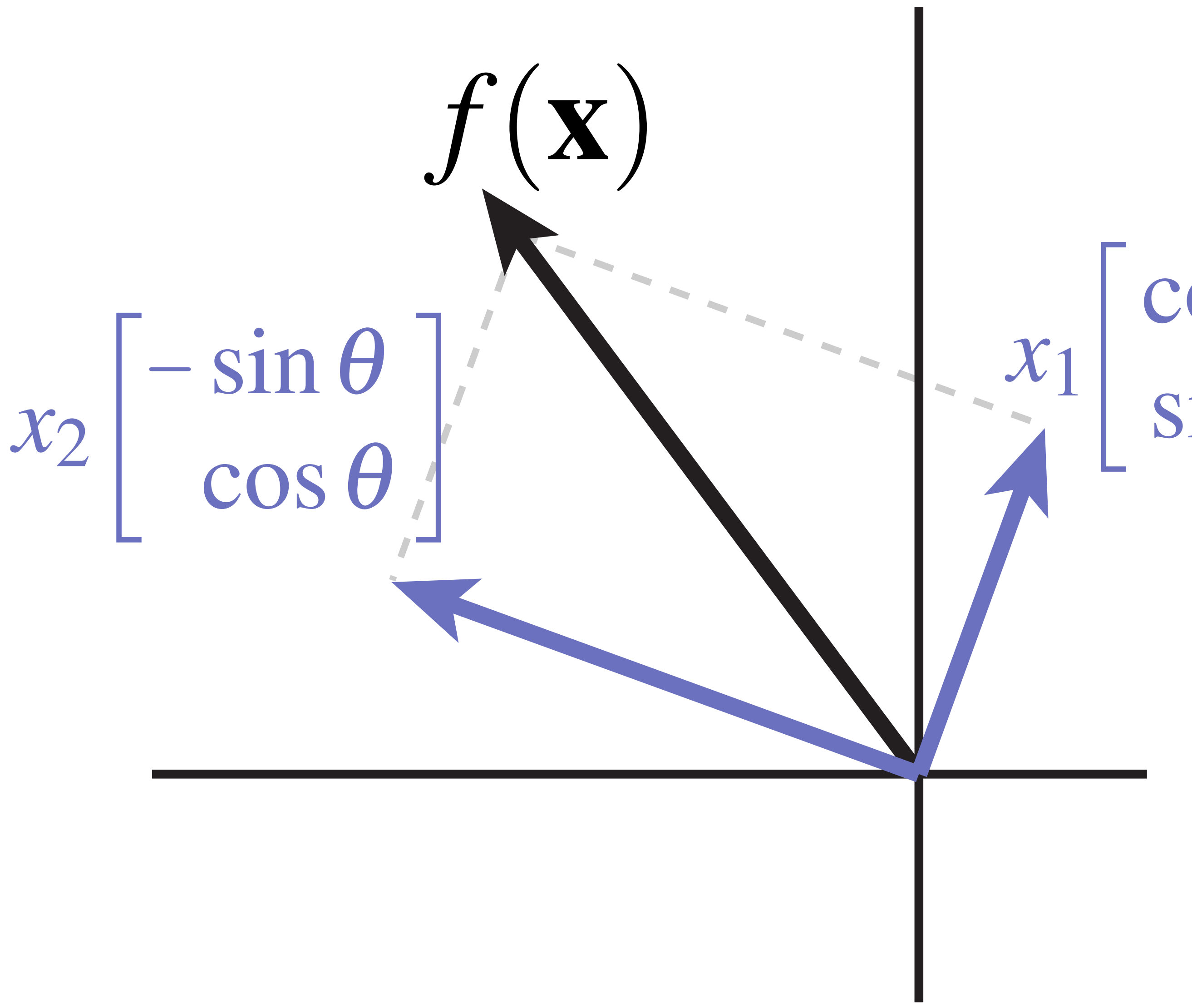


- **Where does $\mathbf{x} = (1,0)$ go if we rotate by θ (counter-clockwise)?**
- **How about $\mathbf{x} = (0,1)$?**

What about a general vector $\mathbf{x} = (x_1, x_2)$?

2D Rotations—Matrix Representation





$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$f(\mathbf{x}) = x_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

So, How do we represent the 2D rotation function $f_\theta(\mathbf{x})$ using a matrix?

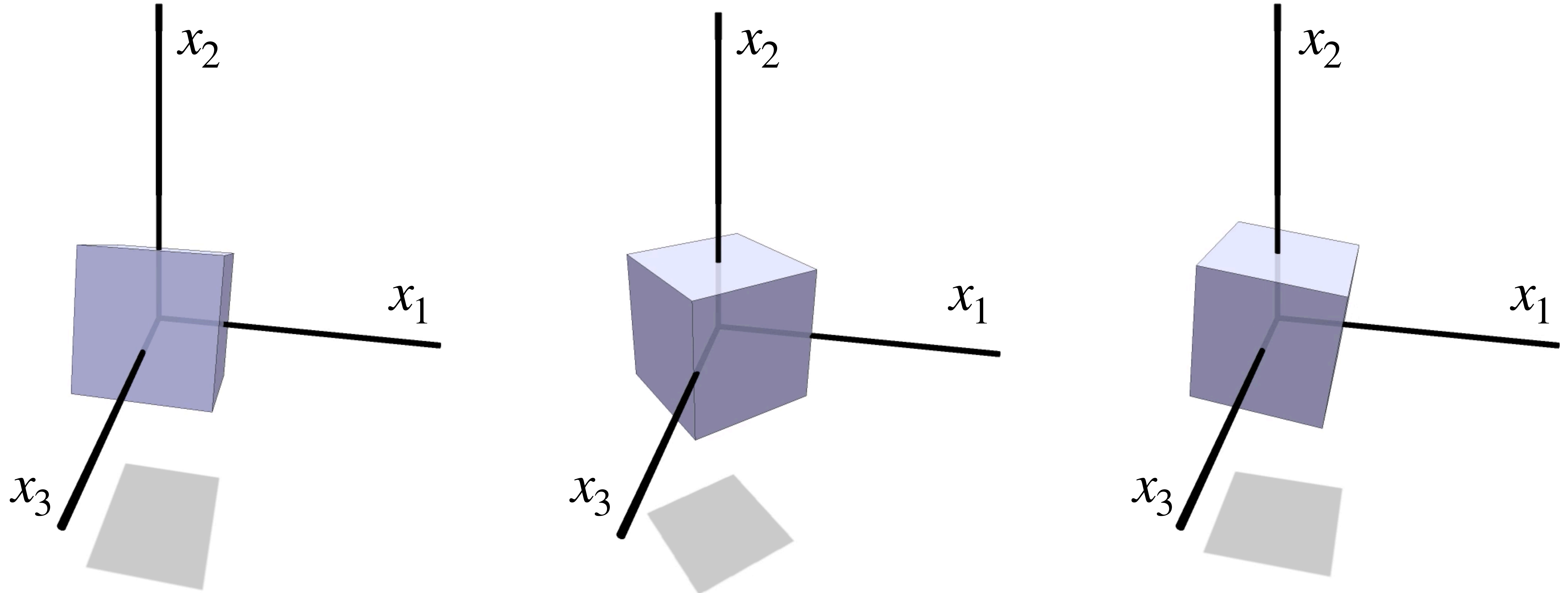
$$f_\theta(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3D Rotations

- **Q: In 3D, how do we rotate around the x_3-axis?**
- **A: Just apply the same transformation of x_1, x_2; keep x_3 fixed**

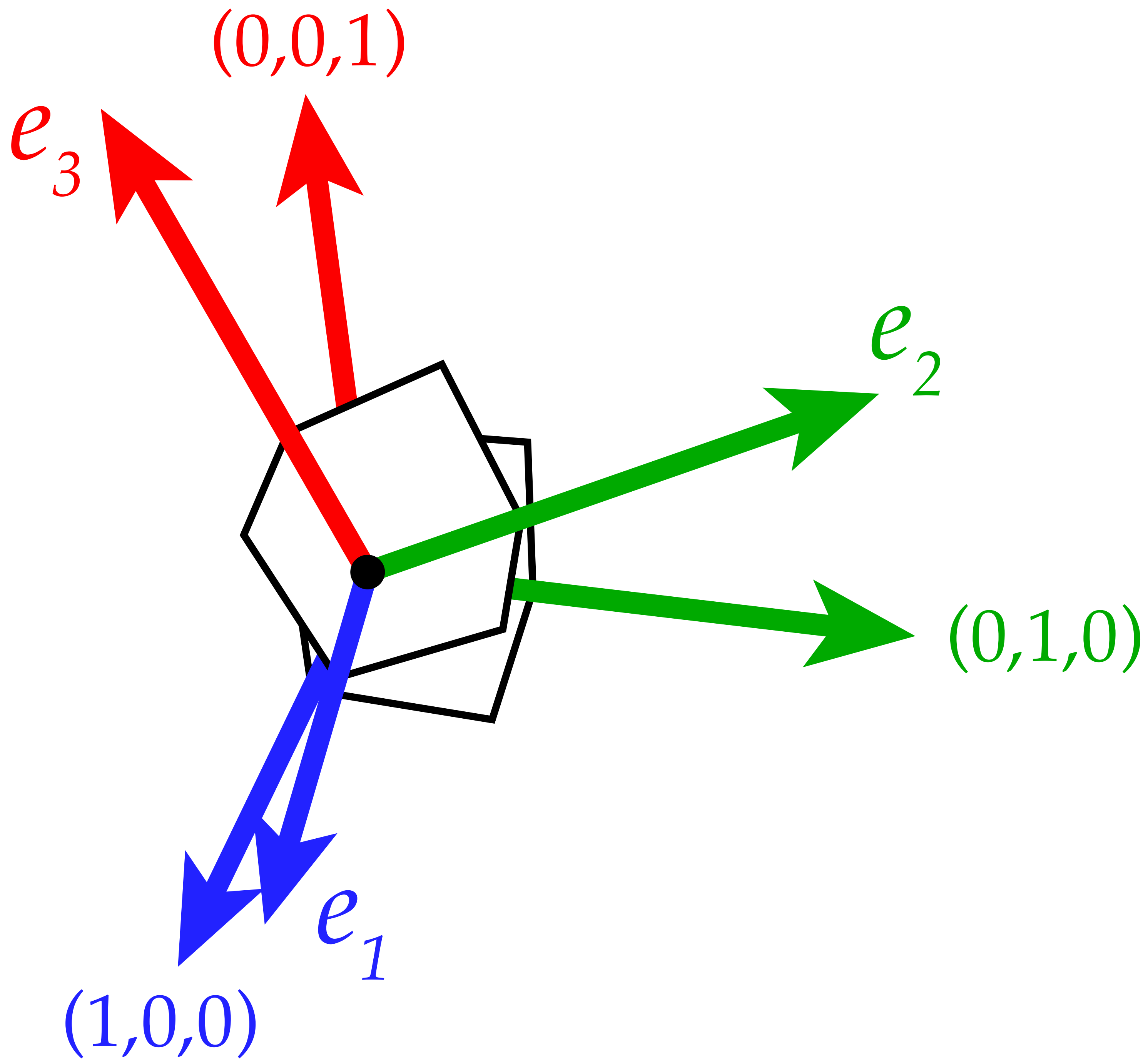
rotate around x_1 | **rotate around x_2** | **rotate around x_3**

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin(\theta) \\ 0 & \sin\theta & \cos(\theta) \end{bmatrix}
\begin{bmatrix} \cos\theta & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos(\theta) \end{bmatrix}
\begin{bmatrix} \cos\theta & -\sin(\theta) & 0 \\ \sin\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$



Rotations—Transpose as Inverse

Rotation will map standard basis to orthonormal basis e_1, e_2, e_3:



$$
\overset{R^\mathsf{T}}{\begin{bmatrix} \text{---} & e_1^T & \text{---} \\ \text{---} & e_2^T & \text{---} \\ \text{---} & e_3^T & \text{---} \end{bmatrix}}
\overset{R}{\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}}
= \underset{I}{\begin{bmatrix} e_1^T e_1 & e_1^T e_2 & e_1^T e_3 \\ e_2^T e_1 & e_2^T e_2 & e_2^T e_3 \\ e_3^T e_1 & e_3^T e_2 & e_3^T e_3 \end{bmatrix}}
$$

Hence, $R^\mathsf{T} R = I$, or equivalently, $R^\mathsf{T} = R^{-1}$.

Reflections

- Q: Does every matrix $Q^\mathsf{T} Q = I$ describe a rotation?
- Remember that rotations must preserve the origin, preserve distances, and preserve orientation
- Consider for instance this matrix:

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q^\mathsf{T} Q = \begin{bmatrix} (-1)^2 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Q: Does this matrix represent a rotation?
(If not, which invariant does it fail to preserve?)

A: No! It represents a reflection across the y-axis
(and hence fails to preserve orientation)

Orthogonal Transformations

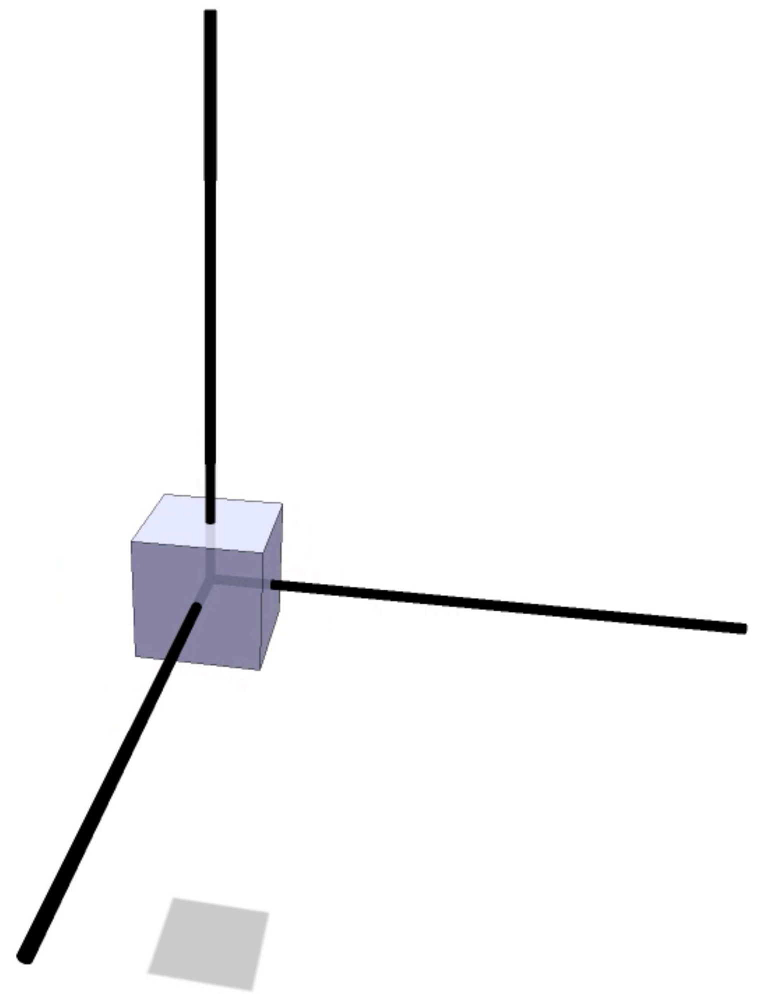
- In general, transformations that preserve <u>distances</u> and the <u>origin</u> are called *orthogonal transformations*
- Represented by matrices $Q^\mathsf{T}Q = I$
  - Rotations additionally <u>preserve</u> orientation: $\det(Q) > 0$
  - Reflections <u>reverse</u> orientation: $\det(Q) < 0$

rotation

reflection

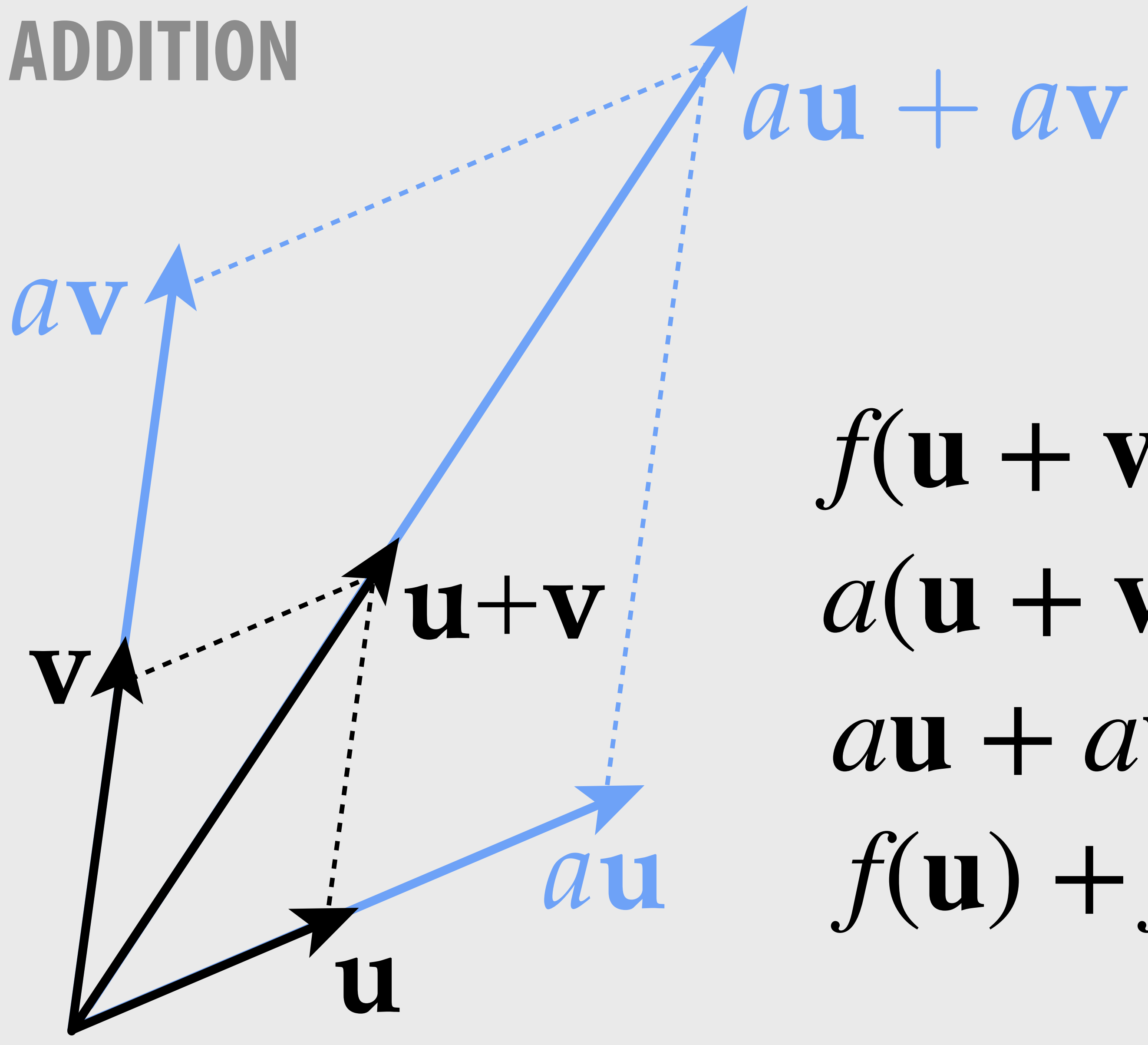
Scaling

*assuming $a \neq 0,\ \mathbf{u} \neq 0$



- Each vector $\mathbf{u}$ gets mapped to a scalar multiple
  - $f(\mathbf{u}) = a\mathbf{u}, \quad a \in \mathbb{R}$
- Preserves the <u>direction</u> of all vectors*
  - $\dfrac{\mathbf{u}}{|\mathbf{u}|} = \dfrac{a\mathbf{u}}{|a\mathbf{u}|}$
- Q: Is scaling a linear transformation? A: Yes!

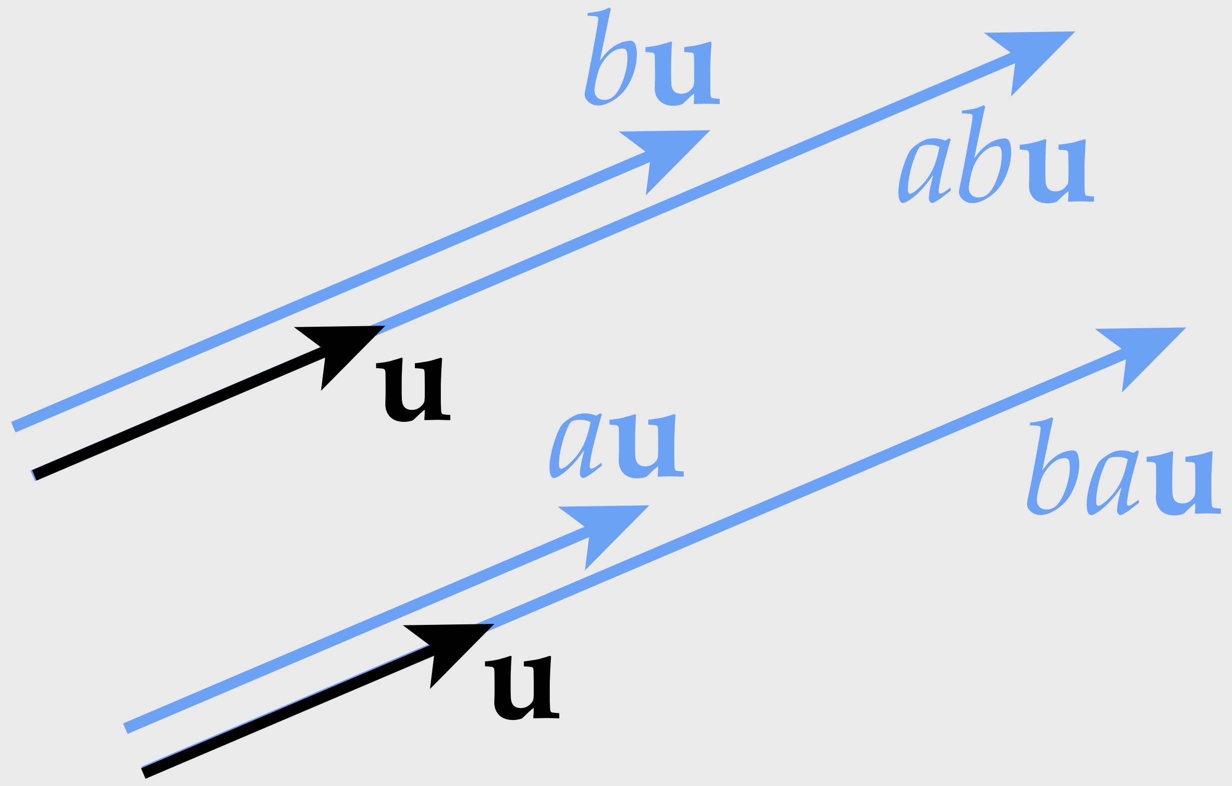
ADDITION



$$f(\mathbf{u}+\mathbf{v}) = a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v} = f(\mathbf{u}) + f(\mathbf{v})$$

SCALAR MULTIPLICATION

$$f(b\mathbf{u}) = ab\mathbf{u} = ba\mathbf{u} = bf(\mathbf{u})$$



Scaling — Matrix Representation

Q: Suppose we want to scale a vector $\mathbf{u} = (u_1, u_2, u_3)$ by a .
How would we represent this operation via a <u>matrix</u>?

A: Just build a *diagonal* matrix D, with a along the diagonal:

$$\underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}}_{a\mathbf{u}}$$

Q: What happens if a is <u>negative</u>?

Negative Scaling

For $a = -1$, can think of scaling by a as sequence of reflections.

E.g., in 2D:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since each reflection reverses orientation, orientation is preserved.

What about 3D?

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} =$$

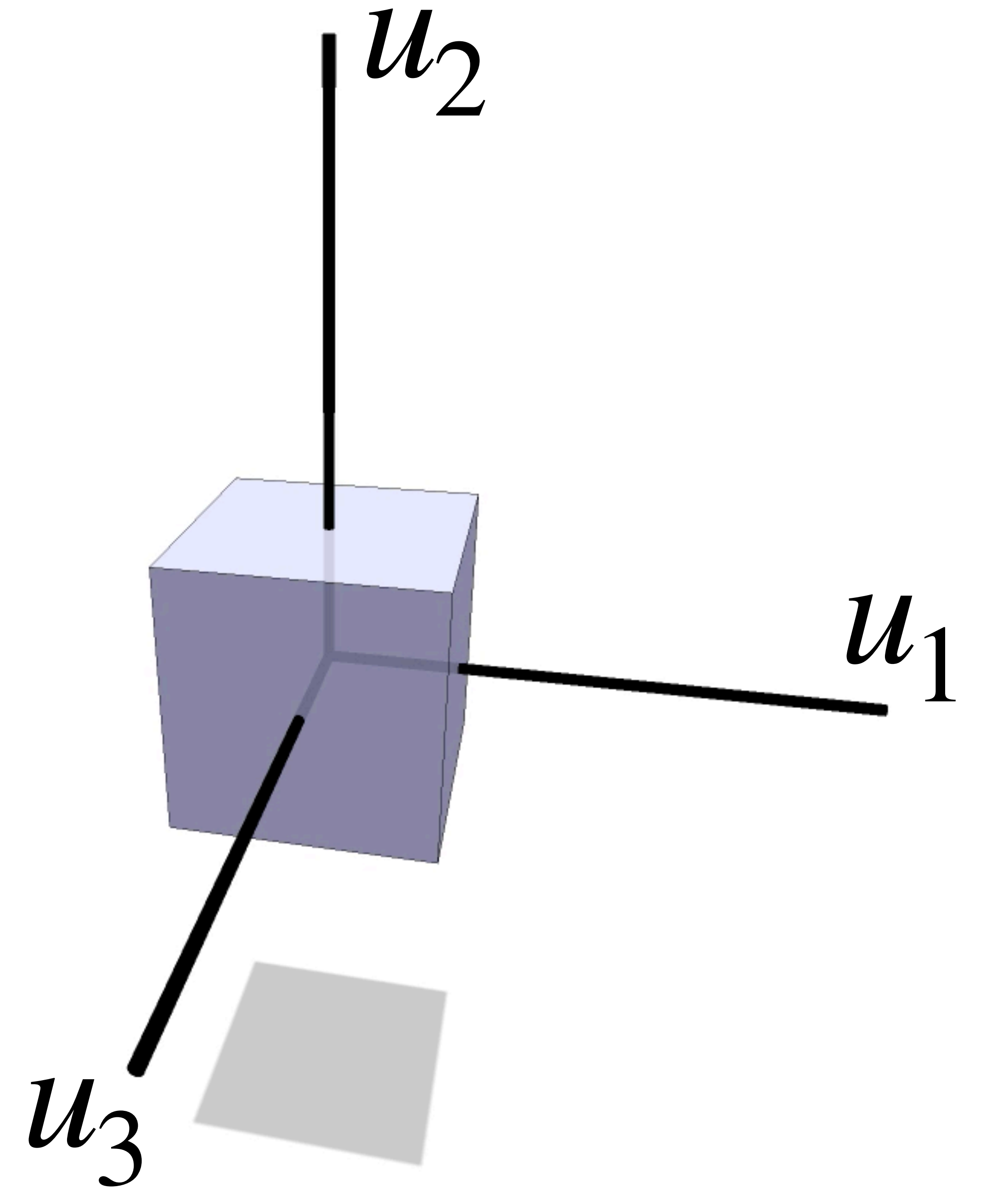
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now we have *three* reflections, and so orientation is reversed!

