Lecture 21:

Dynamics and Time Integration

Computer Graphics
CMU 15-462/15-662

Last time: animation

- Added motion to our model
- Interpolate keyframes
- Still a lot of work!
- Today: physically-based animation
  - often less manual labor
  - often more compute-intensive
- Leverage tools from physics
  - dynamical descriptions
  - numerical integration
- Payoff: beautiful, complex behavior from simple models
- Widely-used techniques in modern film (and games!)

Dynamical Description of Motion

> *"A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."*

—Sir Isaac Newton, 1687

> *"Dynamics is concerned with the study of forces and their effect on motion, as opposed to kinematics, which studies the motion of objects without reference to its causes."*

—Sir Wiki Pedia, 2015

(Q: Is keyframe interpolation *dynamic*, or *kinematic?*)

The Animation Equation

- Already saw the *rendering equation*
- What's the *animation equation?*

mass

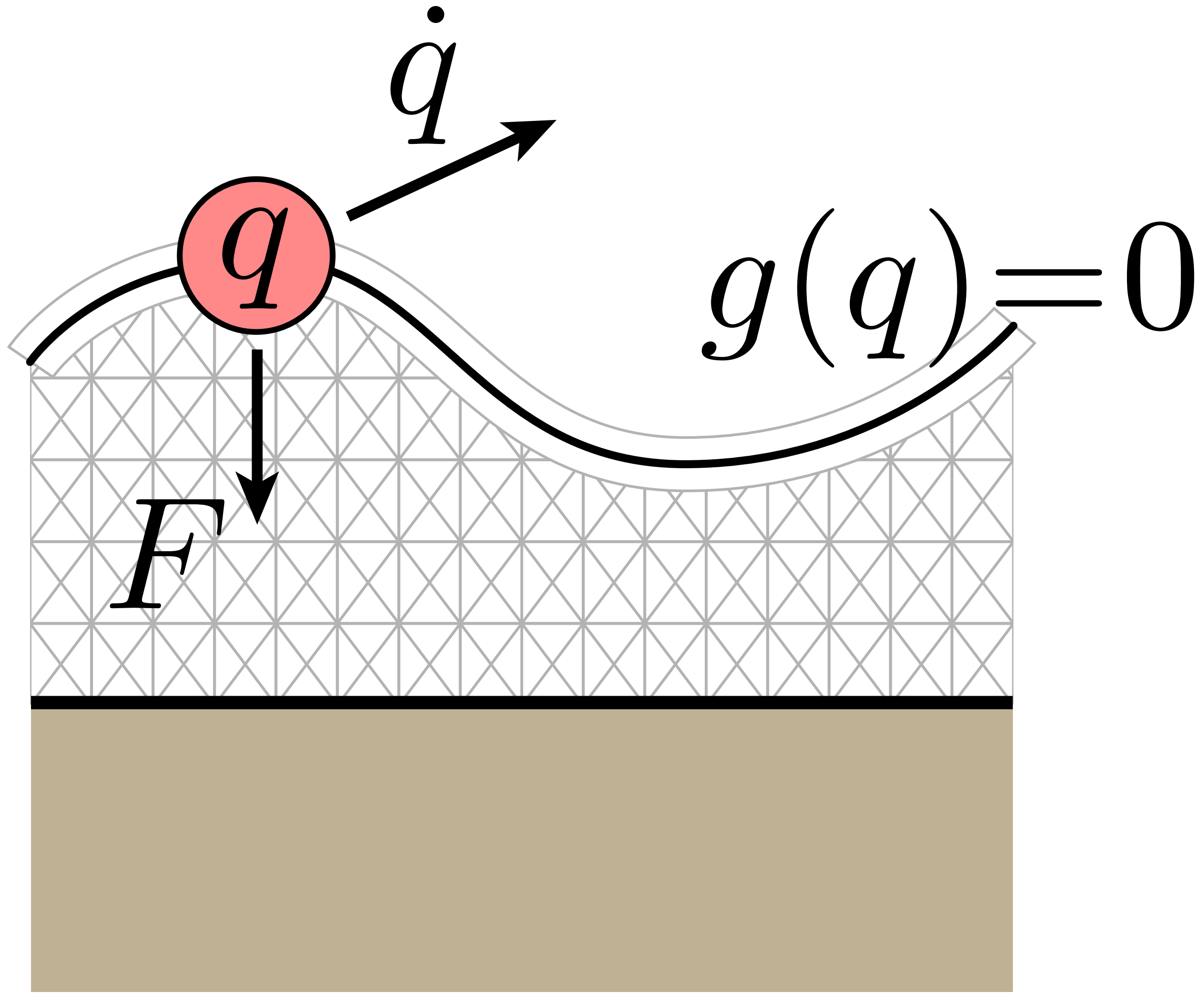
acceleration

$$F = ma$$

force

The “Animation Equation,” revisited

- **Well actually there are some more equations...**
- **Let’s be more careful:**
  - **Any system has a *configuration*** $q(t)$
  - **It also has a *velocity*** $\dot{q} := \frac{d}{dt}q$
  - **And some kind of *mass*** M
  - **There are probably some *forces*** F
  - **And also some *constraints*** $g(q, \dot{q}, t) = 0$

