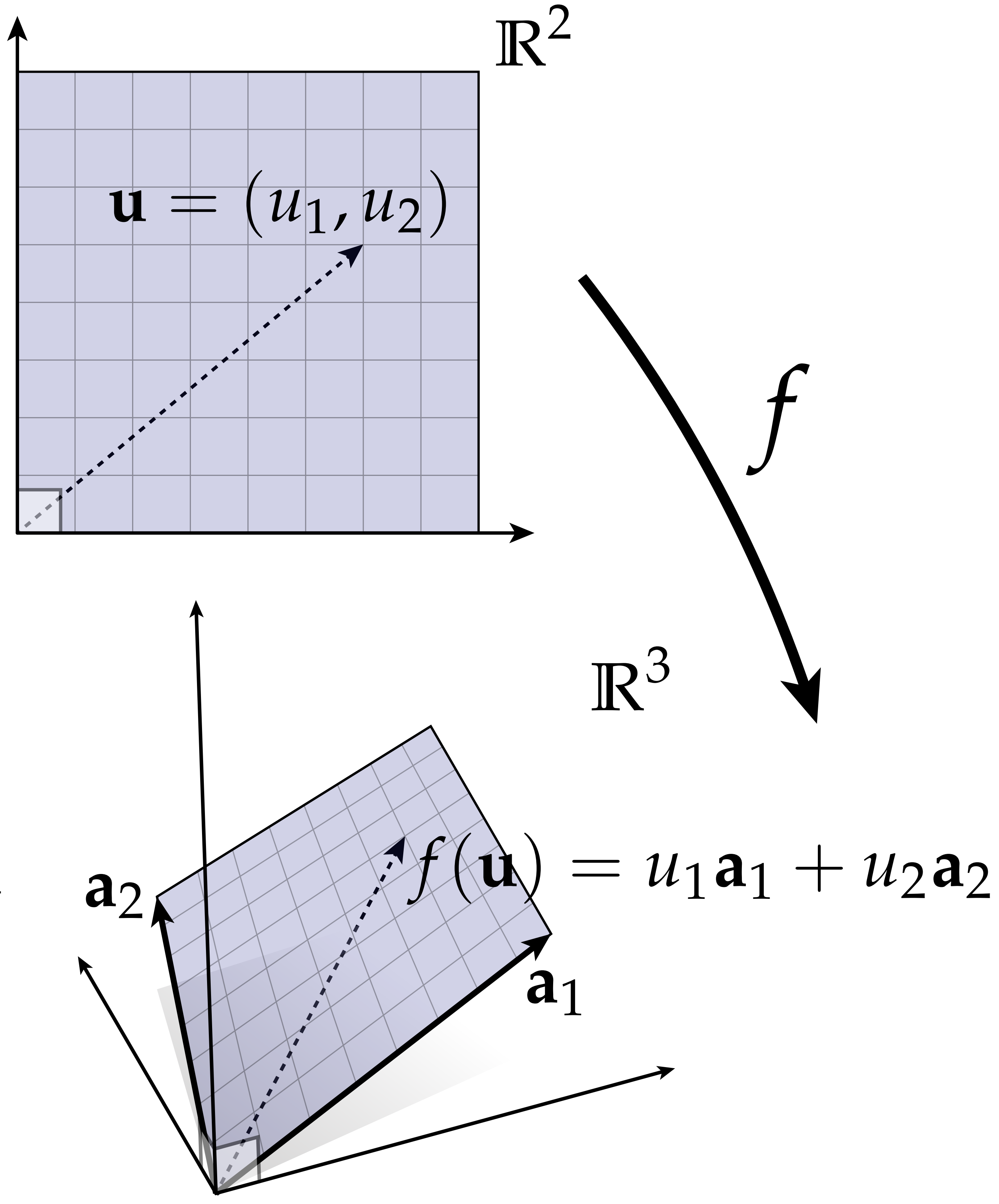
# Math (P)Review Part II:
# Vector Calculus

**Computer Graphics**
**CMU 15-462/662**

# Last Time: Linear Algebra




- **Touched on a variety of topics:**

vectors & vector spaces
norm
$L^2$ norm/inner product
span
Gram-Schmidt
linear systems
quadratic forms
...

vectors as functions
inner product
linear maps
basis
frequency decomposition
bilinear forms
matrices
...

- **Don't have time to cover everything!**
- **But there are some fantastic lectures online:**


**3Blue1Brown — Essence of Linear Algebra**

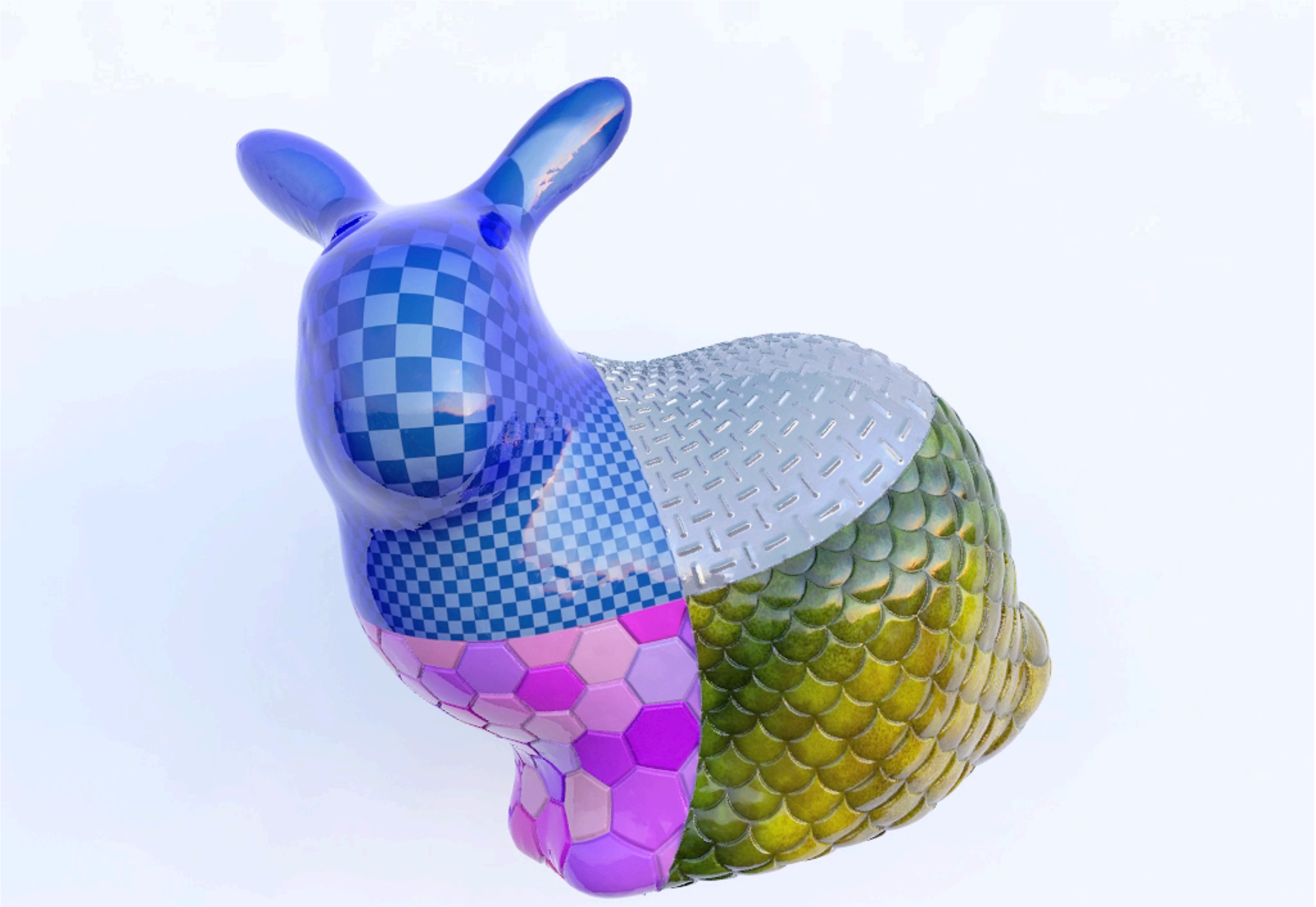
**Robert Ghrist — Calculus Blue**

**...**

**(Let us know about others online!)**

# Vector Calculus in Computer Graphics

- **Today's topic: vector calculus.**
- **Why is vector calculus important for computer graphics?**
  - **Basic language for talking about spatial relationships, transformations, etc.**
  - **Much of modern graphics (physically-based animation, geometry processing, etc.) formulated in terms of *partial differential equations* (PDEs) that use div, curl, Laplacian...**
  - **As we saw last time, vector-valued data is everywhere in graphics!**

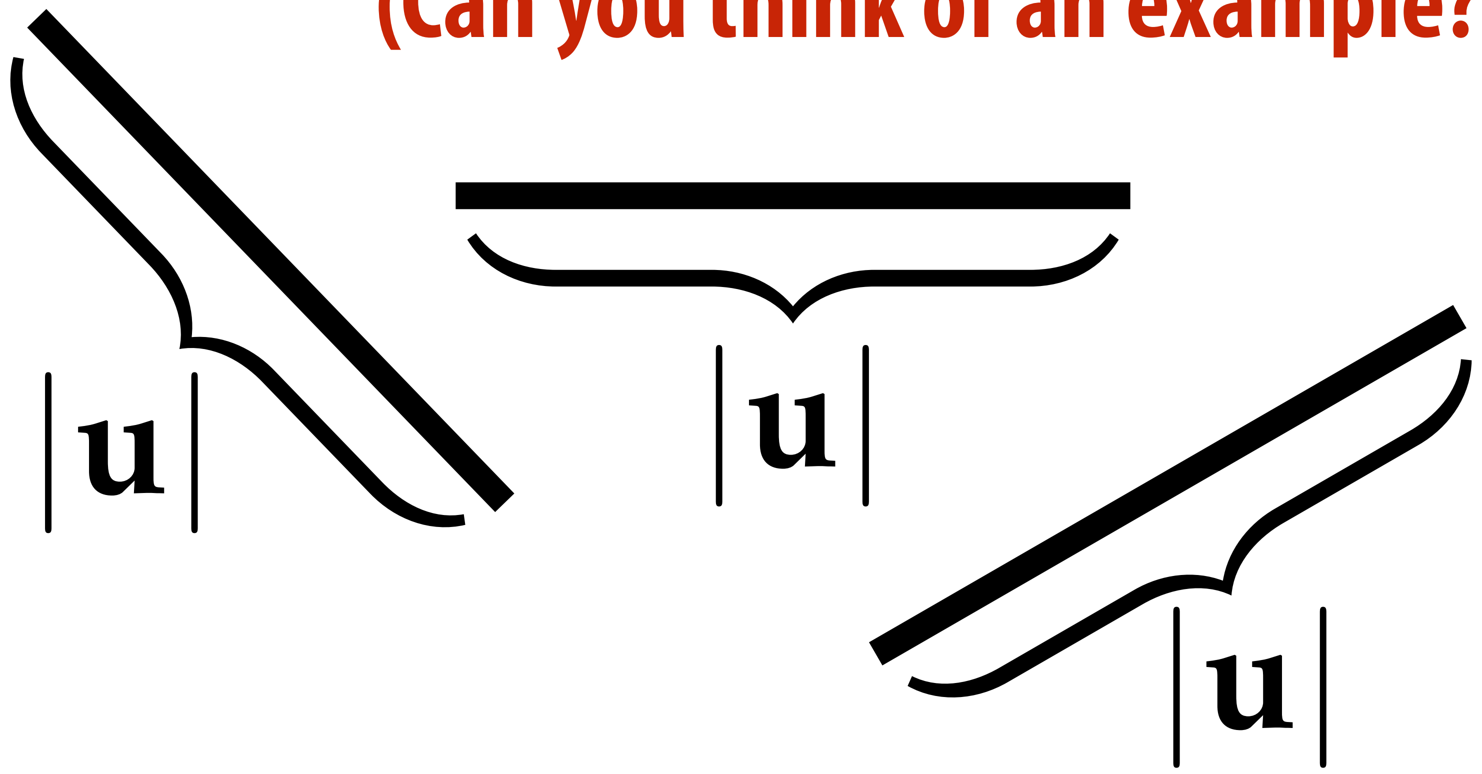


# Euclidean Norm

- Last time, developed idea of *norm*, which measures total size, length, volume, intensity, etc.
- For geometric calculations, the norm we most often care about is the Euclidean norm
- Euclidean norm is any notion of length preserved by rotations/translations/reflections of space.
- In *orthonormal* coordinates:

$$|\mathbf{u}| := \sqrt{u_1^2 + \cdots + u_n^2}$$

Not true for all norms!
(Can you think of an example?)

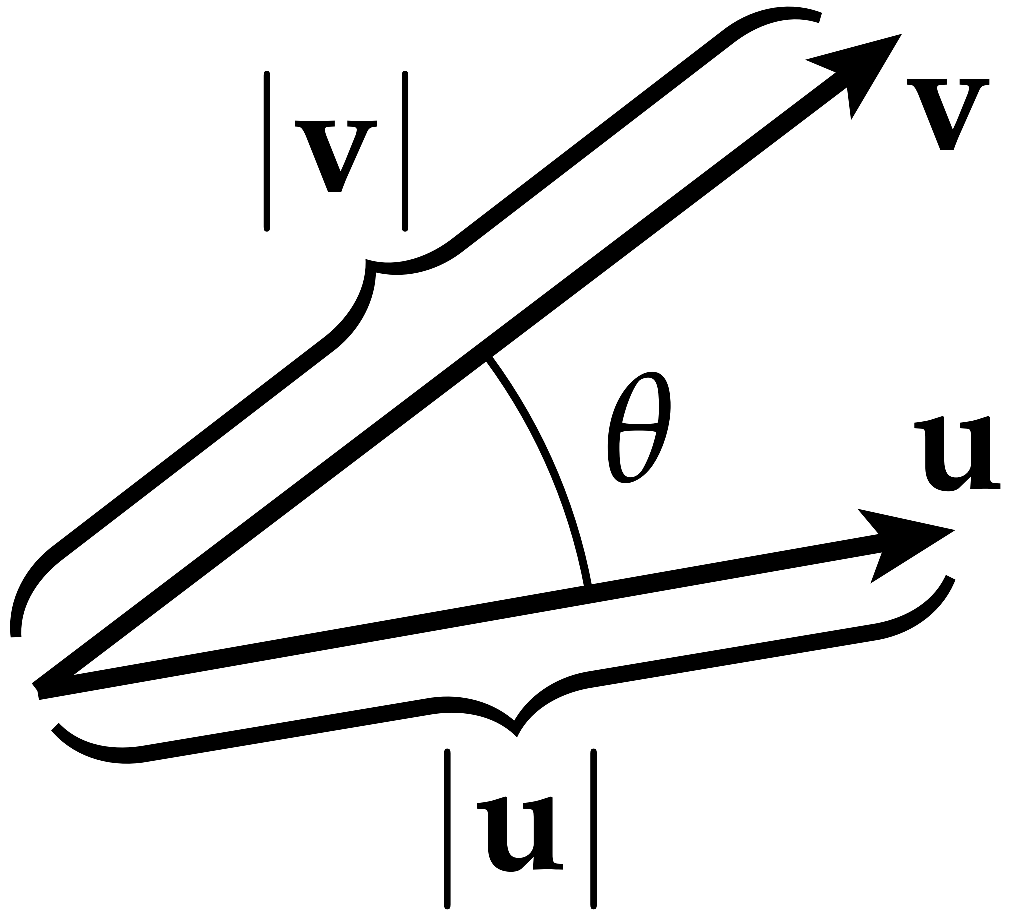



**WARNING:** This quantity does *not* encode geometric length unless vectors are encoded in an orthonormal basis. (Common source of bugs!)

# Euclidean Inner Product / Dot Product

- Likewise, lots of possible inner products—intuitively, measure some notion of "alignment."
- For *geometric* calculations, want to use inner product that captures something about geometry!
- For n-dimensional vectors, Euclidean inner product defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle := |\mathbf{u}||\mathbf{v}|\cos(\theta)$$



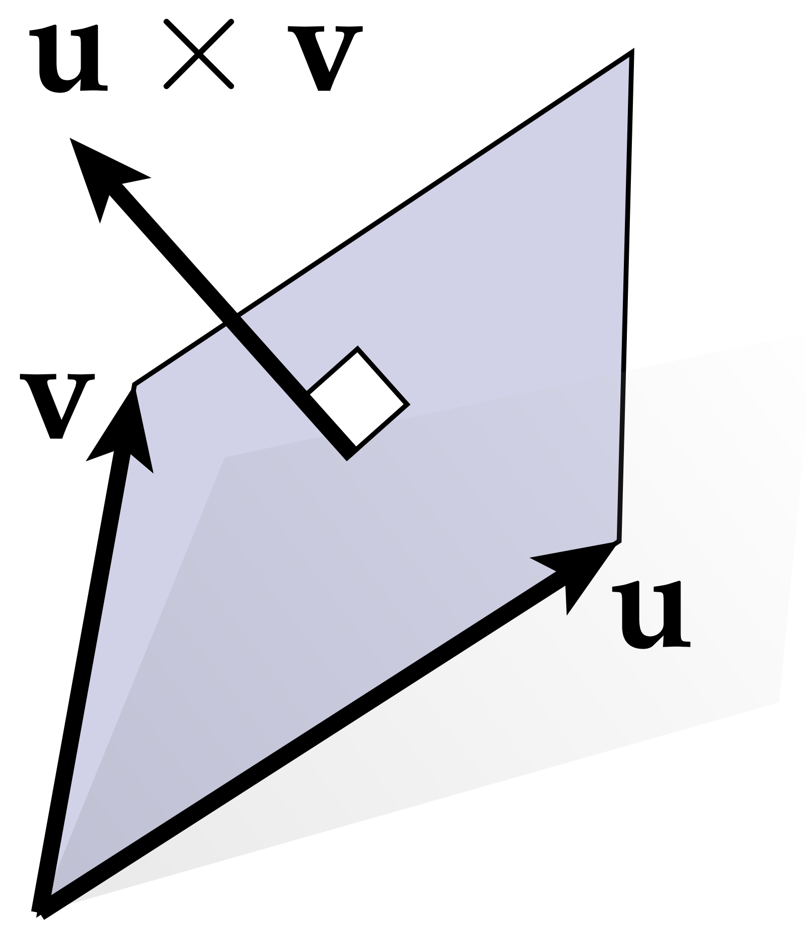

- In orthonormal Cartesian coordinates, can be represented via the dot product

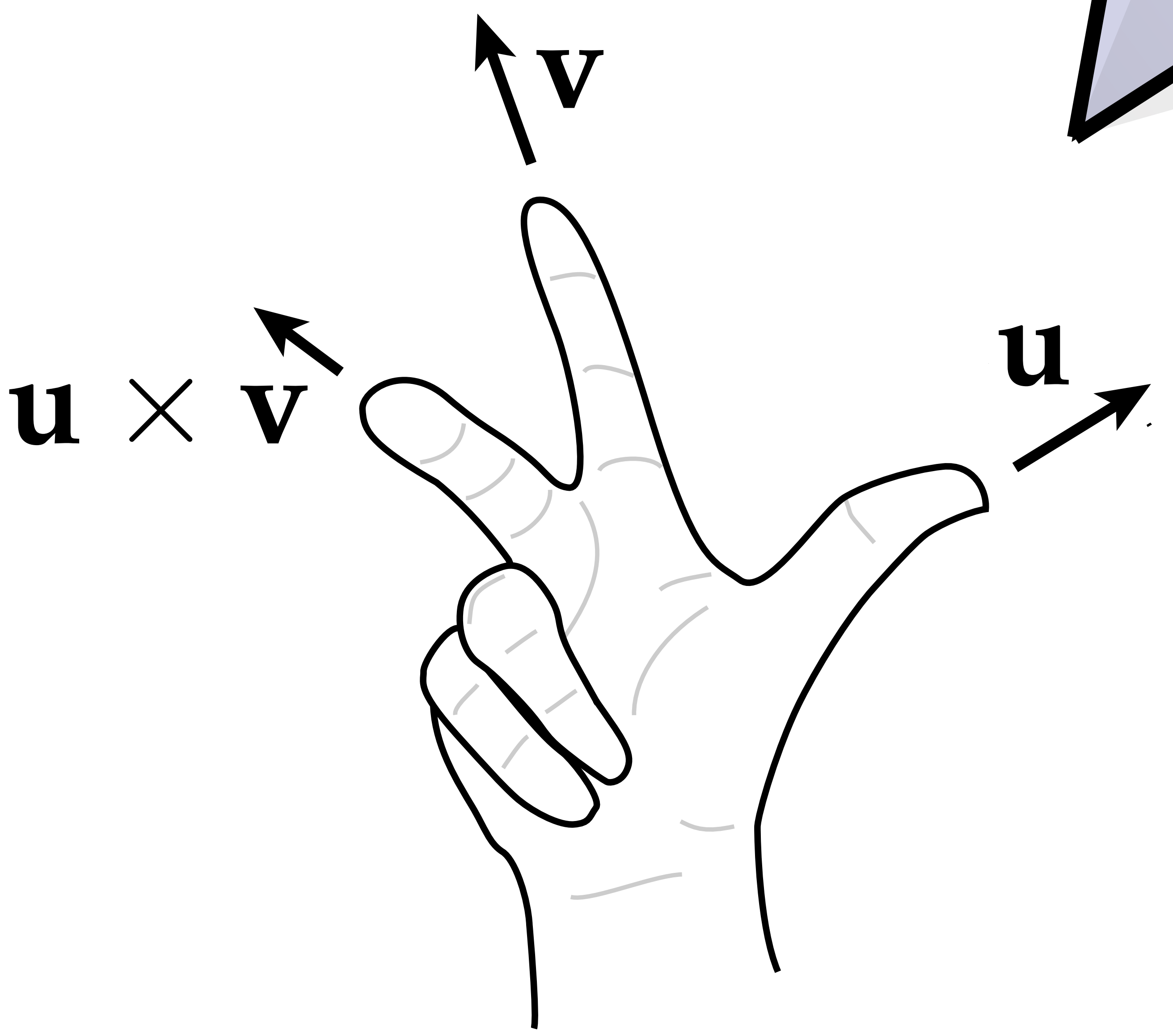
$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n$$

- WARNING: As with Euclidean norm, no geometric meaning unless coordinates come from an orthonormal basis.