Nonuniform Scaling (Axis-Aligned)

- **We can also scale each axis by a different amount**
  - $f(u_1, u_2, u_3) = (au_1, bu_2, cu_3), \quad a, b, c \in \mathbb{R}$
- **Q: What's the matrix representation?**
- **A: Just put a, b, c on the diagonal:**

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ bu_2 \\ cu_3 \end{bmatrix}$$

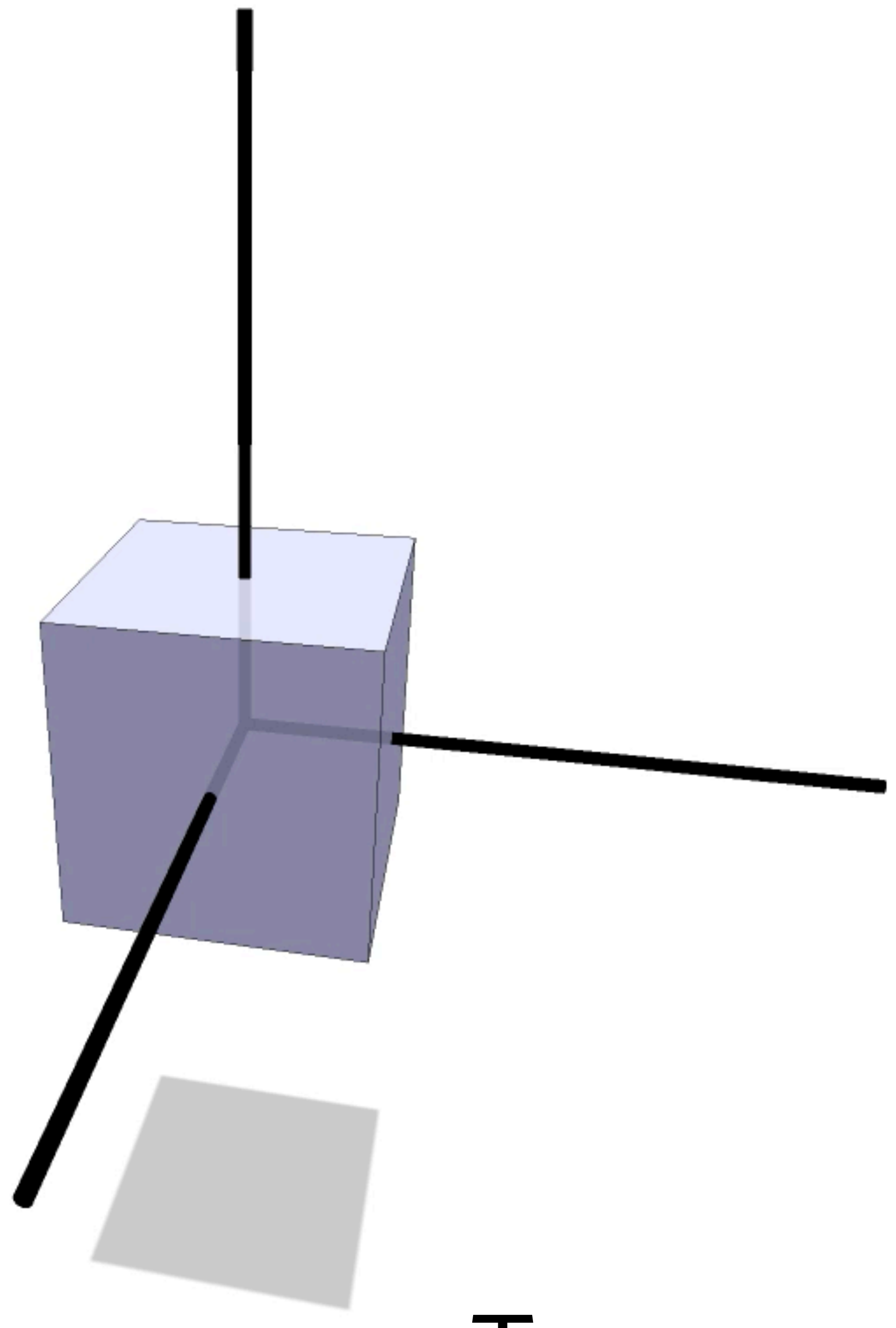


Ok, but what if we want to scale along some *other* axes?

Nonuniform Scaling

- <u>Idea.</u> We could:
  - rotate to the new axes (R)
  - apply a diagonal scaling (D)
  - rotate *back** to the original axes ($R^\top$)
- Notice that the overall transformation is represented by a <u>symmetric</u> matrix

$$A := R^\top D R$$



$$f(\mathbf{x}) = R^\top D R \mathbf{x}$$

Q: Do <u>all</u> symmetric matrices represent nonuniform scaling (for some choice of axes)?

*Recall that for a rotation, the inverse equals the transpose: $R^{-1} = R^\top$

Spectral Theorem

- A: Yes! Spectral theorem says a symmetric matrix $A = A^\top$ has
  - orthonormal eigenvectors $e_1, \ldots, e_n \in \mathbb{R}^n$
  - real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

$$\boxed{Ae_i = \lambda_i e_i}$$

- Can also write this relationship as $AR = RD$, where

$$R = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- Equivalently, $A = RDR^\top$
- Hence, every symmetric matrix performs a non-uniform scaling along some set of orthogonal axes.
- If A is *positive definite* ($\lambda_i > 0$), this scaling is positive.

Shear

- A shear displaces each point $\mathbf{x}$ in a direction $\mathbf{u}$ according to its distance along a fixed vector $\mathbf{v}$:

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

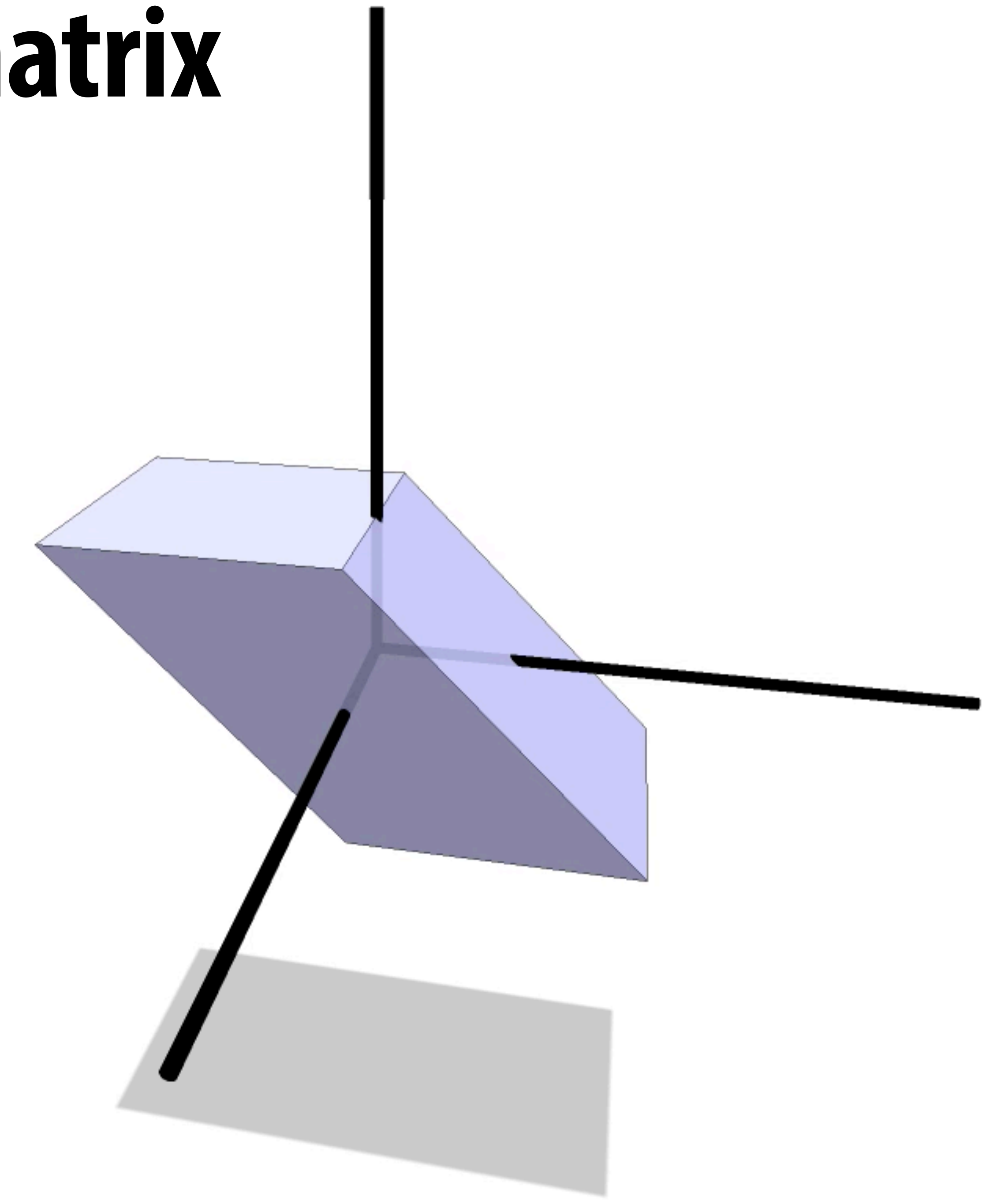
- Q: Is this transformation *linear*?
- A: Yes—for instance, can represent it via a matrix

$$A_{\mathbf{u},\mathbf{v}} = I + \mathbf{u}\mathbf{v}^\top$$

<u>**Example.**</u>

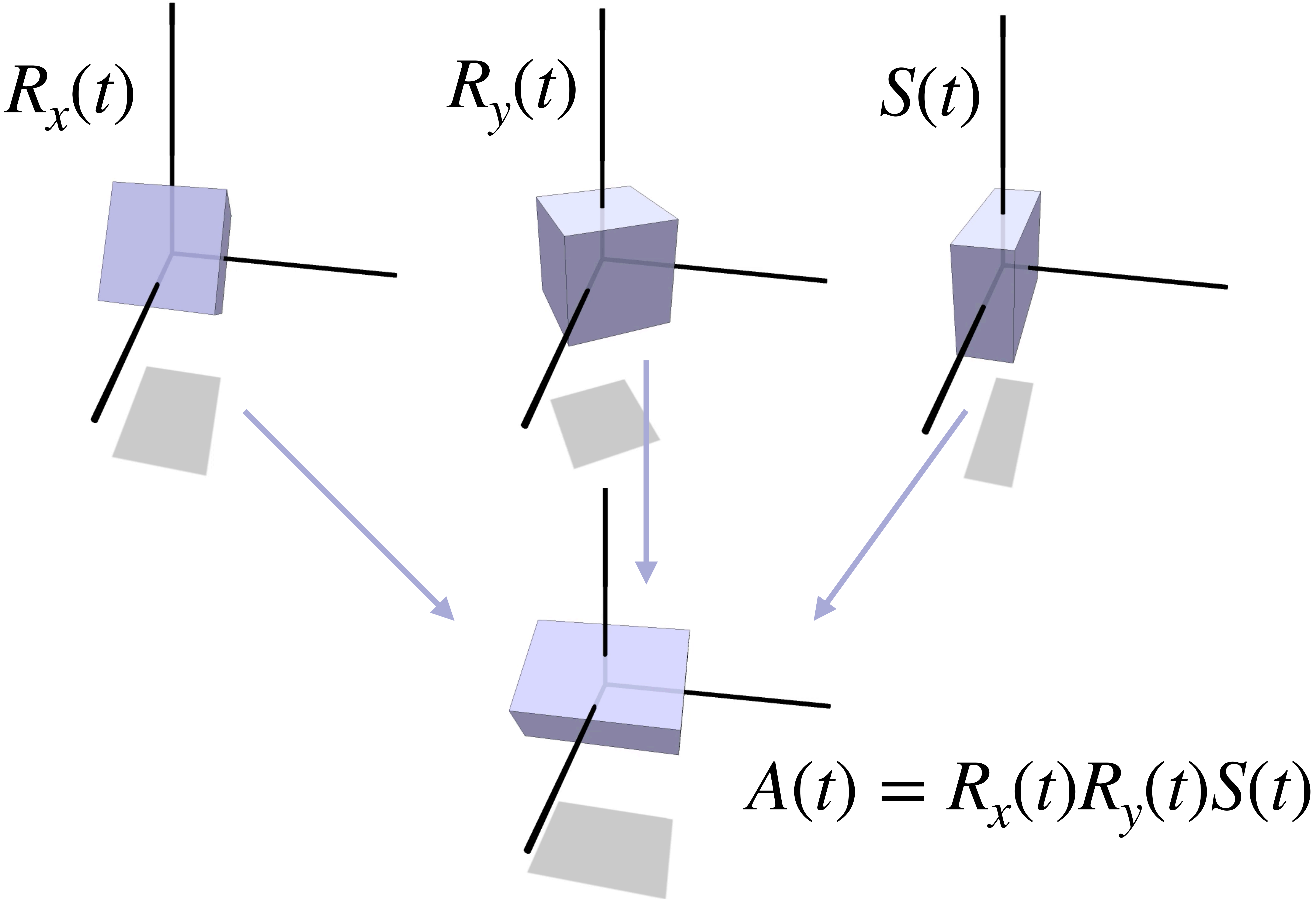
$$\mathbf{u} = (\cos(t),0,0)$$
$$\mathbf{v} = (0,1,0)$$

$$A_{\mathbf{u},\mathbf{v}} = \begin{bmatrix} 1 & \cos(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composite Transformations

From these basic transformations (rotation, reflection, scaling, shear…) we can now build up composite transformations via matrix multiplication:

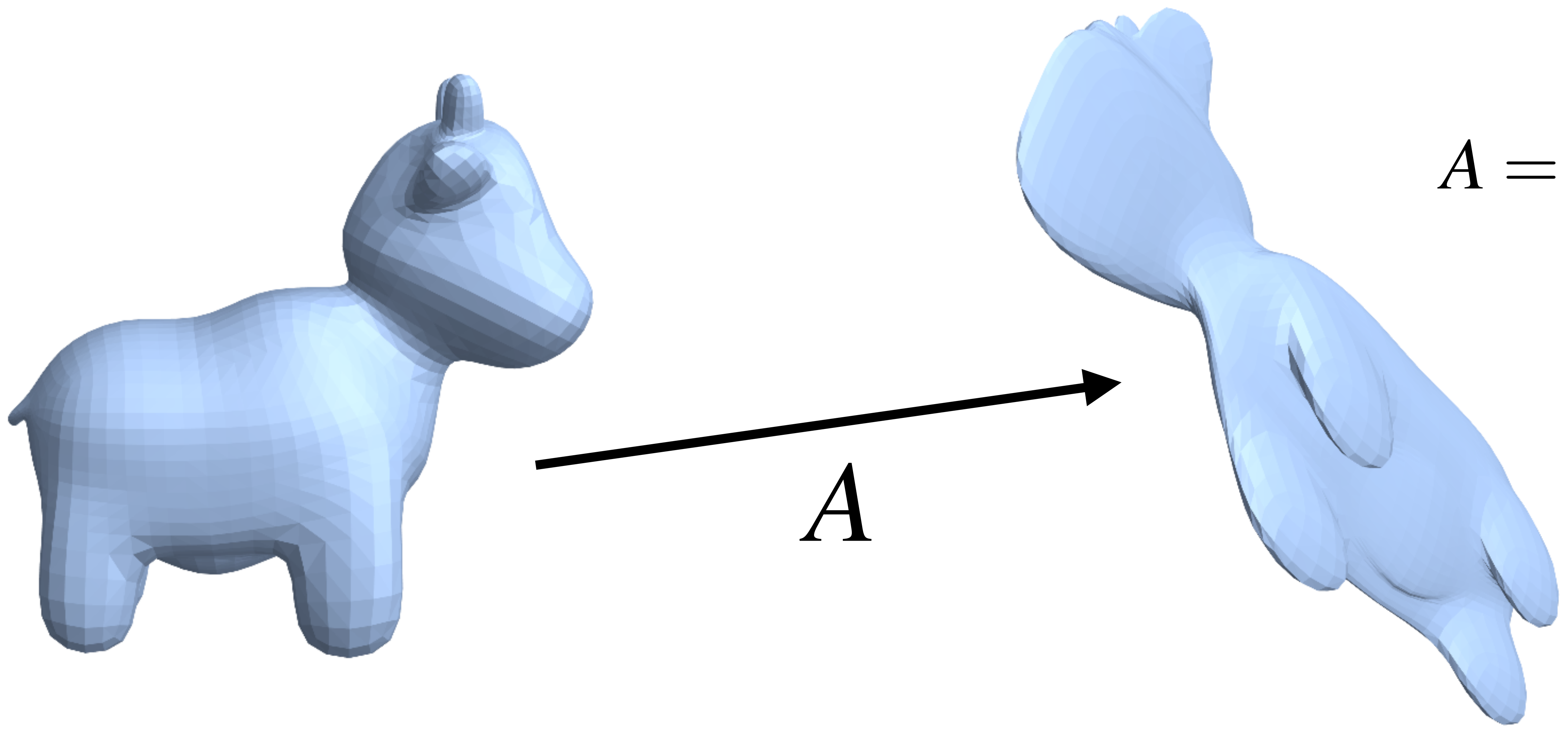


How do we <u>decompose</u> a linear transformation into pieces?

(rotations, reflections, scaling, …)

Decomposition of Linear Transformations

- In general, no **unique** way to write a given linear transformation as a composition of basic transformations!
- However, there are *many* useful decompositions:
  - singular value decomposition (good for signal processing)
  - LU factorization (good for solving linear systems)
  - polar decomposition (good for spatial transformations)
  - ...
- Consider for instance this linear transformation:



$$A = \begin{bmatrix} .34 & -.11 & -.89 \\ -.65 & .52 & -.70 \\ .25 & .23 & -.69 \end{bmatrix}$$

Polar & Singular Value Decomposition

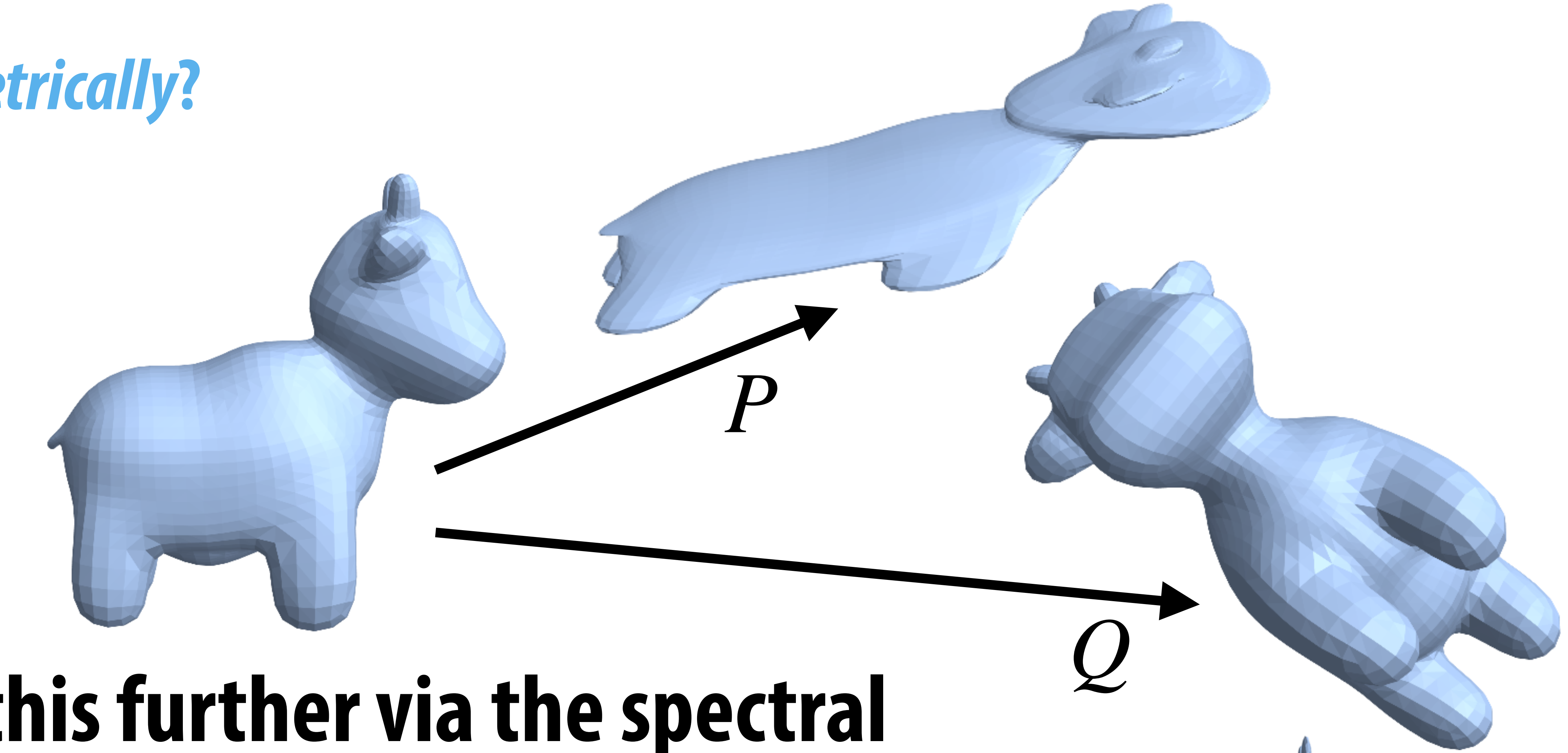
For example, polar decomposition decomposes any matrix A into orthogonal matrix Q and symmetric positive-semidefinite matrix P:

Q: What do each of the parts mean *geometrically*?

rotation/reflection

nonnegative, nonuniform scaling

$$A = QP$$



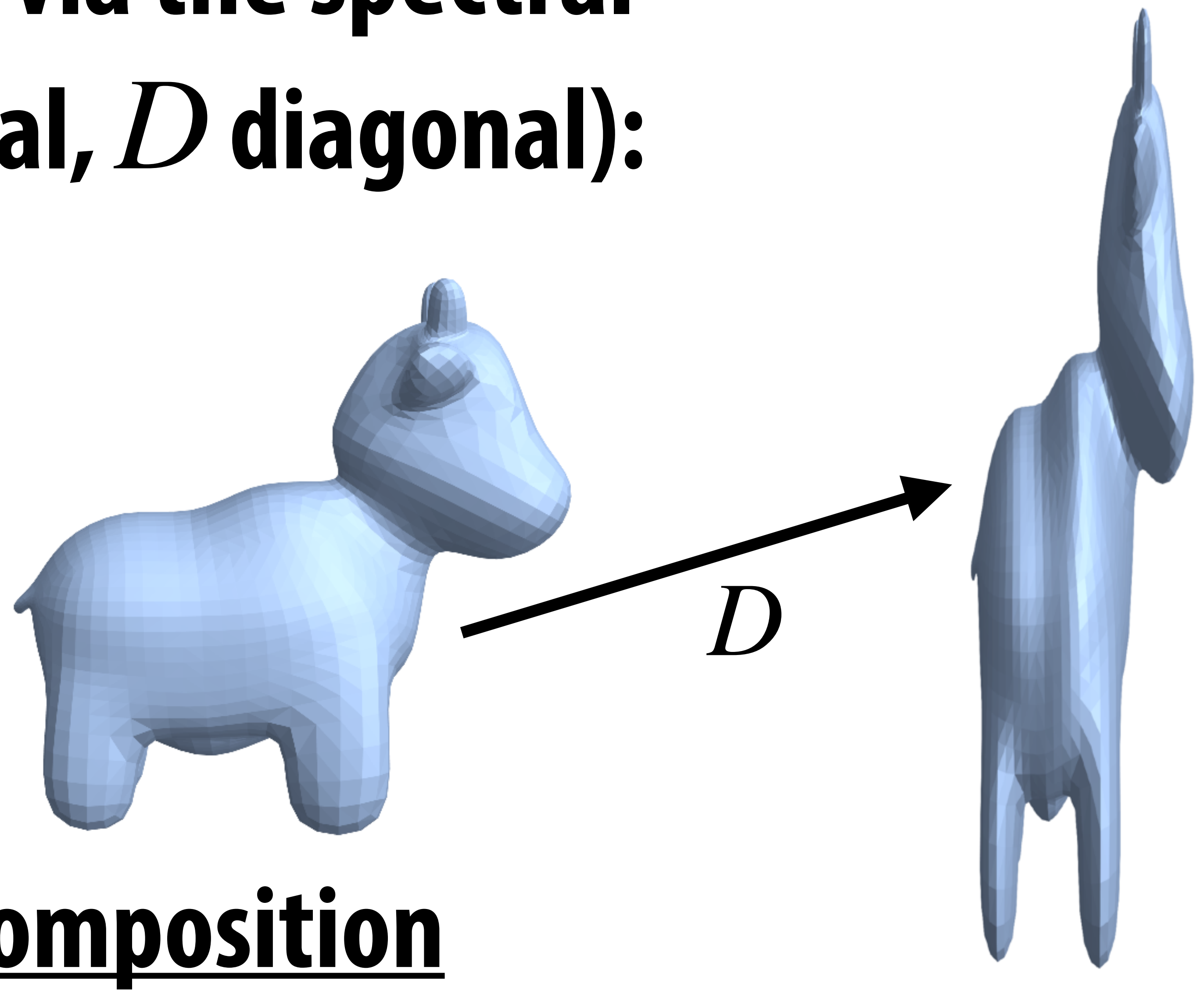
Since P is symmetric, can take this further via the spectral decomposition $P = VDV^\top$ (V orthogonal, D diagonal):