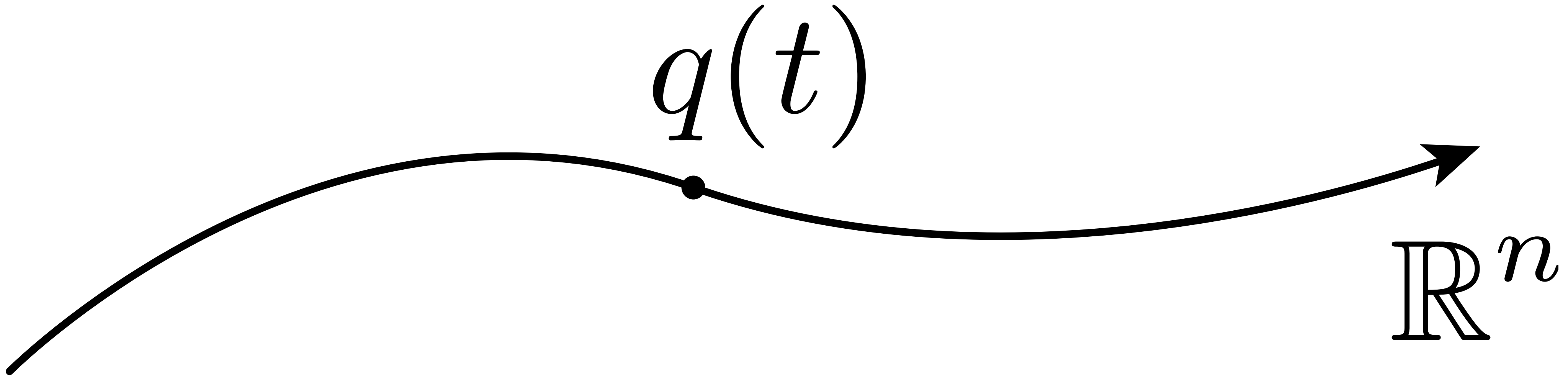


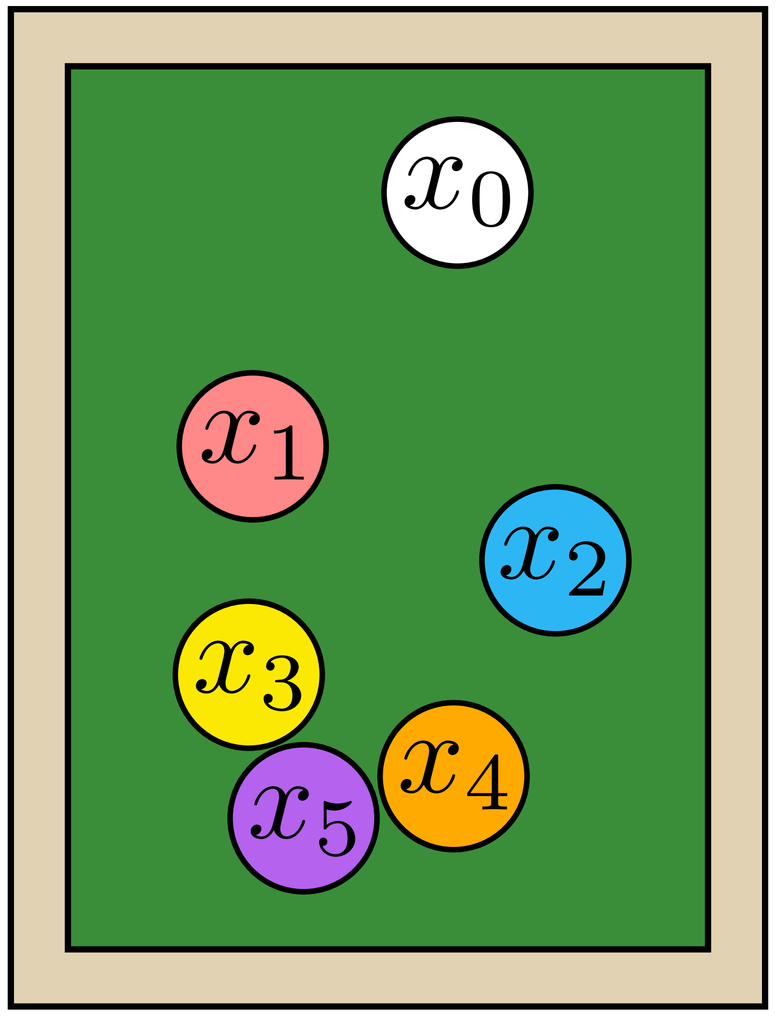
- **E.g., could write Newton’s 2nd law as** $\ddot{q} = F/m$
- **Makes two things clear:**
  - **acceleration is 2nd time derivative of configuration**
  - **ultimately, we want to solve for the configuration *q***