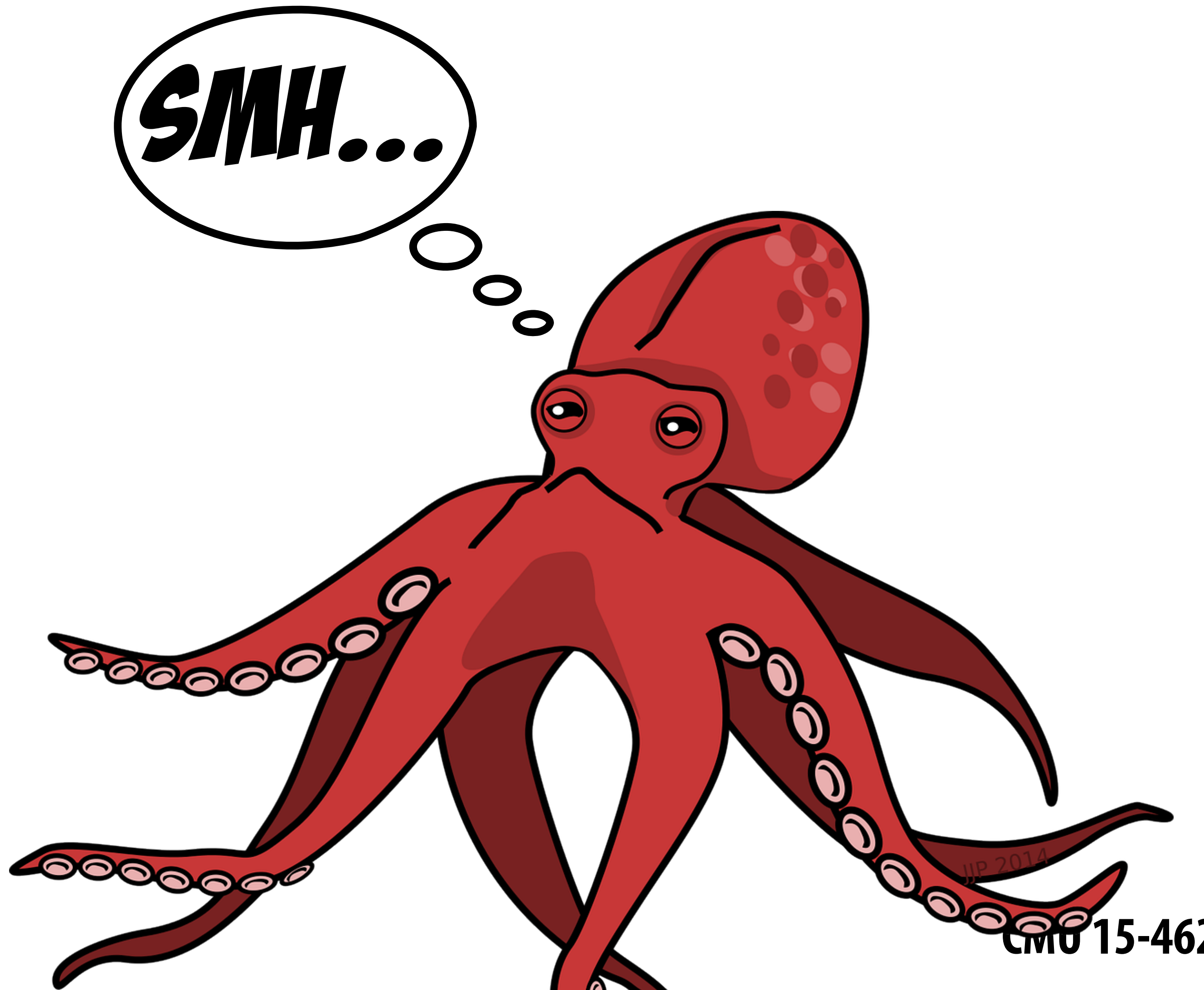
# Cross Product

- Inner product takes two vectors and produces a *scalar*
- In 3D, cross product is a natural way to take two vectors and get a *vector*, written as "u x v"
- Geometrically:
  - magnitude equal to parallelogram area
  - direction orthogonal to both vectors
  - …but which way?
- Use "right hand rule"



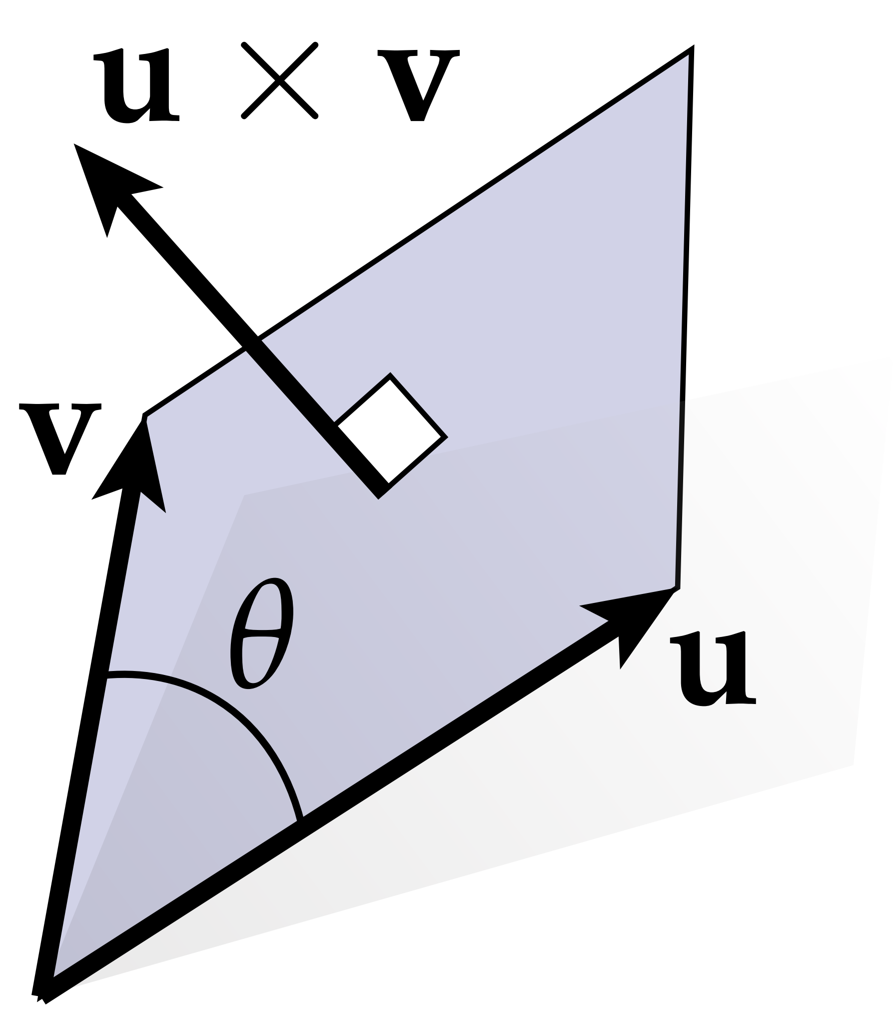

(Q: Why only 3D?)



# Cross Product, Determinant, and Angle

- **More precise definition (that does not require hands):**

$$\sqrt{\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})} = |\mathbf{u}||\mathbf{v}|\sin(\theta)$$




- **$\theta$ is angle between u and v**
- **"det" is determinant of three column vectors**
- **Uniquely determines coordinate formula:**

$$\mathbf{u} \times \mathbf{v} := \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

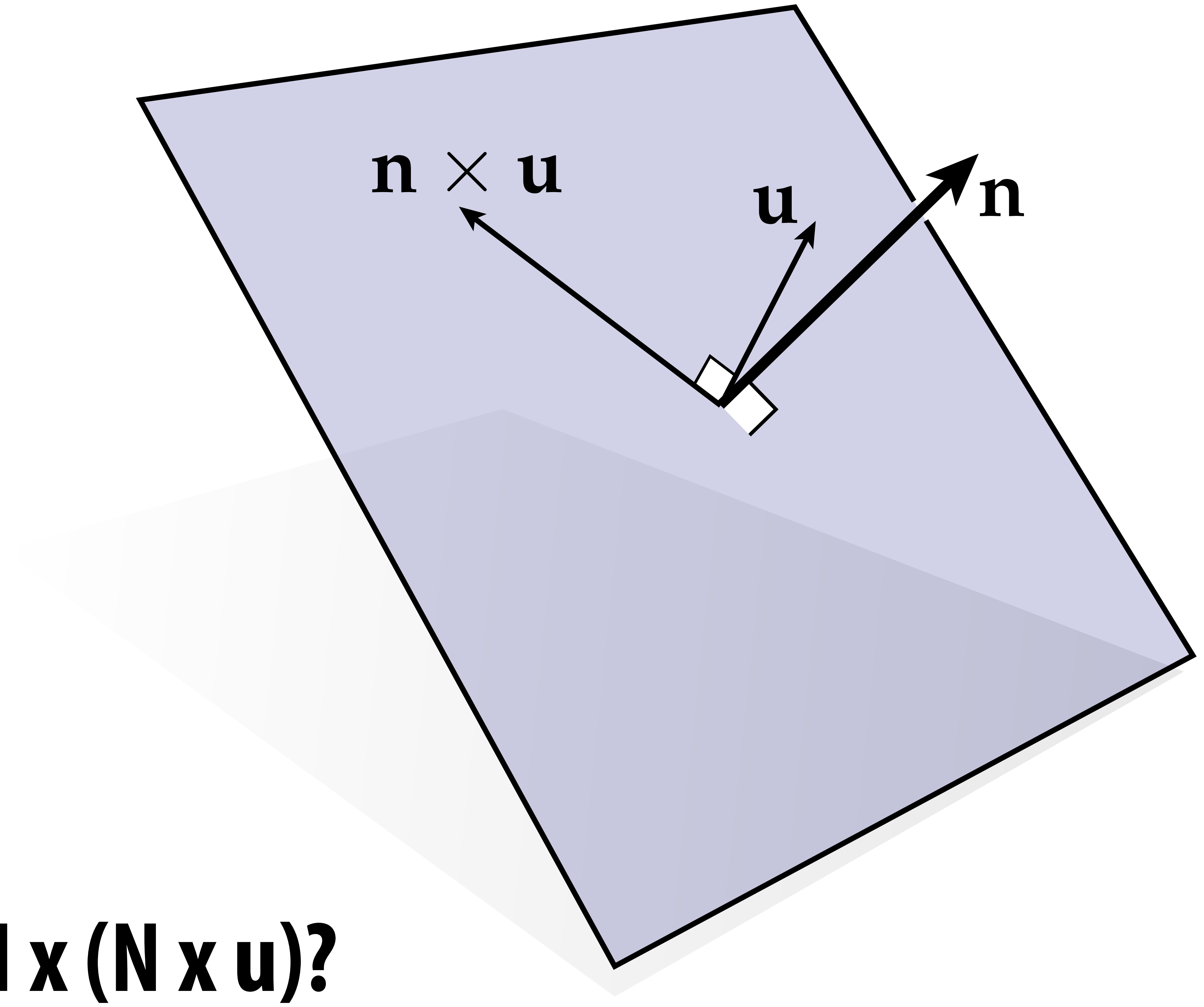
$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

(mnemonic)

- **Useful abuse of notation in 2D:** $\mathbf{u} \times \mathbf{v} := u_1 v_2 - u_2 v_1$

# Cross Product as Quarter Rotation

- Simple but useful observation for manipulating vectors in 3D: cross product with a unit vector N is equivalent to a quarter-rotation in the plane with normal N:




- Q: What is N x (N x u)?
- Q: If you have u and N x u, how do you get a rotation by some arbitrary angle θ?

# Matrix Representation of Dot Product

- Often convenient to express dot product via matrix product:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^\mathsf{T}\mathbf{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^n u_i v_i$$

- By the way, what about some other inner product?
- E.g., <u,v> := 2 u1 v1 + u1 v2 + u2 v1 + 3 u2 v2

$$\underbrace{\begin{bmatrix} u_1 & u_2 \end{bmatrix}}_{\mathbf{u}^\mathsf{T}}\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} 2v_1 + v_2 \\ v_1 + 3v_2 \end{bmatrix}$$

$$= (2u_1v_1 + u_1v_2) + (u_2v_1 + 3u_2v_2). \quad \checkmark$$

Q: Why is matrix representing inner product always *symmetric* ($A^\mathsf{T}=A$)?

# Matrix Representation of Cross Product

- Can also represent cross product via matrix multiplication:

$$\mathbf{u} := (u_1, u_2, u_3) \qquad \Rightarrow \widehat{\mathbf{u}} := \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

$$\mathbf{u} \times \mathbf{v} = \widehat{\mathbf{u}}\mathbf{v} \;\; = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

(Did we get it right?)

- Q: Without building a new matrix, how can we express v x u?
- A: Useful to notice that v x u = -u x v (why?). Hence,

$$\mathbf{v} \times \mathbf{u} = -\widehat{\mathbf{u}}\mathbf{v} \;\; = \widehat{\mathbf{u}}^{\mathsf{T}}\mathbf{v}$$

# Determinant

- **Q: How do you compute the determinant of a matrix?**

$$\mathbf{A} := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- **A: Apply some algorithm somebody told me once upon a time:**

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

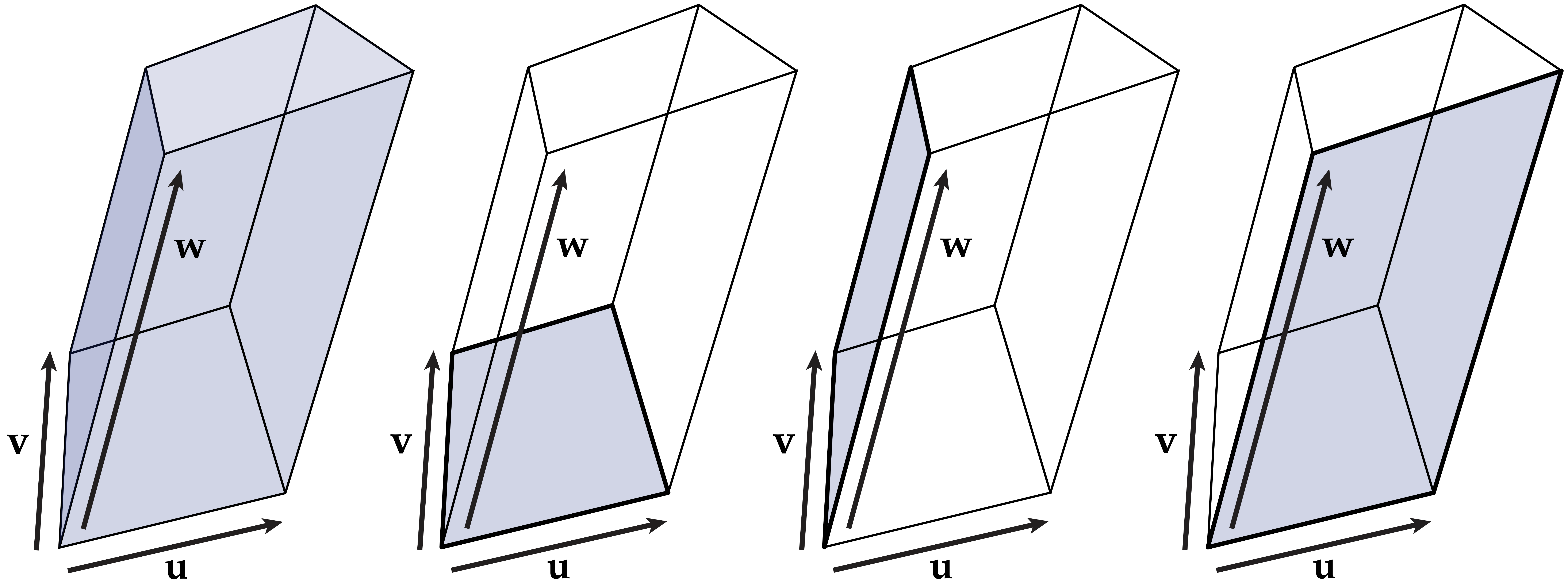
$$\det(\mathbf{A}) = a(ei - fh) + b(fg - di) + c(dh - eg)$$

**Totally obvious… right?**

- **Q: No! What the heck does this number *mean?!***

# Determinant, Volume and Triple Product

- **Better answer: det(u,v,w) encodes (signed) volume of parallelepiped with edge vectors u, v, w.**




$$\det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

- **Relationship known as a "triple product formula"**
- **(Q: What happens if we reverse order of cross product?)**

# Determinant of a Linear Map

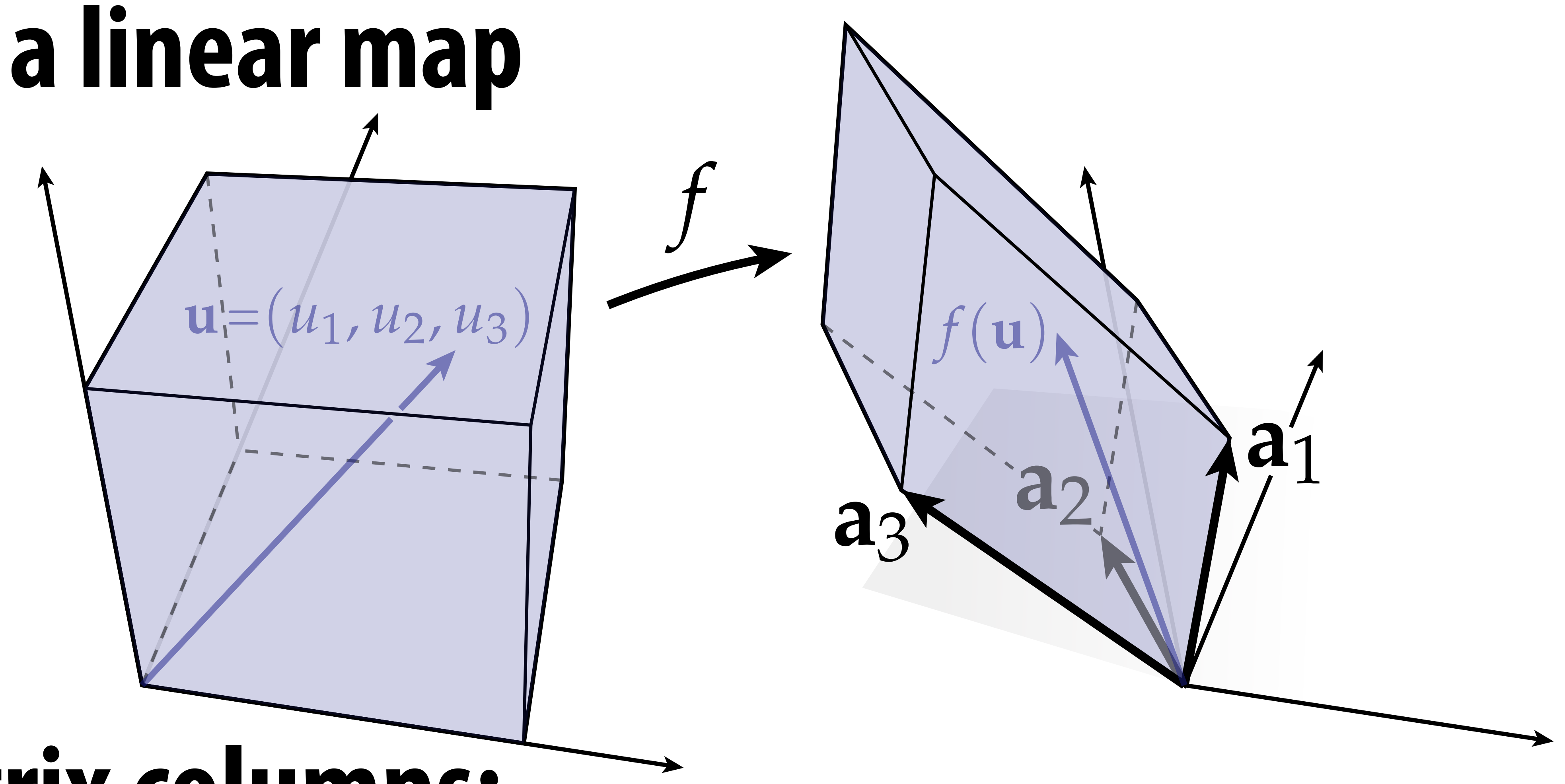
- **Q: If a matrix A encodes a *linear map* f, what does det(A) mean?**

**(First: need to recall how a matrix encodes a linear map!)**

# Representing Linear Maps via Matrices

- **Key example: suppose I have a linear map**

$$f(\mathbf{u}) = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3$$




- **How do I encode as a matrix?**
- **Easy: "a" vectors become matrix columns:**

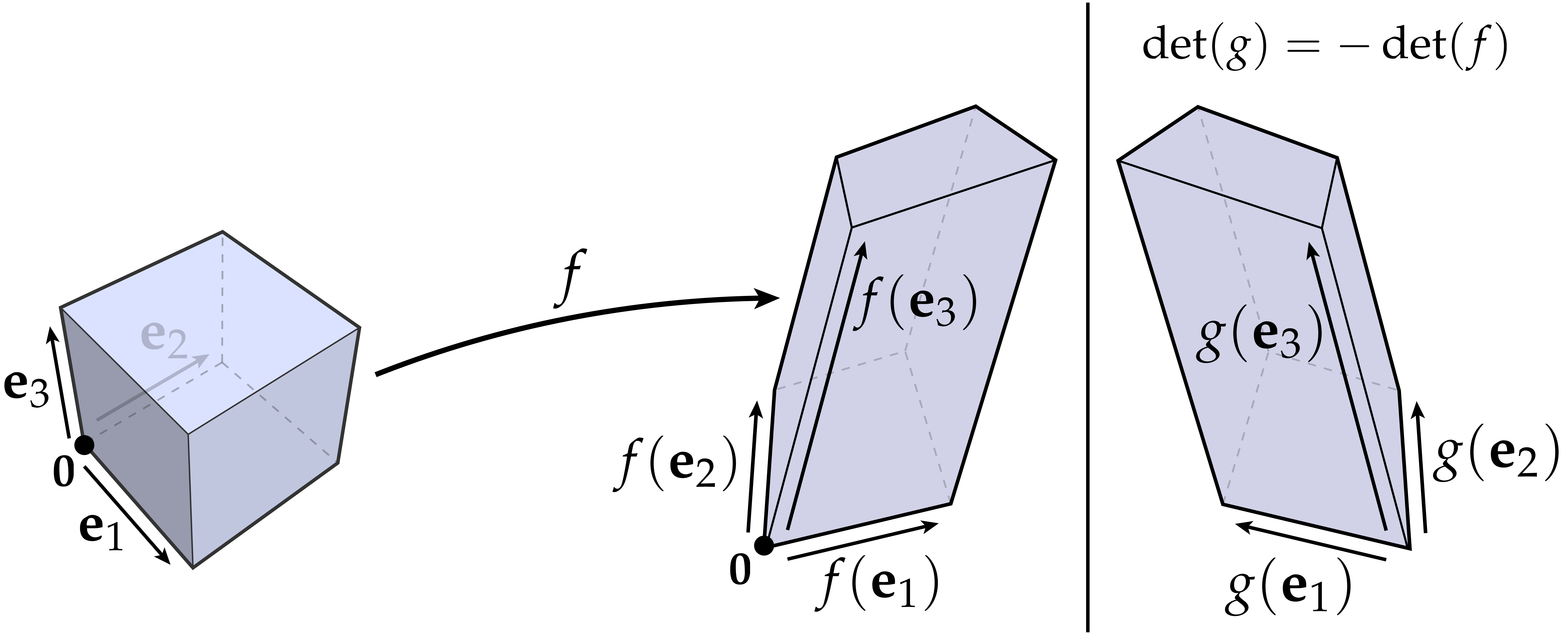
$$A := \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} a_{1,x} & a_{2,x} & a_{3,x} \\ a_{1,y} & a_{2,y} & a_{3,y} \\ a_{1,z} & a_{2,z} & a_{3,z} \end{bmatrix}$$

- **Now, matrix-vector multiply recovers original map:**

$$A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a_{1,x}u_1 + a_{2,x}u_2 + a_{3,x}u_3 \\ a_{1,y}u_1 + a_{2,y}u_2 + a_{3,y}u_3 \\ a_{1,z}u_1 + a_{2,z}u_2 + a_{3,z}u_3 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3$$

# Determinant of a Linear Map

- **Q: If a matrix A encodes a *linear map* f, what does det(A) mean?**



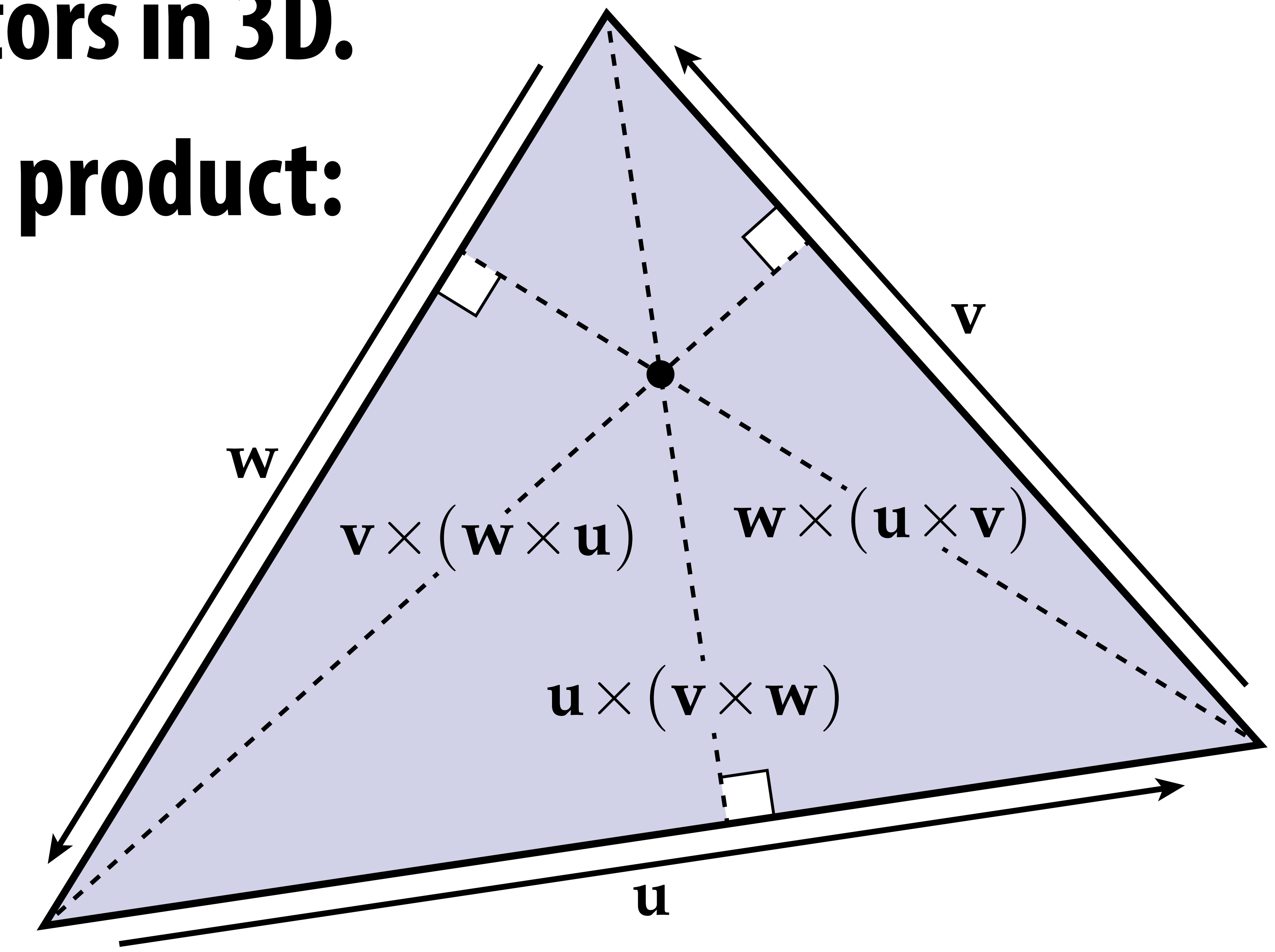

- **A: It measures the *change in volume*.**
- **Q: What does the *sign* of the determinant tell us, in this case?**
- **A: It tells us whether *orientation* was reversed (det(A) < 0)**

**(Do we really need a *matrix* in order to talk about the determinant of a linear map?)**

# Other Triple Products

- Super useful for working w/ vectors in 3D.
- E.g., Jacobi identity for the cross product:

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) & \quad + \\ \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) & \quad + \\ \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) & \quad = \quad 0 \end{aligned}$$