$$A = \underbrace{QV}_{U} DV^\top = UDV^\top$$

rotation

axis-aligned scaling

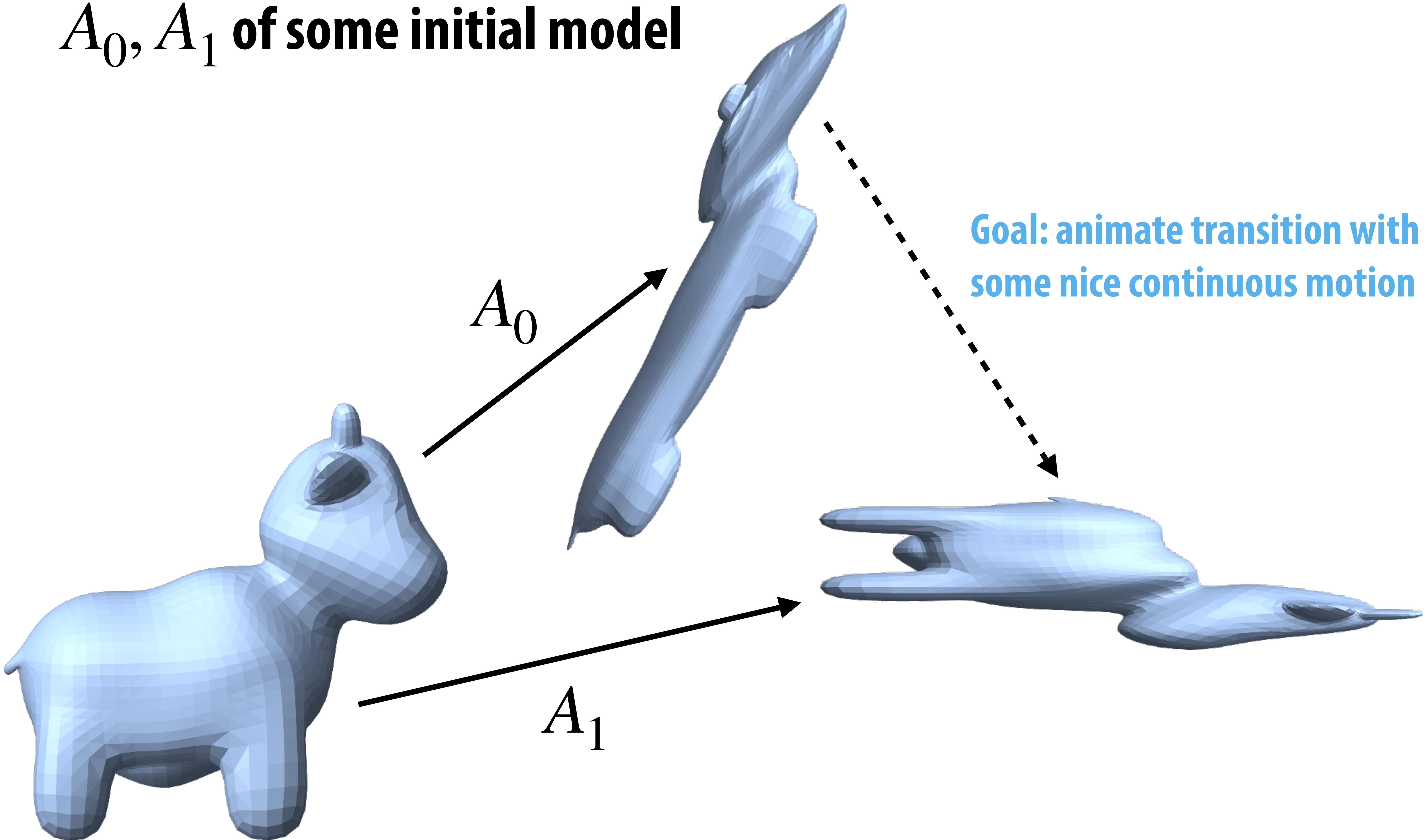
rotation



Result $UDV^\top$ is called the singular value decomposition

Interpolating Transformations

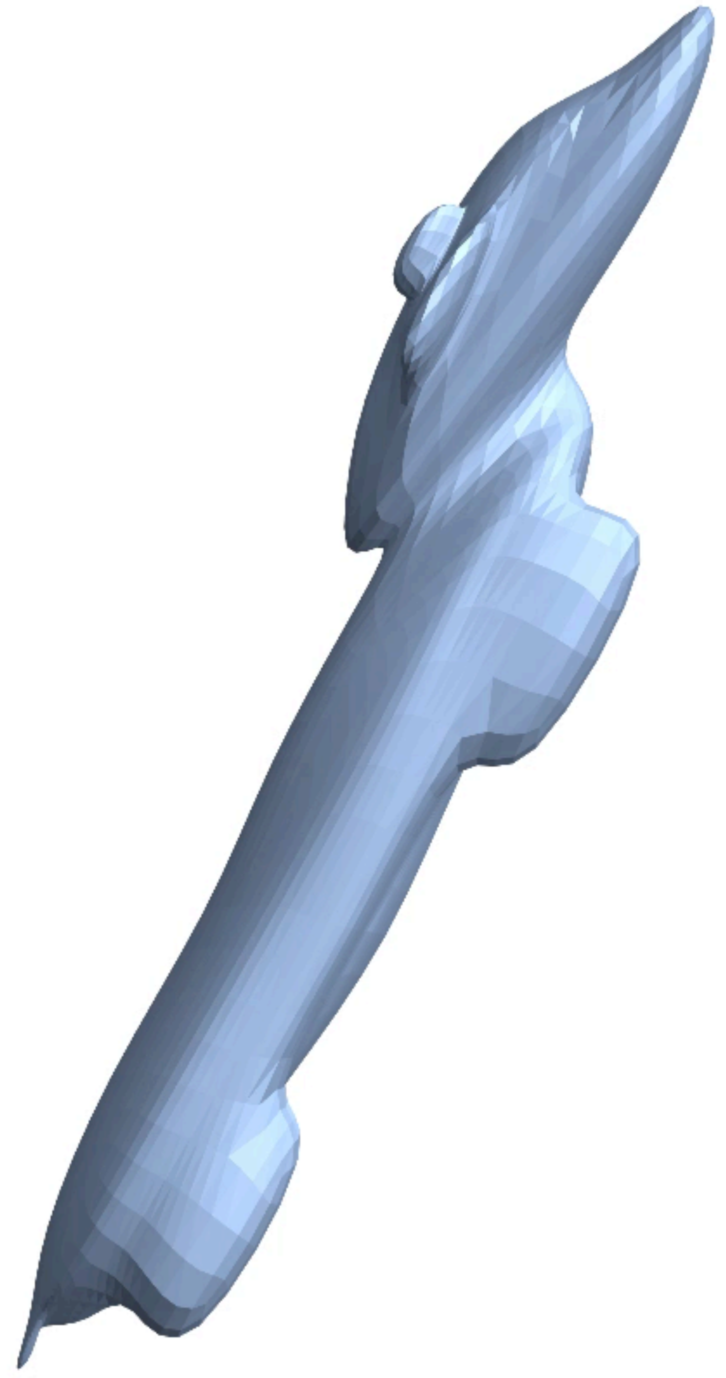
- How are these decompositions useful for graphics?
- Consider interpolating between two linear transformations A_0, A_1 of some initial model



Interpolating Transformations—Linear

One idea: just take a linear combination of the two matrices, weighted by the current time $t \in [0,1]$

$$A(t) = (1-t)A_0 + tA_1$$



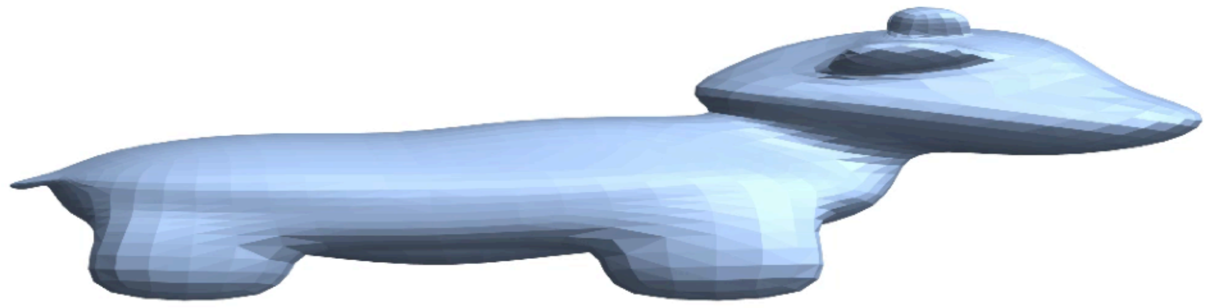
Hits the right start/endpoints… but looks awful in between!

Interpolating Transformations—Polar

Better idea: *separately* interpolate components of polar decomposition.

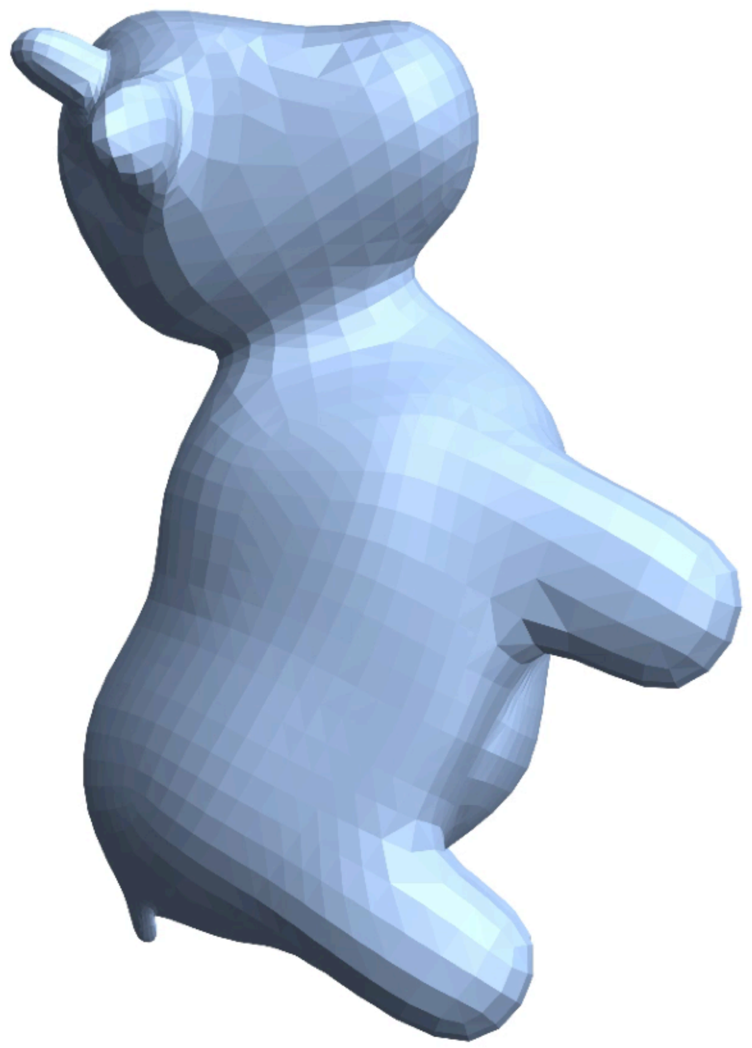
$$A_0 = Q_0 P_0, \quad A_1 = Q_1 P_1$$

scaling



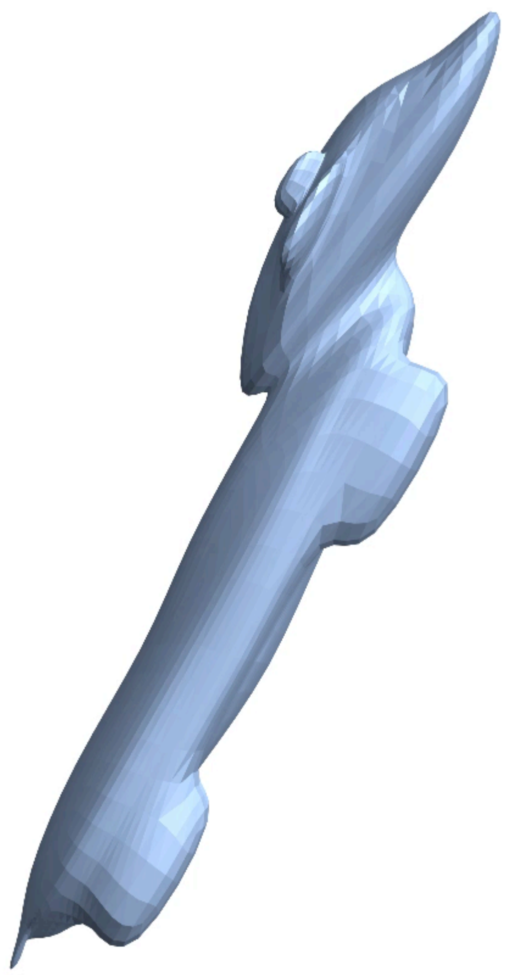
$$P(t) = (1-t)P_0 + tP_1$$

rotation



$$\widetilde{Q}(t) = (1-t)Q_0 + tQ_1$$
$$\widetilde{Q}(t) = Q(t)X(t)$$

final interpolation



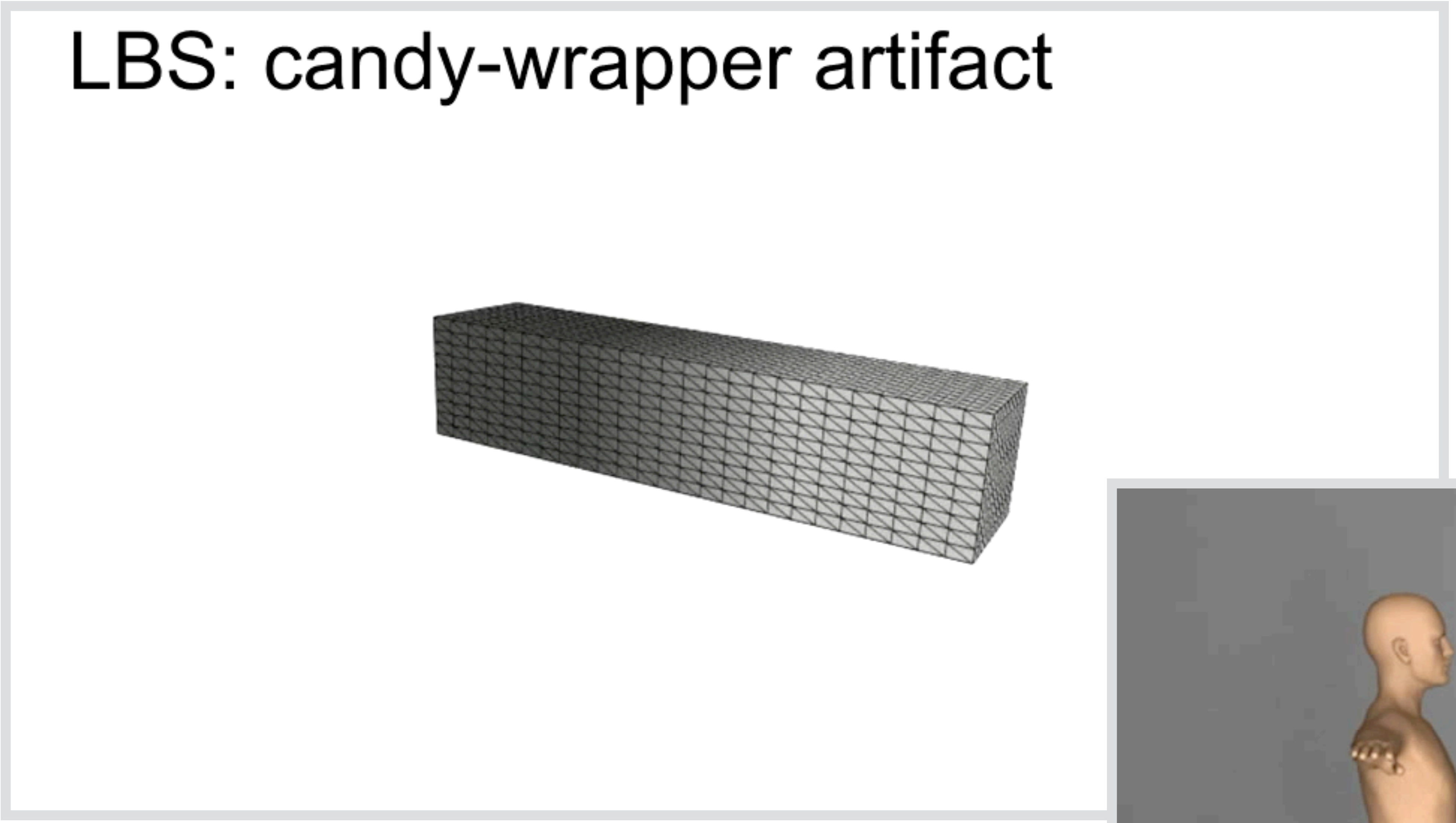
$$A(t) = Q(t)P(t)$$

…looks better!

See: Shoemake & Duff, "Matrix Animation and Polar Decomposition"

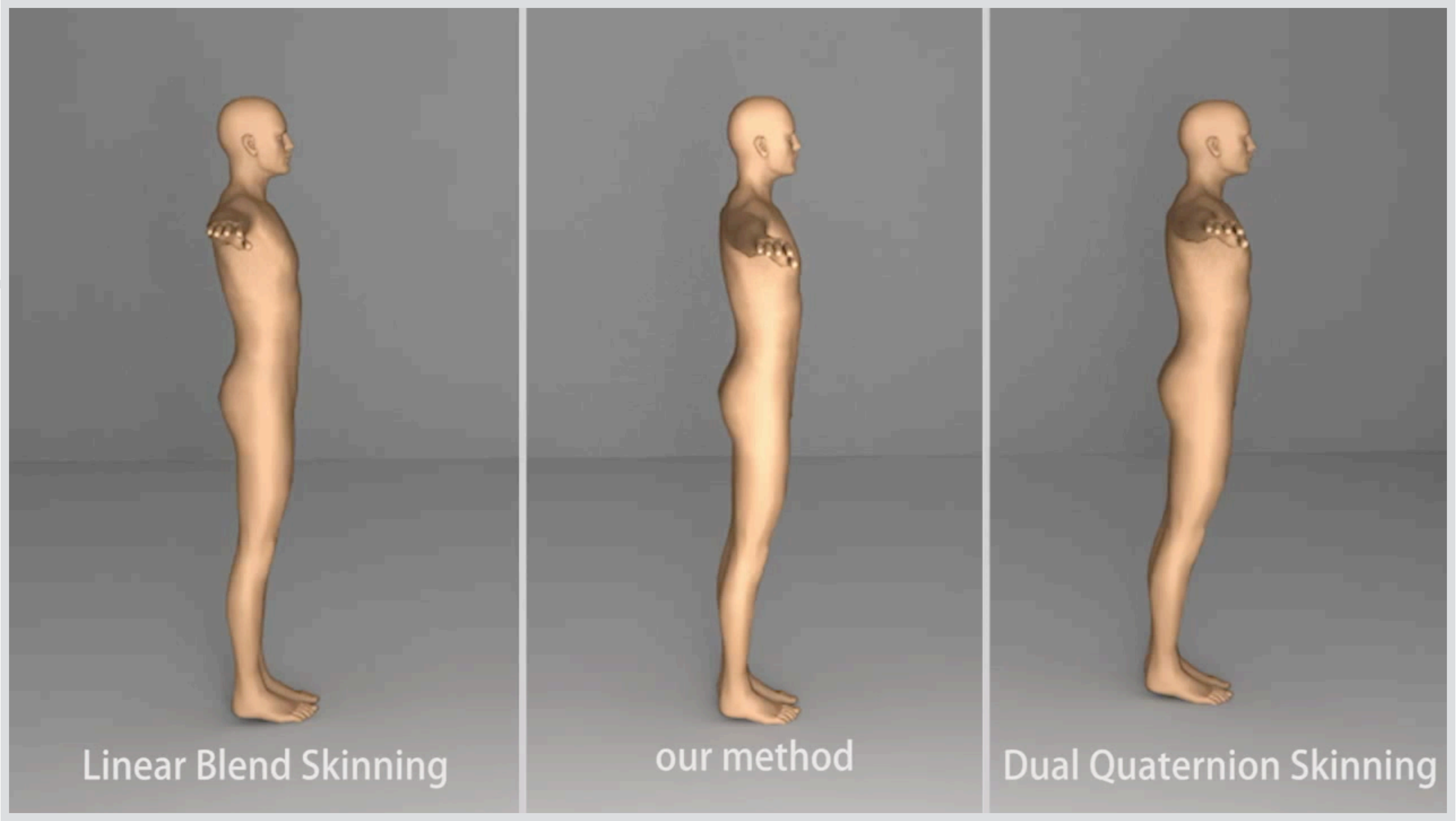
Example: Linear Blend Skinning

- Naïve linear interpolation also causes artifacts when blending between transformations on a character ("candy wrapper effect")
- Lots of research on alternative ways to blend transformations…



Jacobson, Deng, Kavan, & Lewis (2014)
"Skinning: Real-time Shape Deformation"

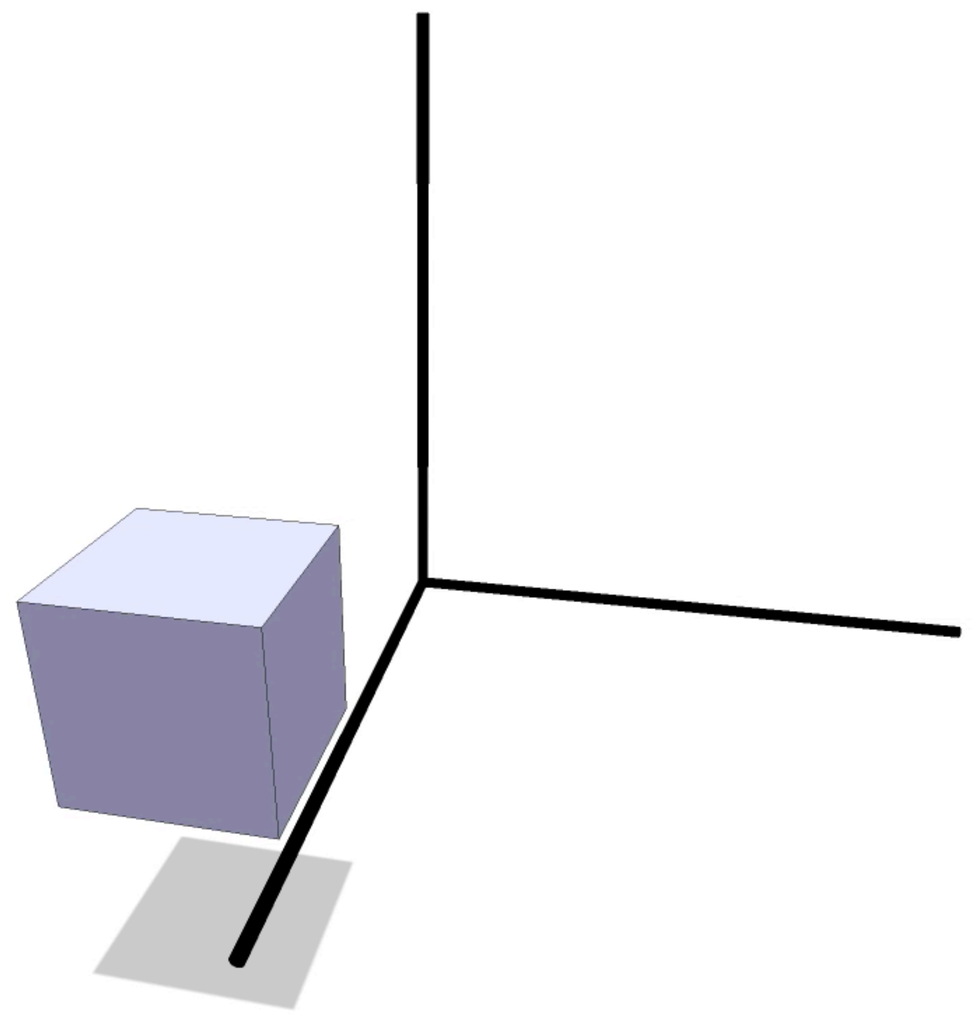
Rumman & Fratarcangeli (2015)
"Position-based Skinning for Soft Articulated Characters"



Translations

- So far we've ignored a basic transformation—translations
- A translation simply adds an offset $\mathbf{u}$ to the given point $\mathbf{x}$:

$$f_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}$$



Q: Is this transformation linear?
(Certainly seems to move us along a line…)

Let's carefully check the definition…

additivity

$$f_{\mathbf{u}}(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{u}$$

$$f_{\mathbf{u}}(\mathbf{x}) + f_{\mathbf{u}}(\mathbf{y}) = \mathbf{x} + \mathbf{y} + 2\mathbf{u}$$

homogeneity

$$f_{\mathbf{u}}(a\mathbf{x}) = a\mathbf{x} + \mathbf{u}$$

$$af_{\mathbf{u}}(\mathbf{x}) = a\mathbf{x} + a\mathbf{u}$$

A: No! Translation is affine, *not* linear!

Composition of Transformations

- **Recall we can compose linear transformations via matrix multiplication:**

$$A_3(A_2(A_1\mathbf{x})) = (A_3A_2A_1)\mathbf{x}$$

- **It's easy enough to compose translations—just add vectors:**

$$f_{\mathbf{u}_3}(f_{\mathbf{u}_2}(f_{\mathbf{u}_1}(\mathbf{x}))) = f_{\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3}(\mathbf{x})$$

- **What if we want to intermingle translations and linear transformations (rotation, scale, shear, etc.)?**

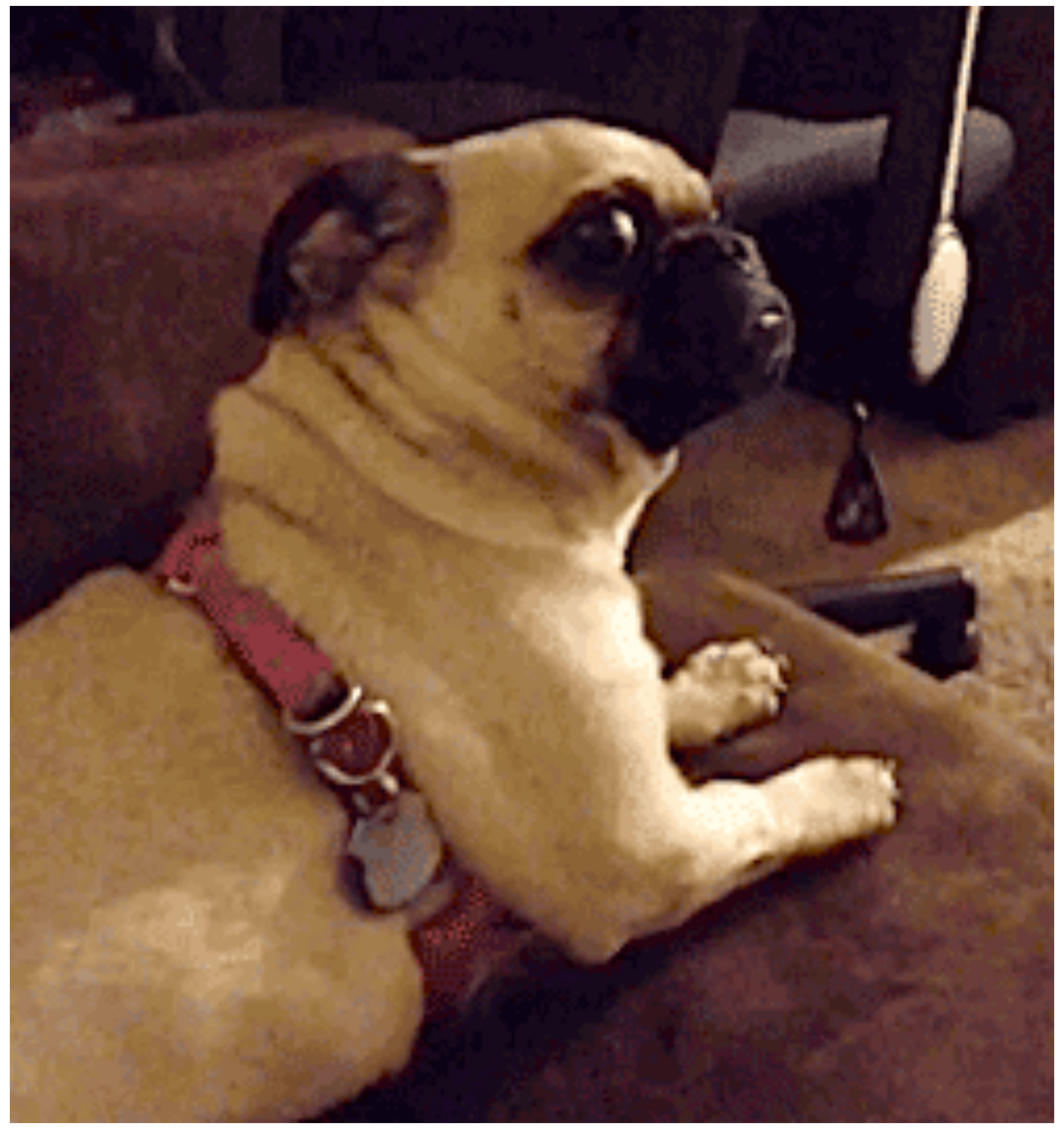
$$A_2(A_1\mathbf{x}+\mathbf{b}_1)+\mathbf{b}_2 = (A_2A_1)\mathbf{x}+(A_2\mathbf{b}_1+\mathbf{b}_2)$$

- **Now we have to keep track of a matrix *and* a vector**
- **Moreover, we'll see (later) that this encoding won't work for other important cases, such as perspective transformations**

But there is a better way…

Strange idea:

Maybe translations turn into <u>linear</u> transformations if we go into the 4th dimension…!