Generalized Coordinates

- **Often describing systems with many, many moving pieces**
- **E.g., a collection of billiard balls, each with position x_i**
- **Collect them all into a single vector of *generalized coordinates:***

$$q = (x_0, x_1, \ldots, x_n)$$



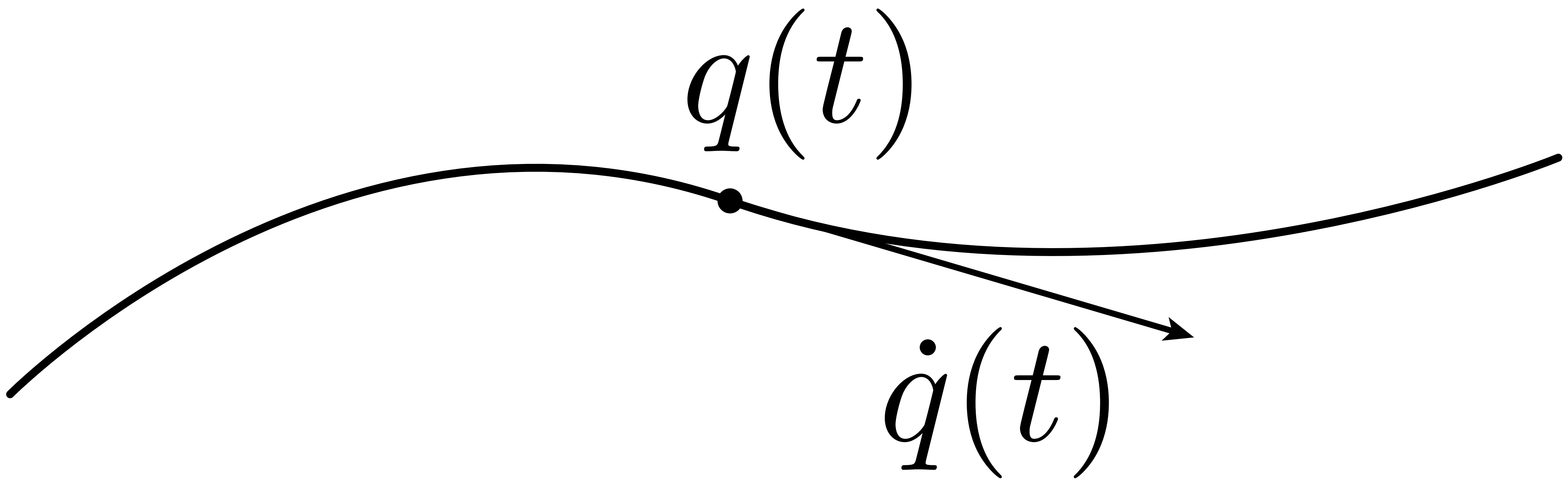


- **Can think of q as a *single point* moving along a trajectory in R^n**
- **This way of thinking naturally maps to the way we actually solve equations on a computer: all variables are often "stacked" into a big long vector and handed to a solver.**
- **(…So why not write things down this way in the first place?)**