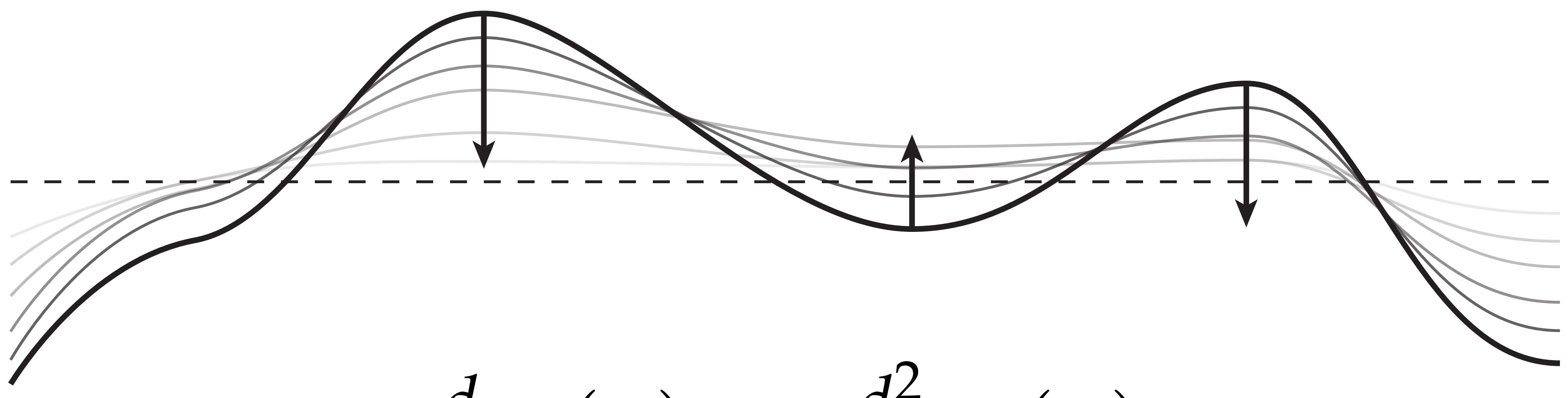

- Why is it true, geometrically?
- There *is* a geometric reason, but not nearly as obvious as det: has to do w/ fact that triangle's altitudes meet at a point.
- Yet another triple product: Lagrange's identity

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})$$

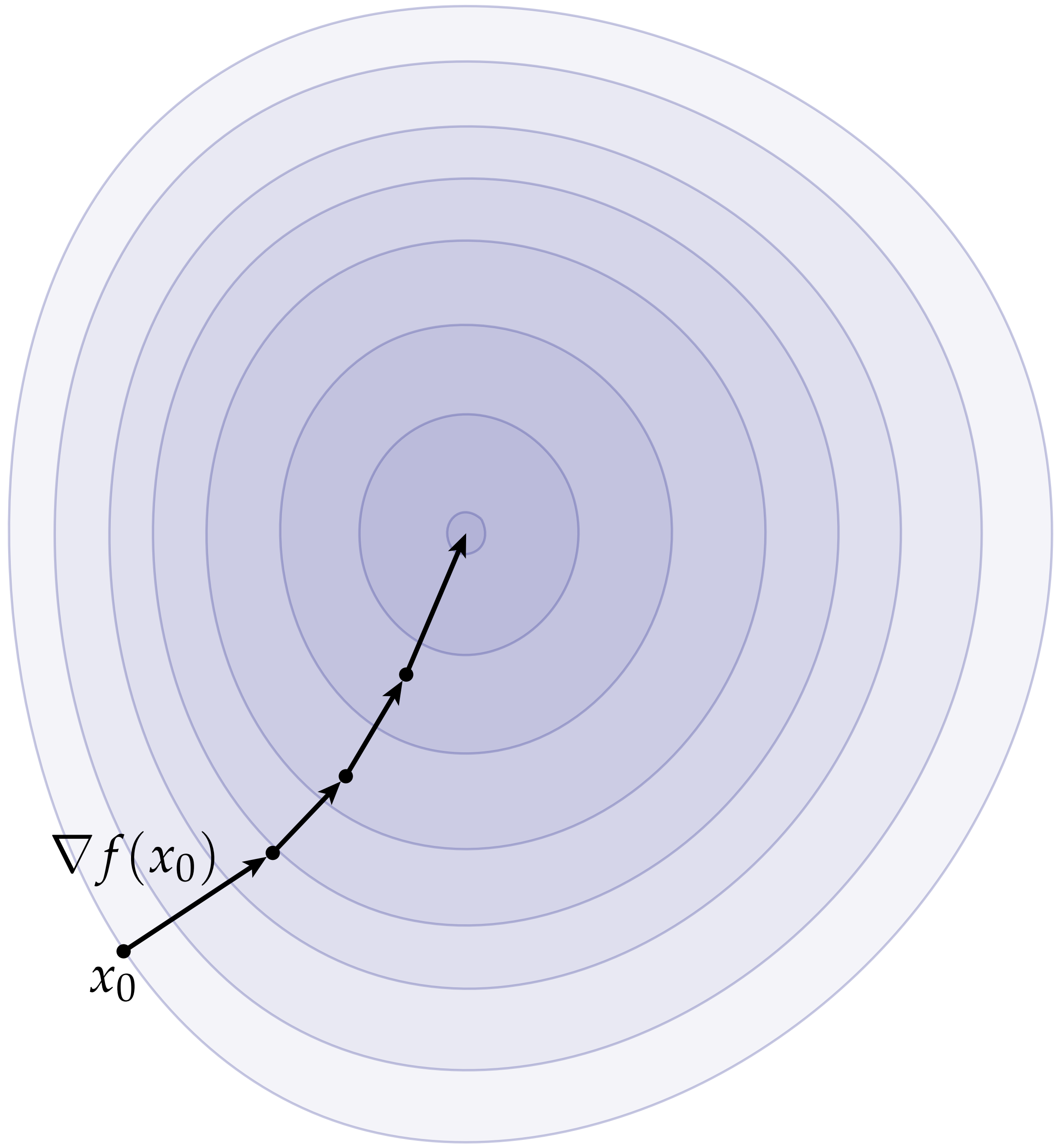
(Can you come up with a geometric interpretation?)

# Differential Operators - Overview

- Next up: differential operators and vector fields.
- Why is this useful for computer graphics?
  - Many physical/geometric problems expressed in terms of *relative rates of change* (ODEs, PDEs).
  - These tools also provide foundation for numerical optimization—e.g., minimize cost by following the *gradient* of some objective.



$$\frac{d}{dt}\phi(x) = \frac{d^2}{dx^2}\phi(x)$$



# Derivative as Slope

- Consider a function f(x): R → R
- What does its derivative f' mean?
- One interpretation "rise over run"
- Corresponds to standard definition:

$$f'(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon}$$

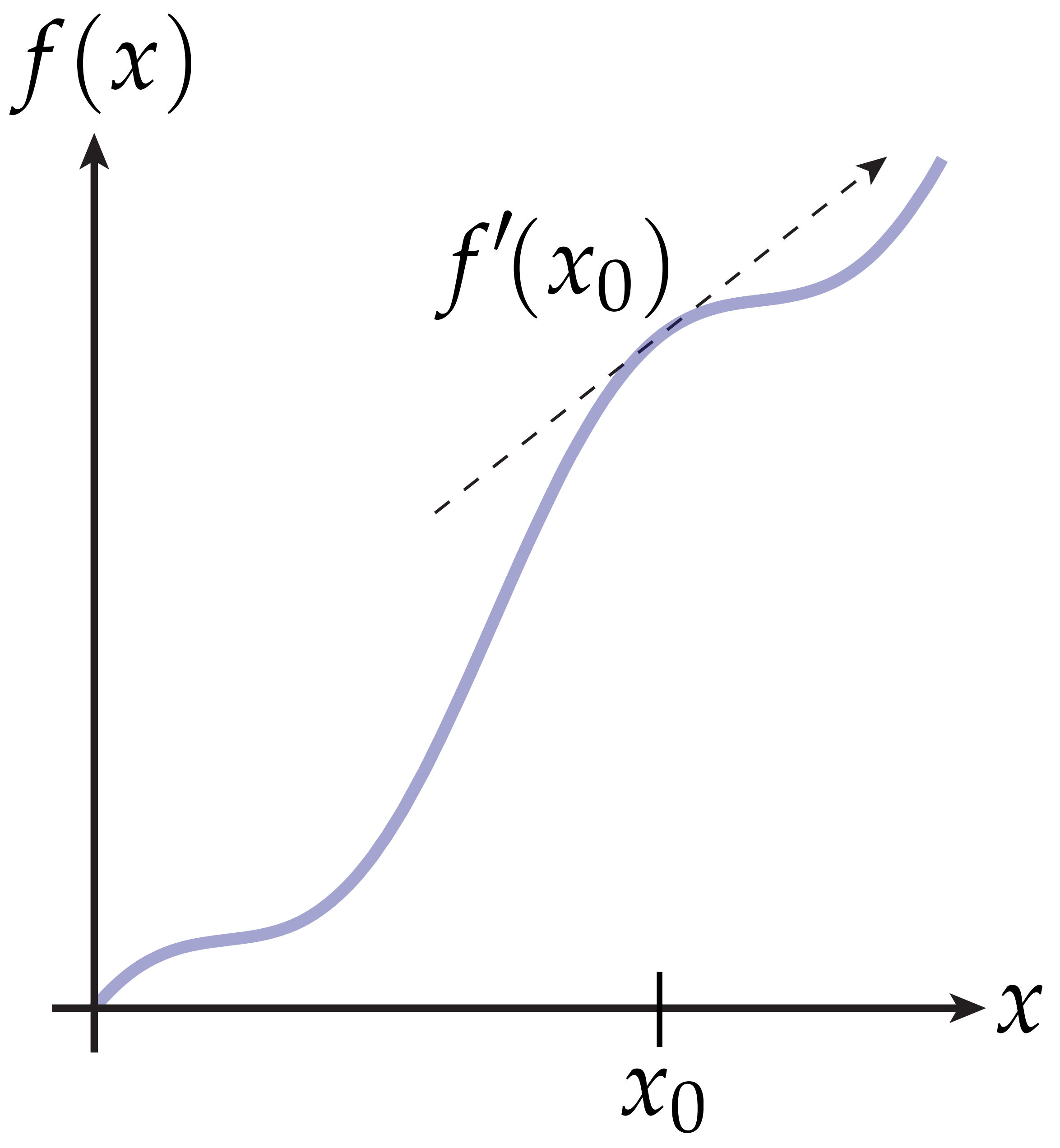
- Careful! What if slope is different when we walk in opposite direction?

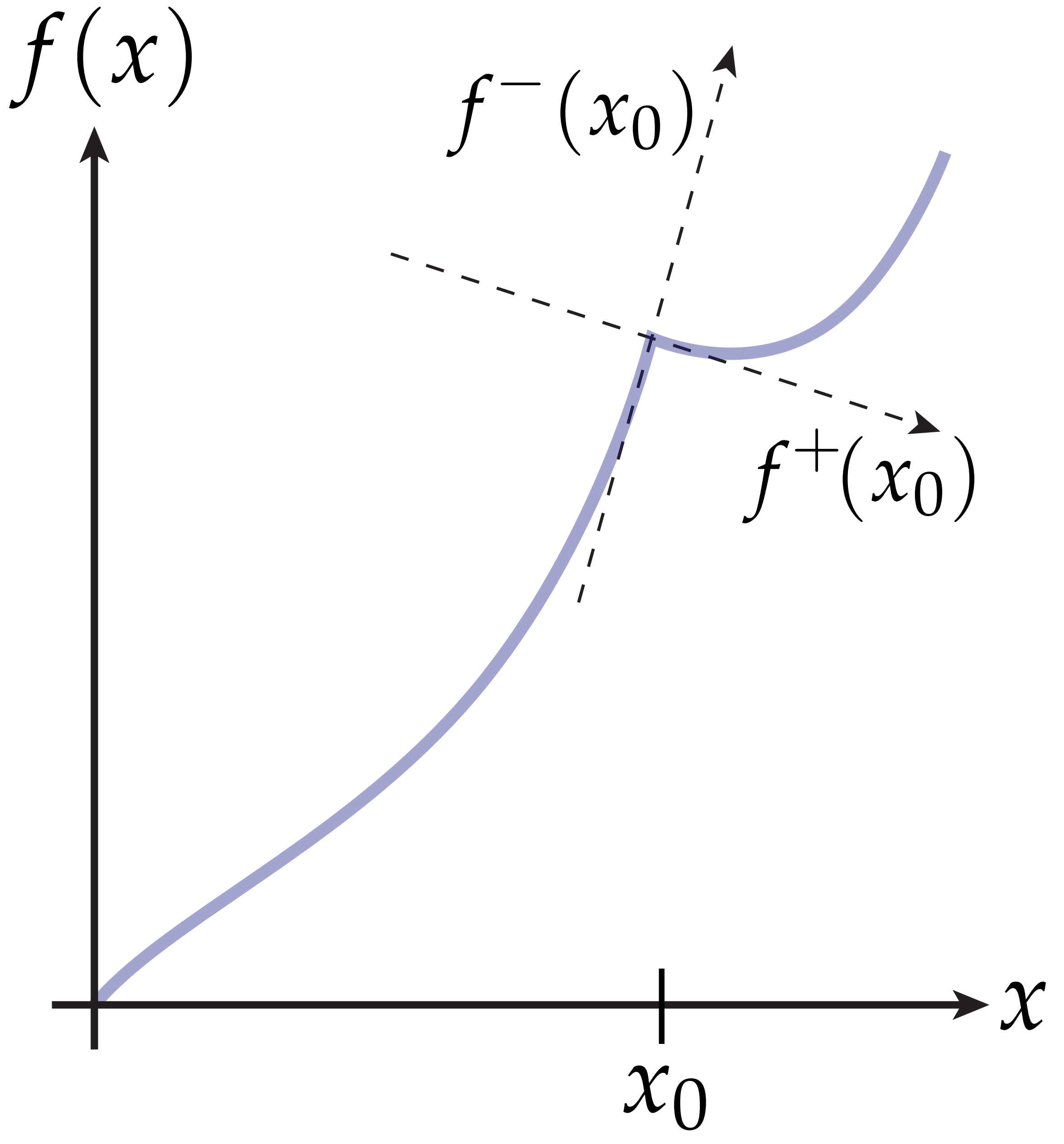
$$f^+(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon}$$

$$f^-(x_0) := \lim_{\varepsilon \to 0} \frac{f(x_0) - f(x_0-\varepsilon)}{\varepsilon}$$

- Differentiable at x0 if $f^+ = f^-$.

Many functions in graphics are NOT differentiable!



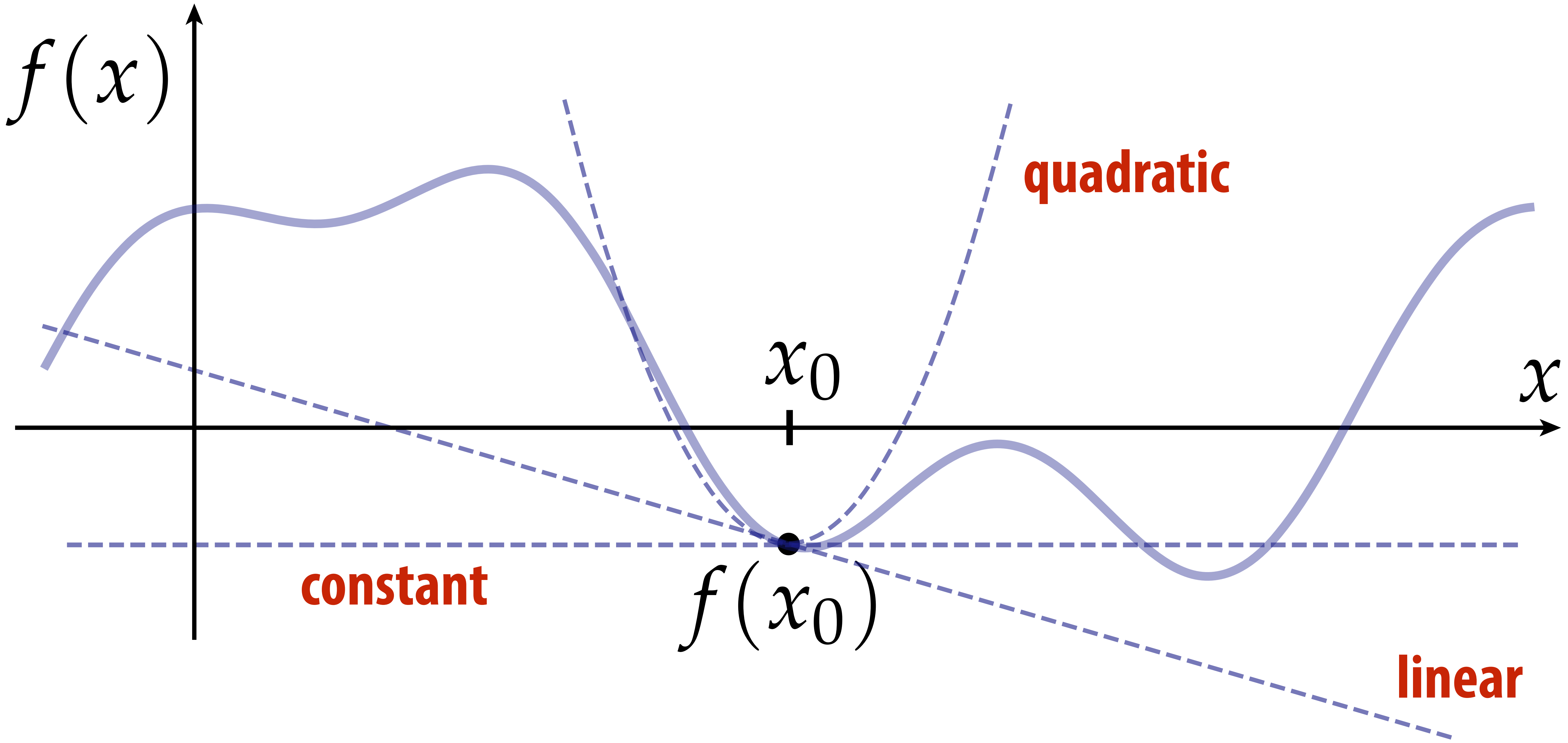



# Derivative as Best Linear Approximation

- Any smooth function f(x) can be expressed as a *Taylor series:*

constant — linear — quadratic

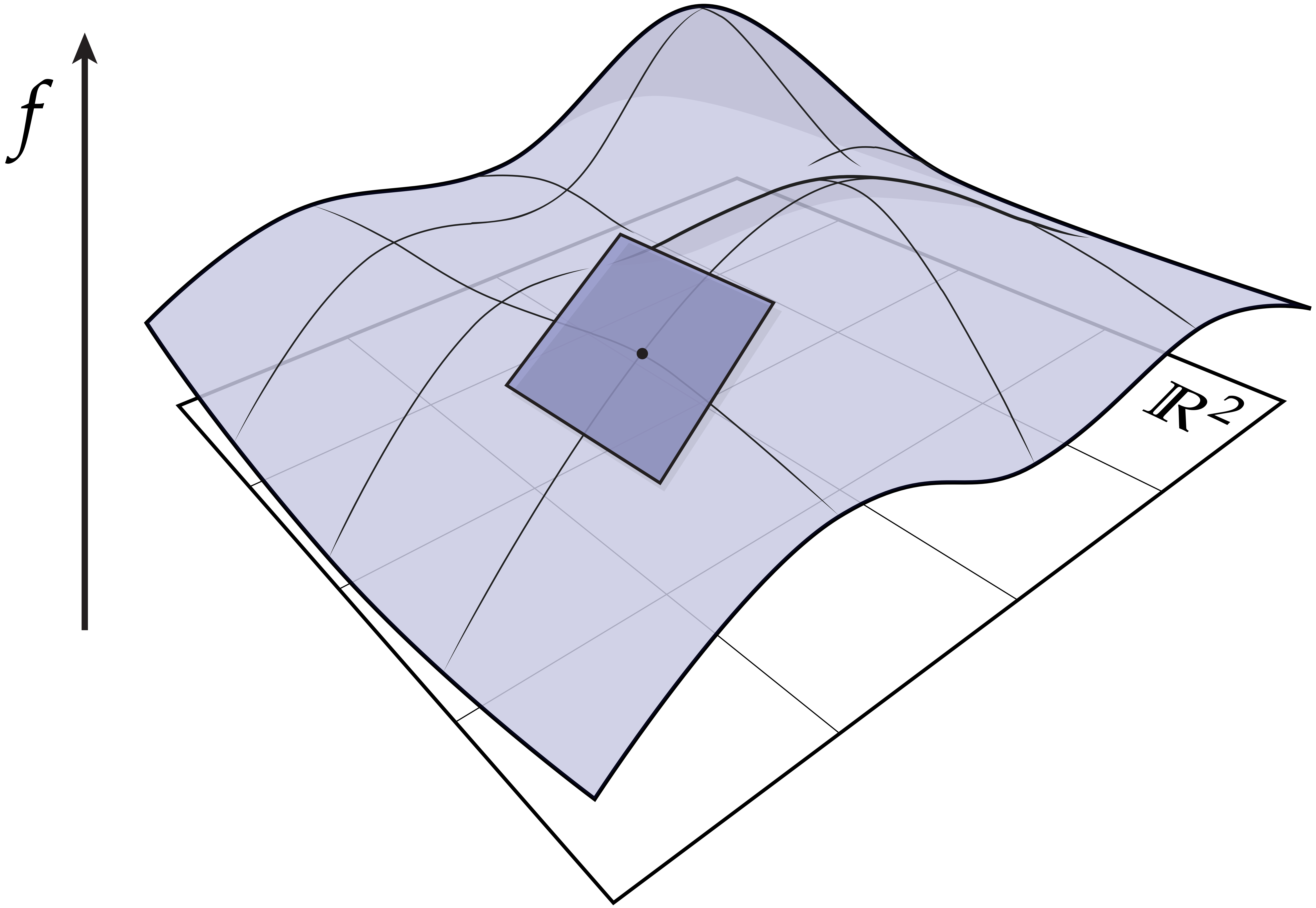
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$




- Replacing complicated functions with a linear (and sometimes quadratic) approximation is a powerful trick in graphics algorithms—we'll see many examples.