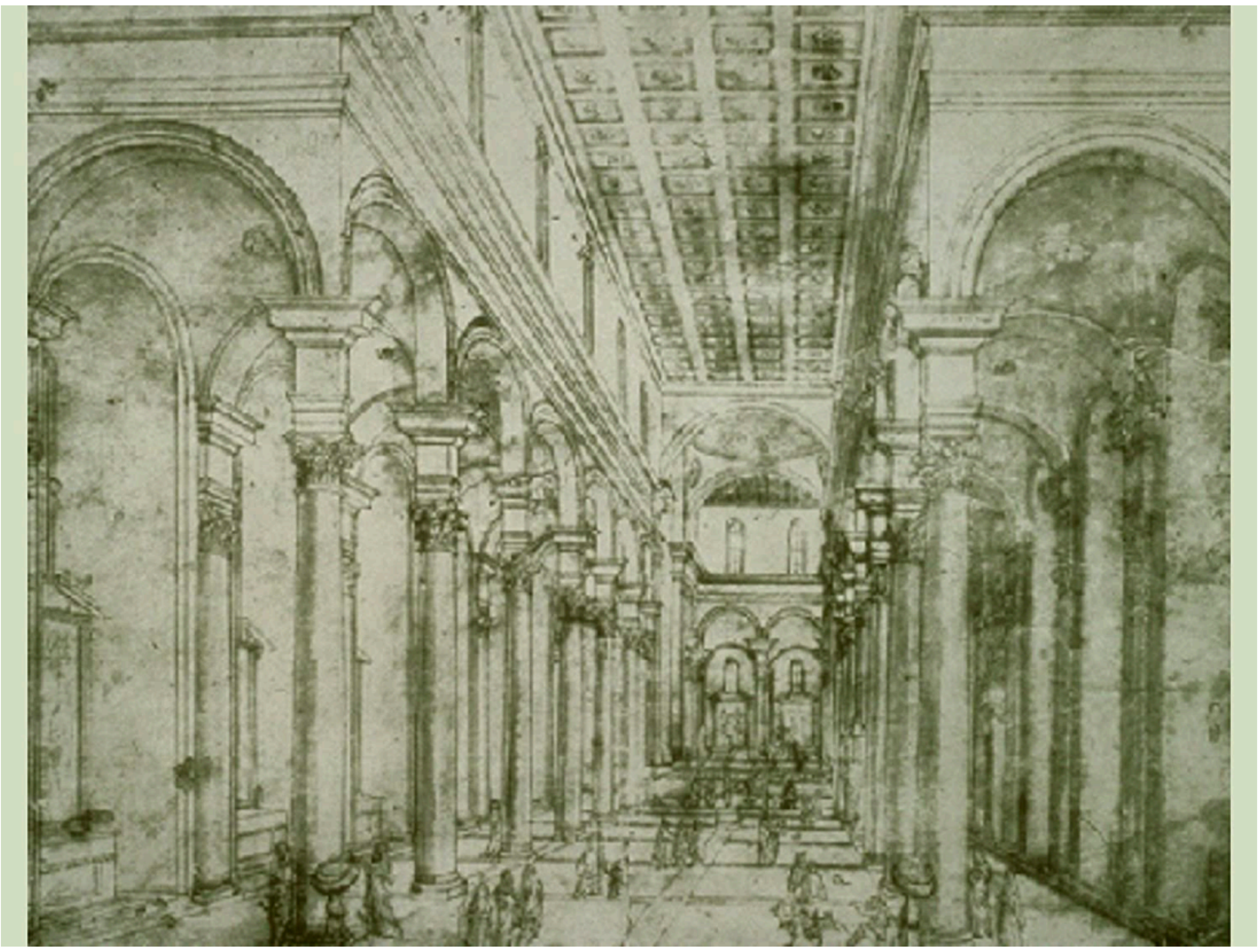


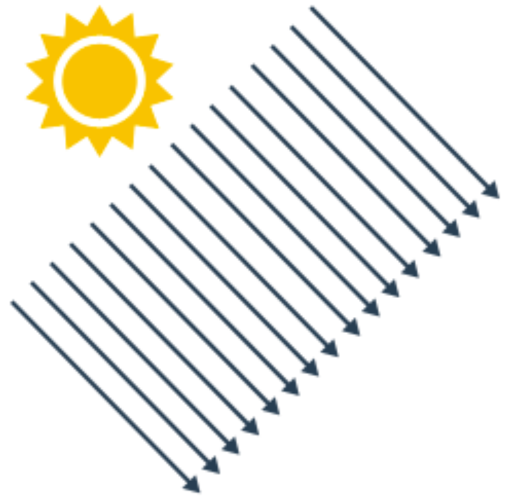
Homogeneous Coordinates

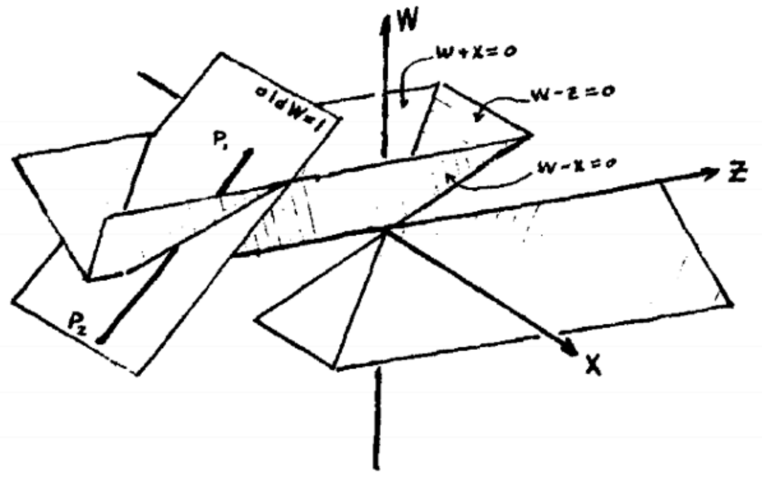
- Came from efforts to study <u>perspective</u>
- Introduced by Möbius as a natural way of assigning coordinates to <u>lines</u>
- Show up naturally in a surprising large number of places in computer graphics:
  - 3D transformations
  - perspective projection
  - quadric error simplification
  - premultiplied alpha
  - shadow mapping
  - projective texture mapping
  - discrete conformal geometry
  - hyperbolic geometry
  - clipping
  - directional lights
  - …

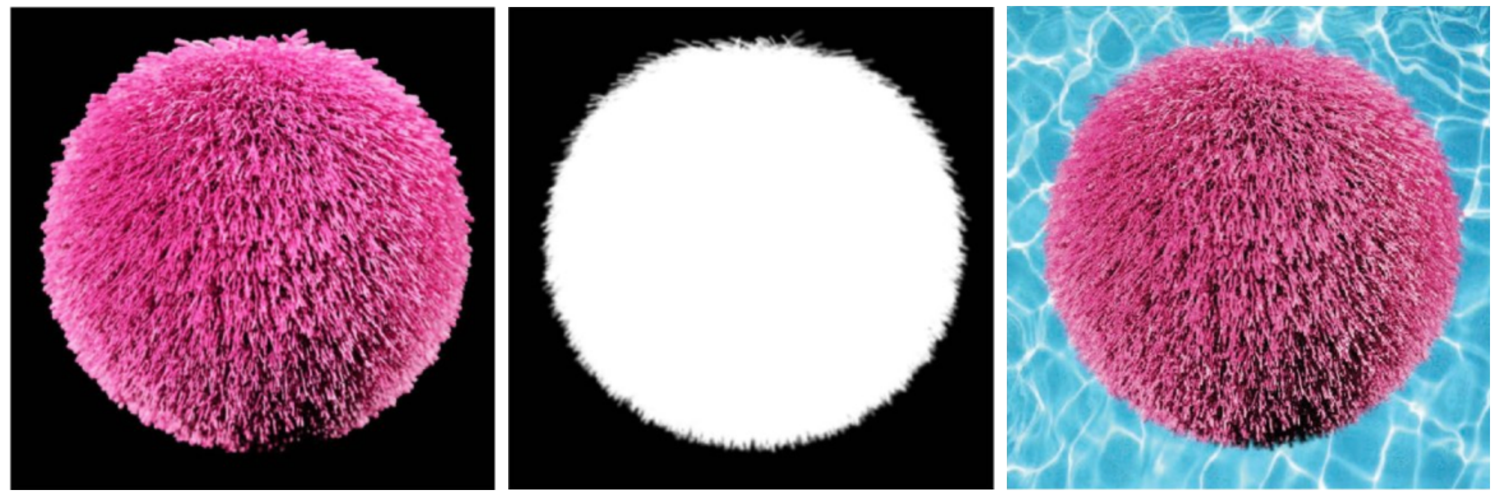
Probably worth understanding!

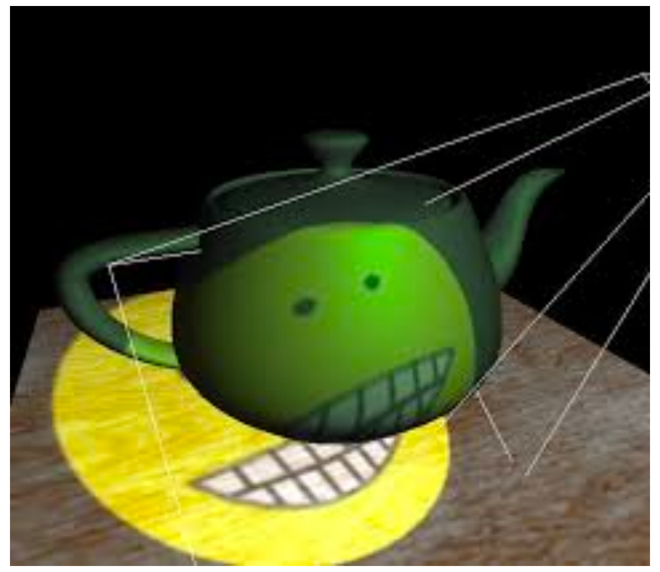
Filippo Brunelleschi, 1428

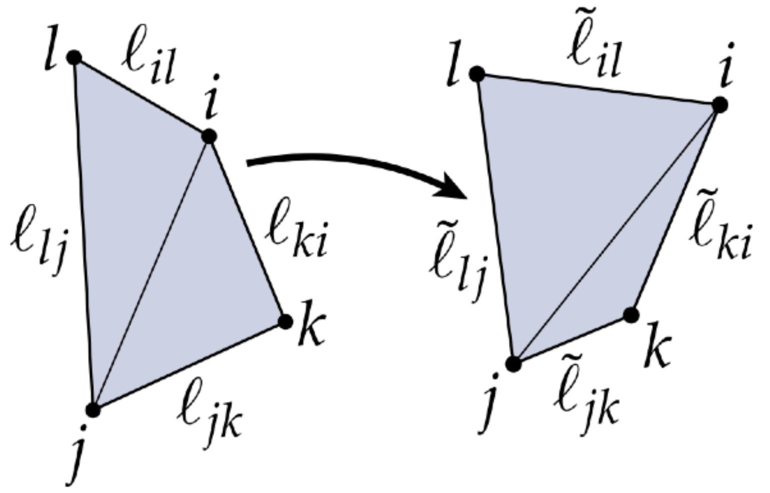






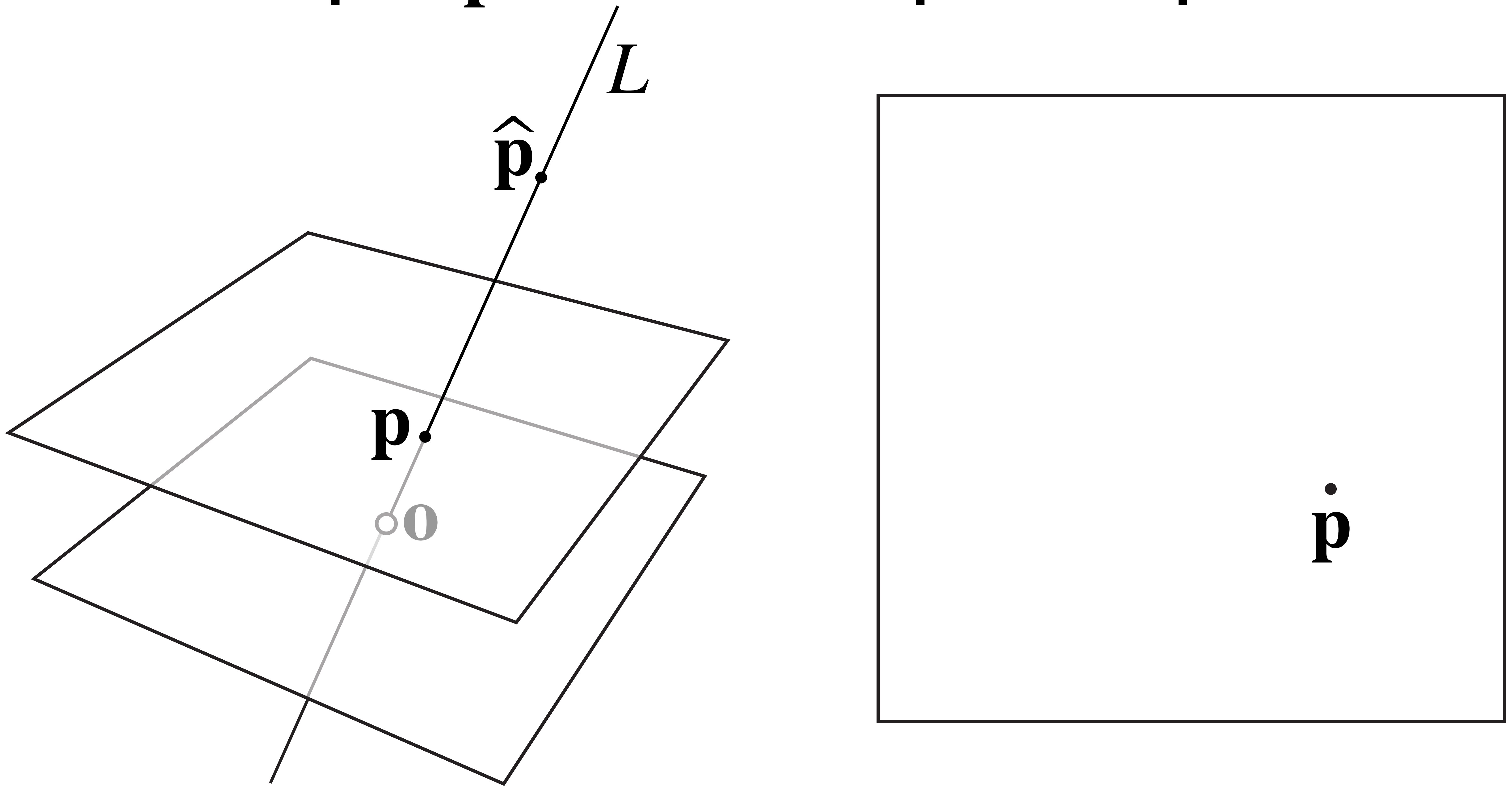






Homogeneous Coordinates—Basic Idea

- Consider any 2D plane that does not pass through the origin $\mathbf{o}$ in 3D
- Every line through the origin in 3D corresponds to a point in the 2D plane
  - Just find the point $\mathbf{p}$ where the line L pierces the plane

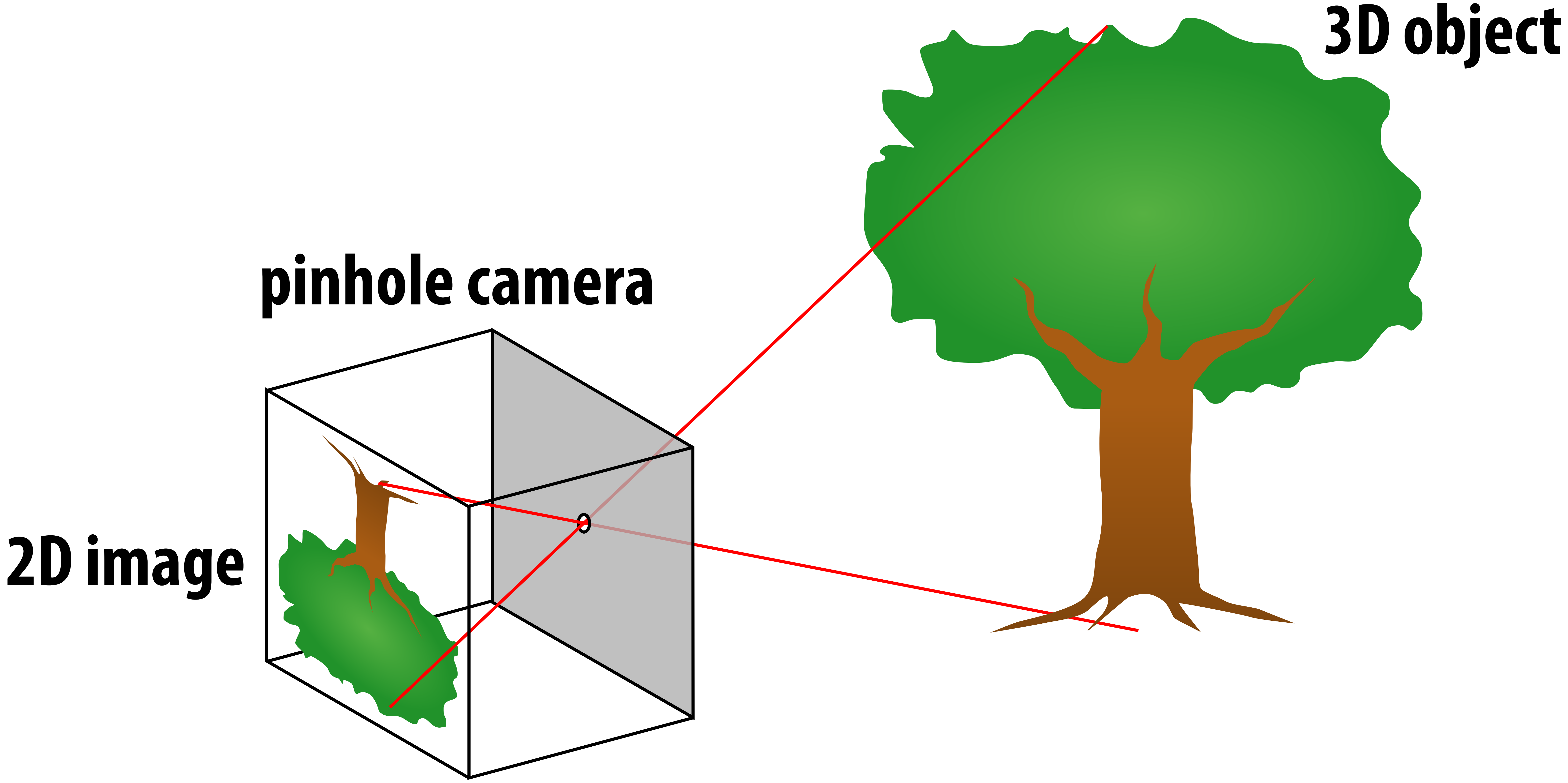


Hence, any point $\hat{\mathbf{p}}$ on the line L can be used to represent the point $\mathbf{p}$.

Q: What does this story remind you of?

Review: Perspective projection

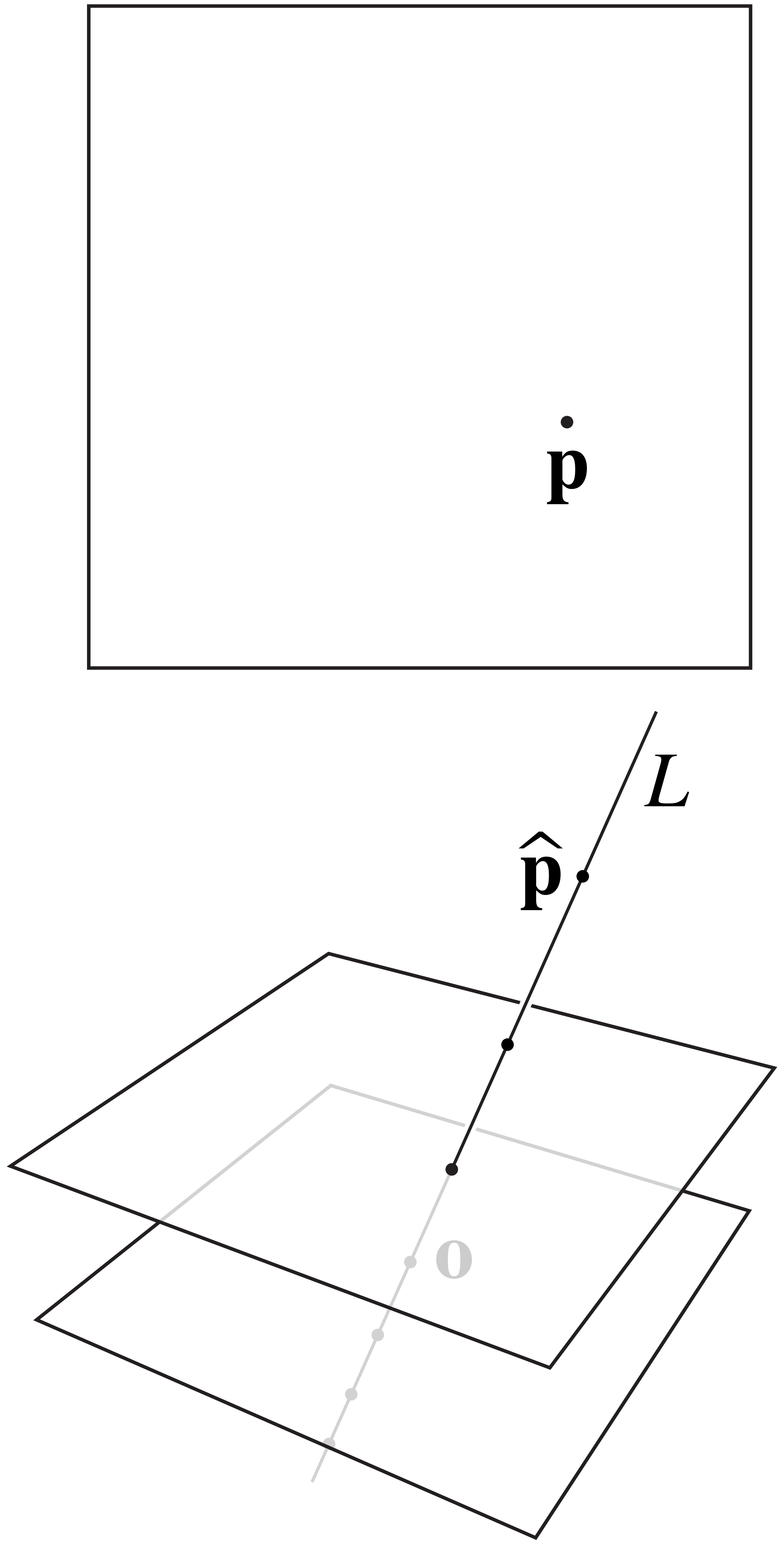
- Hopefully it reminds you of our "pinhole camera"
- Objects along the same line project to the same point



If you have an image of a single dot, can't know where it is! Only which line it belongs to.

Homogeneous Coordinates (2D)

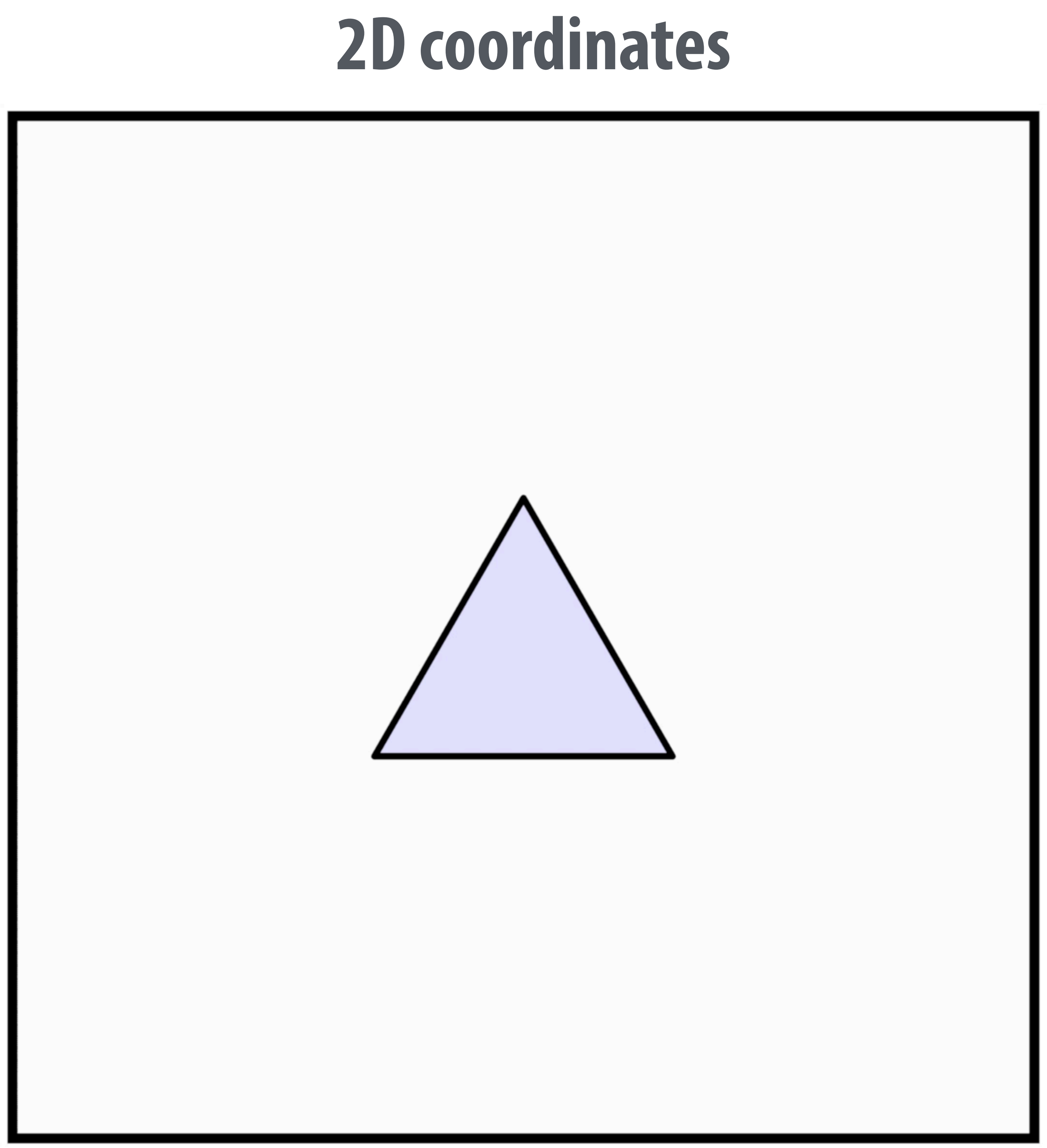
- More explicitly, consider a point $\mathbf{p} = (x, y)$, and the plane $z = 1$ in 3D
- Any three numbers $\hat{\mathbf{p}} = (a, b, c)$ such that $(a/c, b/c) = (x, y)$ are <u>homogeneous coordinates</u> for $\mathbf{p}$
  - E.g., $(x, y, 1)$
  - In general: (cx, cy, c) for $c \neq 0$
- Hence, two points $\hat{\mathbf{p}}, \hat{\mathbf{q}} \in \mathbb{R}^3 \backslash \{O\}$ describe the same point in 2D (and line in 3D) if $\hat{\mathbf{p}} = \lambda \hat{\mathbf{q}}$ for some $\lambda \neq 0$



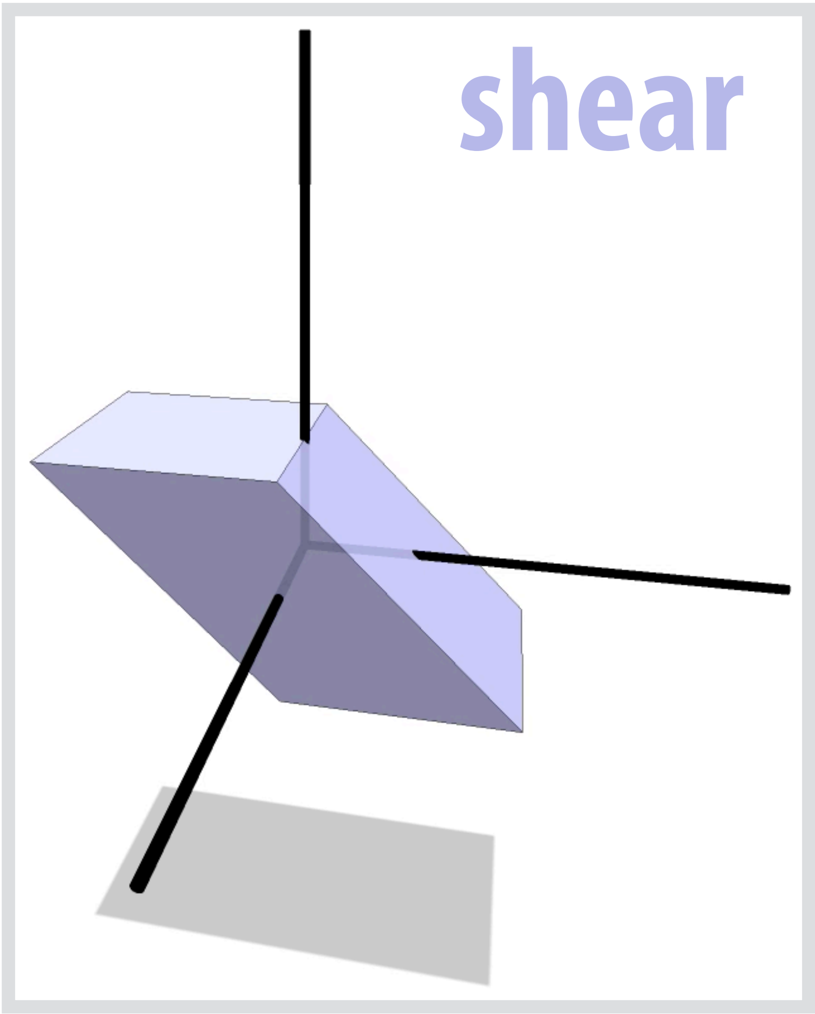
Great… but how does this help us with transformations?