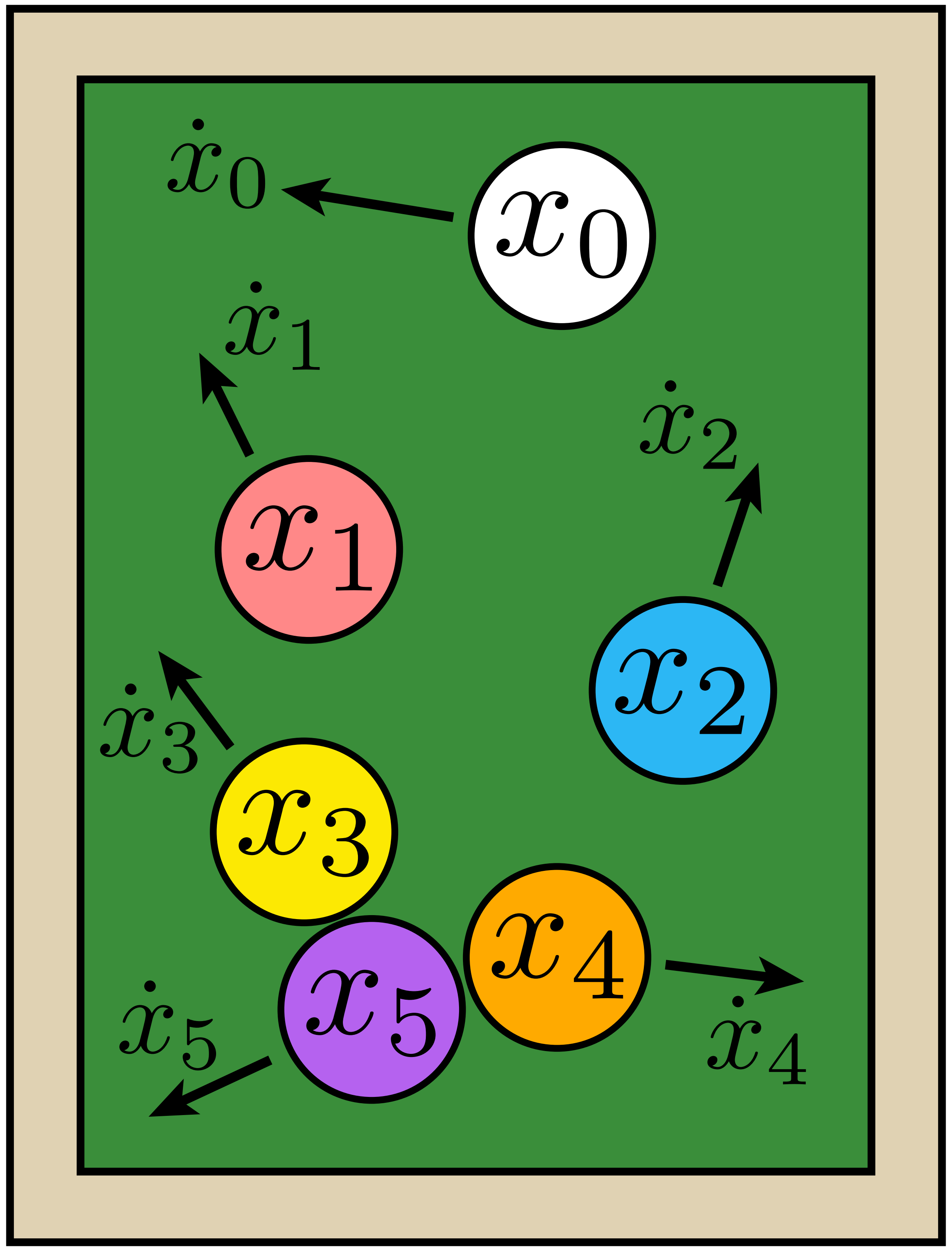
Generalized Velocity

- **Not much more to say about generalized velocity: it's the time derivative of the generalized coordinates!**

$$\dot{q} = (\dot{x}_0, \dot{x}_1, \ldots, \dot{x}_n)$$



All of life (and physics) is just traveling along a curve...



Ordinary Differential Equations

- Many dynamical systems can be described via an *ordinary differential equation (ODE)* in generalized coordinates:

change in configuration over time

velocity function

$$\frac{d}{dt}q = f(q, \dot{q}, t)$$

- ODE doesn't have to describe mechanical phenomenon, e.g.,

$$\frac{d}{dt}u(t) = au$$

"rate of growth is proportional to value"

- Solution? $u(t) = be^{at}$
- Describes exponential decay ($a < 1$), or really great stock ($a > 1$)
- "Ordinary" means "involves derivatives in time but not space"
- We'll talk about spatial derivatives (PDEs) in another lecture...

Dynamics via ODEs

- **Another key example: Newton's 2nd law!**

$$\ddot{q} = F/m$$

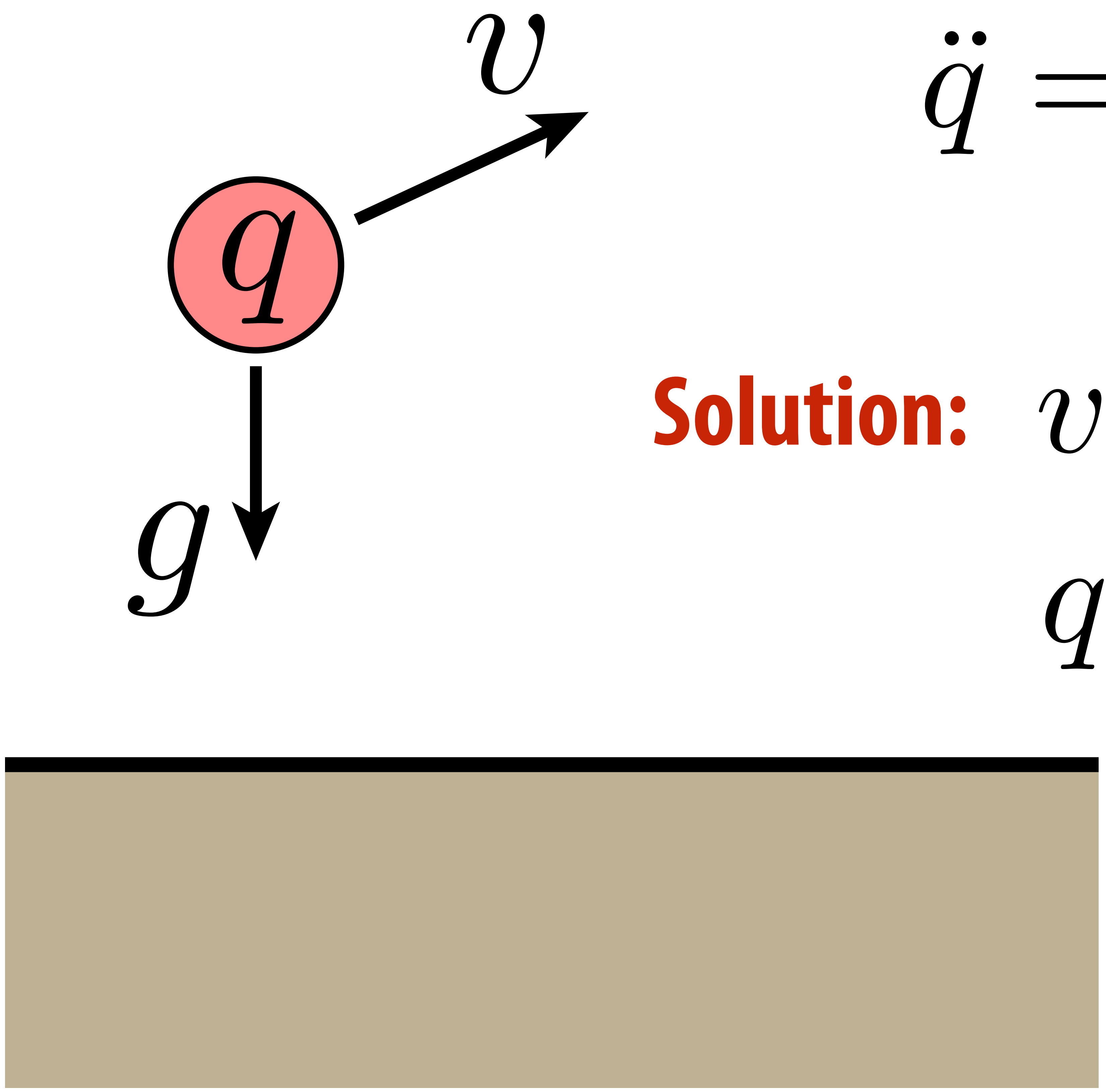
- **"Second order" ODE since we take *two* time derivatives**
- **Can also write as a *system* of two *first order* ODEs, by introducing new "dummy" variable for velocity:**