# Derivative as Best Linear Approximation

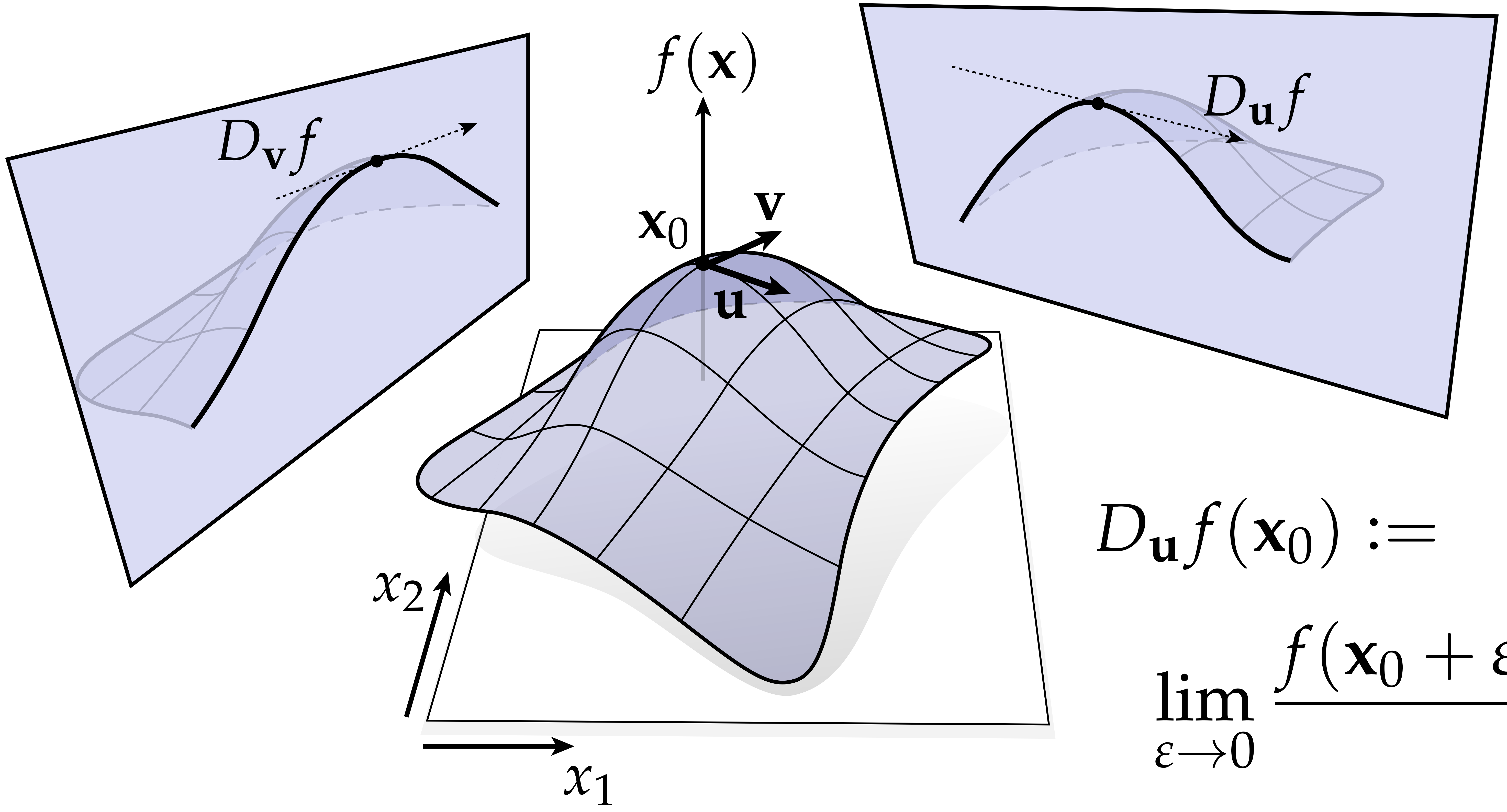
- **Intuitively, same idea applies for functions of multiple variables:**



**How do we think about derivatives for a function that has multiple variables?**

# Directional Derivative

- One way: suppose we have a function f(x1,x2)
  - Take a “slice” through the function along some line
  - Then just apply the usual derivative!
  - Called the directional derivative



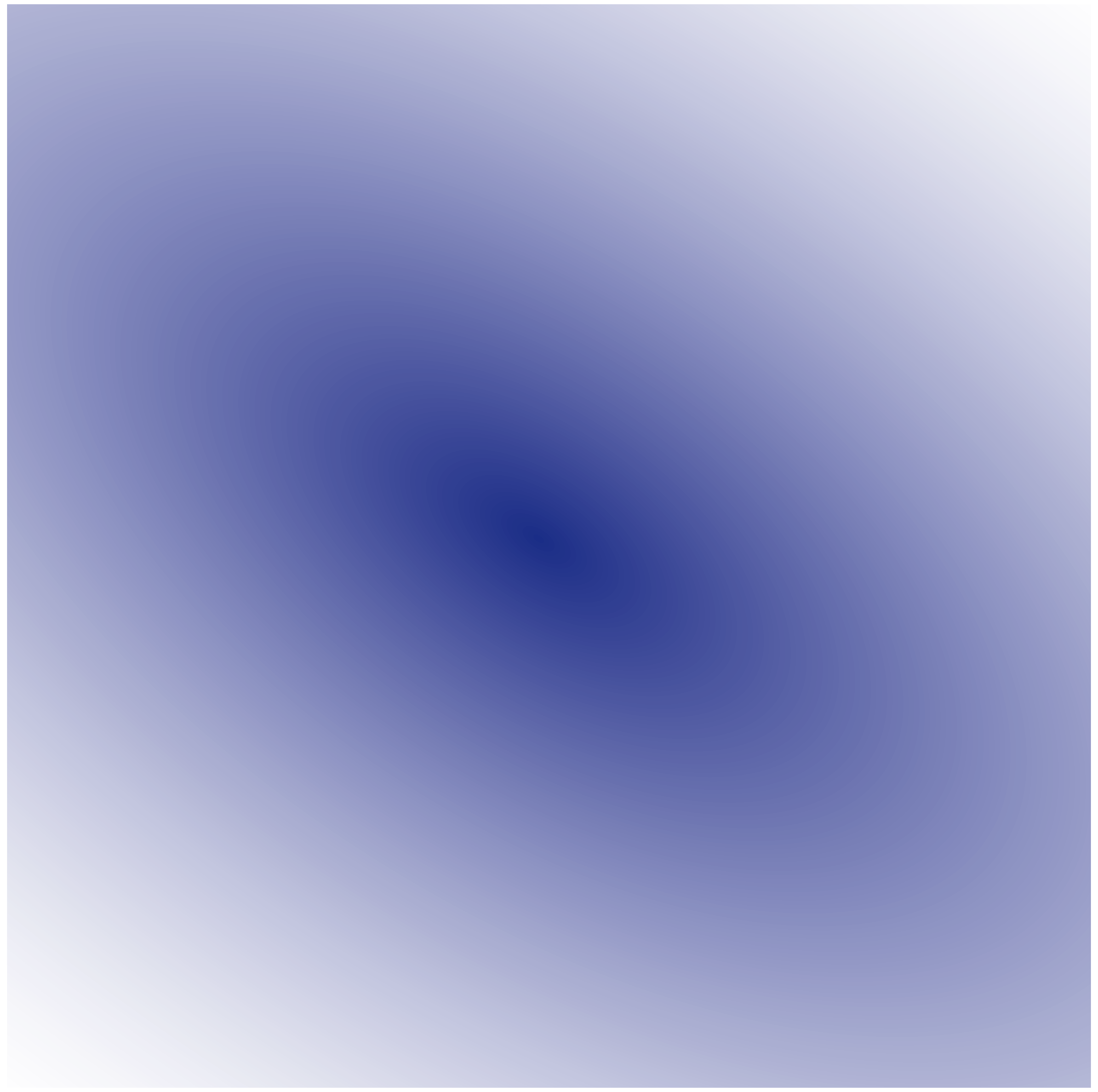

take a small step along u

$$D_{\mathbf{u}}f(\mathbf{x}_0) := \lim_{\varepsilon \to 0} \frac{f(\mathbf{x}_0 + \varepsilon \mathbf{u}) - f(\mathbf{x}_0)}{\varepsilon}$$

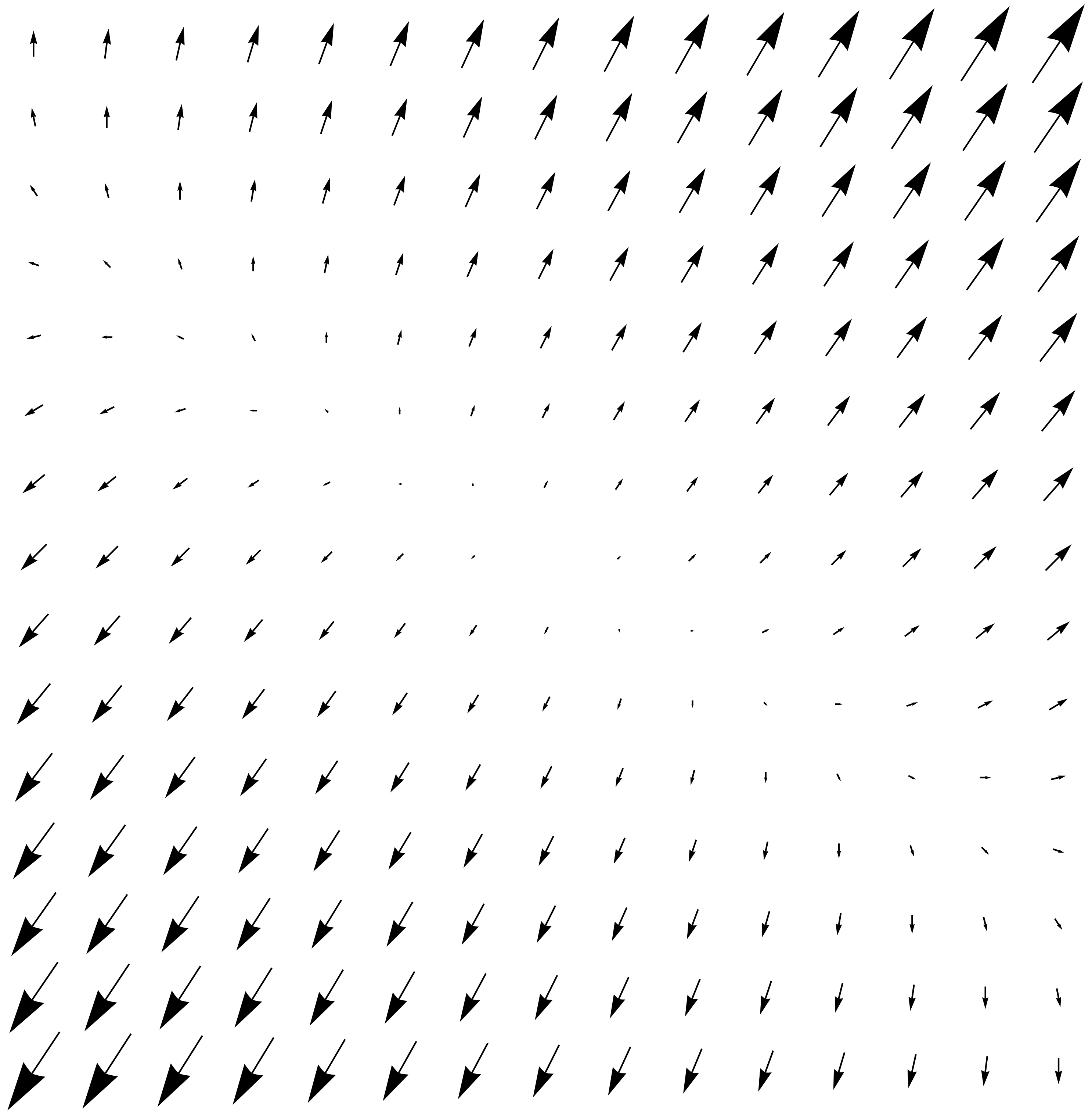
# Gradient

"nabla"

- Given a multivariable function $f(\mathbf{x})$, gradient $\nabla f(\mathbf{x})$ assigns a vector at each point:



$f(\mathbf{x})$



$\nabla f(\mathbf{x})$

- (Ok, but which vectors, exactly?)

# Gradient in Coordinates

- **Most familiar definition: list of *partial derivatives***
- **I.e., imagine that all but one of the coordinates are just constant values, and take the usual derivative**

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$

- **Two potential problems:**
  - **Role of inner product is not clear (more later!)**
  - **No way to differentiate *functions of functions* F(f) since we don't have a finite list of coordinates $x_1, \ldots, x_n$**
- **Still, extremely common way to calculate the gradient…**

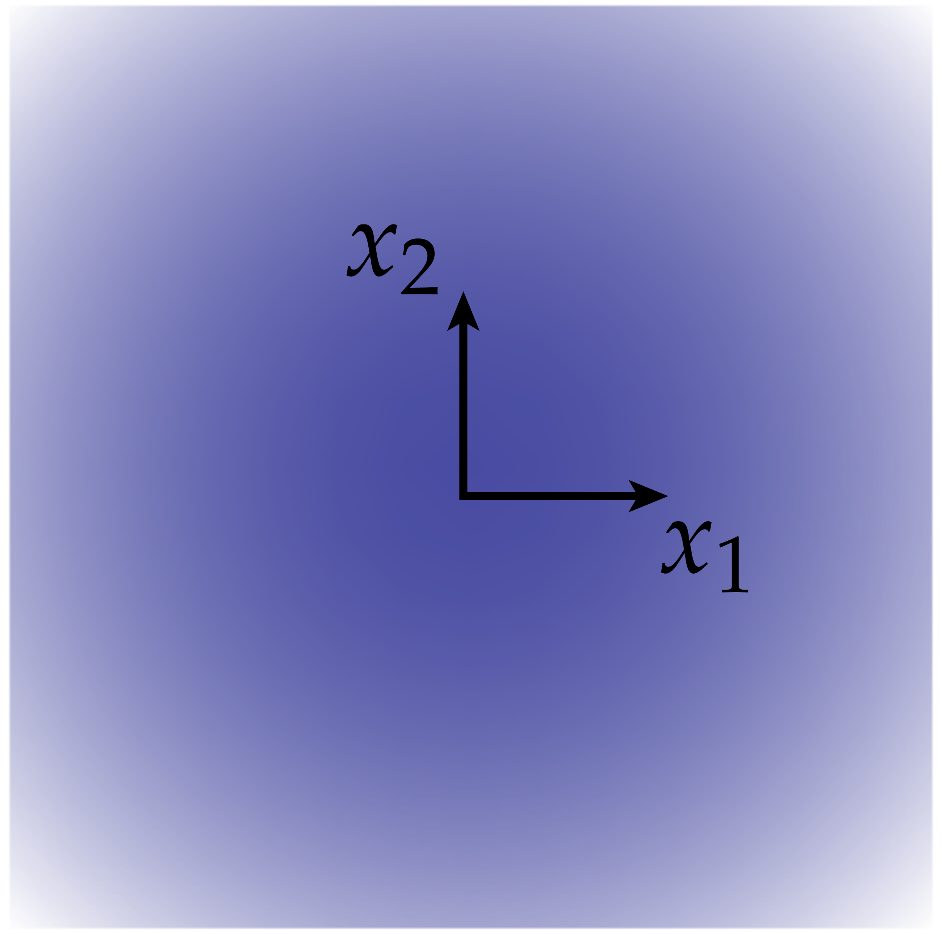
# Example: Gradient in Coordinates

$$f(\mathbf{x}) := x_1^2 + x_2^2$$

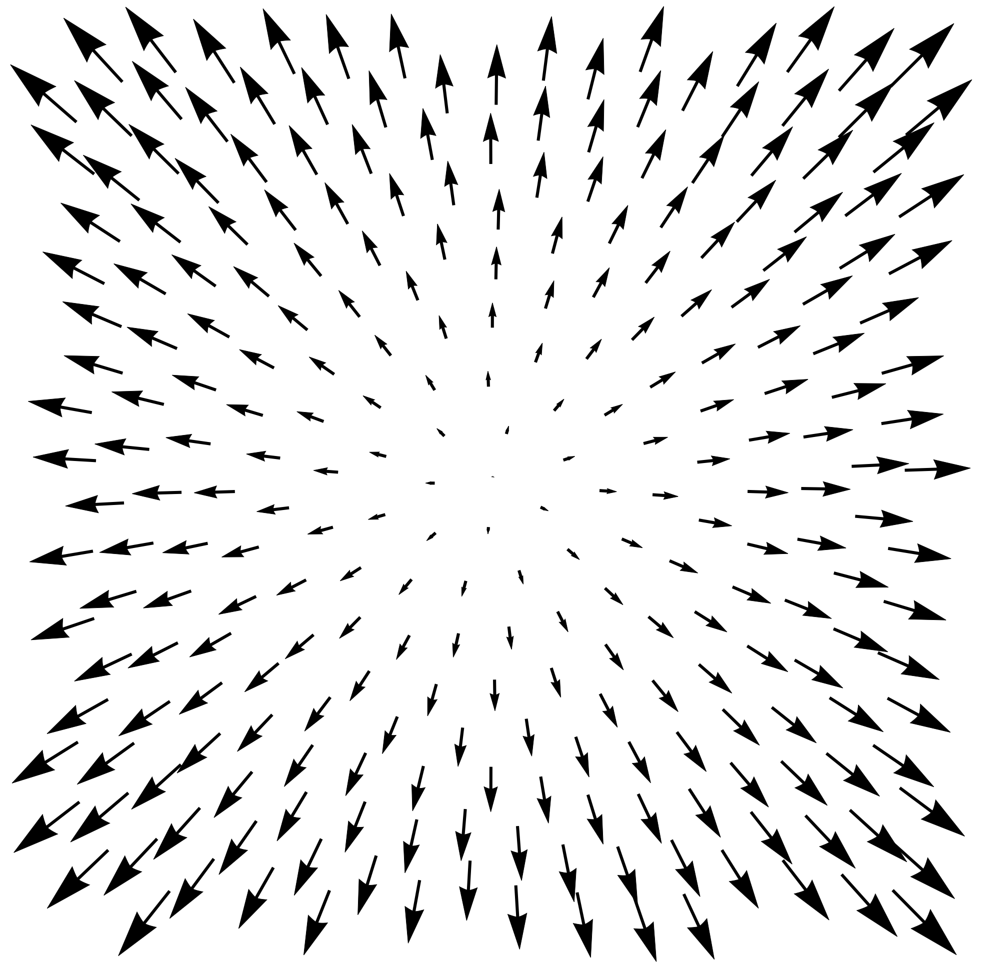
$$\frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1} x_1^2 + \frac{\partial}{\partial x_1} x_2^2 = 2x_1 + 0$$

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial x_2} x_1^2 + \frac{\partial}{\partial x_2} x_2^2 = 0 + 2x_2$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 2\mathbf{x}$$




$f(\mathbf{x})$

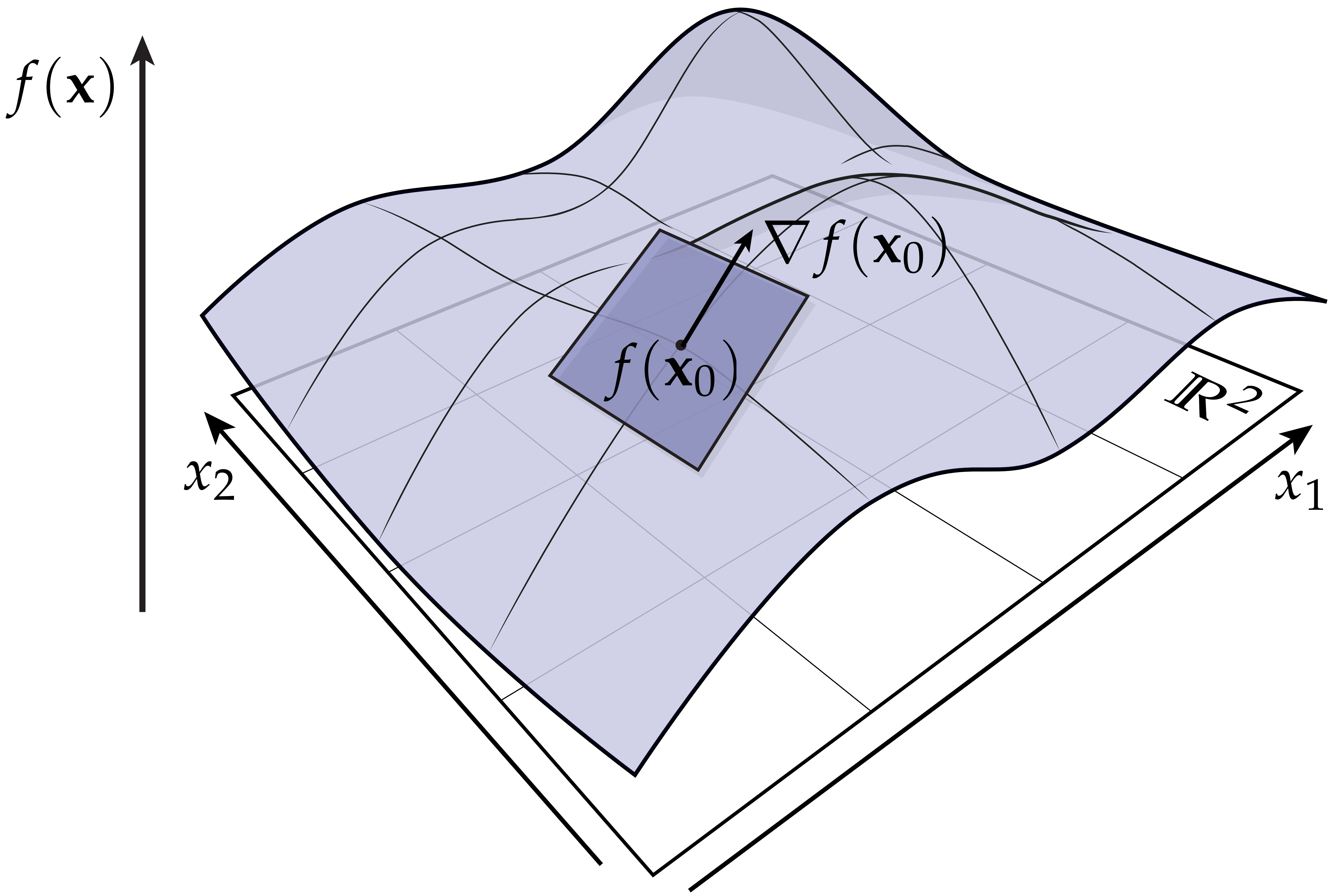


$\nabla f(\mathbf{x})$

# Gradient as Best Linear Approximation

**Another way to think about it: at each point x0, gradient is the vector $\nabla f(\mathbf{x}_0)$ that leads to the best possible approximation**

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle$$

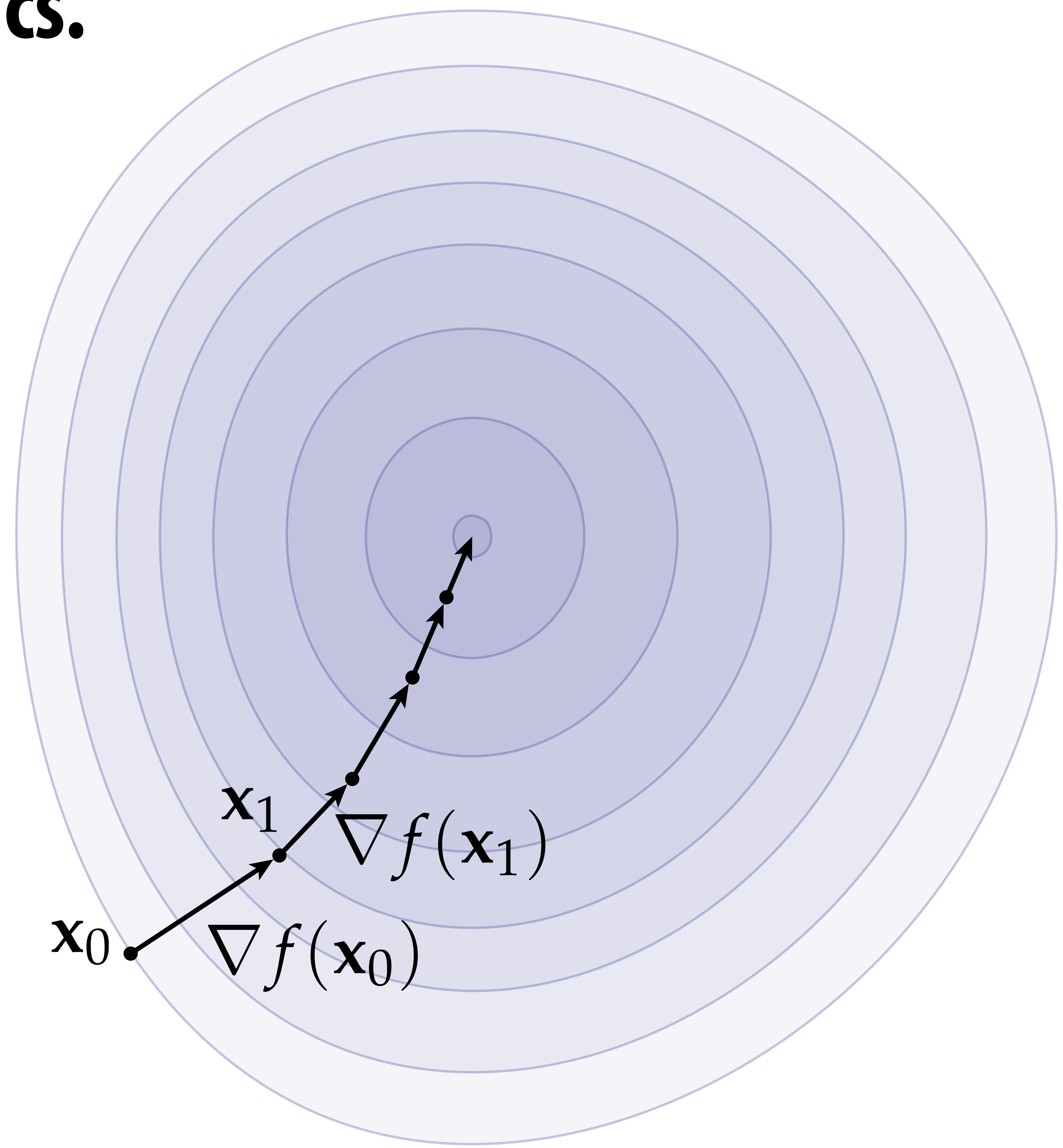



Starting at $x_0$, this term gets:

- bigger if we move in the direction of the gradient,
- smaller if we move in the opposite direction, and
- doesn't change if we move orthogonal to gradient.

# The gradient takes you uphill…

- Another way to think about it: direction of "steepest ascent"
- I.e., what direction should we travel to increase value of function as quickly as possible?
- This viewpoint leads to algorithms for optimization, commonly used in graphics.



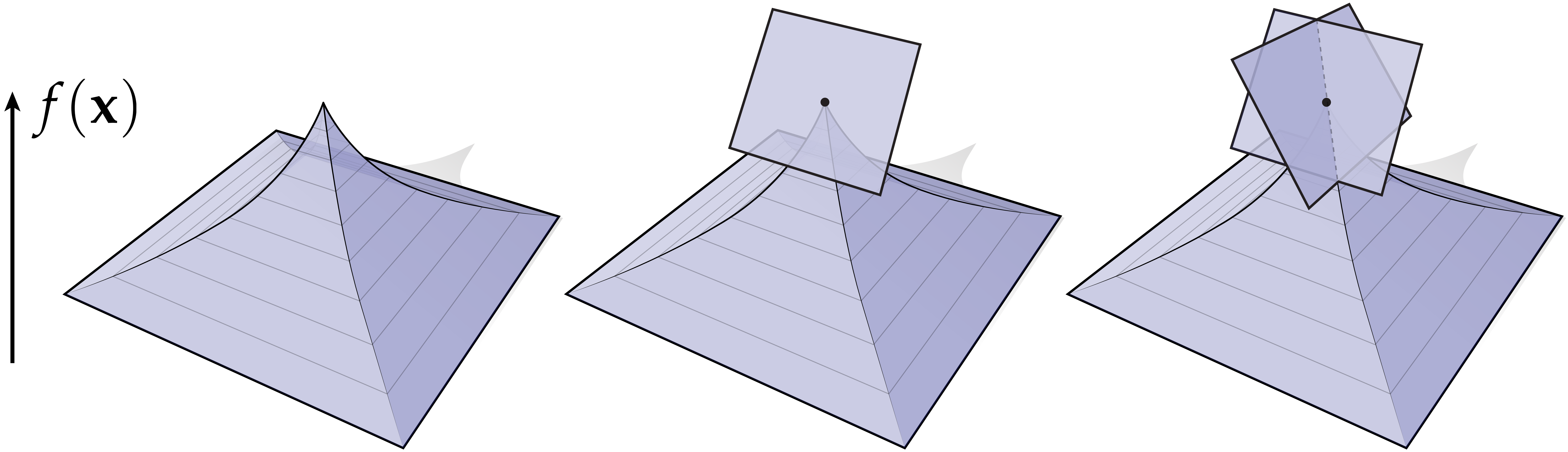
# Gradient and Directional Derivative

**At each point x, gradient is unique vector $\nabla f(\mathbf{x})$ such that**

$$\langle \nabla f(x), \mathbf{u} \rangle = D_{\mathbf{u}} f(\mathbf{x})$$

**for all u. In other words, such that taking the inner product w/ this vector gives you the directional derivative in any direction u.**

**Can't happen if function is not differentiable!**




**(Notice: gradient also depends on choice of *inner product*...)**

# Example: Gradient of Dot Product

- **Consider the dot product expressed in terms of matrices:**

$$f := \mathbf{u}^\mathsf{T}\mathbf{v}$$

- **What is gradient of $f$ with respect to u?**
- **One way: write it out in coordinates:**

$$\mathbf{u}^\mathsf{T}\mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

(equals zero unless i = k)

$$\frac{\partial}{\partial u_k}\sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} \frac{\partial}{\partial u_k}(u_i v_i) = v_k$$

$$\Rightarrow \nabla_{\mathbf{u}} f = \begin{bmatrix} v_1 \\ \dots \\ v_n \end{bmatrix}$$

**In other words:**

$$\boxed{\nabla_{\mathbf{u}}(\mathbf{u}^\mathsf{T}\mathbf{v}) = \mathbf{v}}$$

Not so different from $\frac{d}{dx}(xy) = y$!