Translation in Homogeneous Coordinates

Let's think about what happens to our homogeneous coordinates $\hat{\mathbf{p}}$ if we apply a translation to our 2D coordinates $\mathbf{p}$



Q: What kind of transformation does this look like?



Translation in Homogeneous Coordinates

- But wait a minute—shear is a linear transformation!
- Can this be right? Let's check in coordinates...
- Suppose we translate a point $\mathbf{p} = (p_1, p_2)$ by a vector $\mathbf{u} = (u_1, u_2)$ to get $\mathbf{p}' = (p_1 + u_1, p_2 + u_2)$
- The homogeneous coordinates $\hat{\mathbf{p}} = (cp_1, cp_2, c)$ then become $\hat{\mathbf{p}}' = (cp_1 + cu_1, cp_2 + cu_2, c)$
- Notice that we're shifting $\hat{\mathbf{p}}$ by an amount $c\mathbf{u}$ that's proportional to the distance c along the third axis—a shear

Using homogeneous coordinates, we can represent an affine transformation in 2D as a linear transformation in 3D

Homogeneous Translation—Matrix Representation

- **To write as a matrix, recall that a shear in the direction $\mathbf{u} = (u_1, u_2)$ according to the distance along a direction $\mathbf{v}$ is**

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{u}$$

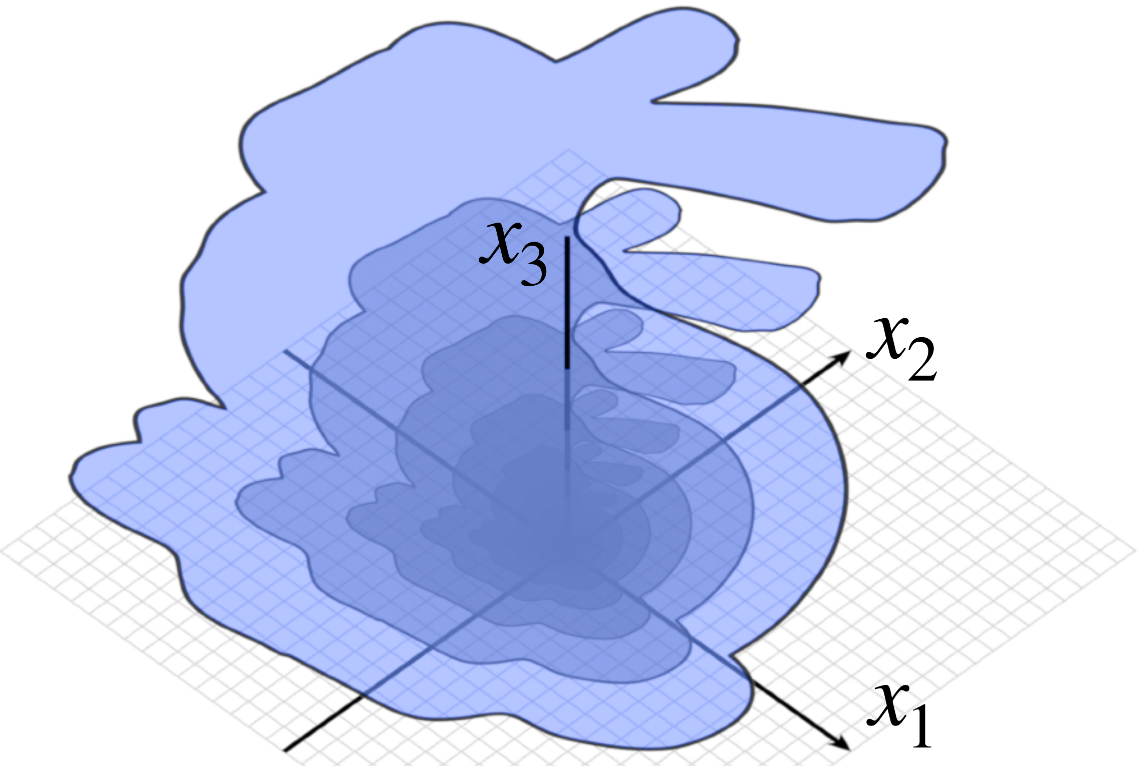
- **In matrix form:**

$$f_{\mathbf{u},\mathbf{v}}(\mathbf{x}) = \left(I + \mathbf{u}\mathbf{v}^{\mathsf{T}} \right) \mathbf{x}$$

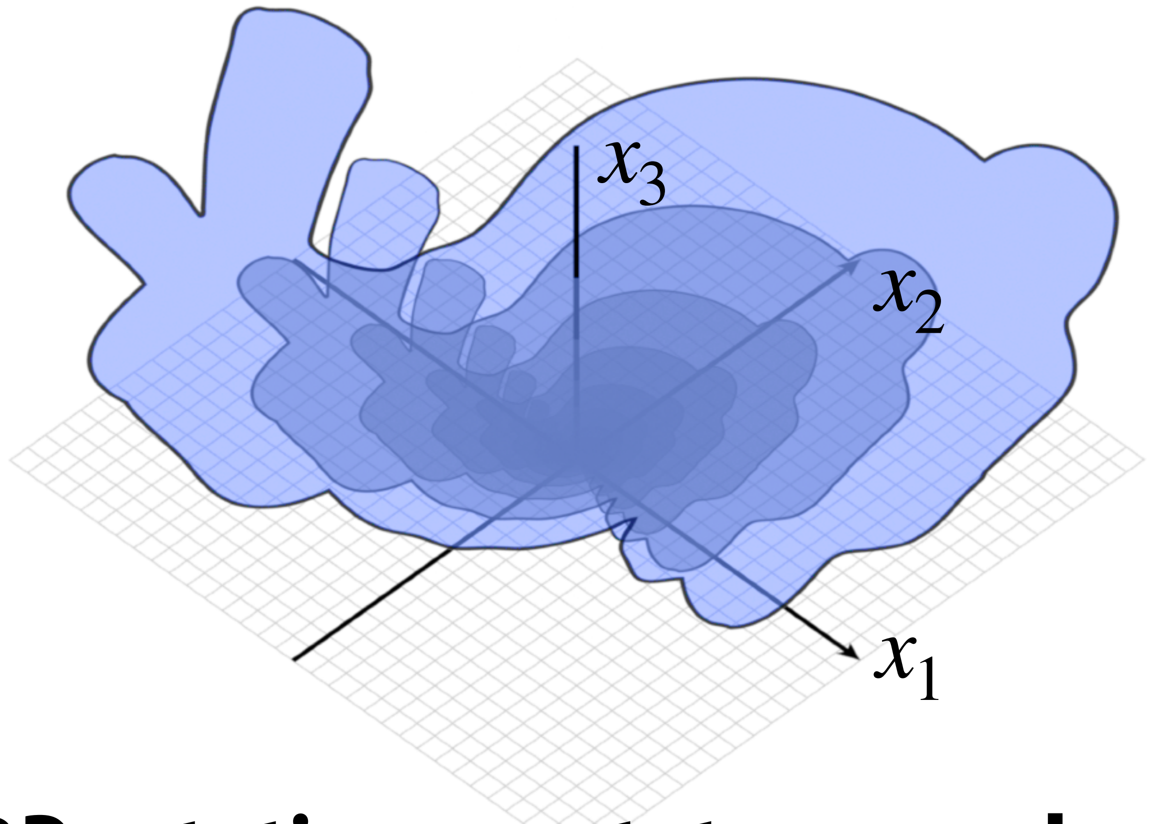
- **In our case, $\mathbf{v} = (0,0,1)$ and so we get a matrix**

$$\begin{bmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cp_1 \\ cp_2 \\ c \end{bmatrix} = \begin{bmatrix} c(p_1 + u_1) \\ c(p_2 + u_2) \\ c \end{bmatrix} \stackrel{1/c}{\Longrightarrow} \begin{bmatrix} p_1 + u_1 \\ p_2 + u_2 \end{bmatrix}$$

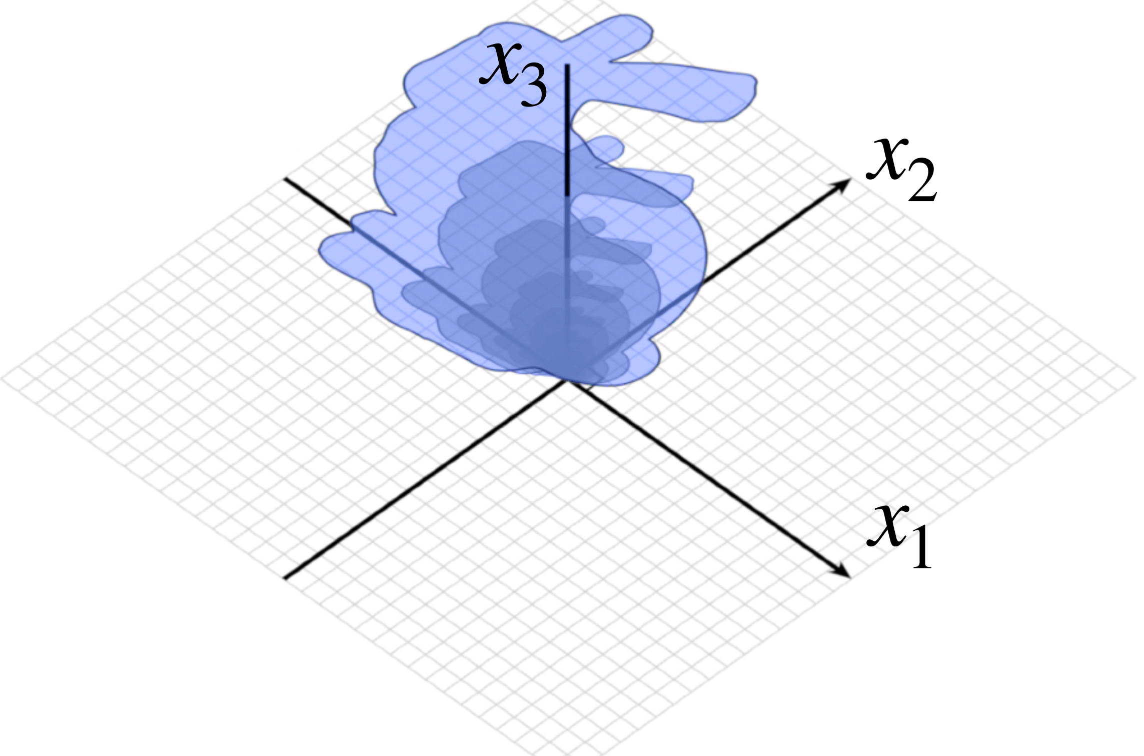
Other 2D Transformations in Homogeneous Coordinates



Original shape in 2D can be viewed as many copies, uniformly scaled by x_3

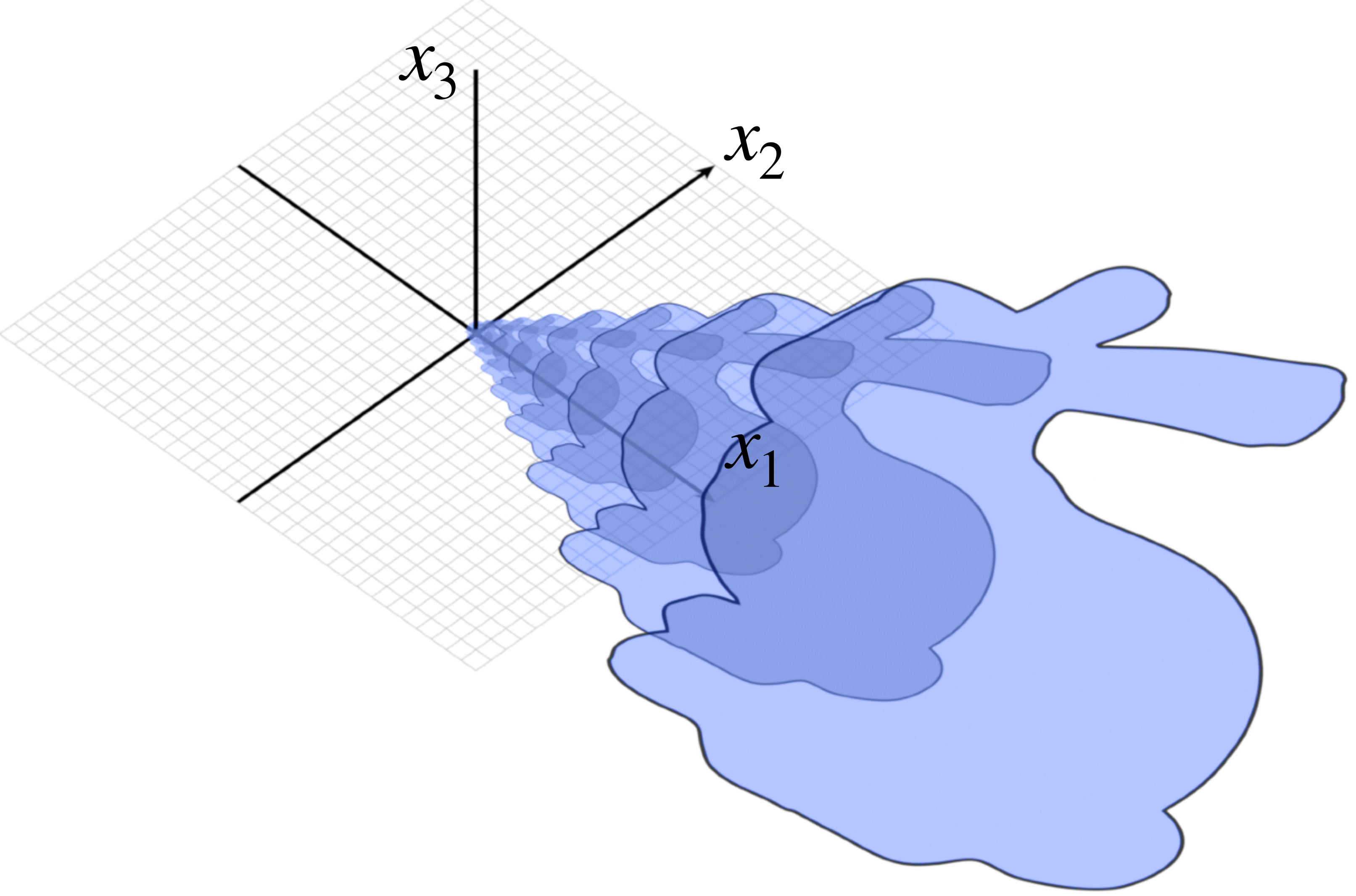


2D rotation ↔ rotate around x_3



2D scale ↔ scale x_1 and x_2; preserve x_3

(Q: what happens to 2D shape if you scale x_1, x_2, and x_3 uniformly?)



2D translate ↔ shear

Now easy to compose all these transformations

3D Transformations in Homogeneous Coordinates

- Not much changes in three (or more) dimensions: just append one "homogeneous coordinate" to the first three
- Matrix representations of 3D linear transformations just get an additional identity row/column; translation is again a shear

point in 3D

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

rotate (x, y, z) around y by θ

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

shear (x, y) by z in (s, t) direction

$$\begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

scale x, y, z by a, b, c

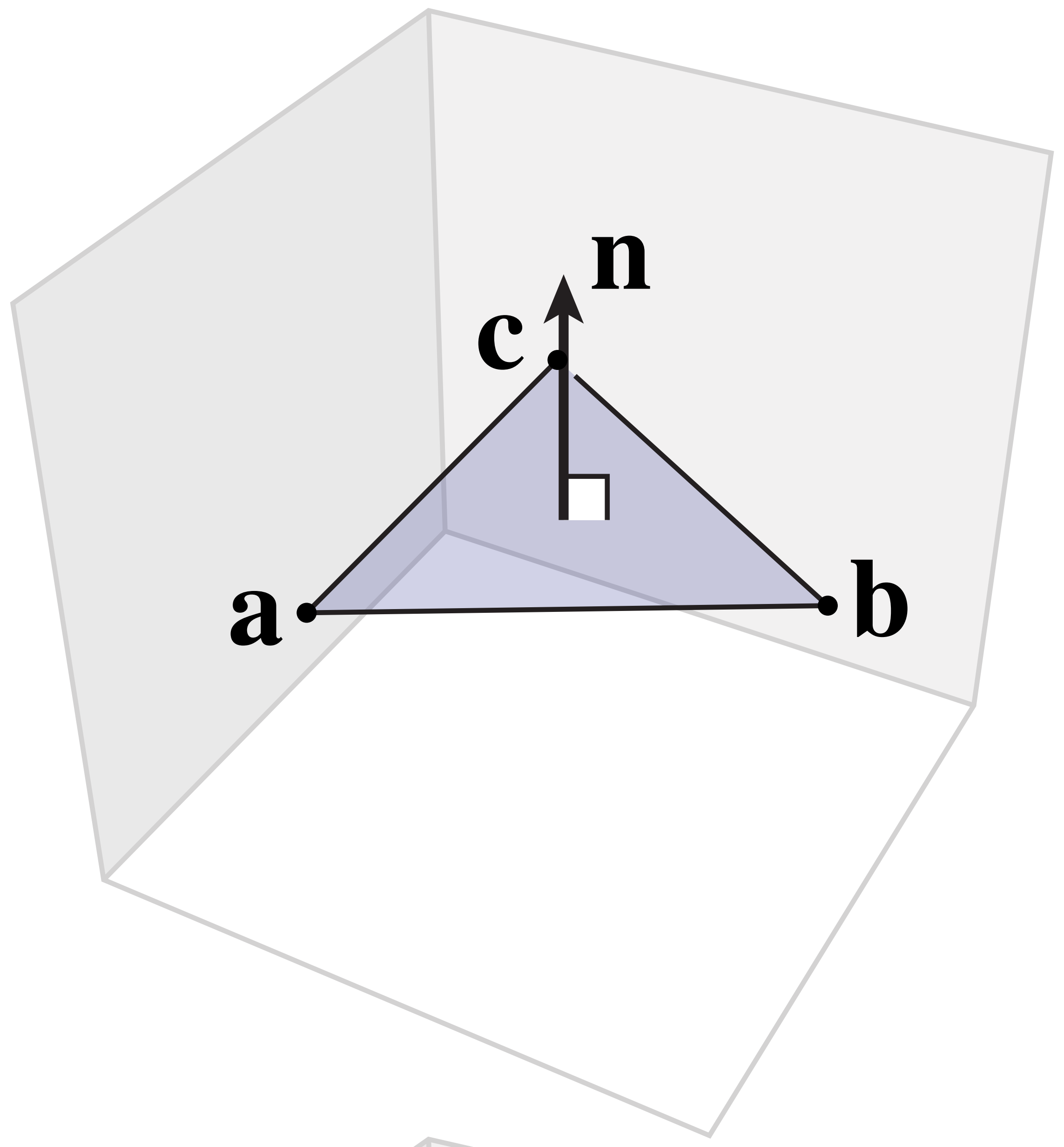
$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

translate (x, y, z) by (u, v, w)

$$\begin{bmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

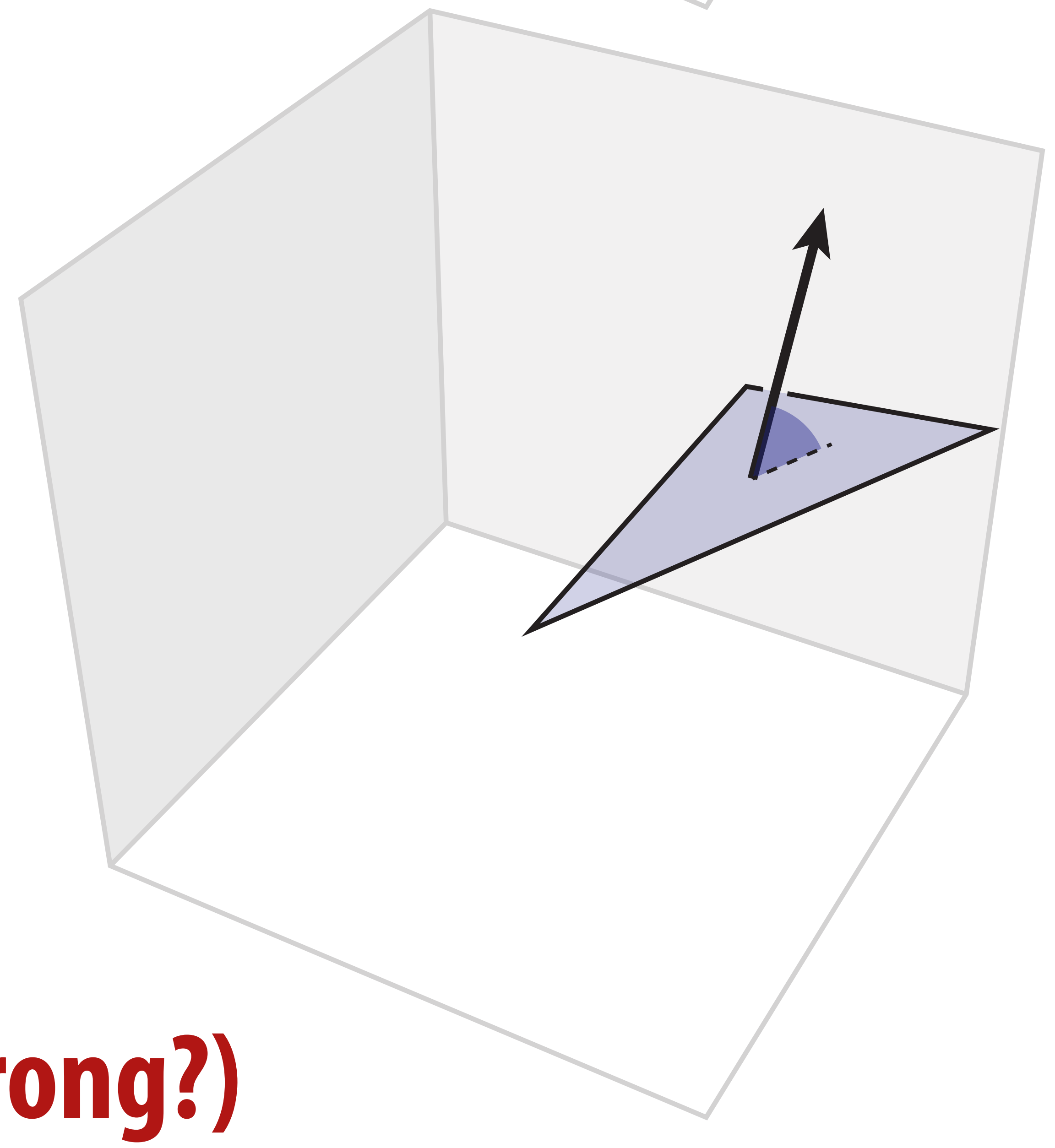
Points vs. Vectors

- Homogeneous coordinates have another useful feature: distinguish between <u>points</u> and <u>vectors</u>
- Consider for instance a triangle with:
  - vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$
  - normal vector $\mathbf{n} \in \mathbb{R}^3$



- Suppose we transform the triangle by appending "1" to $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{n}$ and multiplying by this matrix:

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow$$



Normal is not orthogonal to triangle! (What went wrong?)