$$\dot{q} = v$$
$$\dot{v} = F/m$$

- **Splitting things up this way will make it easy to talk about solving these equations numerically (among other things)**

Simple Example: Throwing a Rock

- Consider a rock* of mass *m* tossed under force of gravity *g*
- Easy to write dynamical equations, since only force is gravity:



$$\ddot{q} = g/m$$

or

$$\dot{q} = v$$
$$\dot{v} = g/m$$

Solution:

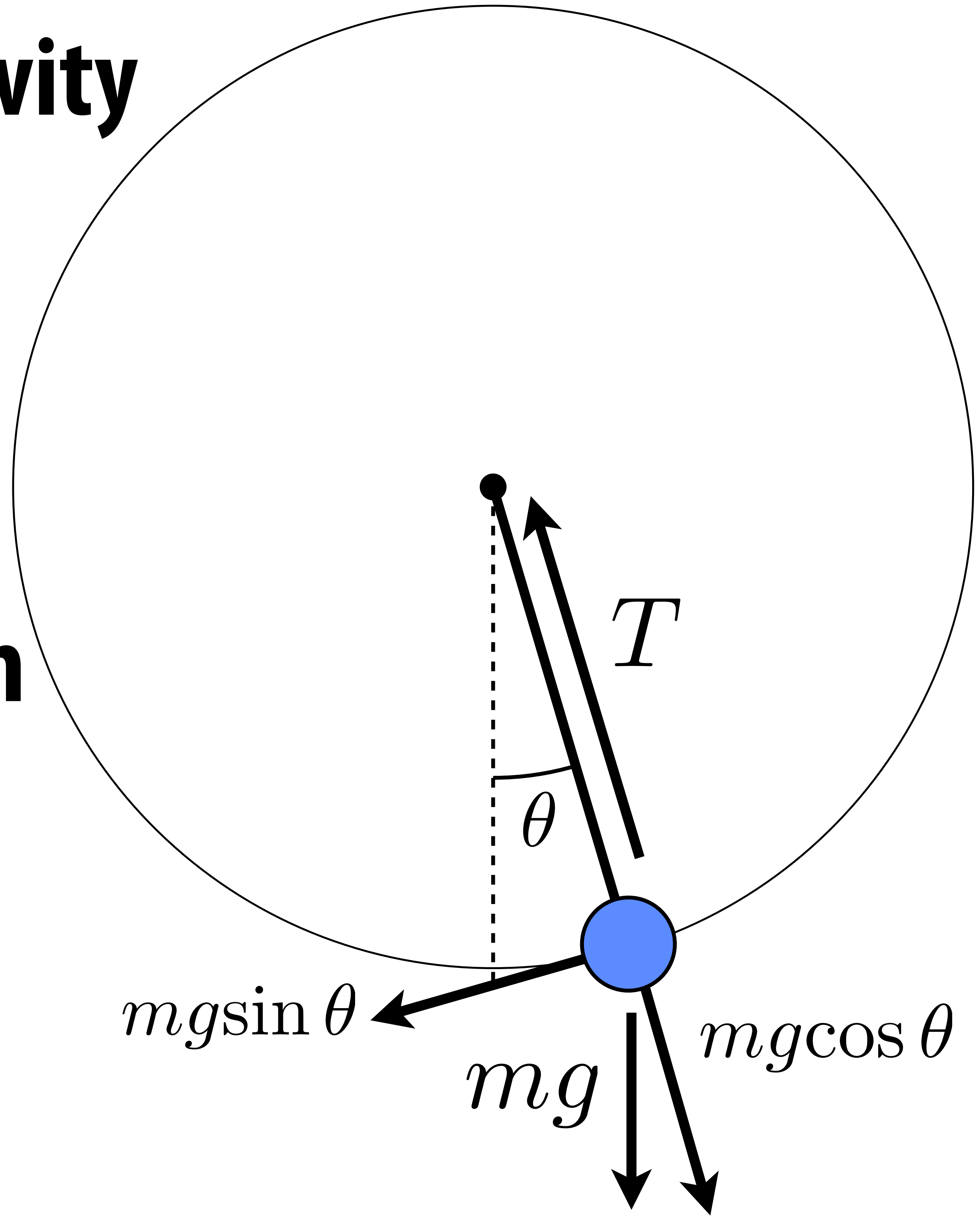
$$v(t) = v_0 + \frac{t}{m}g$$
$$q(t) = q_0 + tv_0 + \frac{t^2}{2m}g$$

(What do we need a computer for?!)