# Gradients of Matrix-Valued Expressions

- **EXTREMELY useful in graphics to be able to differentiate expressions involving matrices**
- **Ultimately, expressions look much like ordinary derivatives**

For any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and **symmetric** matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$:

| MATRIX DERIVATIVE | LOOKS LIKE |
|---|---|
| $\nabla_{\mathbf{x}}(\mathbf{x}^T\mathbf{y}) = \mathbf{y}$ | $\frac{d}{dx}xy = y$ |
| $\nabla_{\mathbf{x}}(\mathbf{x}^T\mathbf{x}) = 2\mathbf{x}$ | $\frac{d}{dx}x^2 = 2x$ |
| $\nabla_{\mathbf{x}}(\mathbf{x}^T A\mathbf{y}) = A\mathbf{y}$ | $\frac{d}{dx}axy = ay$ |
| $\nabla_{\mathbf{x}}(\mathbf{x}^T A\mathbf{x}) = 2A\mathbf{x}$ | $\frac{d}{dx}ax^2 = 2ax$ |
| $\ldots$ | $\ldots$ |

**Excellent resource: Petersen & Pedersen, "The Matrix Cookbook"**

- **At least once in your life, work these out meticulously in coordinates (to convince yourself they're true).**
- **Then… forget about coordinates altogether!**

# Advanced*: $L^2$ Gradient

- Consider a function of a function $F(f)$
- What is the gradient of F with respect to f?
- Can't take partial derivatives anymore!
- Instead, look for function $\nabla F$ such that for all functions u,

$$\langle\langle \nabla F, u \rangle\rangle = D_u F$$

- What is *directional derivative of a function of a function??*
- Don't freak out—just return to good old-fashioned limit:

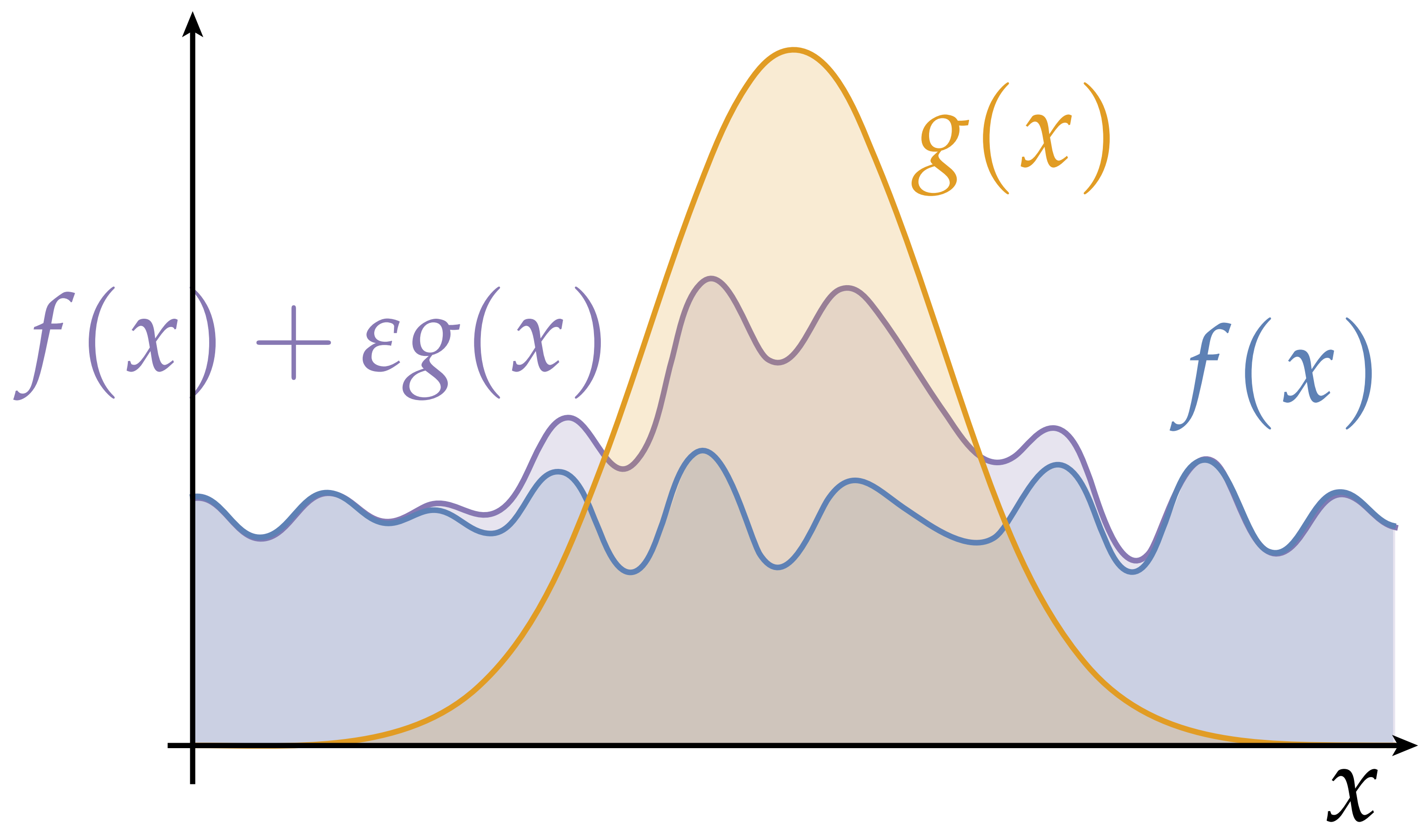
$$D_u F(f) = \lim_{\varepsilon \to 0} \frac{F(f + \varepsilon u) - F(f)}{\varepsilon}$$

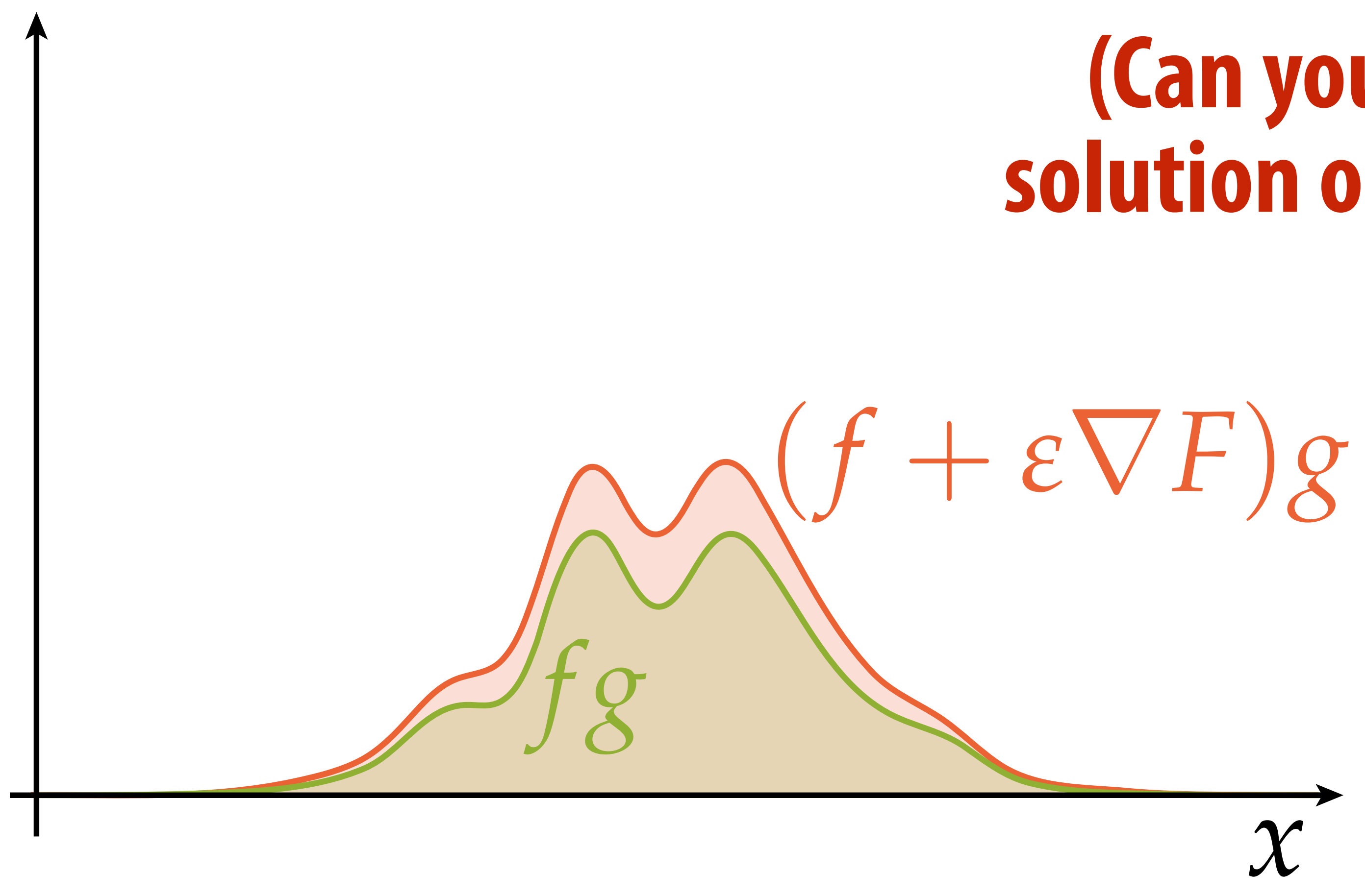
- This strategy becomes much clearer w/ a concrete example...

*as in, NOT on the test! (But perhaps somewhere in the test of life…)

# Advanced Visual Example: $L^2$ Gradient

- Consider function $F(f) := \langle\langle f, g \rangle\rangle$ for f,g: [0,1]→R
- I claim the gradient is: $\nabla F = g$
- Does this make sense intuitively? How can we increase inner product with g as quickly as possible?
  - inner product measures how well functions are "aligned"
  - g is definitely function best-aligned with g!
  - so to increase inner product, add a little bit of g to f




(Can you work this solution out formally?)

# Advanced Example: L² Gradient

- Consider function $F(f) := ||f||^2$ for arguments f: [0,1]→R
- At each "point" f0, we want function ∇F such that for all functions u

$$\langle\langle \nabla F(f_0), u \rangle\rangle = \lim_{\varepsilon \to 0} \frac{F(f_0 + \varepsilon u) - F(f_0)}{\varepsilon}$$

- Expanding 1st term in numerator, we get

$$||f_0 + \varepsilon u||^2 = ||f_0||^2 + \varepsilon^2 ||u||^2 + 2\varepsilon \langle\langle f_0, u \rangle\rangle$$

- Hence, limit becomes

$$\lim_{\varepsilon \to 0} (\varepsilon ||u||^2 + 2\langle\langle f_0, u \rangle\rangle) = 2\langle\langle f_0, u \rangle\rangle$$

- The only solution to $\langle\langle \nabla F(f_0), u \rangle\rangle = 2\langle\langle f_0, u \rangle\rangle$ for all u is

$$\boxed{\nabla F(f_0) = 2f_0}$$

not much different from $\frac{d}{dx} x^2 = 2x$!

**Key idea:**

**Once you get the hang of taking the gradient of ordinary functions, it's *(superficially)* not much harder for more exotic objects like matrices, functions of functions, …**

# Vector Fields

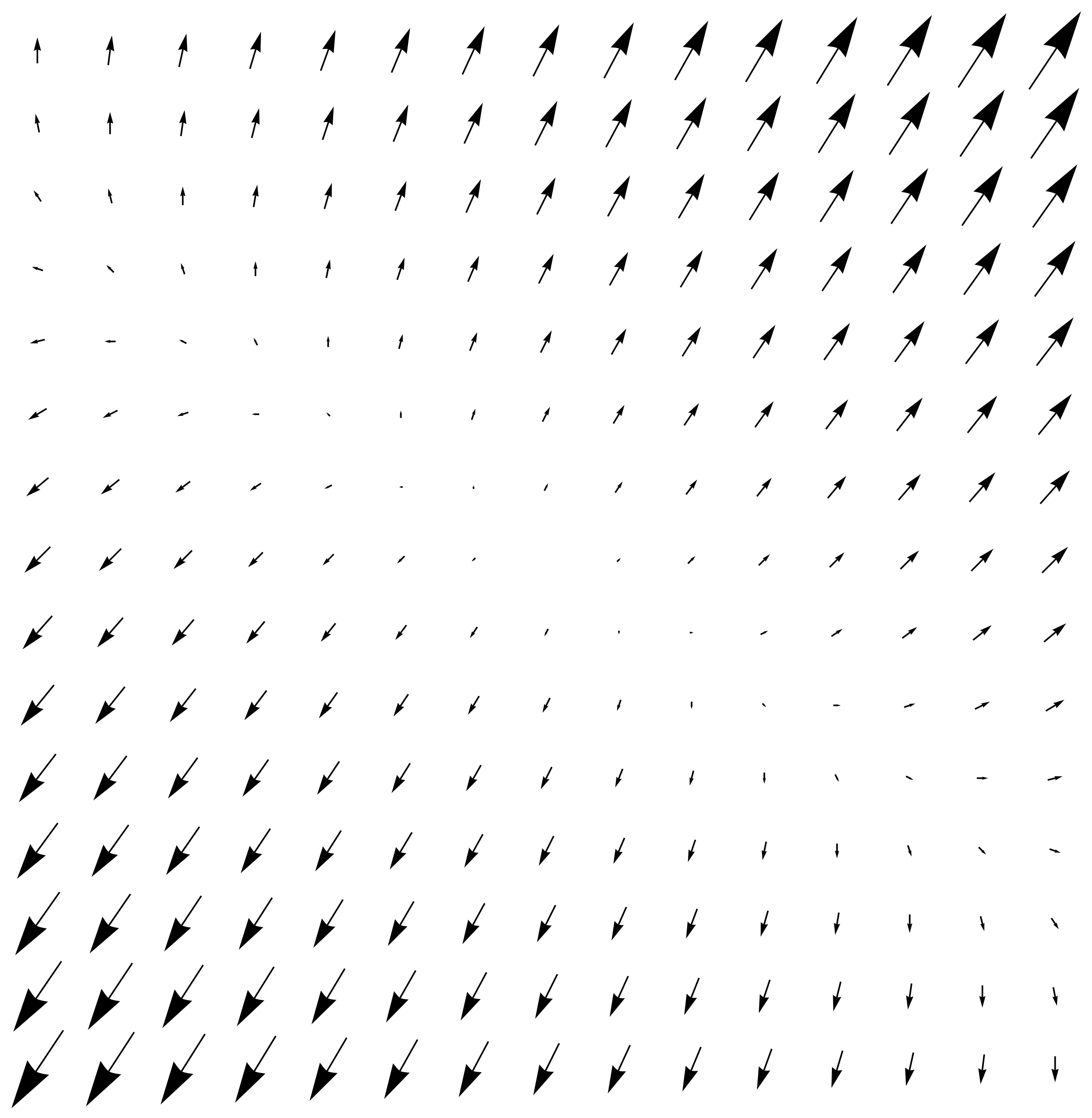
- Gradient was our first example of a vector field
- In general, a vector field assigns a vector to each point in space
- E.g., can think of a 2-vector field in the plane as a map

$$X : \mathbb{R}^2 \to \mathbb{R}^2$$

- For example, we saw a gradient field

$$\nabla f(x,y) = (2x, 2y)$$

(for the function $f(x,y) = x^2 + y^2$)



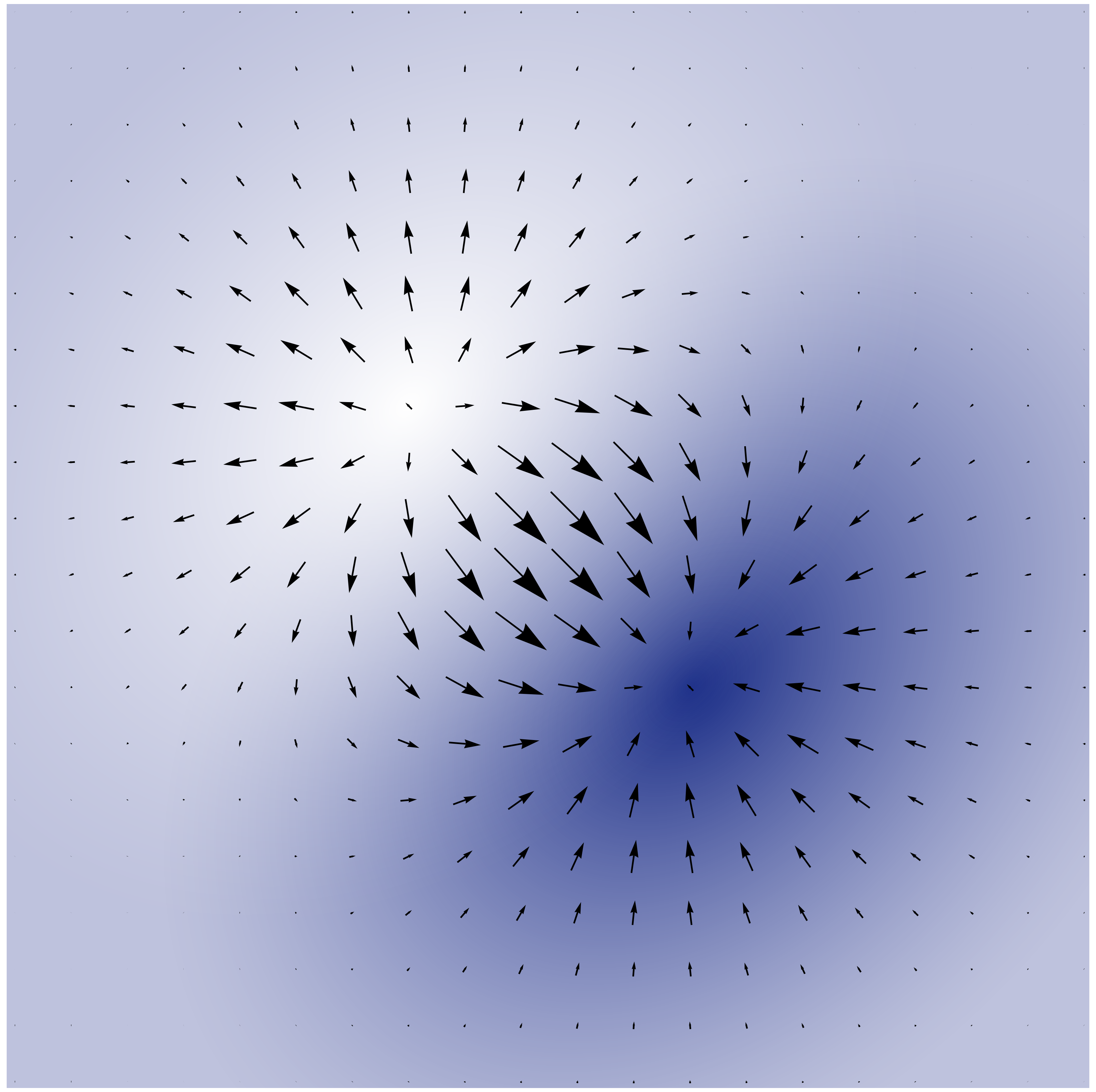
$\nabla f(\mathbf{x})$

**Q: How do we measure the *change* in a vector field?**

# Divergence and Curl

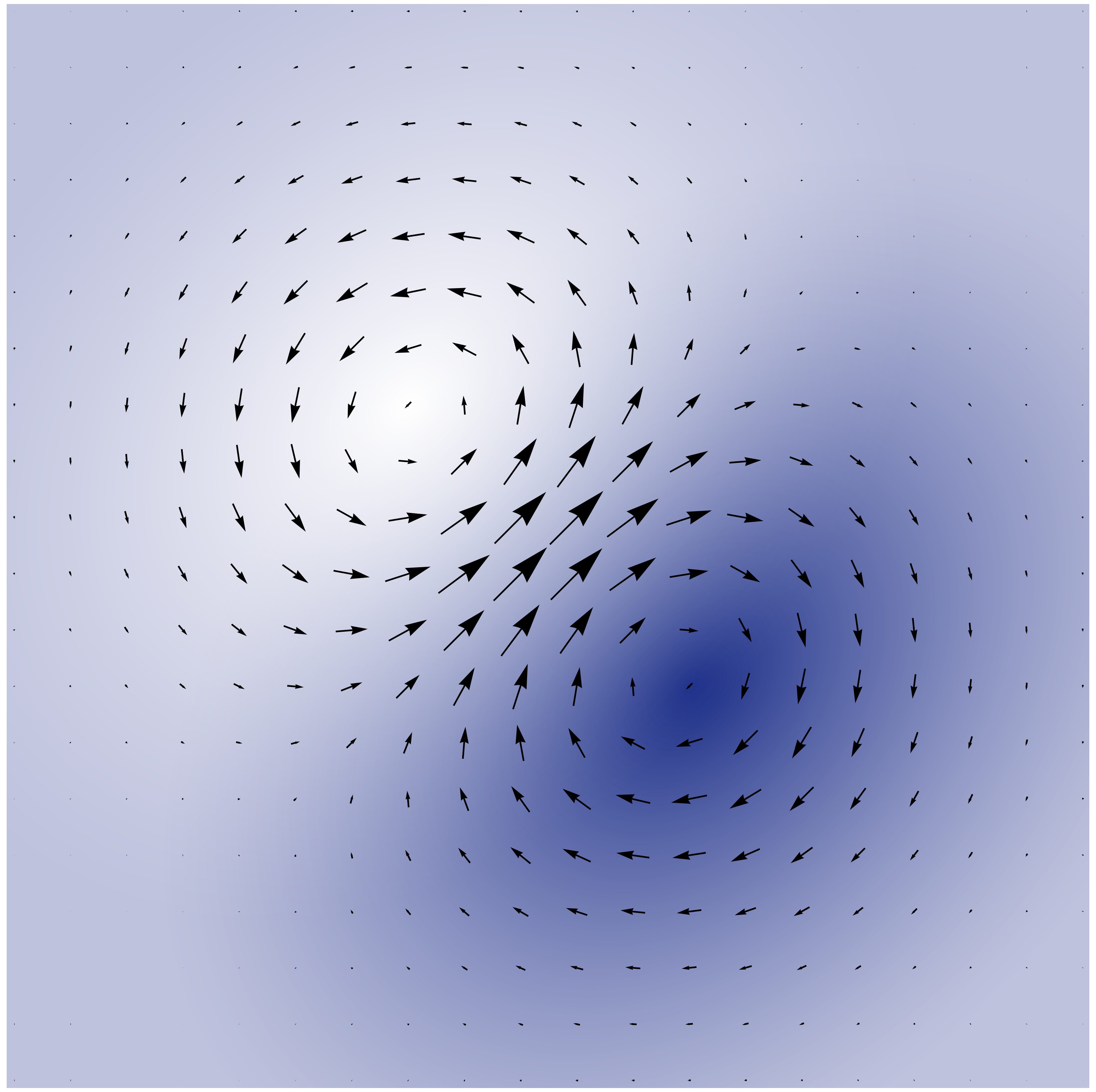
- Two basic derivatives for vector fields:

“How much is field shrinking/expanding?”



$\mathrm{div}\, X$

“How much is field spinning?”