Points vs. Vectors (continued)

- **Let's think about what happens when we multiply the normal vector $\mathbf{n}$ by our matrix:**

rotate normal around y by θ

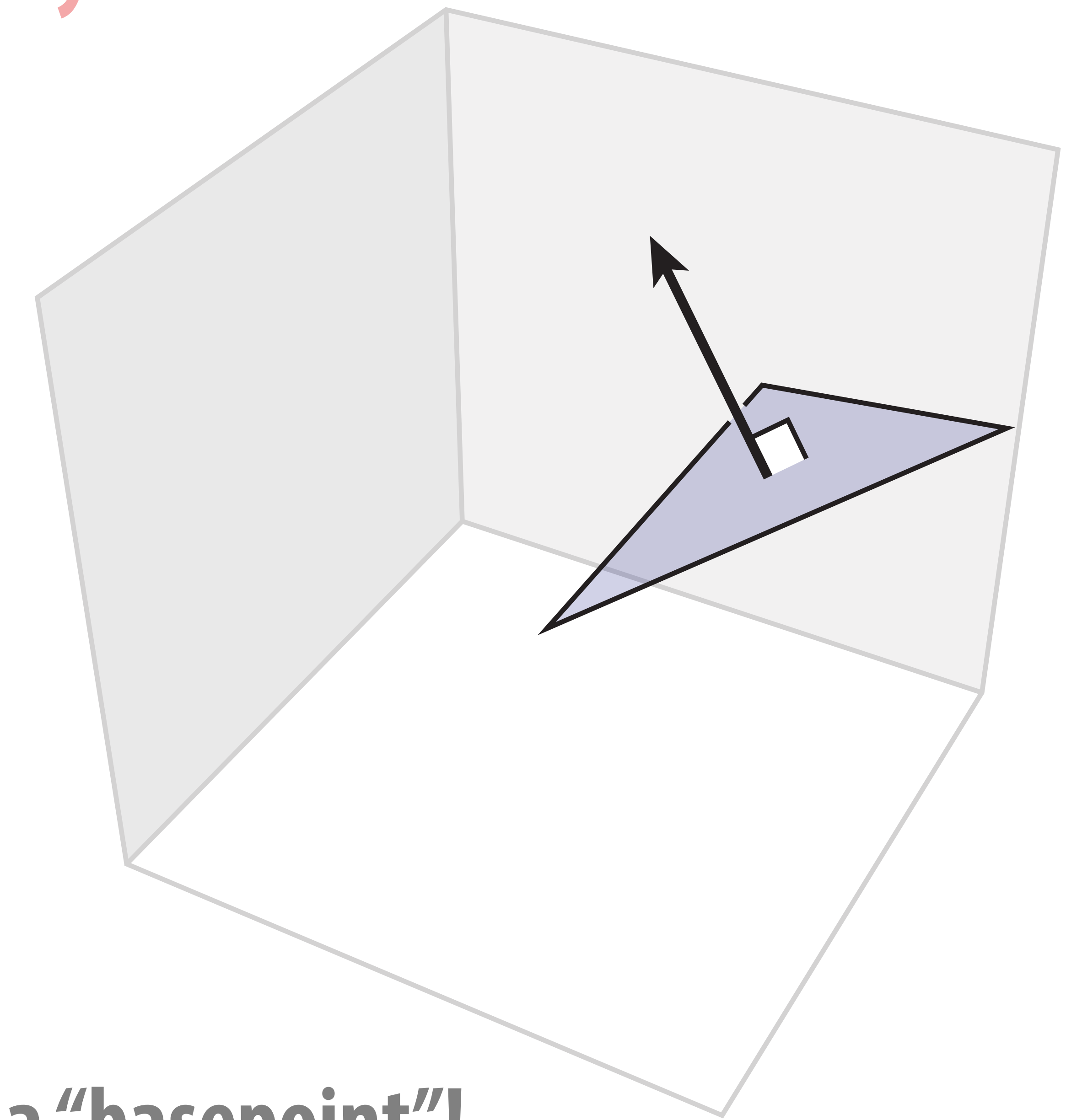
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & u \\ 0 & 1 & 0 & v \\ -\sin\theta & 0 & \cos\theta & w \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 1 \end{bmatrix}$$

translate normal by (u, v, w)

- **But when we rotate/translate a triangle, its normal should just rotate!***
- **Solution? Just set homogeneous coordinate to zero!**

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ 0 \end{bmatrix}$$

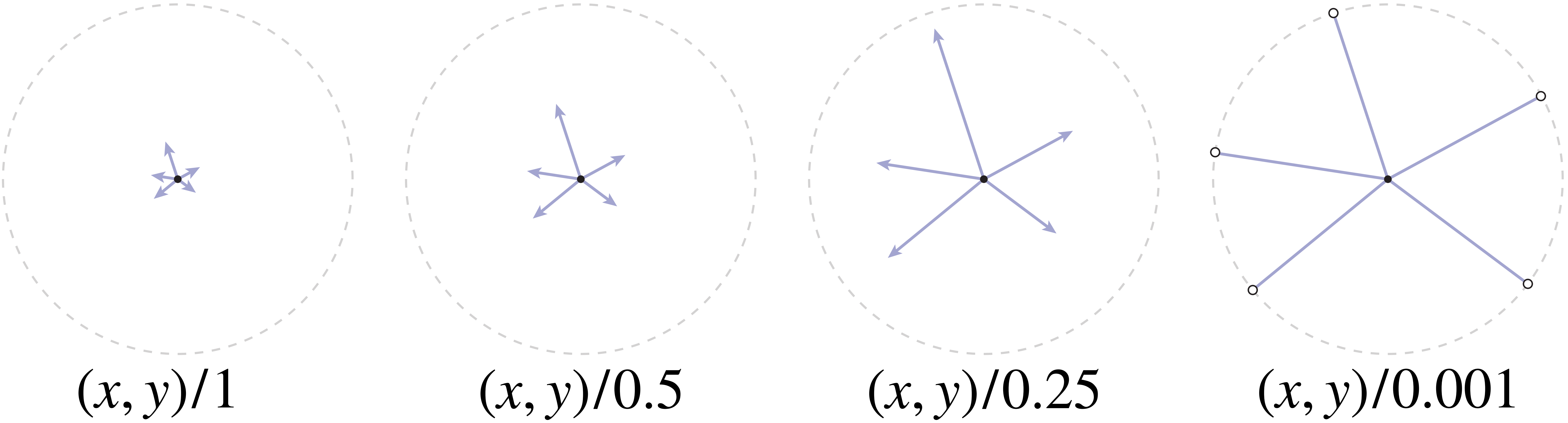
- **Translation now gets ignored; normal is orthogonal to triangle**



*Recall that vectors just have direction and magnitude—they don't have a "basepoint"!

Points vs. Vectors in Homogeneous Coordinates

- In general:
  - A *point* has a nonzero homogeneous coordinate $(c = 1)$
  - A *vector* has a zero homogeneous coordinate $(c = 0)$
- But wait… what division by c mean when it's equal to zero?
- Well consider what happens as $c \to 0$…

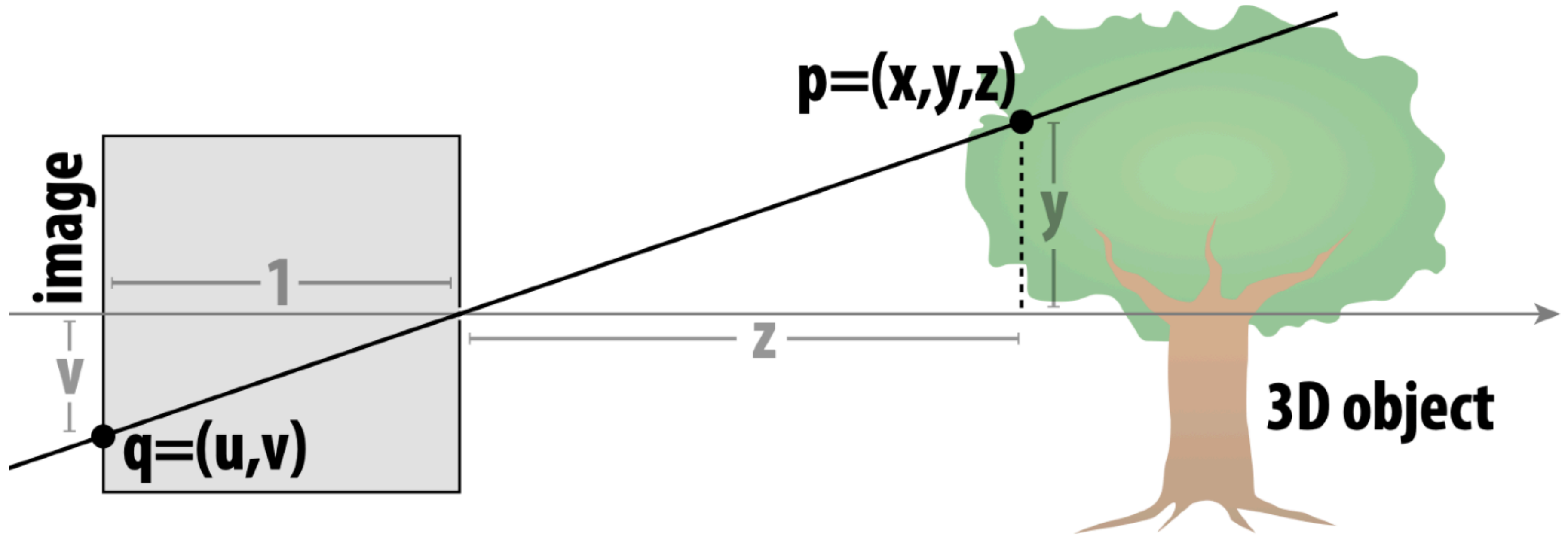


Can think of vectors as "points at infinity" (sometimes called "ideal points")

(In practice: still need to check for divide by zero!)

Perspective Projection in Homogeneous Coordinates

- Q: How can we perform perspective projection* using homogeneous coordinates?
- Remember from our pinhole camera model that the basic idea was to "divide by z"
- So, we can build a matrix that "copies" the z coordinate into the homogeneous coordinate
- Division by the homogeneous coordinate now gives us perspective projection onto the plane $z = 1$



$$(x, y, z) \mapsto (x/z, y/z)$$

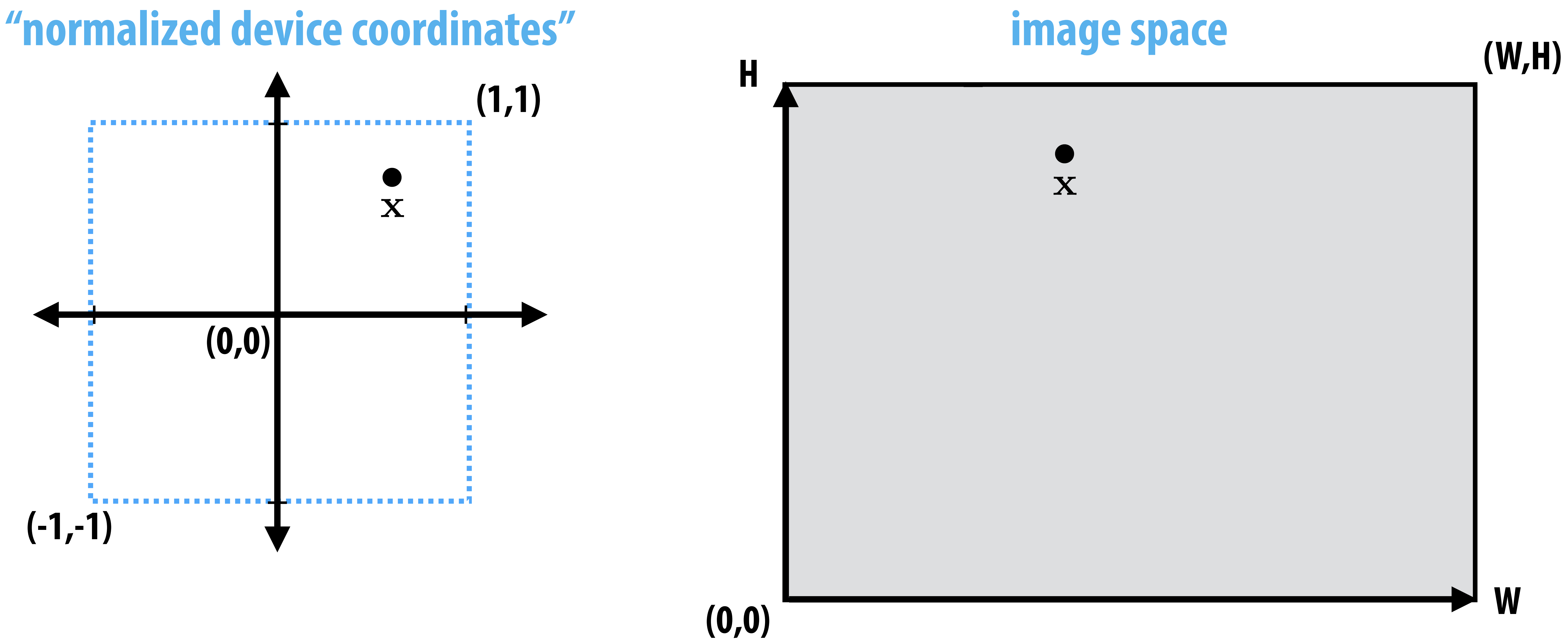
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x/z \\ y/z \\ 1 \end{bmatrix}$$

*Assuming a pinhole camera at (0,0,0) looking down the z-axis

Screen Transformation (OpenGL)

- One last transformation is needed in the rasterization pipeline: transform from viewing plane to pixel coordinates
- E.g., suppose we want to draw all points that fall inside the square [-1,1] x [-1,1] on the z = 1 plane, into a W x H pixel image



Q: What transformation(s) would you apply?

Screen Transformation (Vulkan, Direct3D)

- One last transformation is needed in the rasterization pipeline: transform from viewing plane to pixel coordinates
- E.g., suppose we want to draw all points that fall inside the square [-1,1] x [-1,1] on the z = 1 plane, into a W x H pixel image *with upper-left origin*.

"normalized device coordinates"

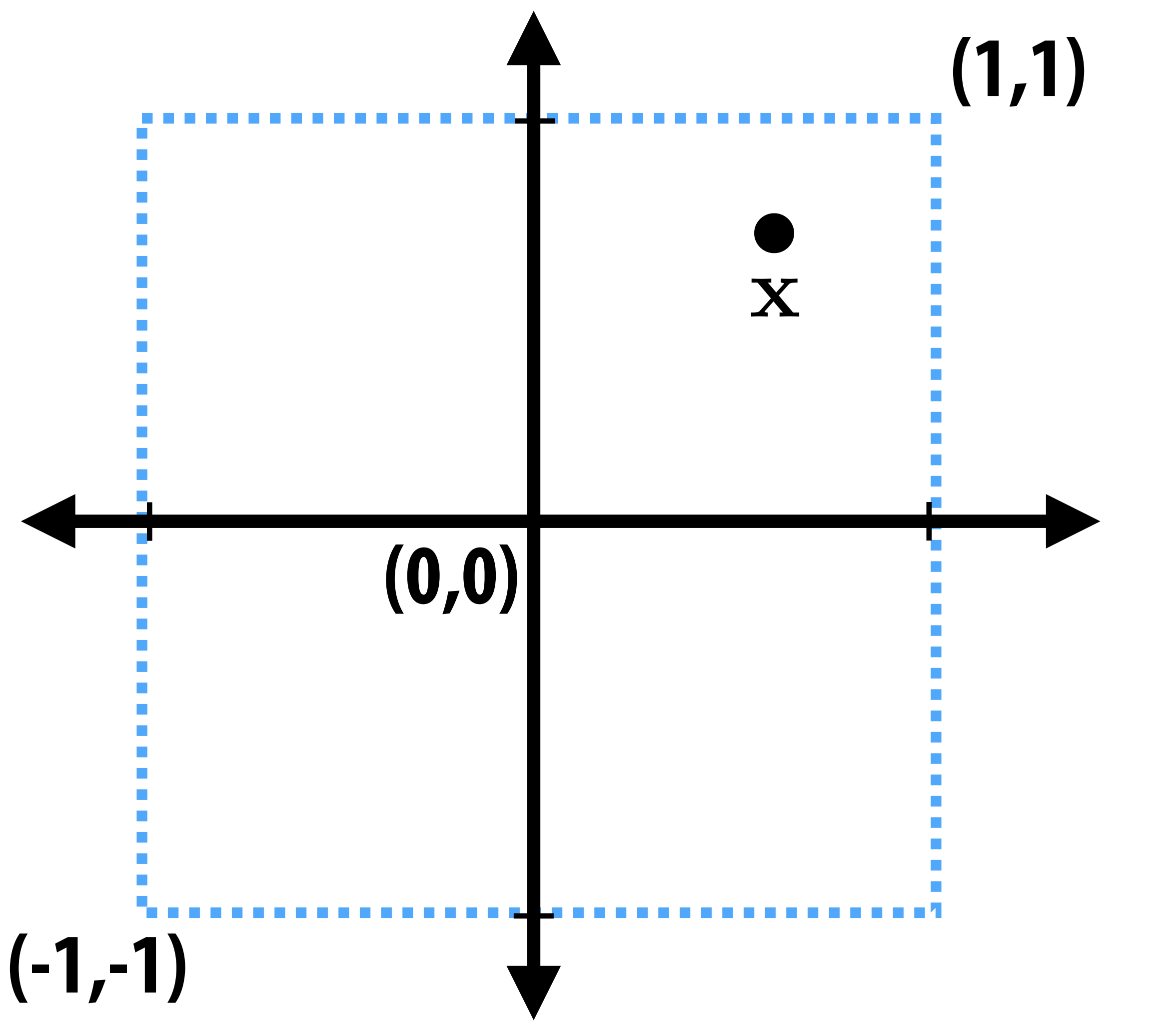
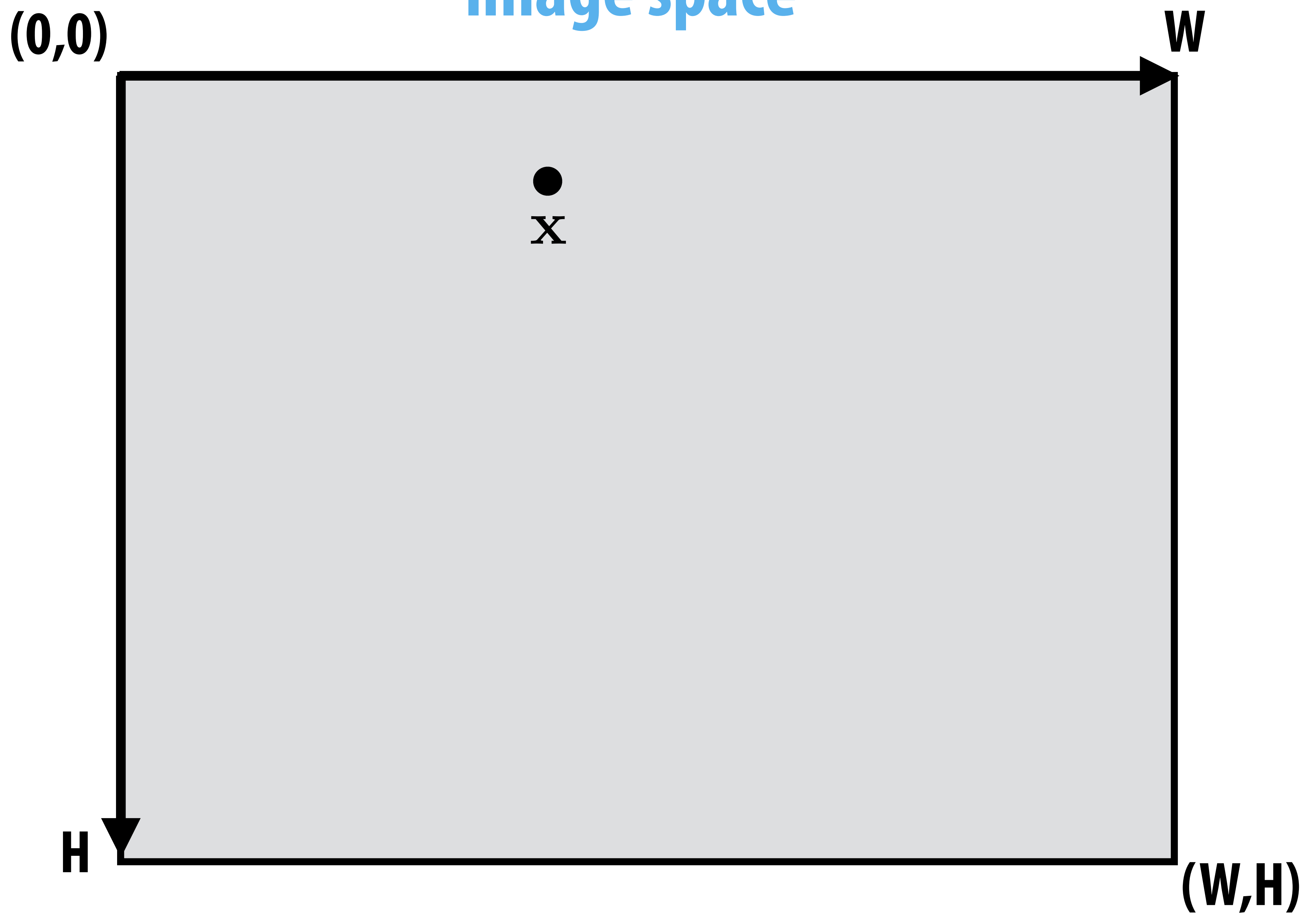
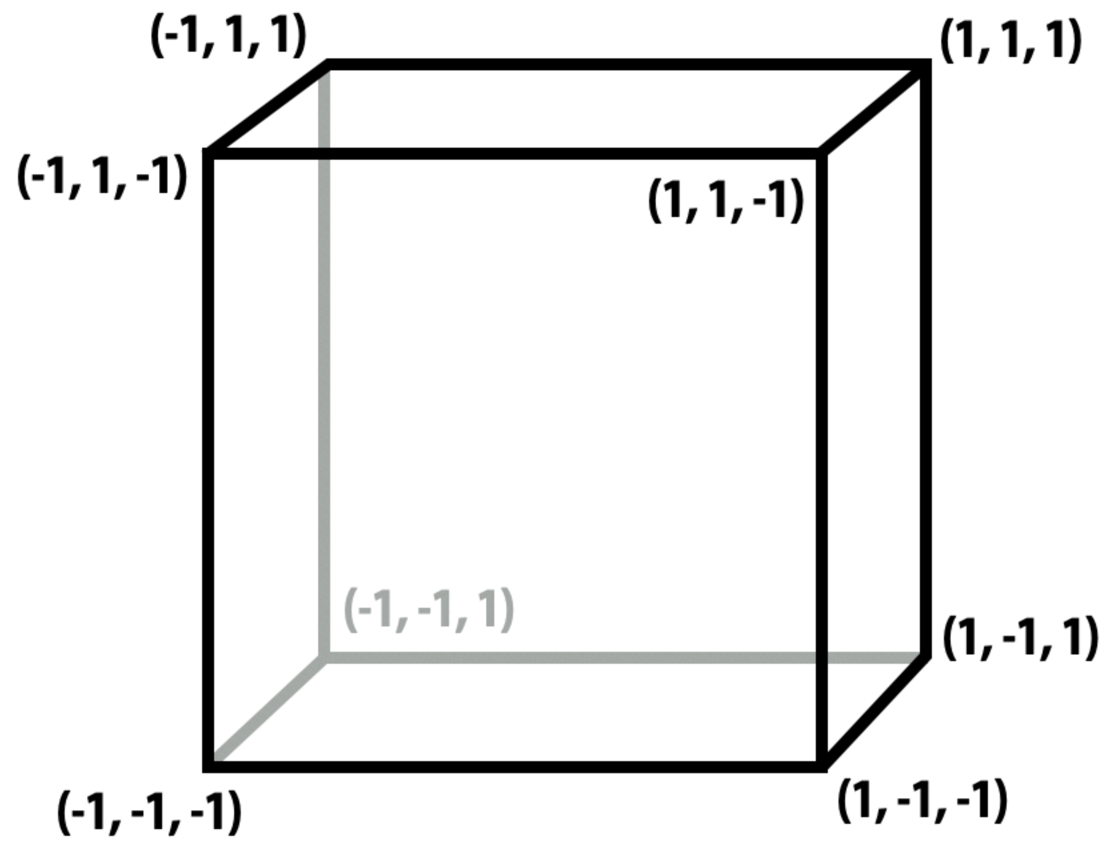


image space

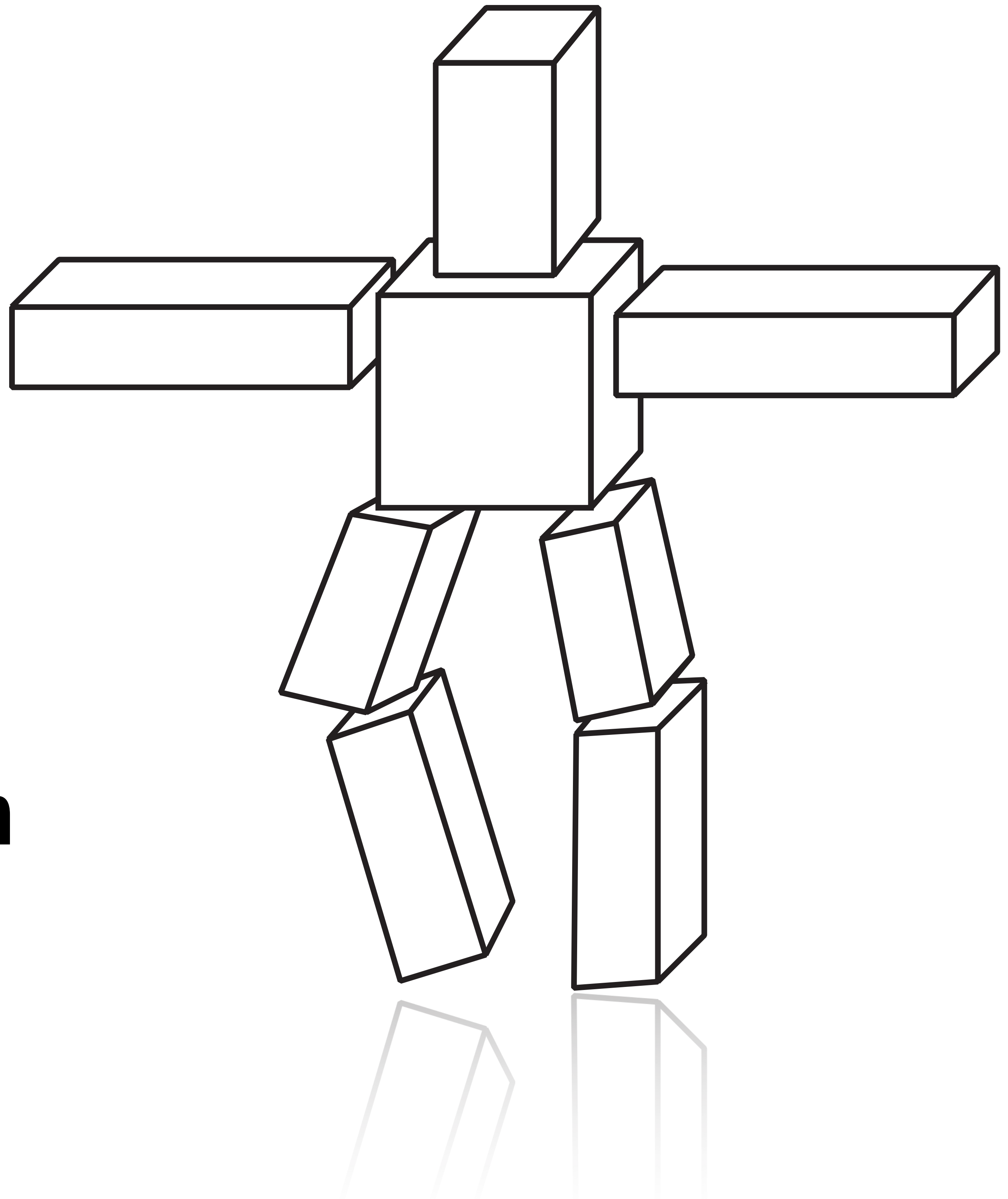


Q: What transformation(s) would you apply? (Careful: y is now *down!*)