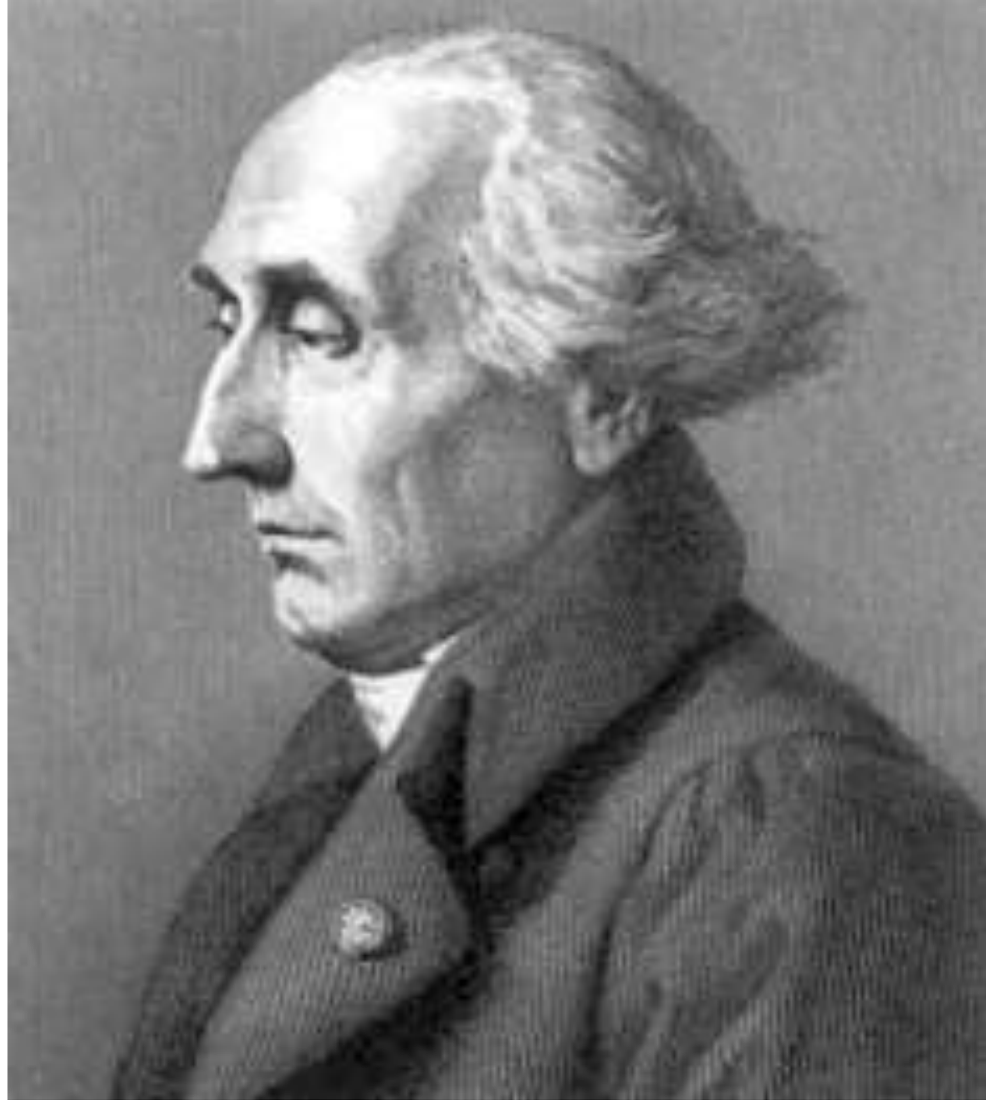
*Yes, this rock is spherical and has uniform density.

Slightly Harder Example: Pendulum

- Mass on end of a bar, swinging under gravity
- What are the equations of motion?
- Same as "rock" problem, but *constrained*
- Could use a *"force diagram"*
  - You probably did this for many hours in high school/college
  - Let's do something new & different!



Lagrangian Mechanics



Joe Lagrange

- Beautifully simple recipe:
  1. Write down kinetic energy K
  2. Write down potential energy U
  3. Write down *Lagrangian* $\mathcal{L} := K - U$
  4. Dynamics then given by *Euler-Lagrange equation*