$\mathrm{curl}\, Y$

# Divergence

- Also commonly written as $\nabla \cdot X$
- Suggests a coordinate definition for divergence
- Think of $\nabla$ as a "vector of derivatives"

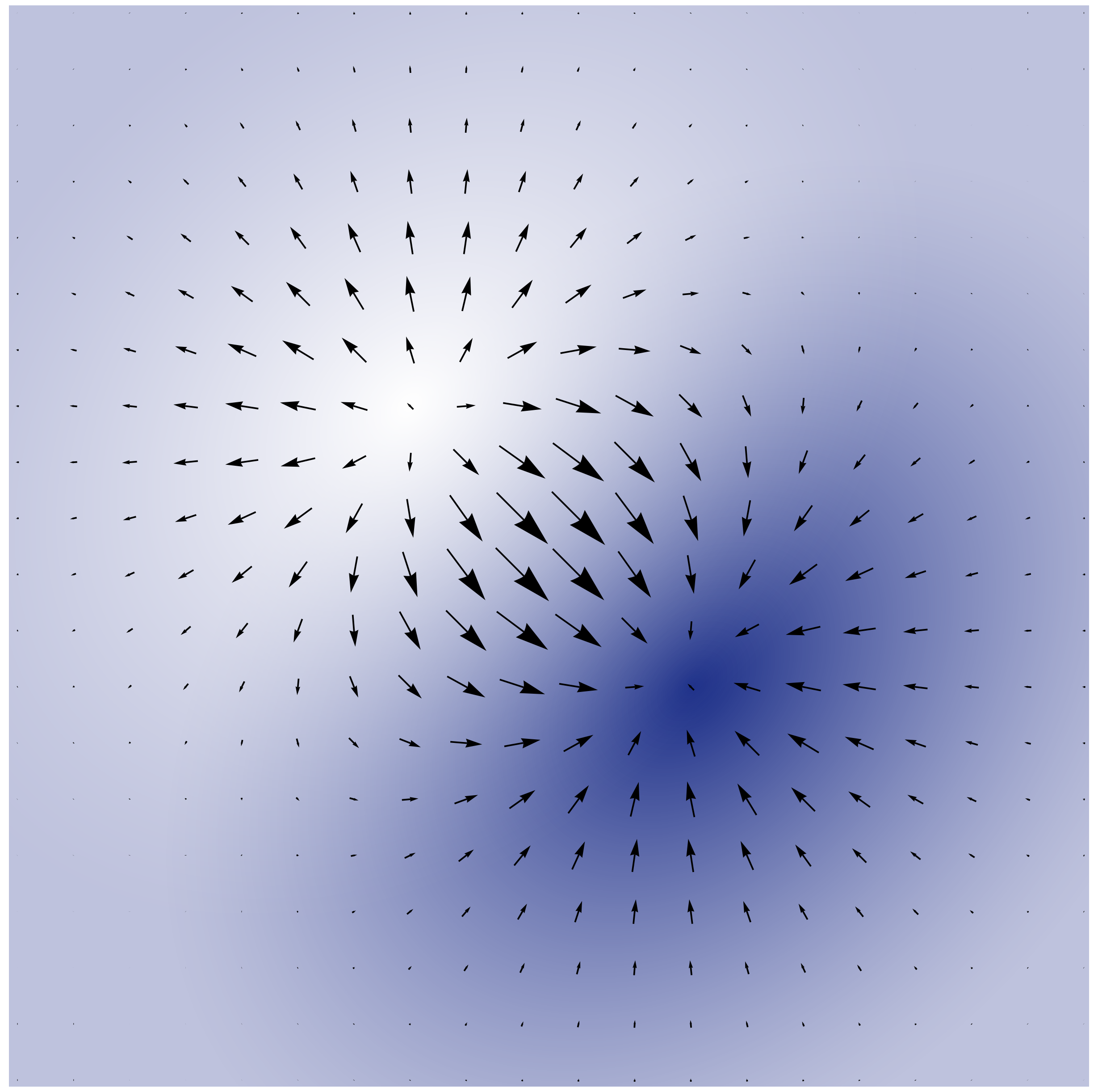
$$\nabla = \left(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\right)$$

- Think of X as a "vector of functions"

$$X(\mathbf{u}) = (X_1(\mathbf{u}), \ldots, X_n(\mathbf{u}))$$

- Then divergence is

$$\nabla \cdot X := \sum_{i=1}^{n} \partial X_i / \partial u_i$$

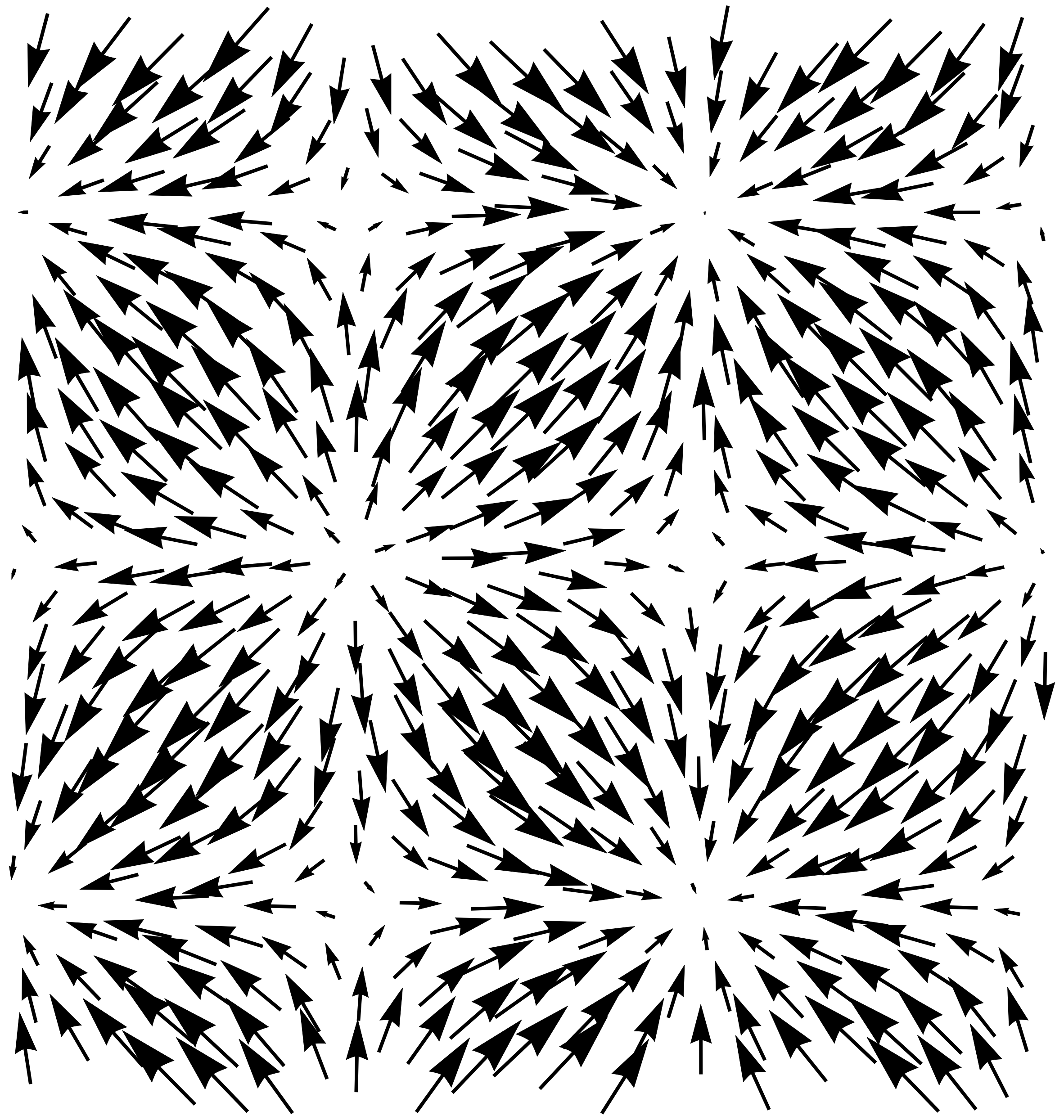


$\nabla \cdot X$

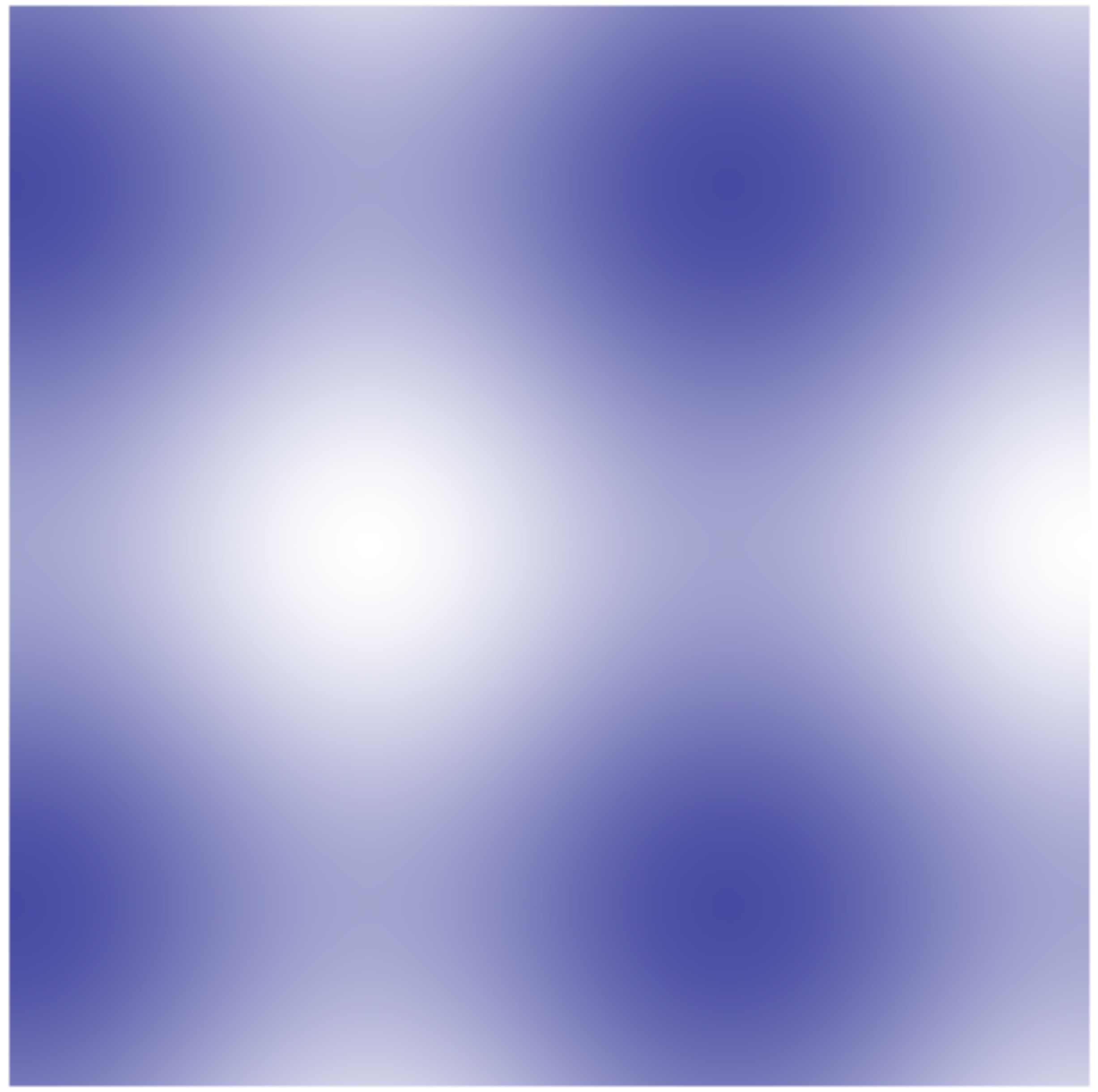
# Divergence - Example

- **Consider the vector field** $X(u,v) := (\cos(u), \sin(v))$
- **Divergence is then**

$$\nabla \cdot X = \frac{\partial}{\partial u}\cos(u) + \frac{\partial}{\partial v}\sin(v) = -\sin(u) + \cos(v).$$



$X$



$\nabla \cdot X$

# Curl

- **Also commonly written as** $\nabla \times X$
- **Suggests a coordinate definition for curl**
- **This time, think of ∇ as a vector of just *three* derivatives:**

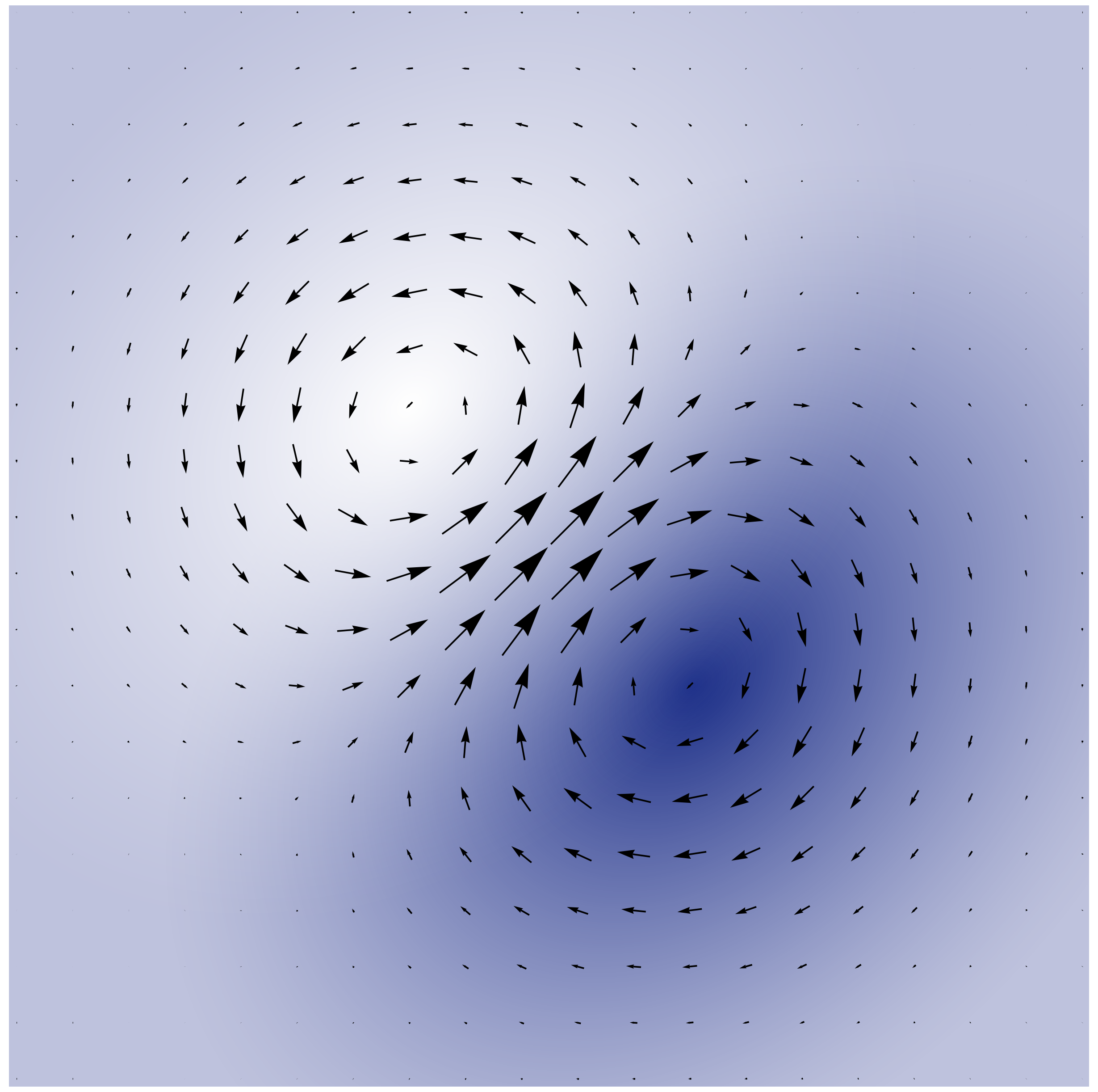
$$\nabla = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}\right)$$

- **Think of X as vector of *three* functions:**

$$X(\mathbf{u}) = (X_1(\mathbf{u}), X_2(\mathbf{u}), X_3(\mathbf{u}))$$

- **Then curl is**

$$\nabla \times X := \begin{bmatrix} \partial X_3/\partial u_2 - \partial X_2/\partial u_3 \\ \partial X_1/\partial u_3 - \partial X_3/\partial u_1 \\ \partial X_2/\partial u_1 - \partial X_1/\partial u_2 \end{bmatrix}$$



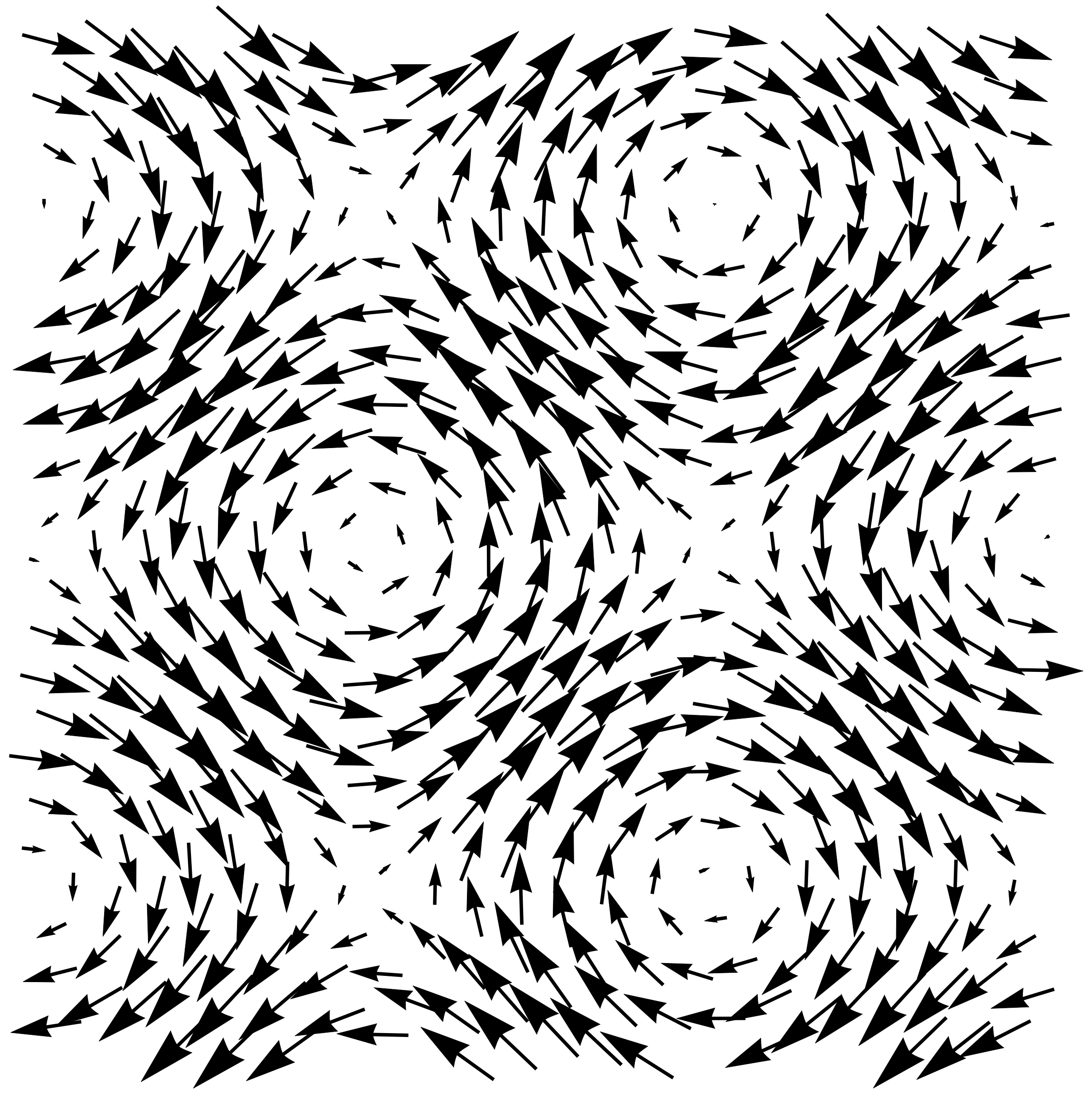
$\nabla \times X$

**(2D "curl":** $\nabla \times X := \partial X_2/\partial u_1 - \partial X_1/\partial u_2$**)**

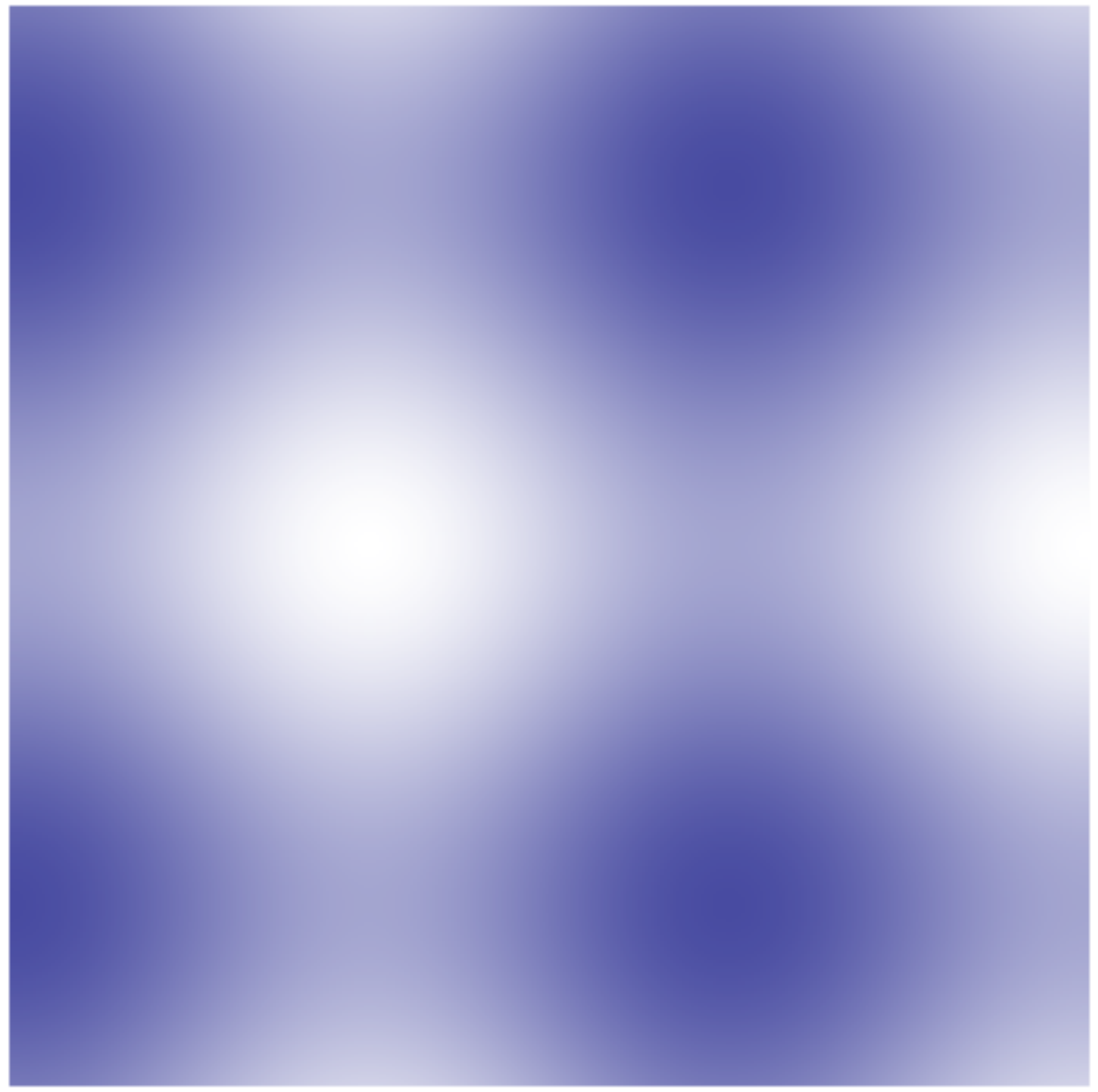
# Curl - Example

- **Consider the vector field** $X(u,v) := (-\sin(v), \cos(u))$
- **(2D) Curl is then**