Scene Graph

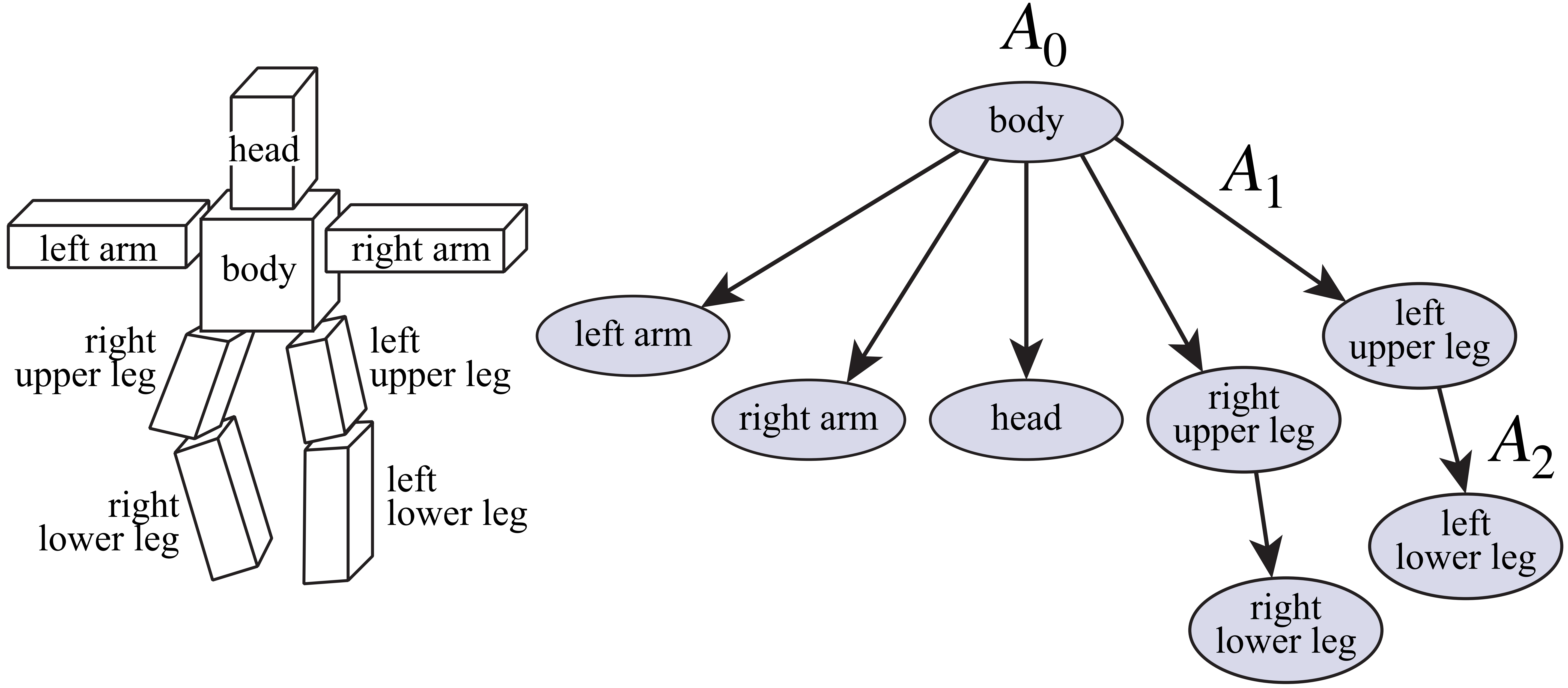


- For complex scenes (e.g., more than just a cube!) <u>scene graph</u> can help organize transformations
- Motivation: suppose we want to build a "cube creature" by transforming copies of the unit cube
- Difficult to specify each transformation directly
- Instead, build up transformations of "lower" parts from transformations of "upper" parts
  - E.g., first position the body
  - Then transform upper arm <u>relative to</u> the body
  - Then transform lower arm relative to upper arm
  - …



Scene Graph (continued)

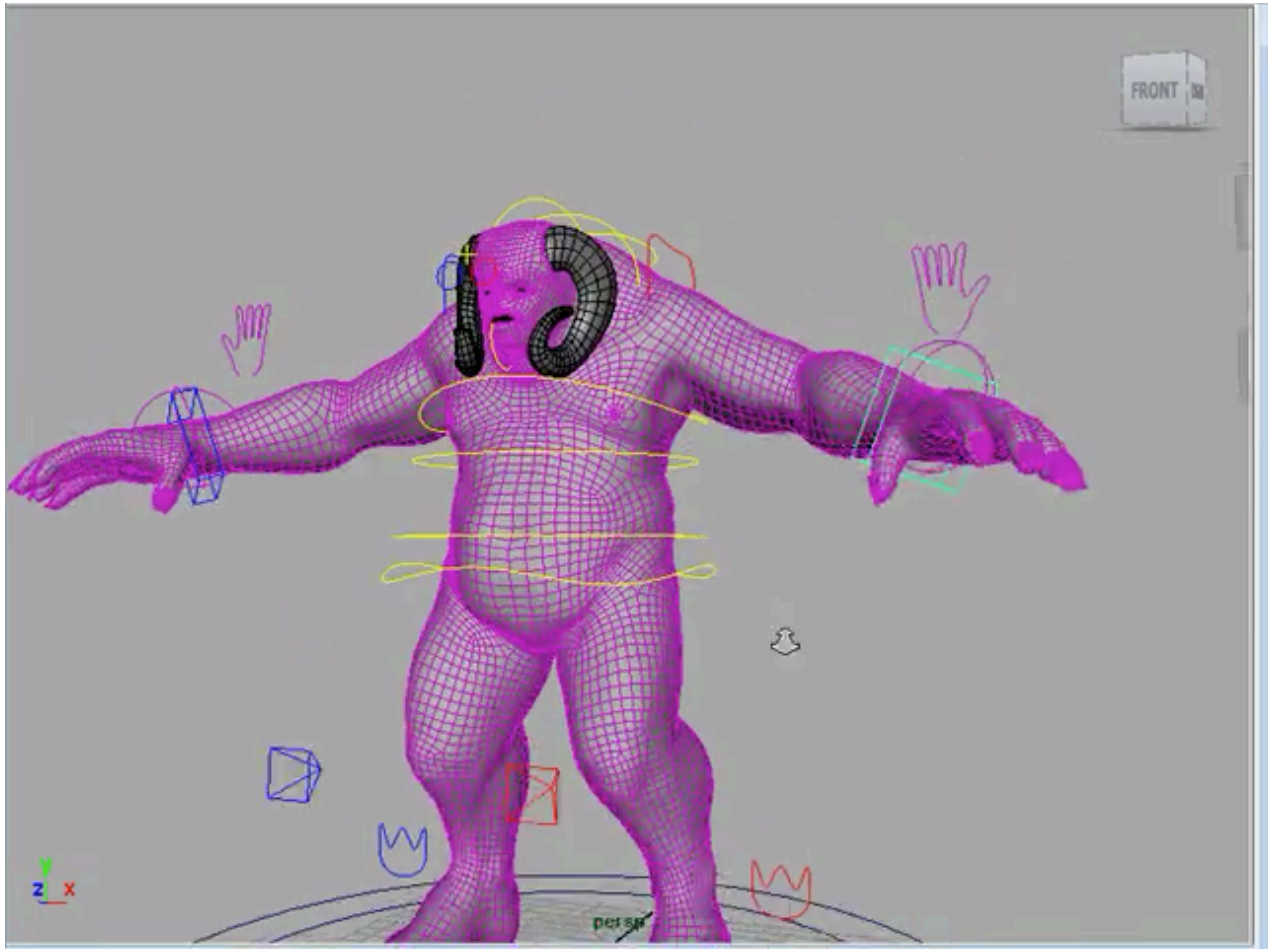
- Scene graph stores relative transformations in directed graph
- Each edge (+root) stores a linear transformation (e.g., a 4x4 matrix)
- Composition of transformations gets applied to nodes



- E.g., $A_1 A_0$ gets applied to left upper leg; $A_2 A_1 A_0$ to left lower leg
- Keep transformations on a stack to reduce redundant multiplication

Scene Graph—Example

Often used to build up complex "rig":

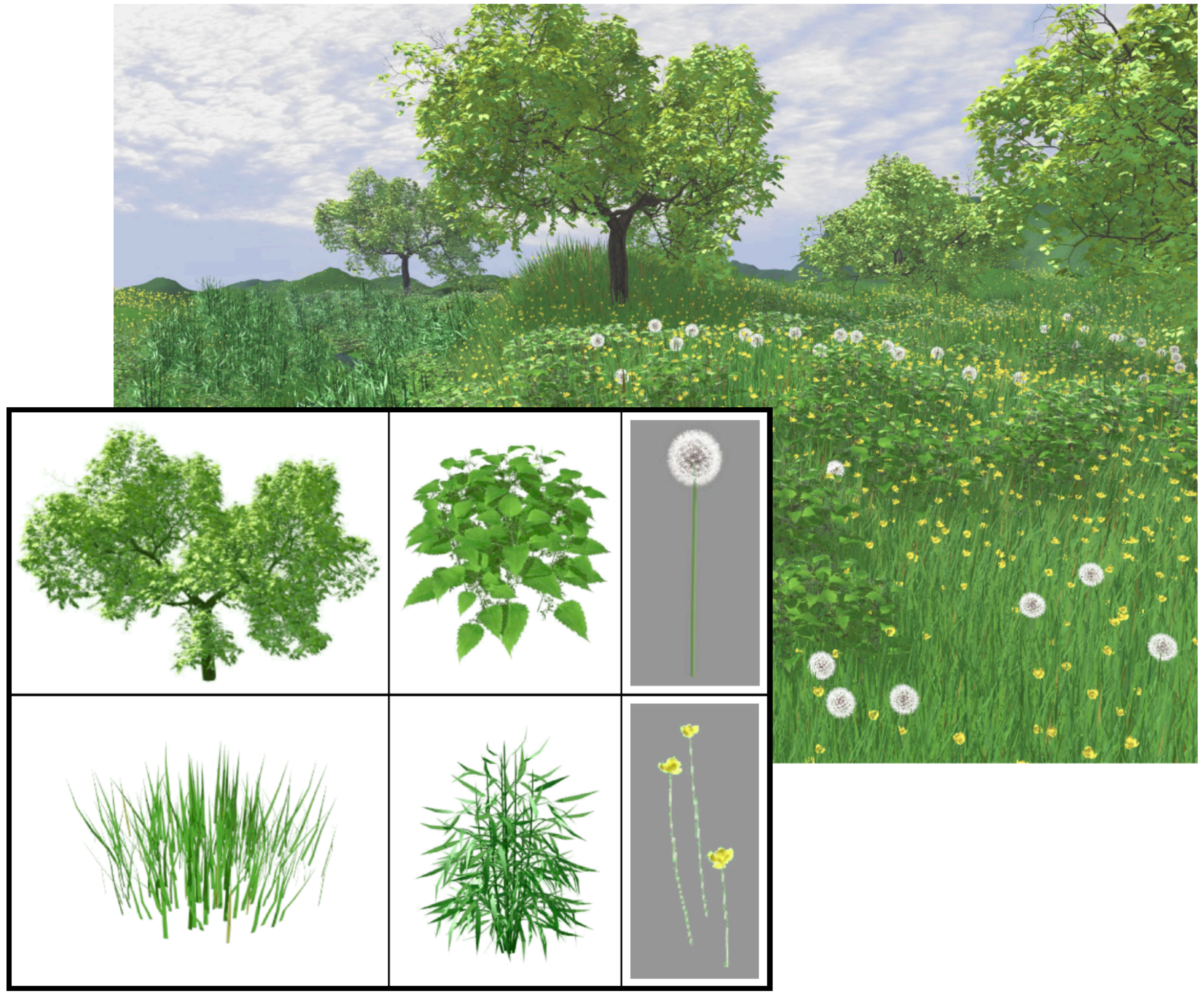


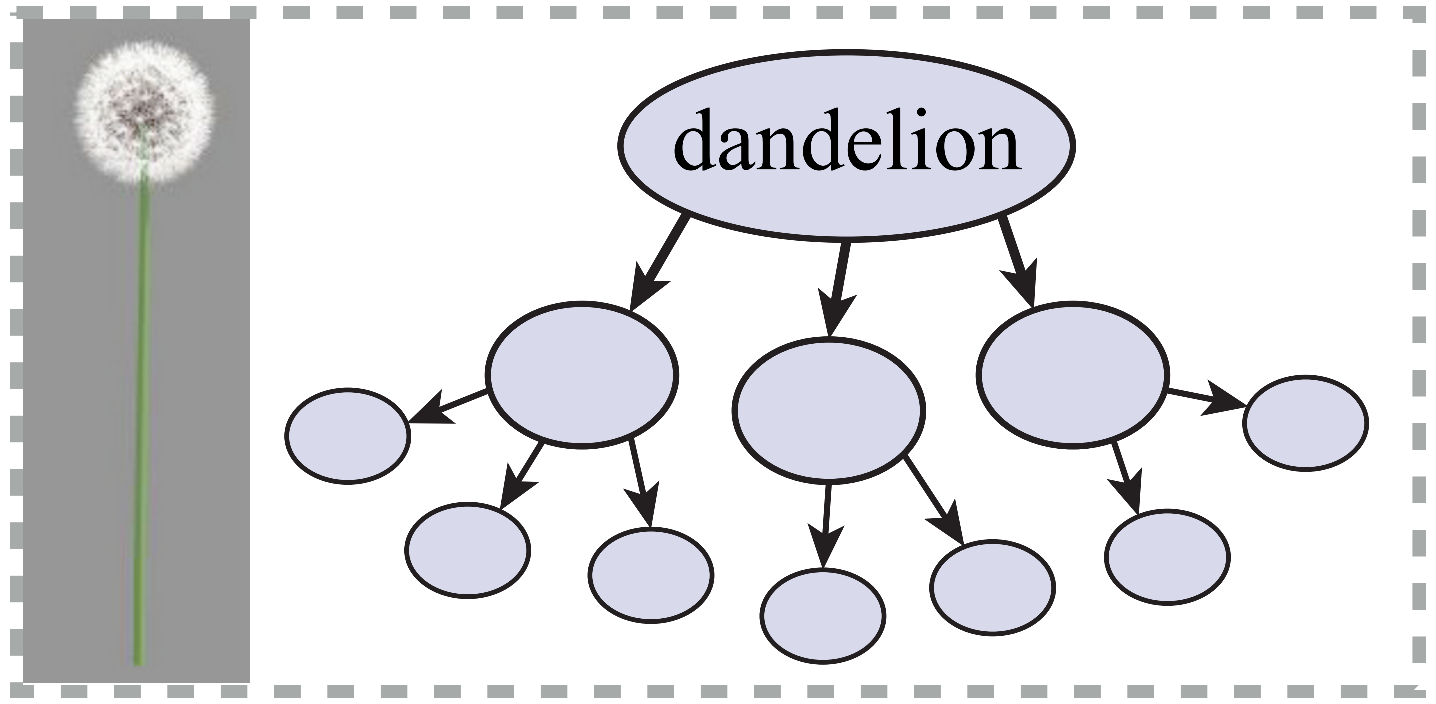
In general, scene graph also includes other models, lights, cameras, . . .

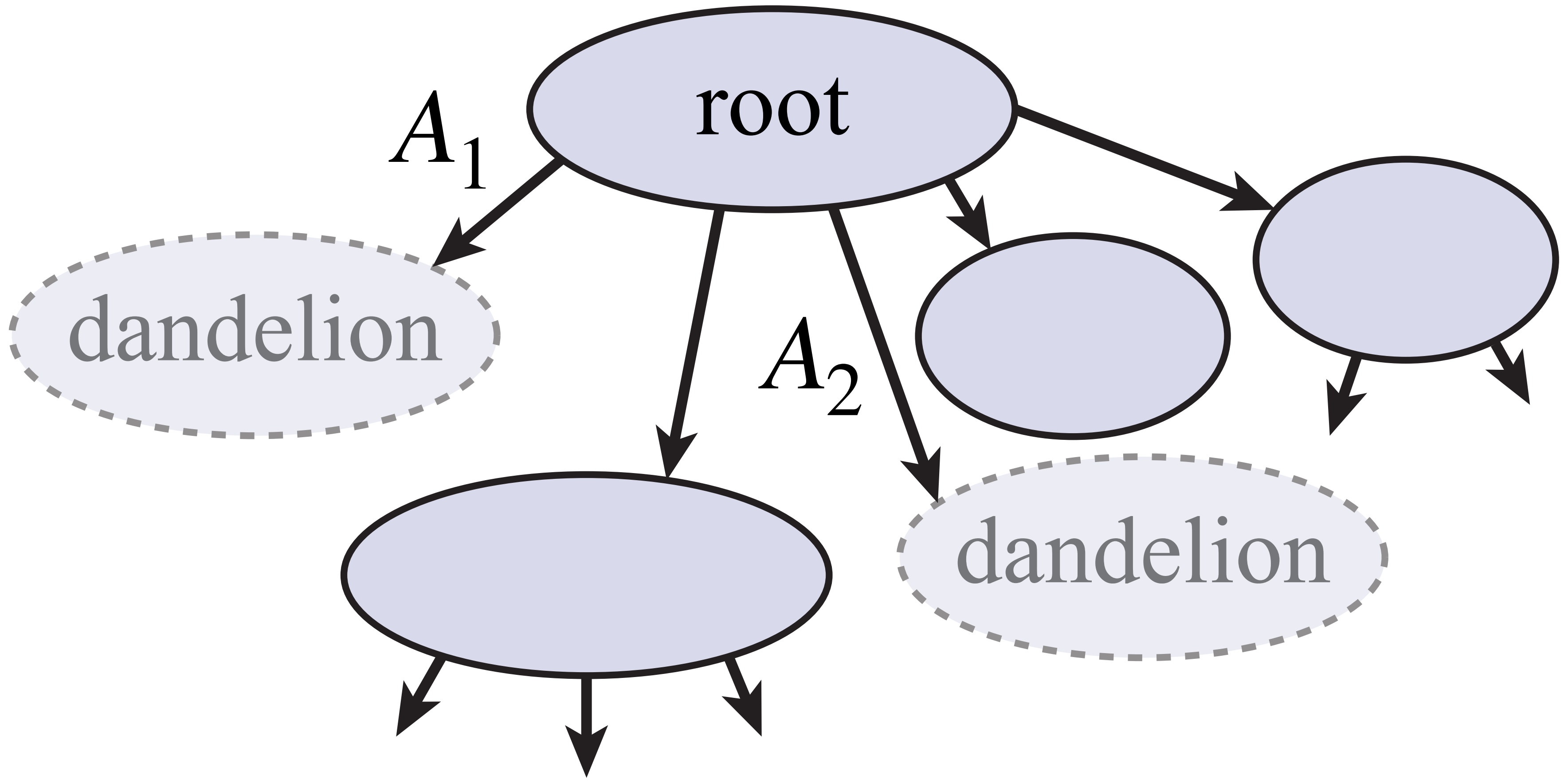
Instancing

- What if we want many copies of the same object in a scene?
- Rather than have many copies of the geometry, scene graph, etc., can just put a "pointer" node in our scene graph
- Like any other node, can specify a different transformation on each incoming edge

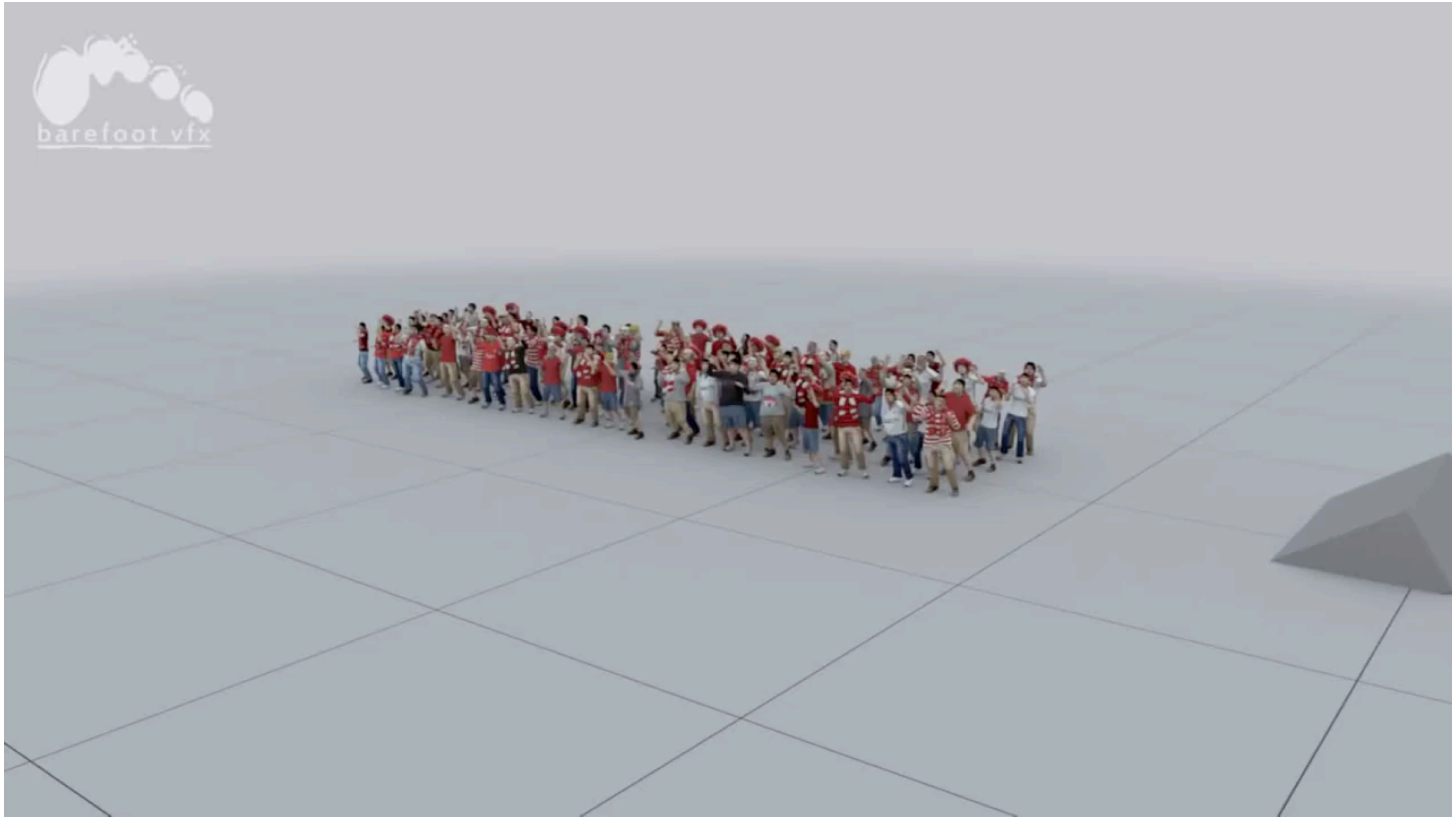
Deussen et al, "Realistic modeling and rendering of plant ecosystems"