becomes (generalized) "MASS TIMES ACCELERATION" →

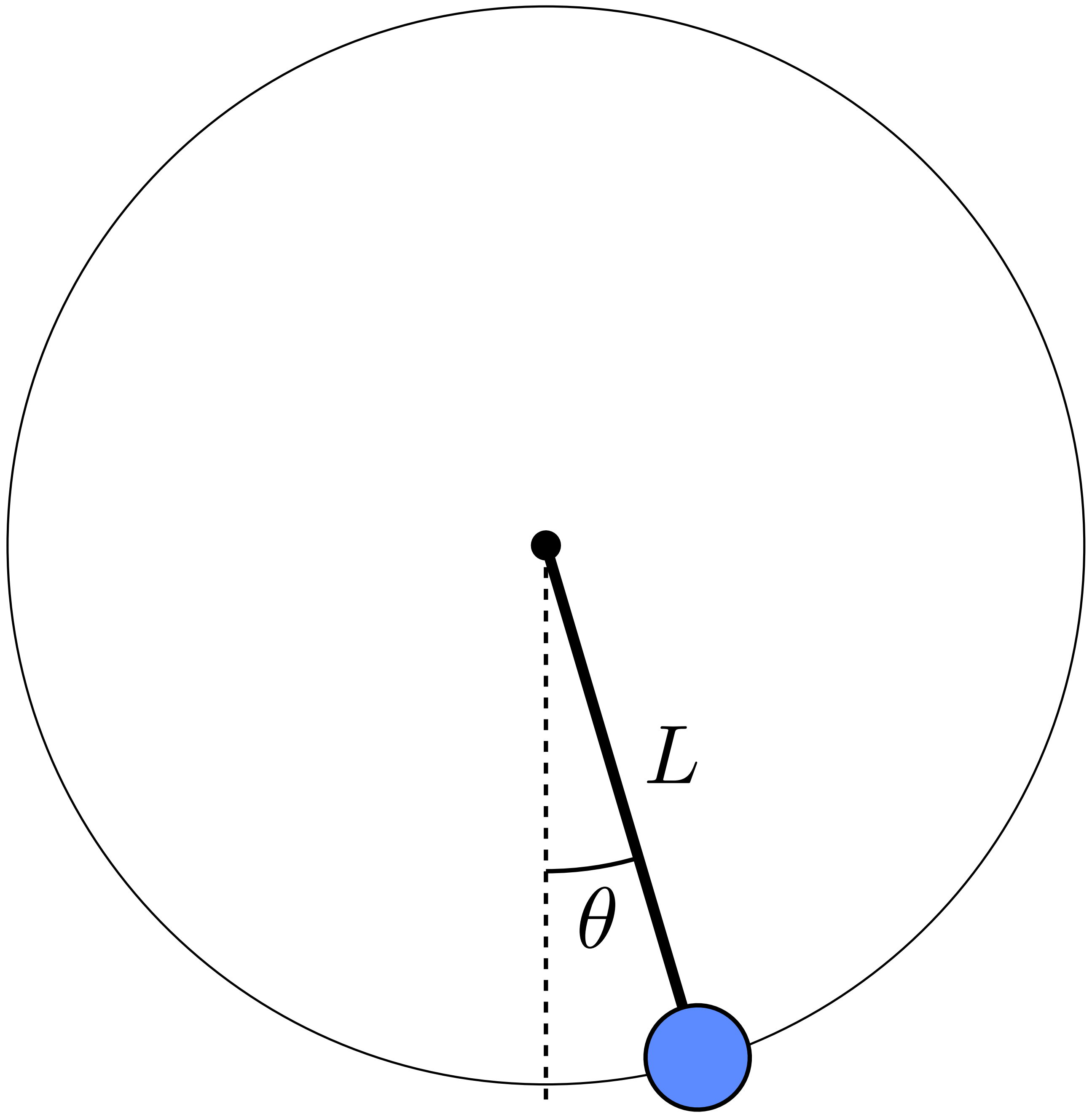
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

← becomes (generalized) "FORCE"

- Why is this useful?
  - often easier to come up with (scalar) energies than forces
  - very general, works in any kind of generalized coordinates
  - helps develop nice class of numerical integrators (symplectic)

Great reference: Sussman & Wisdom, "Structure and Interpretation of Classical Mechanics"

Lagrangian Mechanics - Example



- **Generalized coordinates for pendulum?**

$$q = \theta \longleftarrow$$

just one coordinate:
angle with the vertical direction

- **Kinetic energy (mass *m*)?**

$$K = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} m L^2 \dot{\theta}^2$$

- **Potential energy?**

$$U = mgh = -mgL\cos\theta$$

- **Euler-Lagrange equations?**

(from here, just "plug and chug"—even a computer could do it!)

$$\mathcal{L} = K - U = m(\tfrac{1}{2} L^2 \dot{\theta}^2 + gL\cos\theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{L}}{\partial \theta} = -mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \qquad \Rightarrow \qquad \boxed{\ddot{\theta} = -\frac{g}{L}\sin\theta}$$