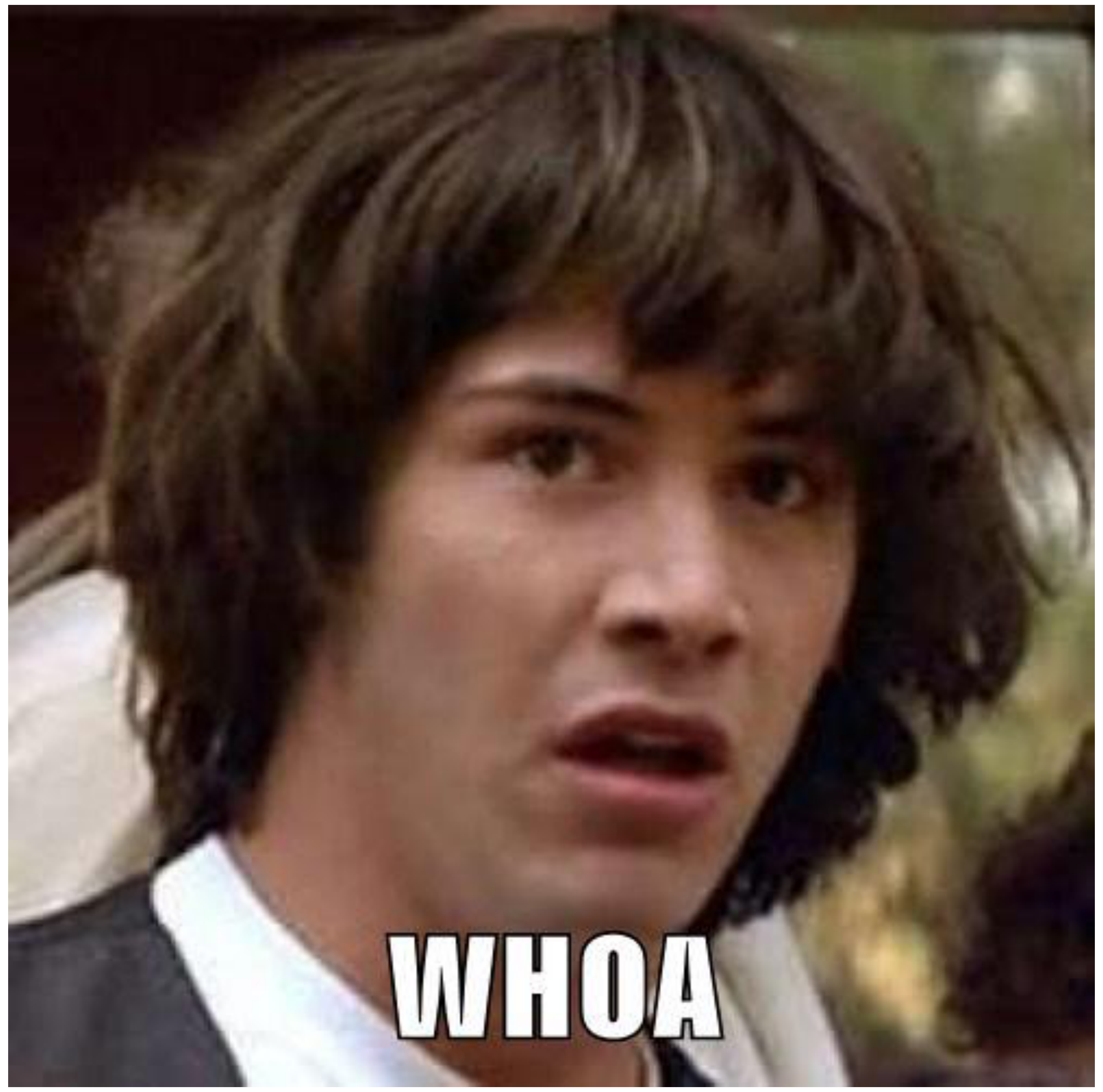
$$\nabla \times X = \frac{\partial}{\partial u}\cos(u) - \frac{\partial}{\partial v}(-\sin(v)) = -\sin(u) + \cos(v).$$



$X$



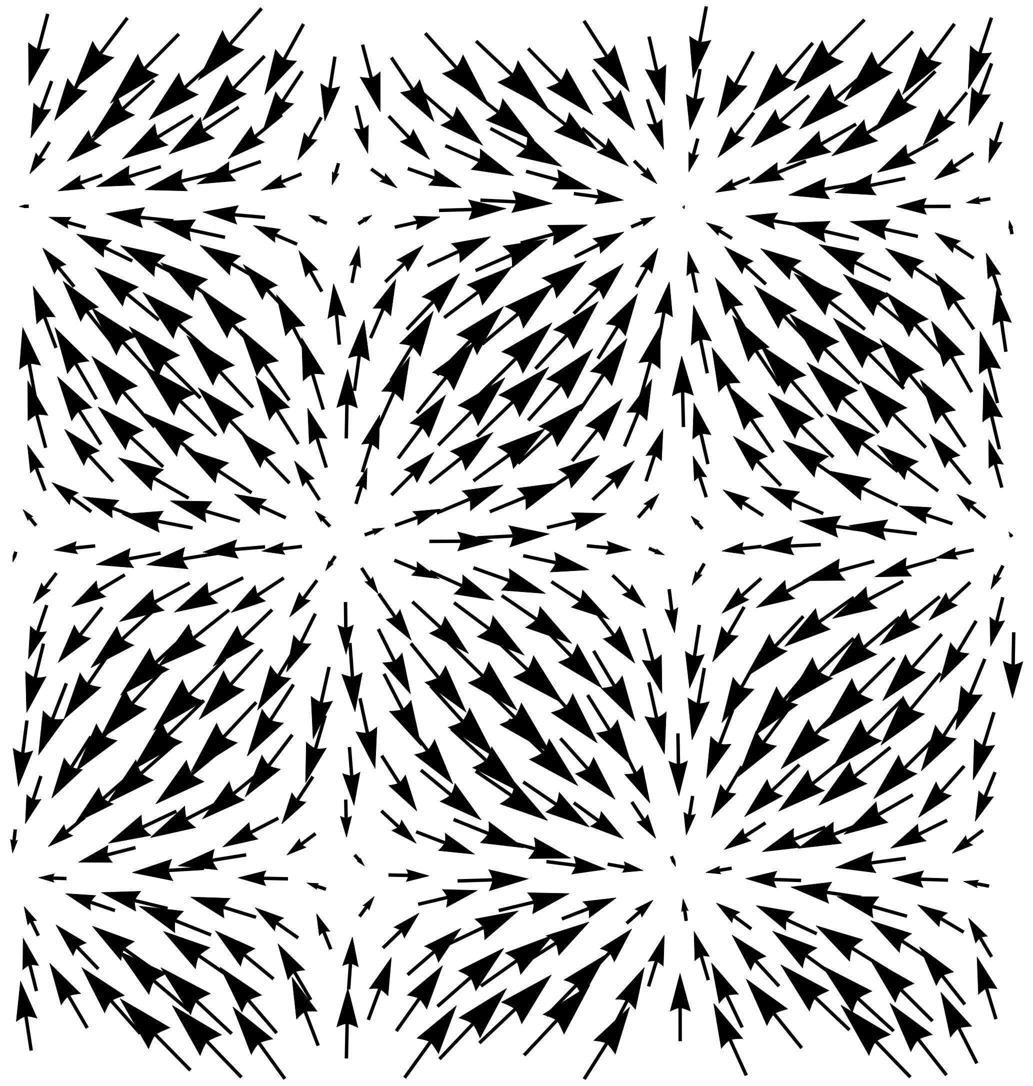
$\nabla \times X$



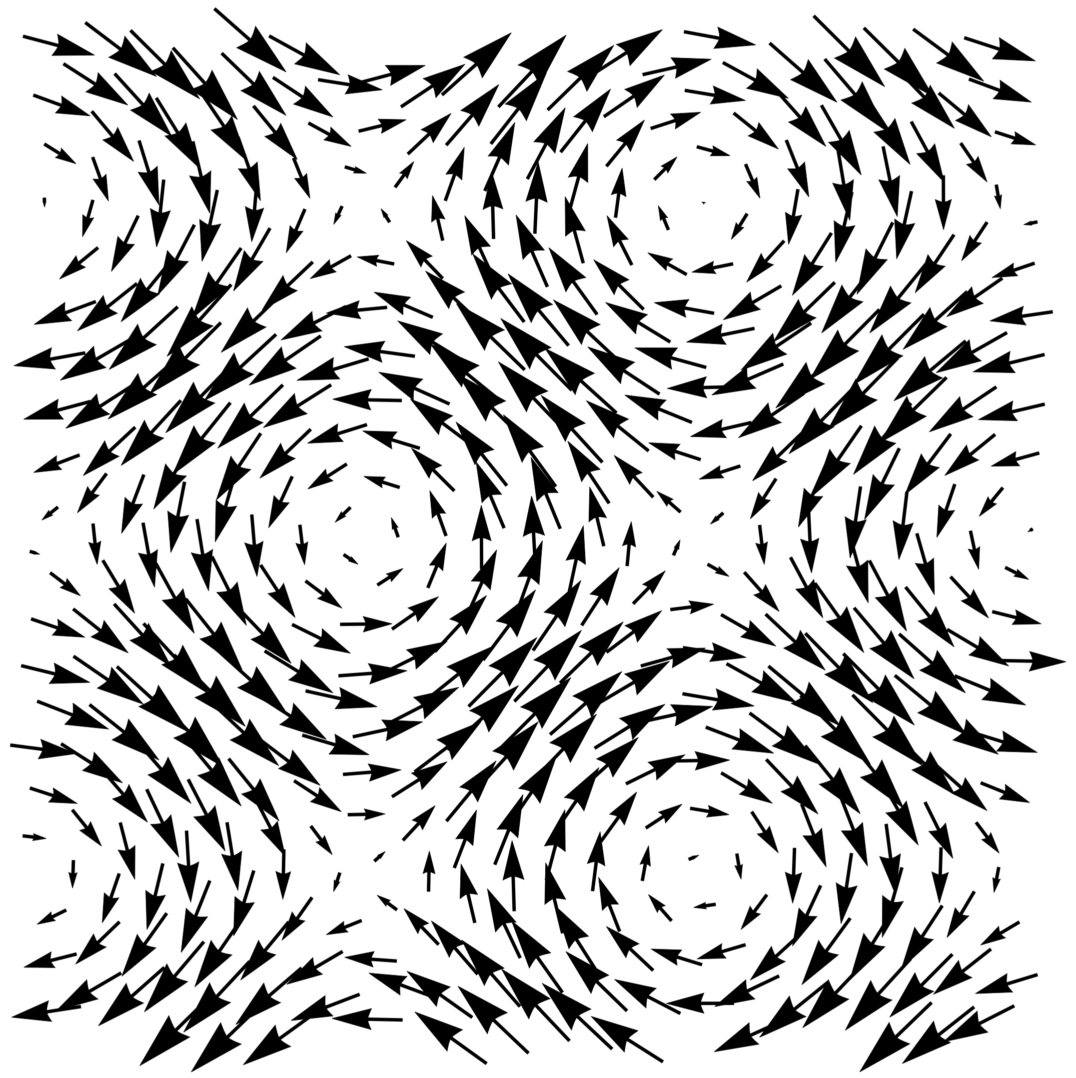

**Notice anything about the relationship between curl and divergence?**

# Divergence vs. Curl (2D)

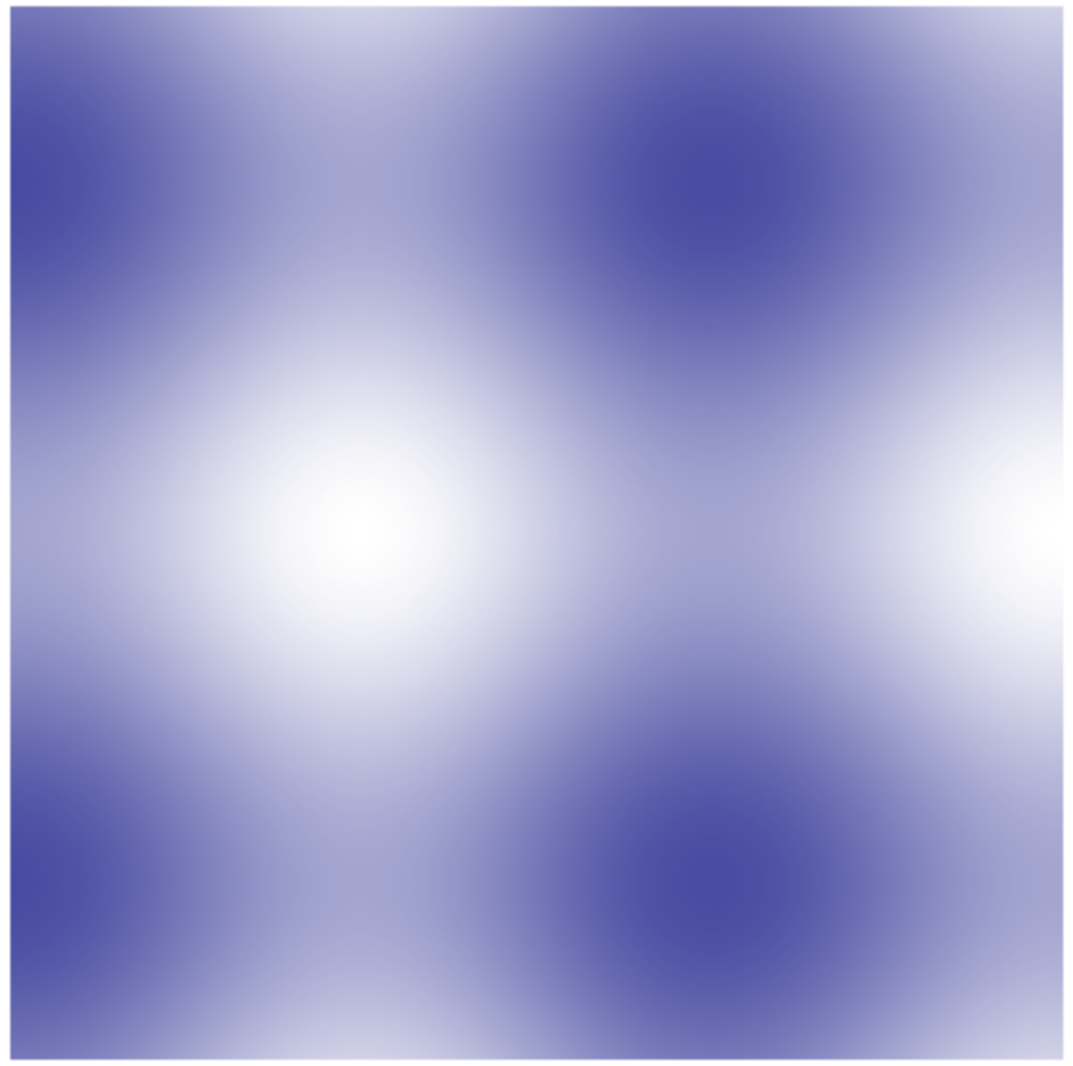
- **Divergence of X is the same as curl of 90-degree rotation of X:**



$X$



$X^{\perp}$

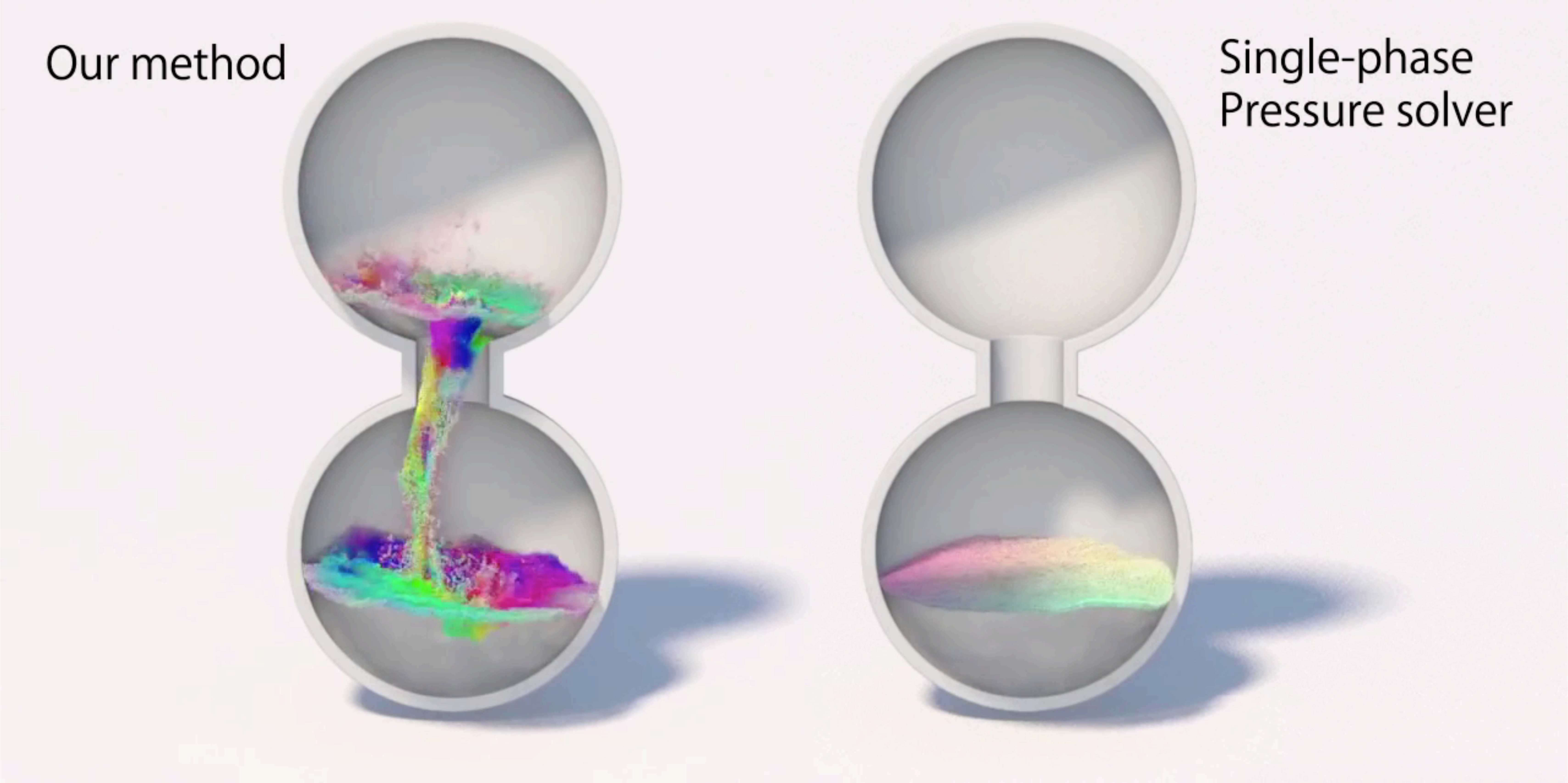


$\nabla \cdot X = \nabla \times X^{\perp}$

- **Playing these kinds of games w/ vector fields plays an important role in algorithms (e.g., fluid simulation)**

- **(Q: Can you come up with an analogous relationship in 3D?)**

# Example: Fluids w/ Stream Function



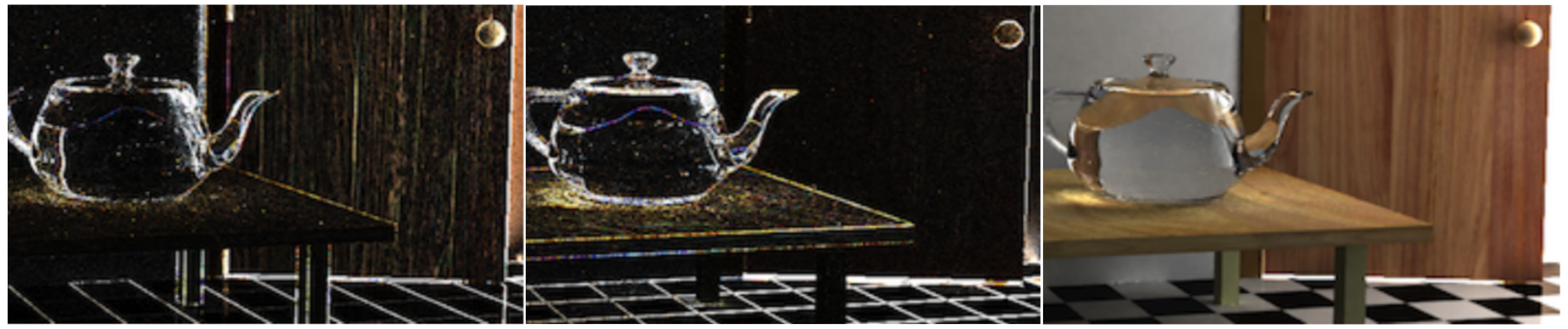

$$\min_{\Psi} ||u^* - \nabla \times \Psi||^2$$
$$u = \nabla \times \Psi$$

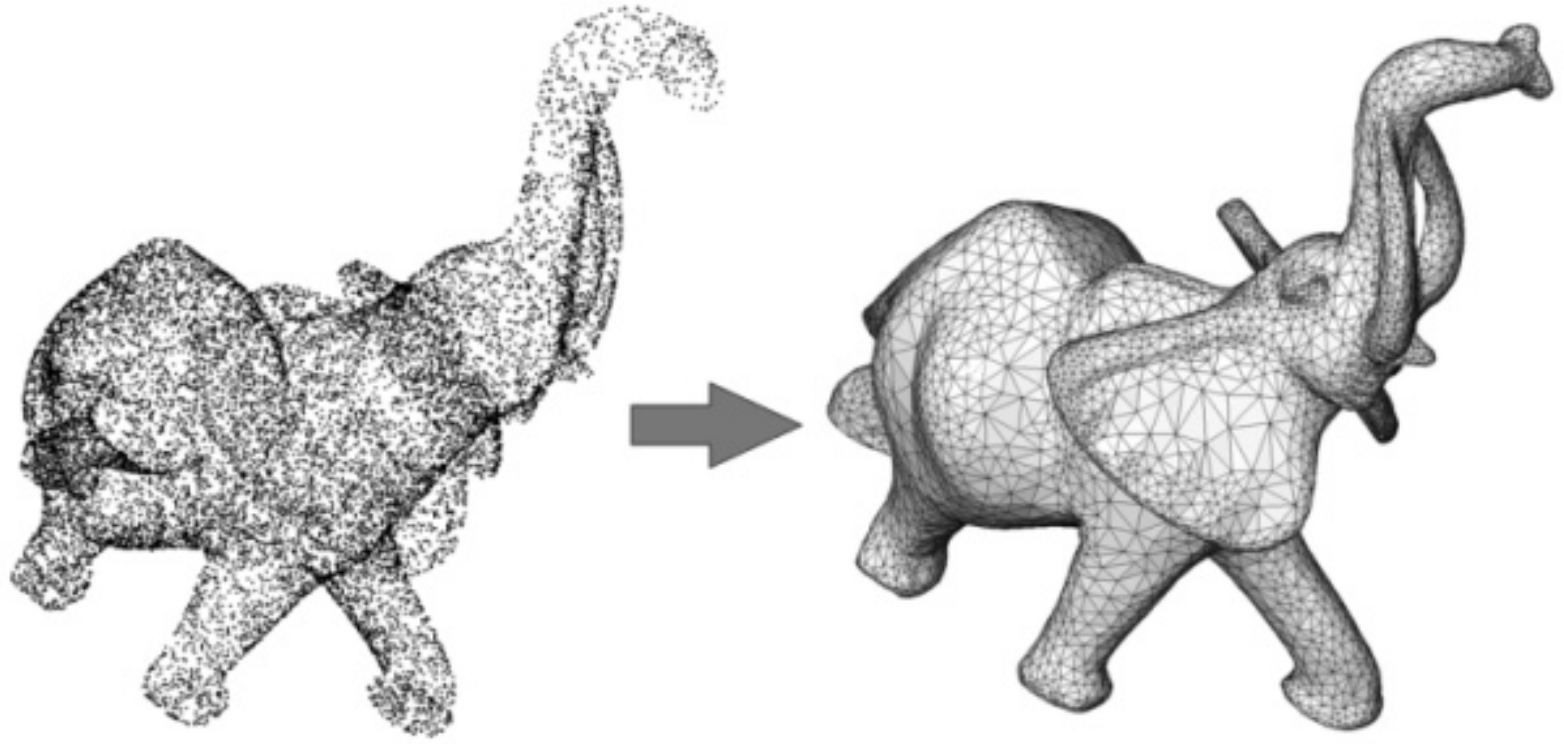
$$\Delta p = \nabla \cdot u^*$$
$$u = u^* - \nabla p$$

Ando et al, “A Stream Function Solver for Liquid Simulations” (SIGGRAPH 2015)

# Laplacian

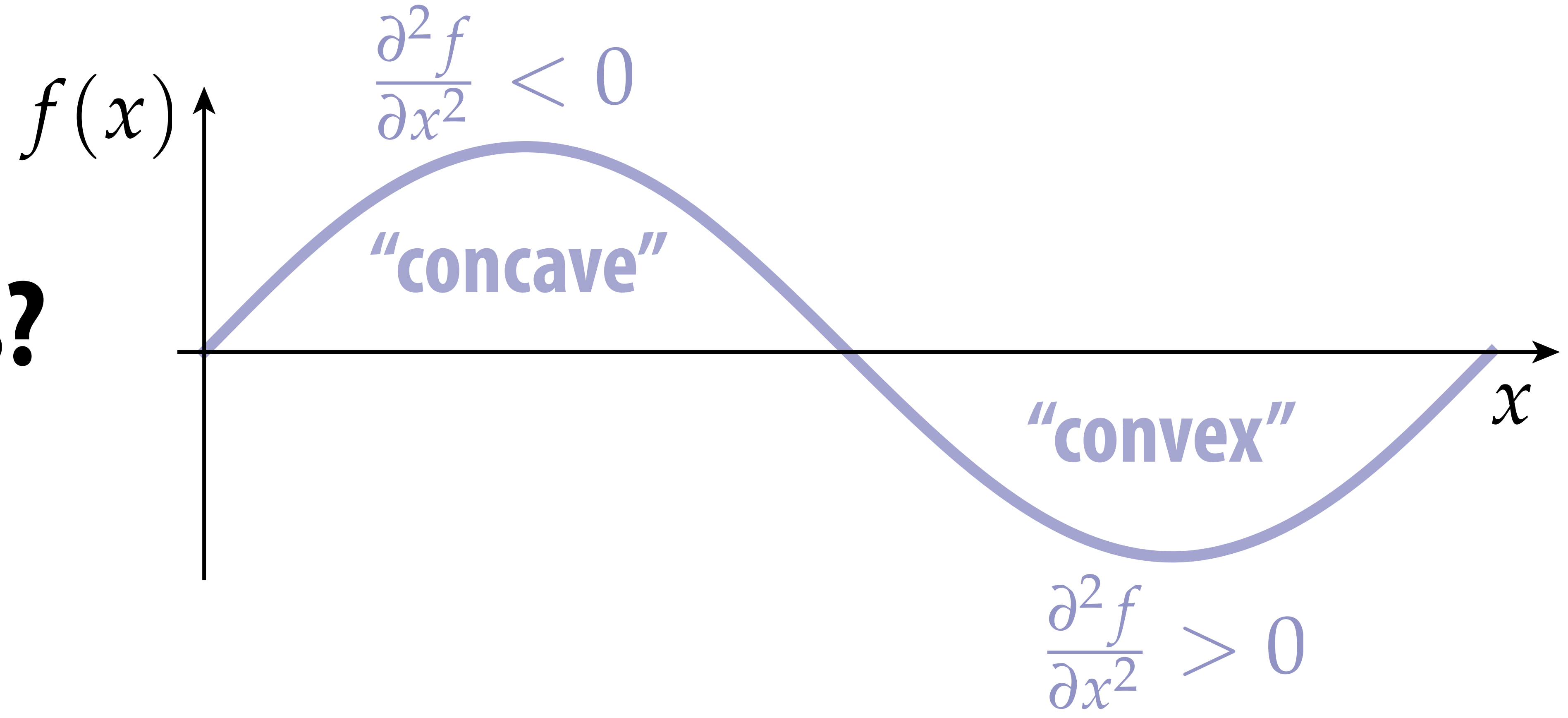
- One more operator we haven't seen yet: the Laplacian
- *Unbelievably* important object in graphics, showing up across geometry, rendering, simulation, imaging
  - basis for Fourier transform / frequency decomposition
  - used to define model PDEs (Laplace, heat, wave equations)
  - encodes rich information about geometry