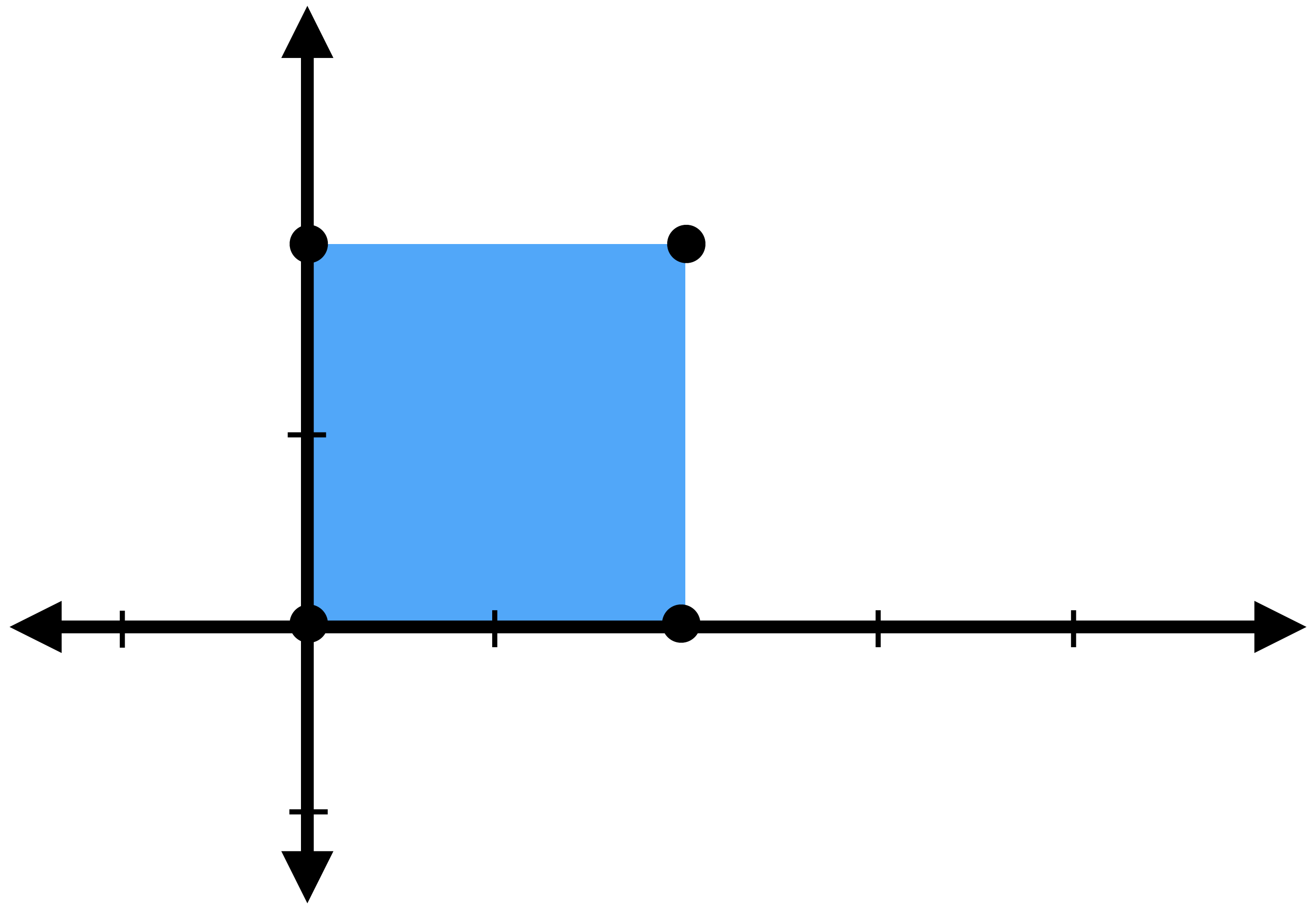




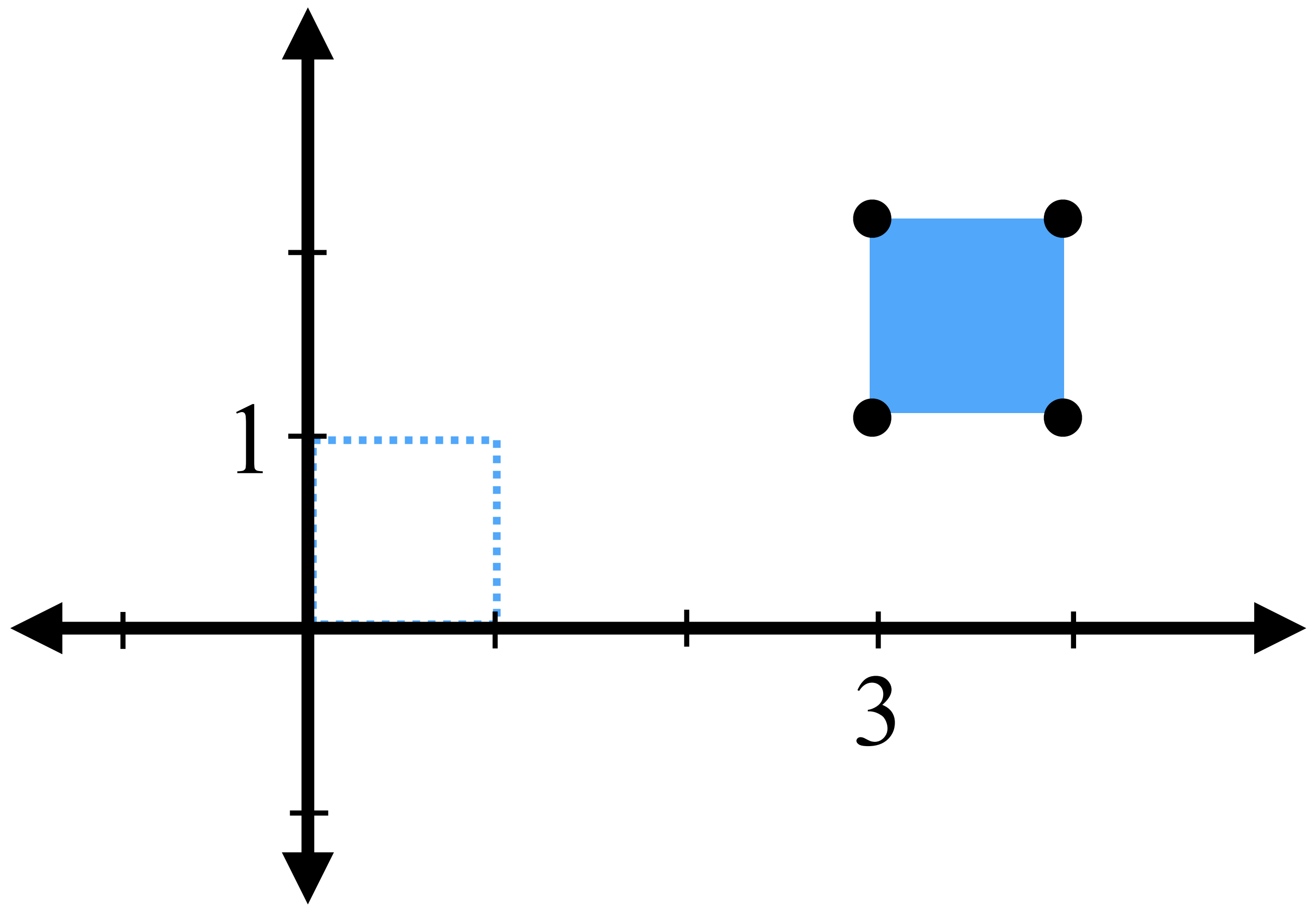
Instancing—Example



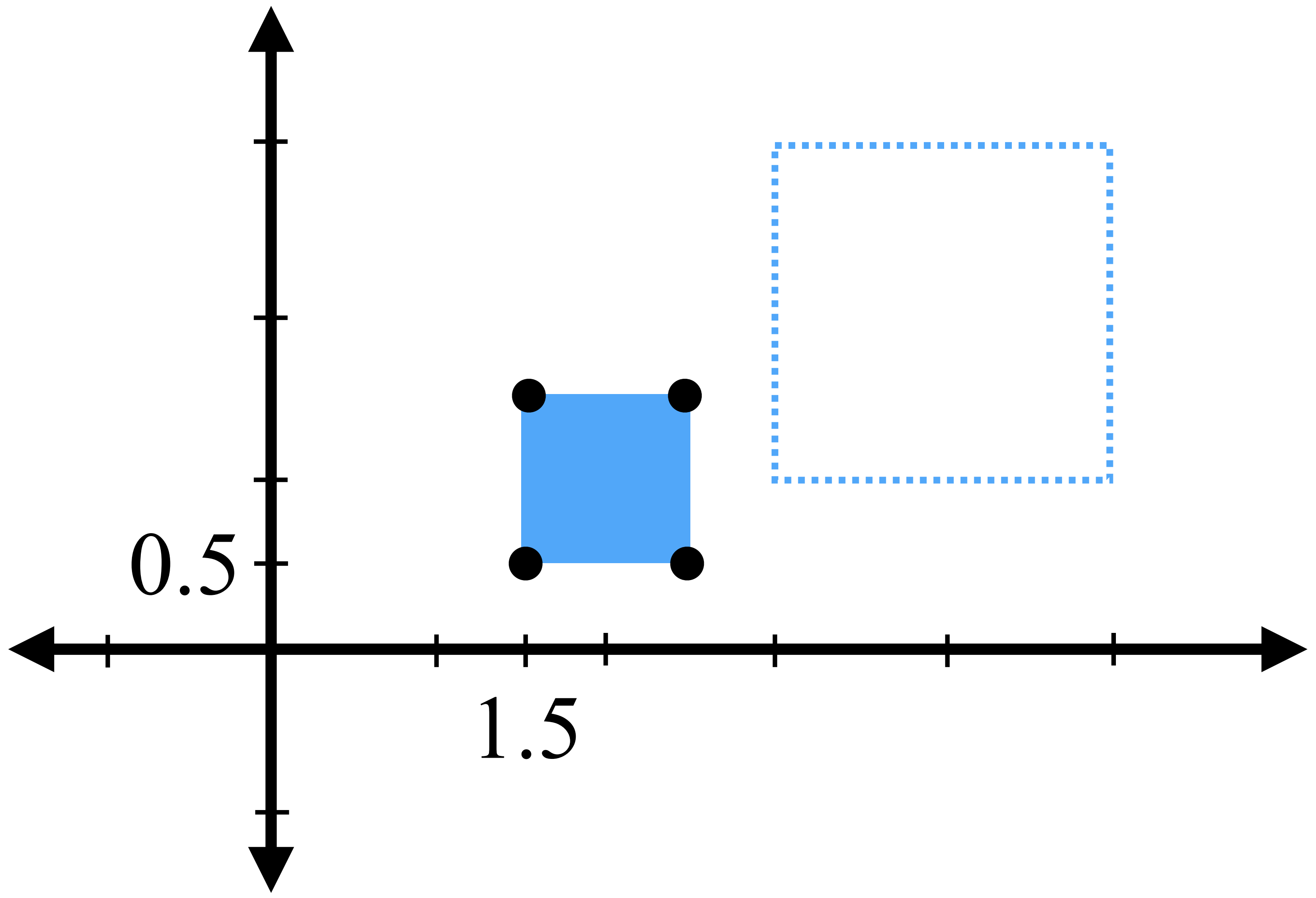
Order matters when composing transformations!



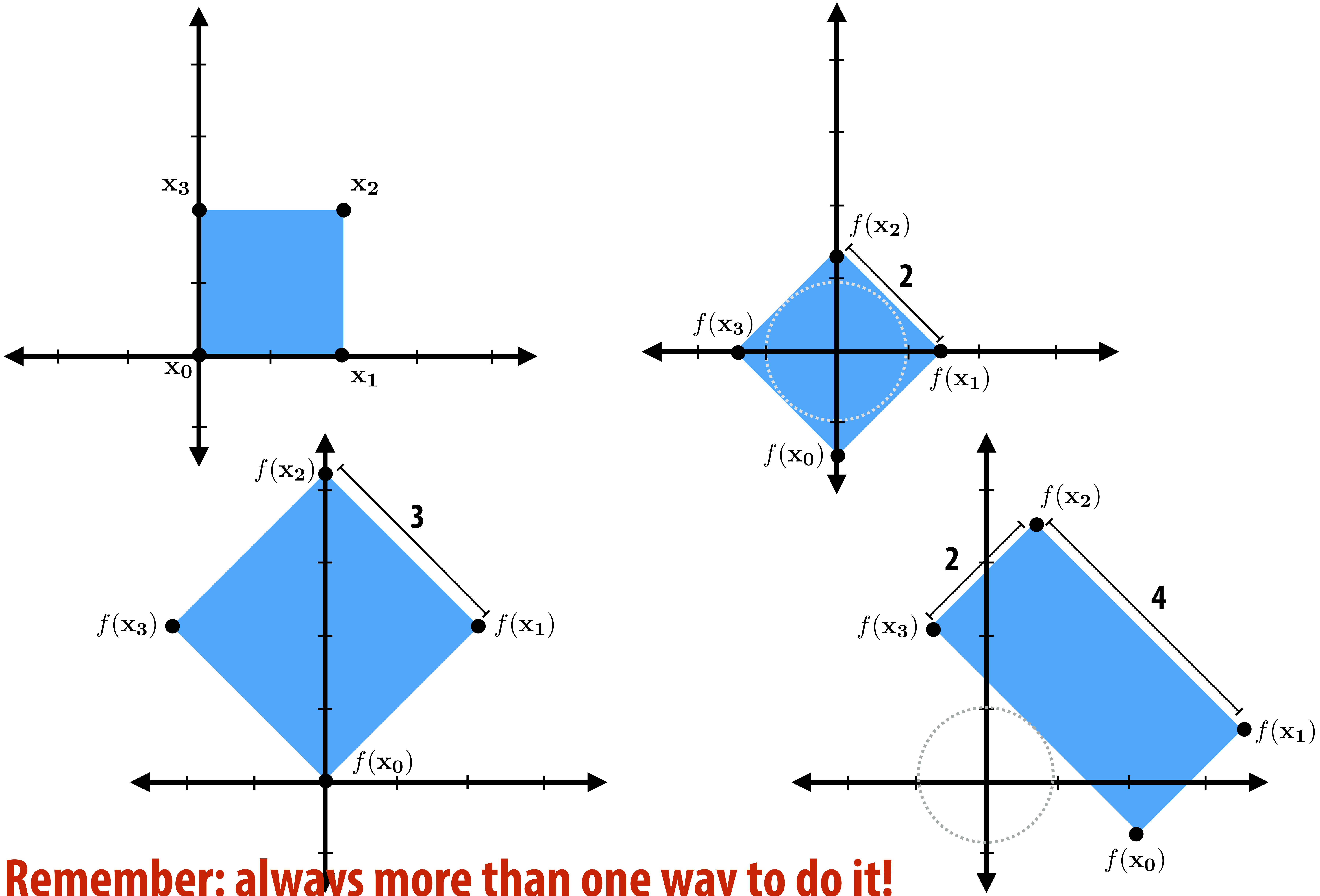
scale by 1/2, then translate by (3,1)



translate by (3,1), then scale by 1/2

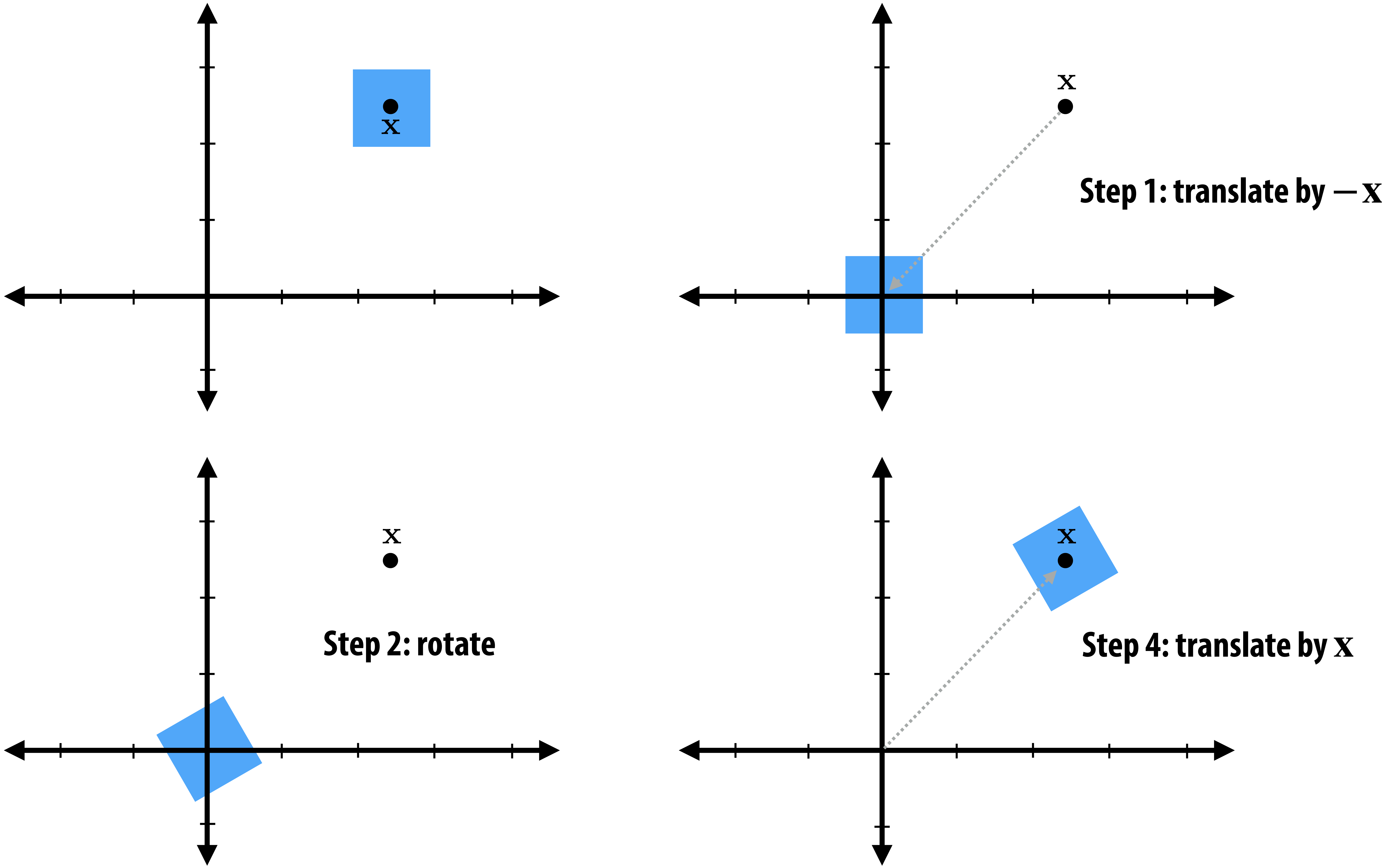


How would you perform these transformations?



Remember: always more than one way to do it!

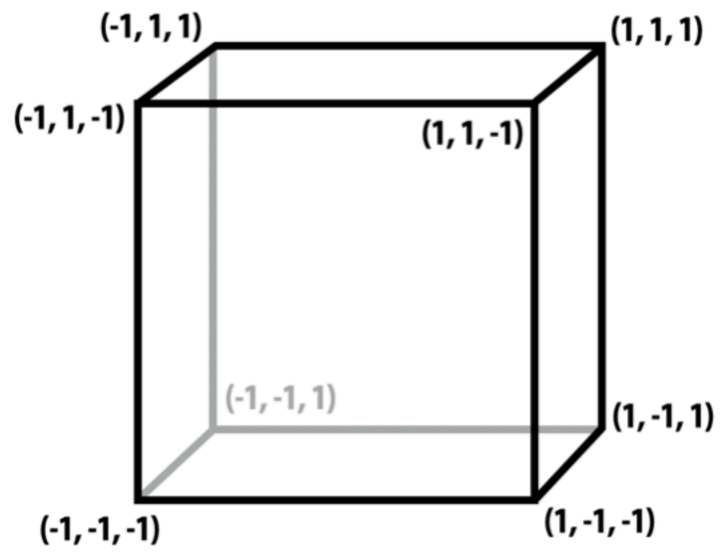
Common task: rotate about a point x

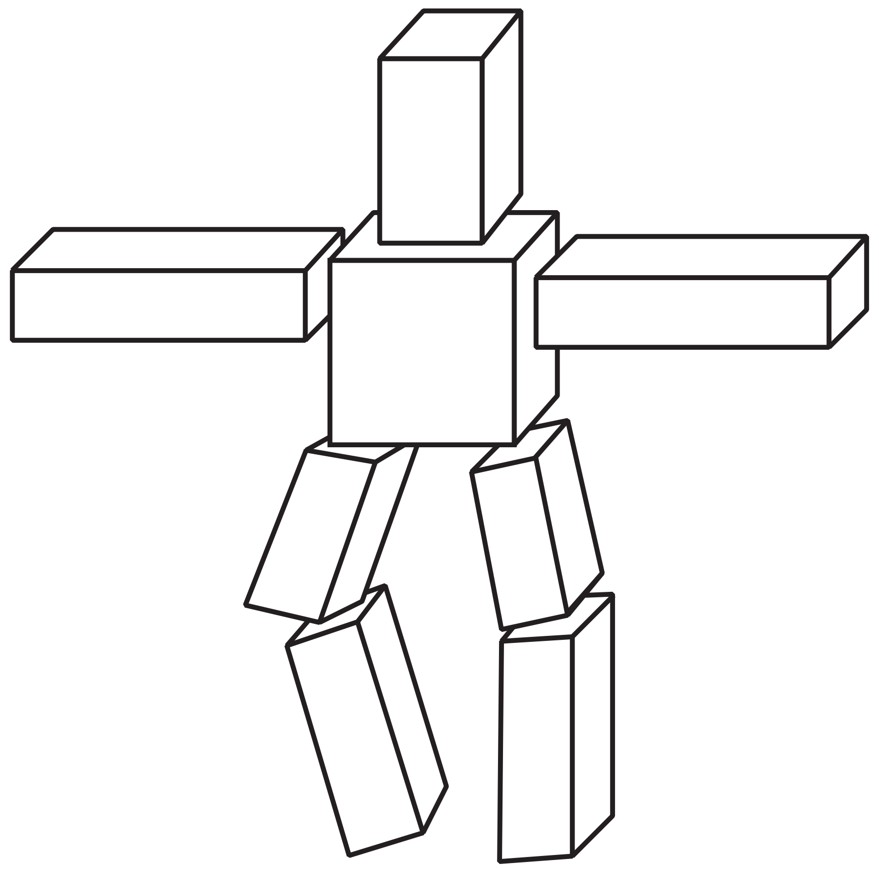


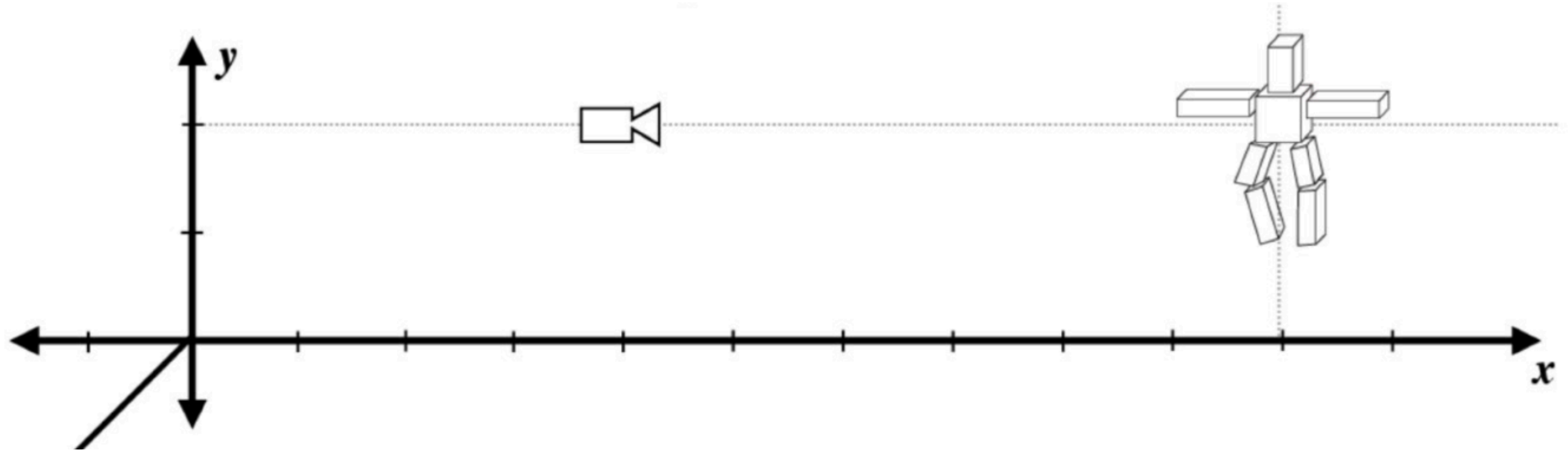
Q: What happens if we just rotate without translating first?

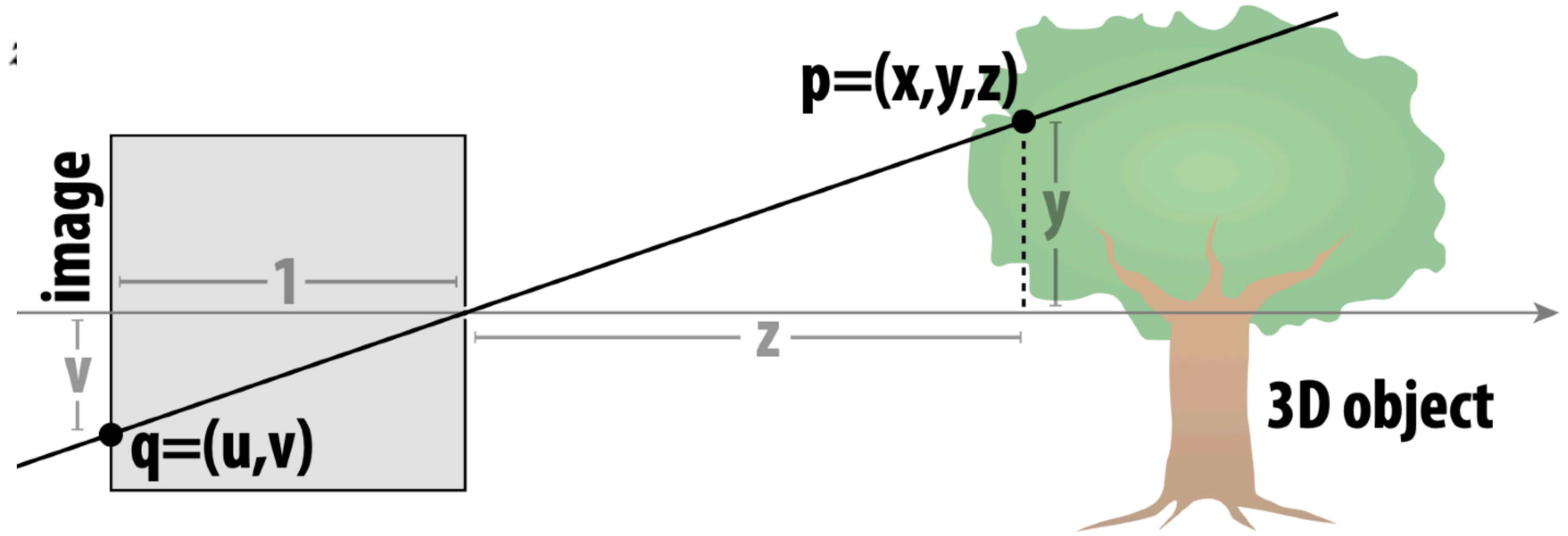
Drawing a Cube Creature

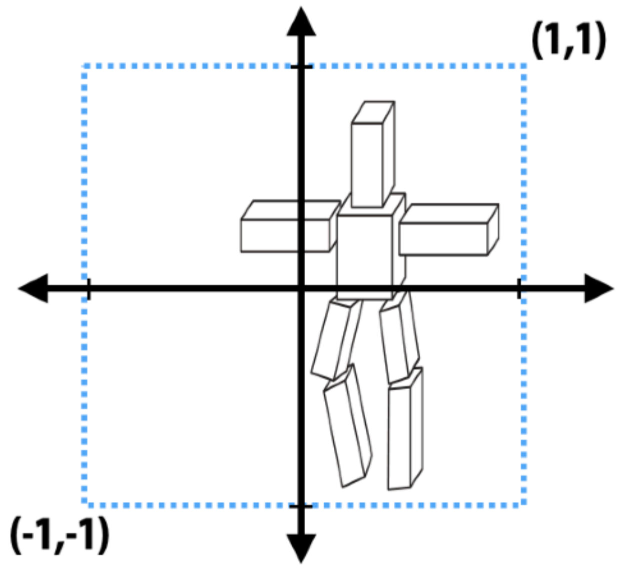
- Let's put this all together: starting with our 3D cube, we want to make a 2D, perspective-correct image of a "cube creature"
- First we use our scene graph to apply 3D transformations to several copies of our cube
- Then we apply a 3D transformation to position our camera
- Then a perspective projection
- Finally we convert to image coordinates (and rasterize)
- …Easy, right? :-)

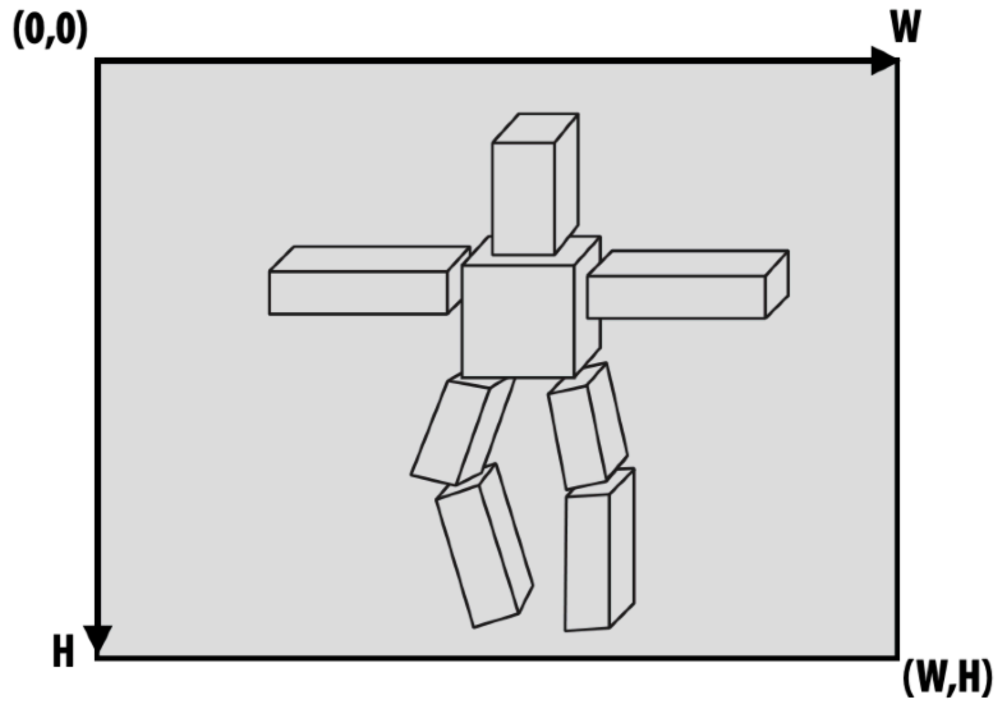












Spatial Transformations—Summary

transformation defined by its <u>invariants</u>

<u>basic linear transformations</u>

scaling
rotation
reflection
shear

<u>basic nonlinear transformations</u>

translation
perspective projection

linear when represented via <u>homogeneous coords</u>

homogeneous coords also distinguish points & vectors

<u>composite transformations</u>

- compose basic transformations to get more interesting ones
- always reduces to a single 4x4 matrix (in homogeneous coordinates)
  - simple, unified representation, efficient implementation
- order of composition matters!
- <u>many</u> ways to decompose a given transformation (polar, SVD, …)
- use *scene graph* to organize transformations
  - use *instancing* to eliminate redundancy

Next time: 3D Rotations