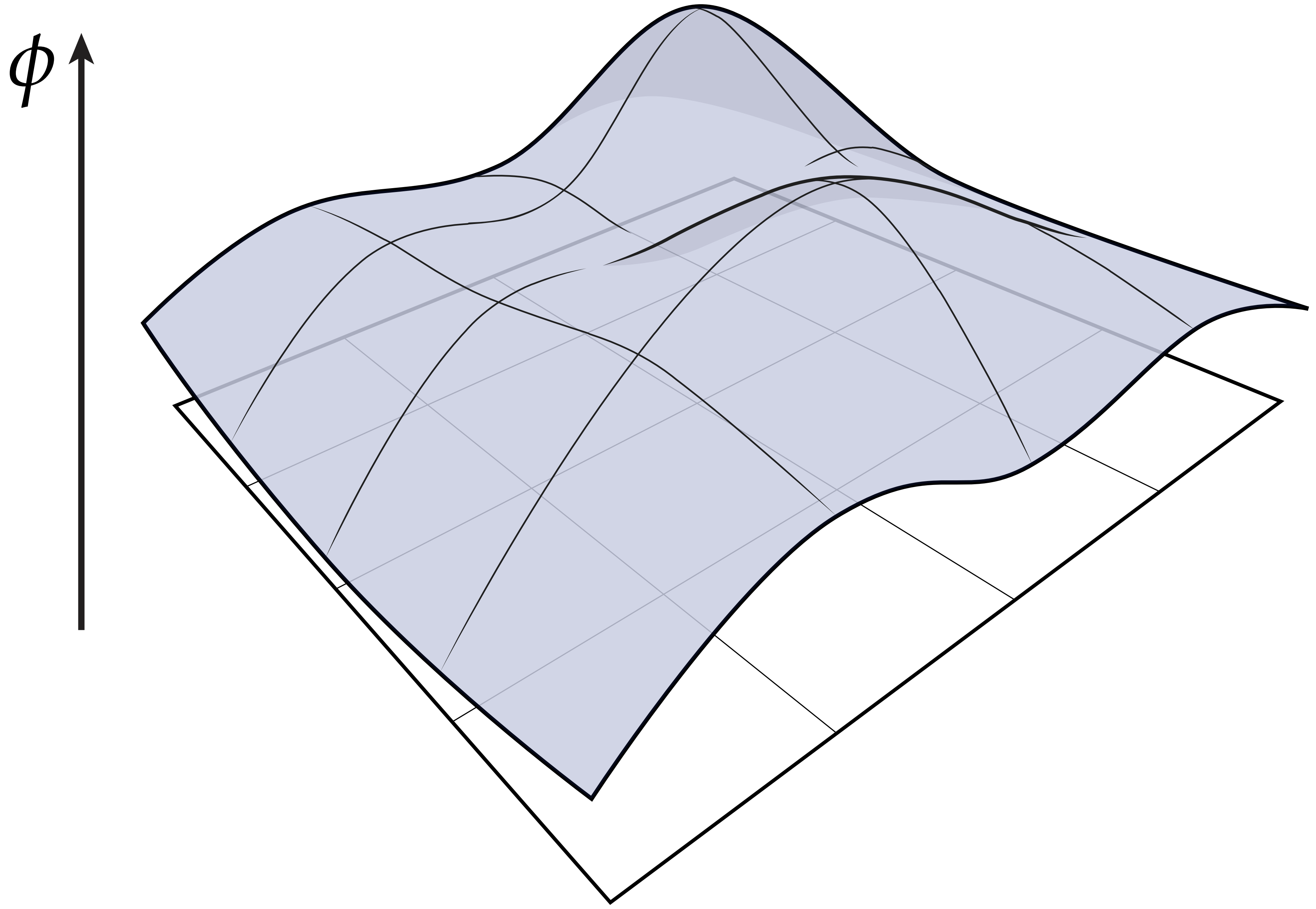


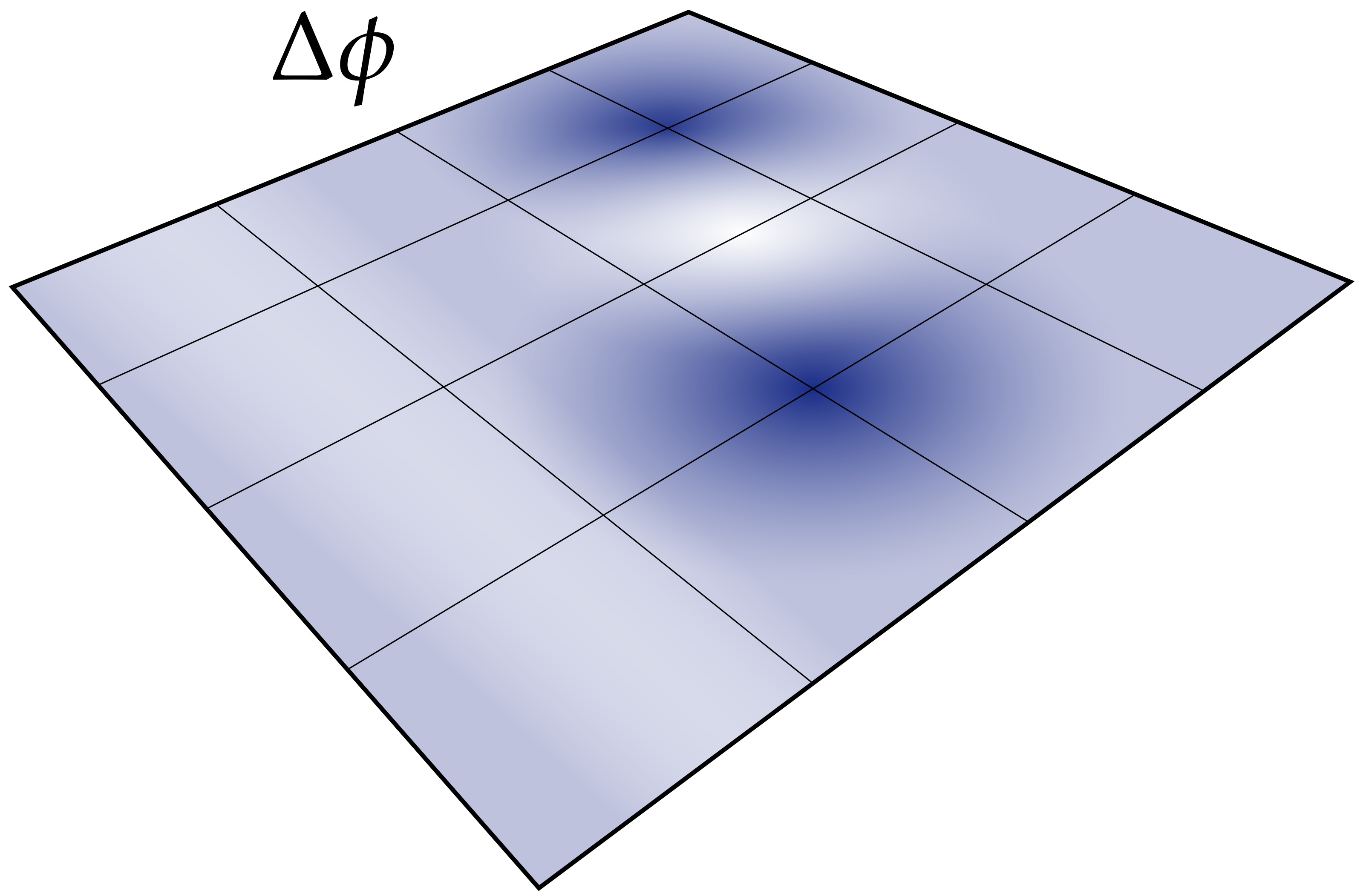


# Laplacian—Visual Intuition

**Q: For ordinary function f(x), what does 2nd derivative tell us?**




**Likewise, Laplacian measures "curvature" of a function.**

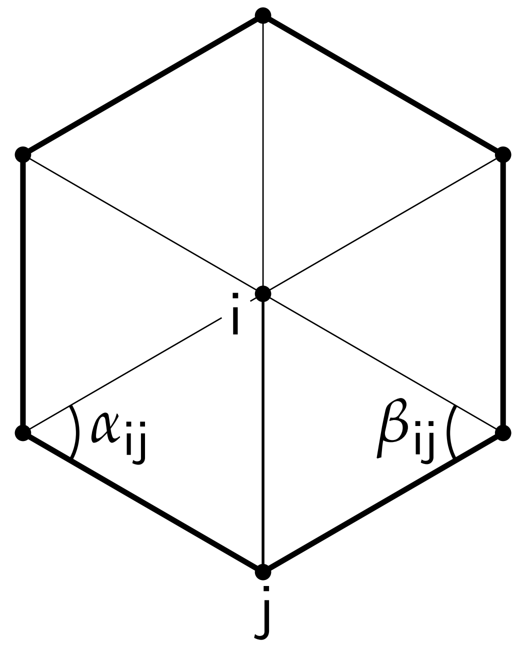
For further interpretations of the Laplacian, see https://youtu.be/oEq9ROI9Umk

# Laplacian—Many Definitions

- Maps a scalar function to another scalar function (*linearly!*)
- Usually* denoted by $\Delta$ "Delta"
- *Many* starting points for Laplacian:
  - divergence of gradient $\Delta f := \nabla \cdot \nabla f = \mathrm{div}(\mathrm{grad} f)$
  - sum of 2nd partial derivatives $\Delta f := \sum_{i=1}^{n} \partial^2 f / \partial x_i^2$
  - gradient of *Dirichlet energy* $\Delta f := -\nabla_f (\frac{1}{2}||\nabla f||^2)$
  - by analogy: *graph Laplacian*
  - variation of surface area
  - trace of Hessian …

| | | |
|---|---|---|
| | 1 | |
| 1 | -4 | 1 |
| | 1 | |

$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2}$$




$$\frac{1}{2}\sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

*Or by $\nabla^2$, but we'll reserve this symbol for the *Hessian*

# Laplacian—Example

- **Let's use coordinate definition:** $\Delta f := \sum_i \partial^2 f / \partial x_i^2$
- **Consider the function** $f(x_1, x_2) := \cos(3x_1) + \sin(3x_2)$
- **We have**

$$\frac{\partial^2}{\partial x_1^2} f = \frac{\partial^2}{\partial x_1^2} \cos(3x_1) + \cancelto{0}{\frac{\partial^2}{\partial x_1^2} \sin(3x_2)} =$$

$$-3 \frac{\partial}{\partial x_1} \sin(3x_1) = -9 \cos(3x_1).$$

**and**

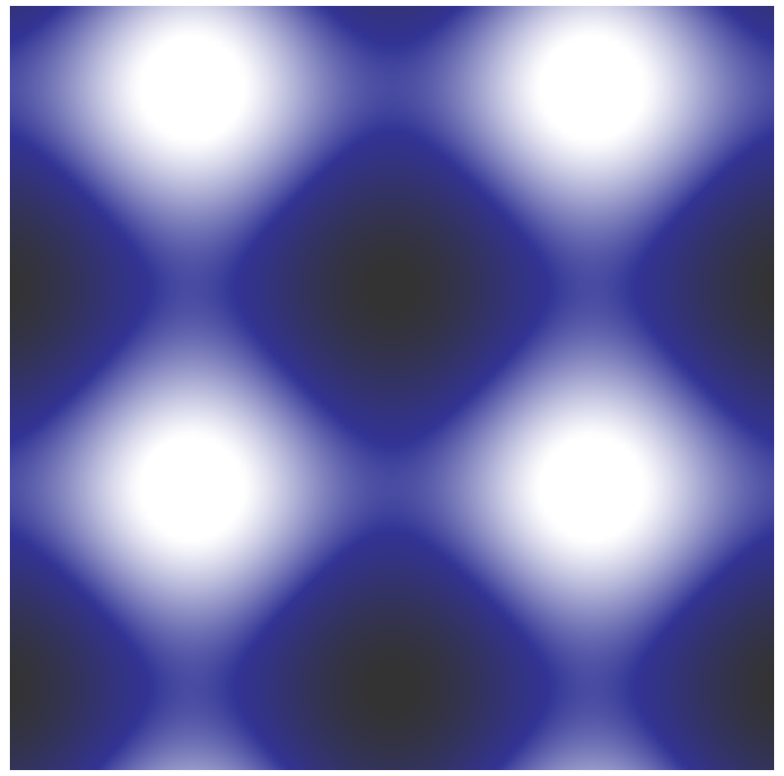
$$\frac{\partial^2}{\partial x_2^2} f = -9 \sin(3x_2).$$

**Hence,**

$$\begin{aligned} \Delta f &= -9(\cos(3x_1) + \sin(3x_2)) \\ &= -9f \end{aligned}$$

**Interesting! Does this always happen?**

$f$

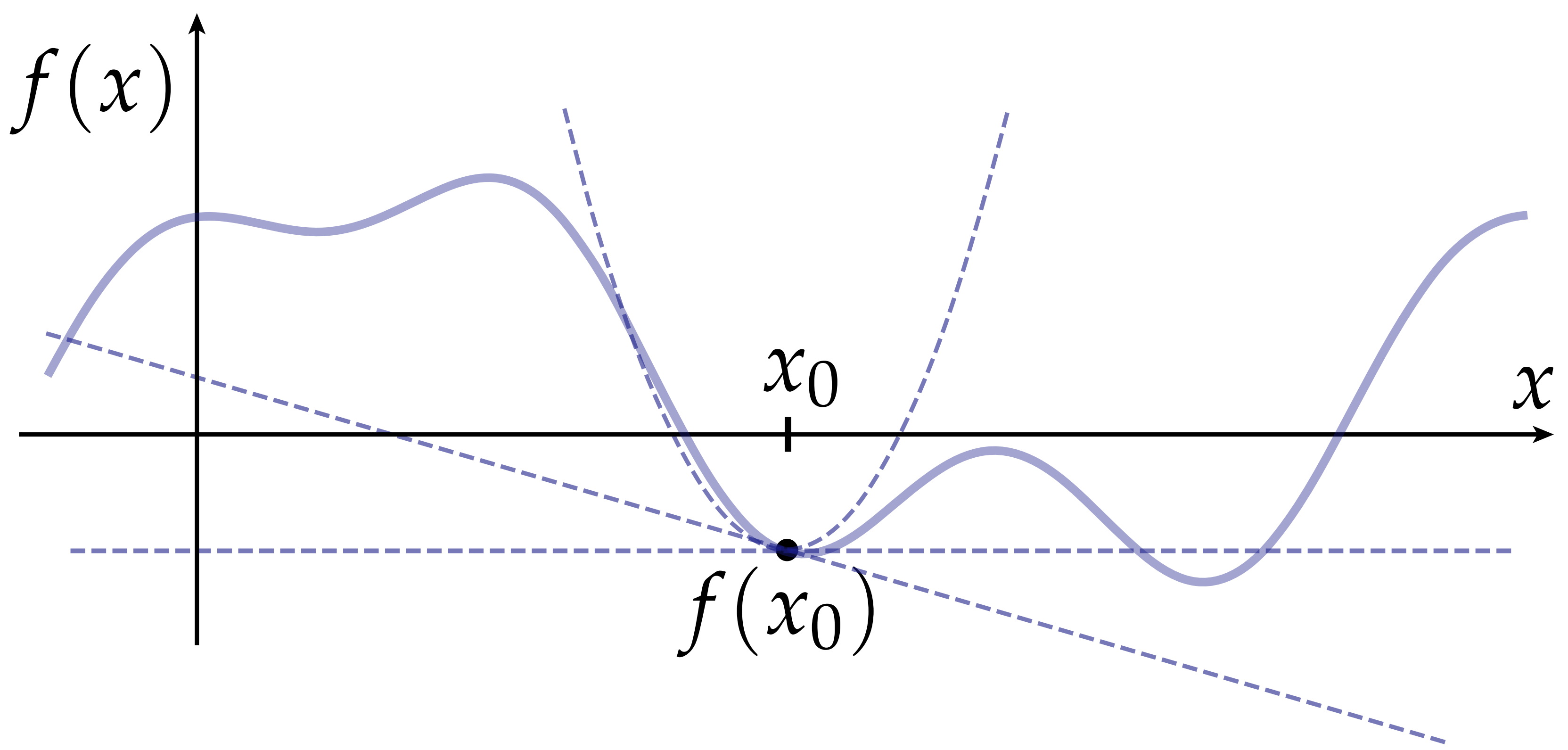


$\Delta f$

# Hessian

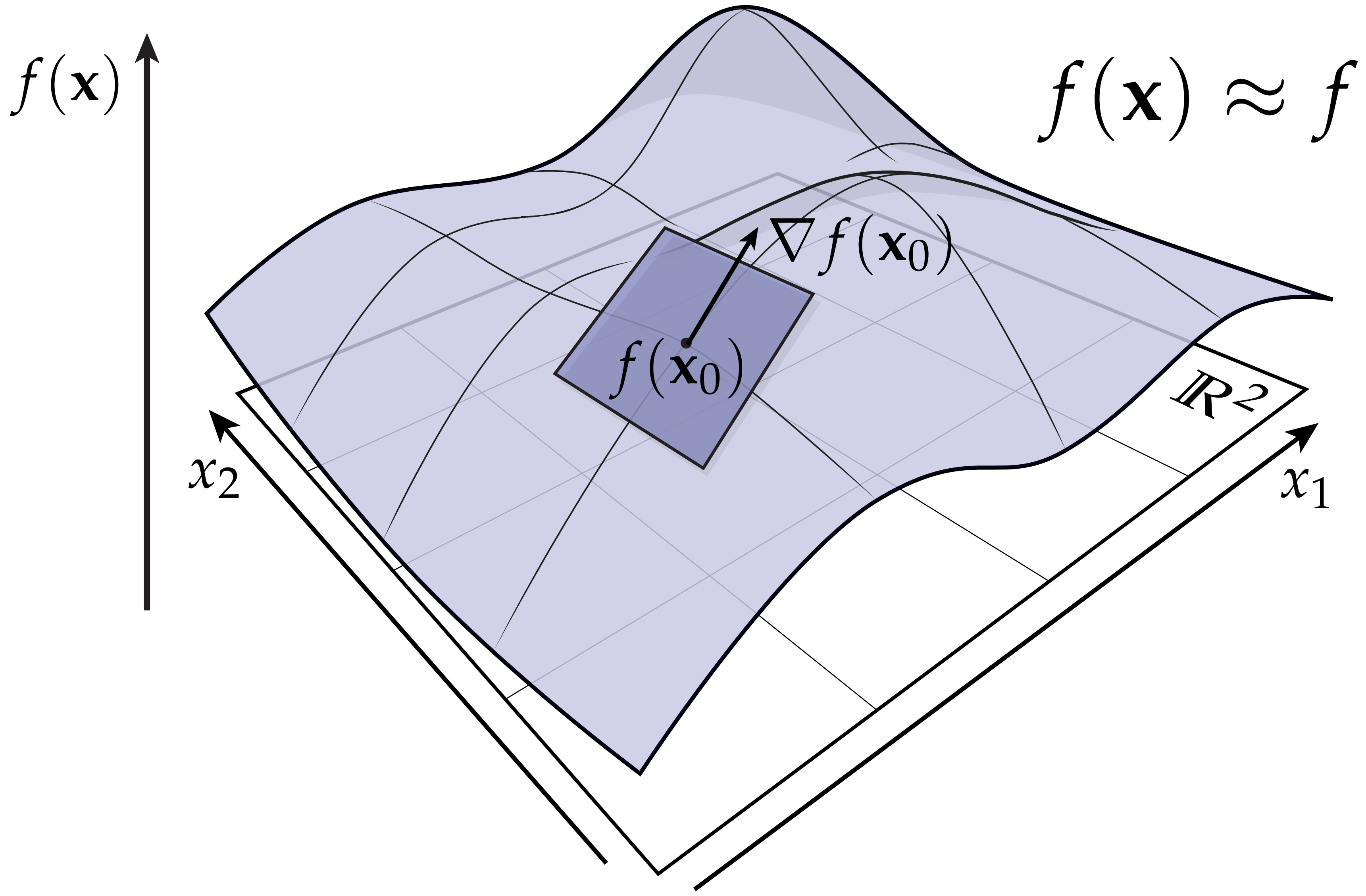
- **Our final differential operator—Hessian will help us locally approximate complicated functions by a few simple terms**
- **Recall our *Taylor series***

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$




- **How do we do this for multivariable functions?**
- **Already talked about best *linear* approximation, using gradient:**

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle$$




**Hessian gives us next, "quadratic" term.**

# Hessian in Coordinates

- **Typically denote Hessian by symbol $\nabla^2$**
- **Just as gradient was "vector that gives us partial derivatives of the *function*," Hessian is "operator that gives us partial derivatives of the *gradient*":**

$$(\nabla^2 f)\mathbf{u} := D_{\mathbf{u}}(\nabla f)$$

- **For a function f(x): $R^n \rightarrow R$, can be more explicit:**

$$\nabla^2 f := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

**Q: Why is this matrix always *symmetric?***

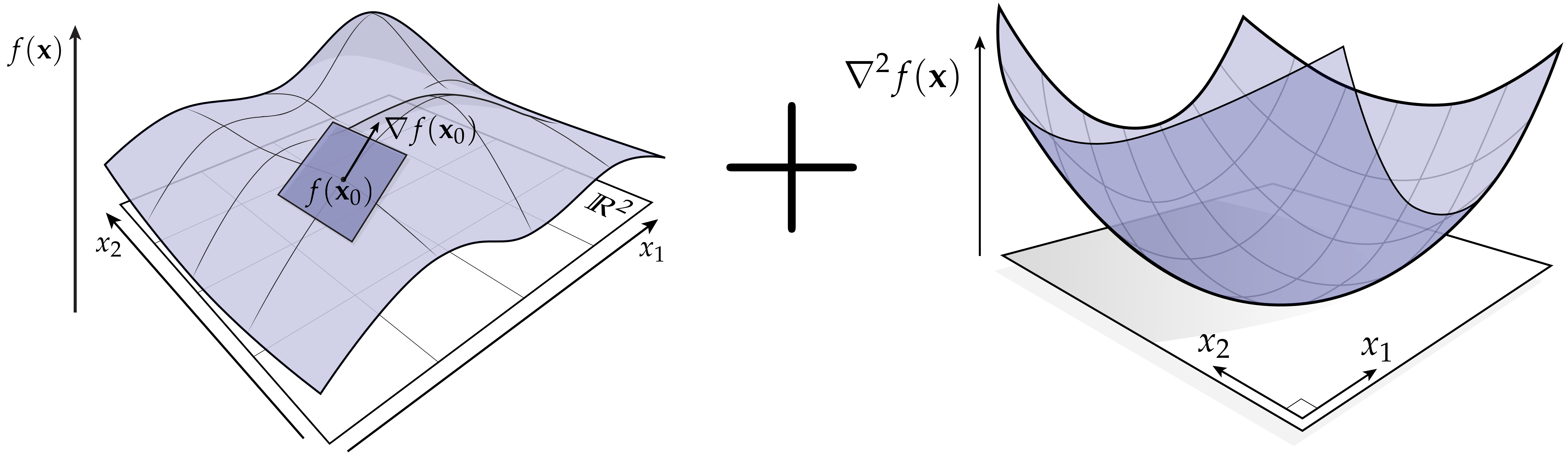
# Taylor Series for Multivariable Functions

- Using Hessian, can now write 2nd-order approximation of any smooth, multivariable function f(x) around some point $\mathbf{x}_0$:

$$f(\mathbf{x}) \approx \underbrace{f(\mathbf{x}_0)}_{c\in\mathbb{R}}^{\text{constant}} + \langle \underbrace{\nabla f(\mathbf{x}_0)}_{\mathbf{b}\in\mathbb{R}^n}^{\text{linear}}, \mathbf{x} - \mathbf{x}_0 \rangle + \langle \underbrace{\nabla^2 f(x_0)}_{\mathbf{A}\in\mathbb{R}^{n\times n}}^{\text{quadratic}} (\mathbf{x} - \mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle / 2$$

- Can write this in matrix form as

$$f(\mathbf{u}) \approx \tfrac{1}{2}\mathbf{u}^\mathsf{T}\mathbf{A}\mathbf{u} + \mathbf{b}^\mathsf{T}\mathbf{u} + c, \quad \mathbf{u} := \mathbf{x} - \mathbf{x}_0$$




Will see later on how this approximation is *very* useful for optimization!

# Next time: Rasterization

- Next time, we'll talk about how to draw triangles
- A lot more interesting (and difficult!) than it might seem…
- Leads to a deep understanding of modern graphics hardware