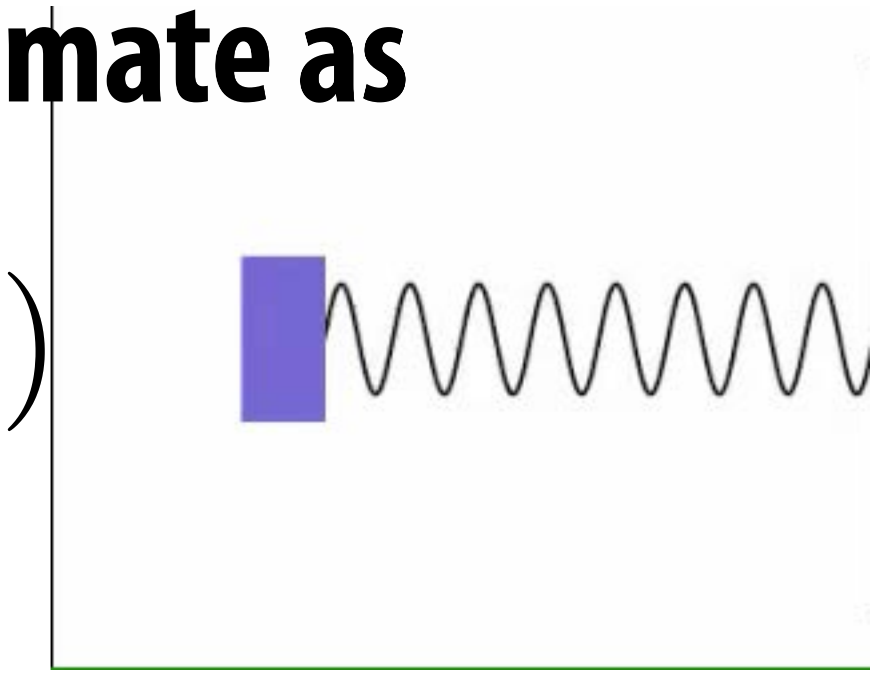
Solving the Pendulum

- Great, now we have a nice simple equation for the pendulum:

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

- For small angles (e.g., clock pendulum) can approximate as

$$\ddot{\theta} = -\frac{g}{L}\theta \qquad \Rightarrow \theta(t) = a\cos(t\sqrt{g/L} + b)$$

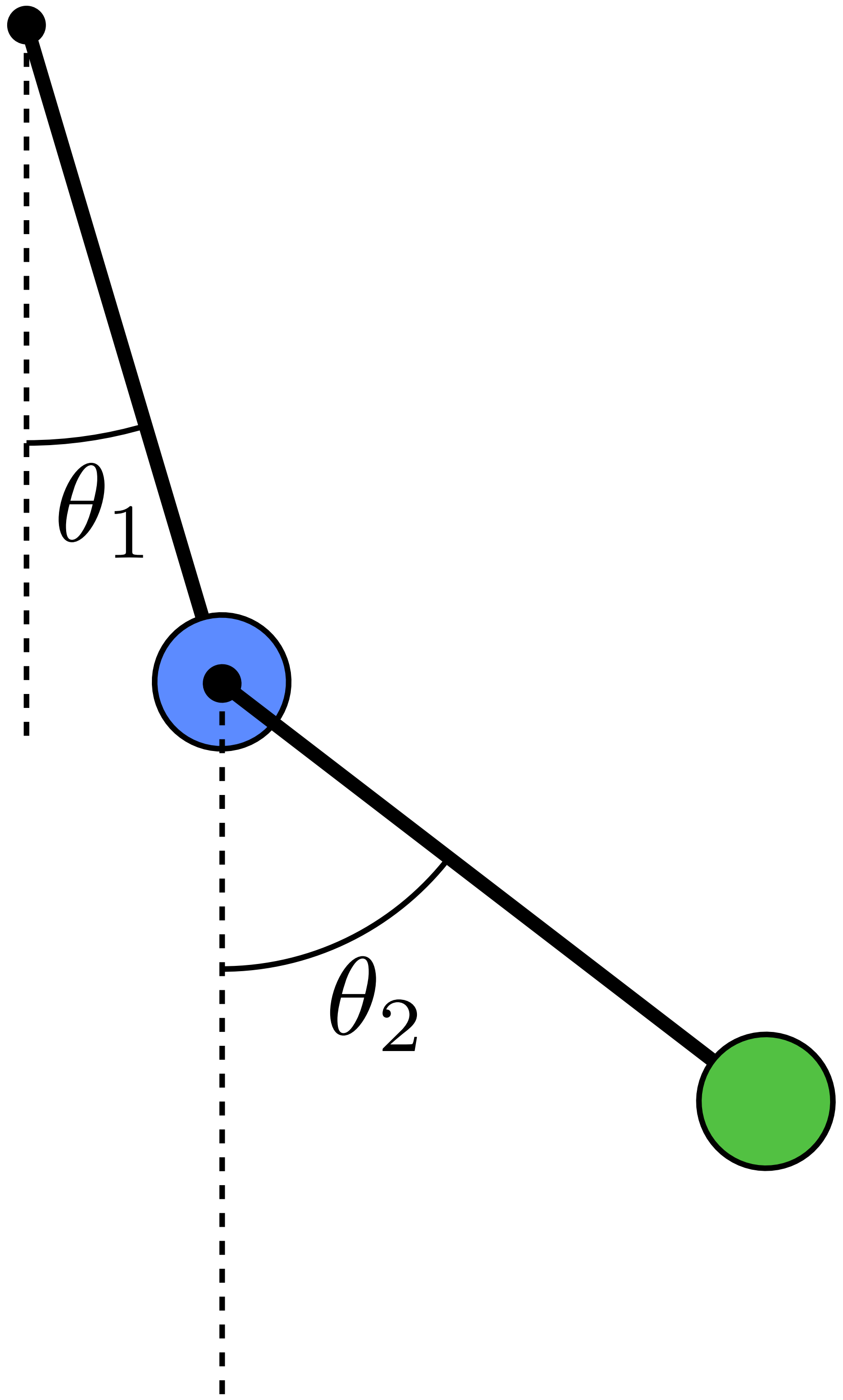


"harmonic oscillator"

- In general, there is *no closed form solution!*
- Hence, we *must* use a numerical approximation
- ...And this was (almost) the simplest system we can think of!
- (What if we want to animate something more interesting?)

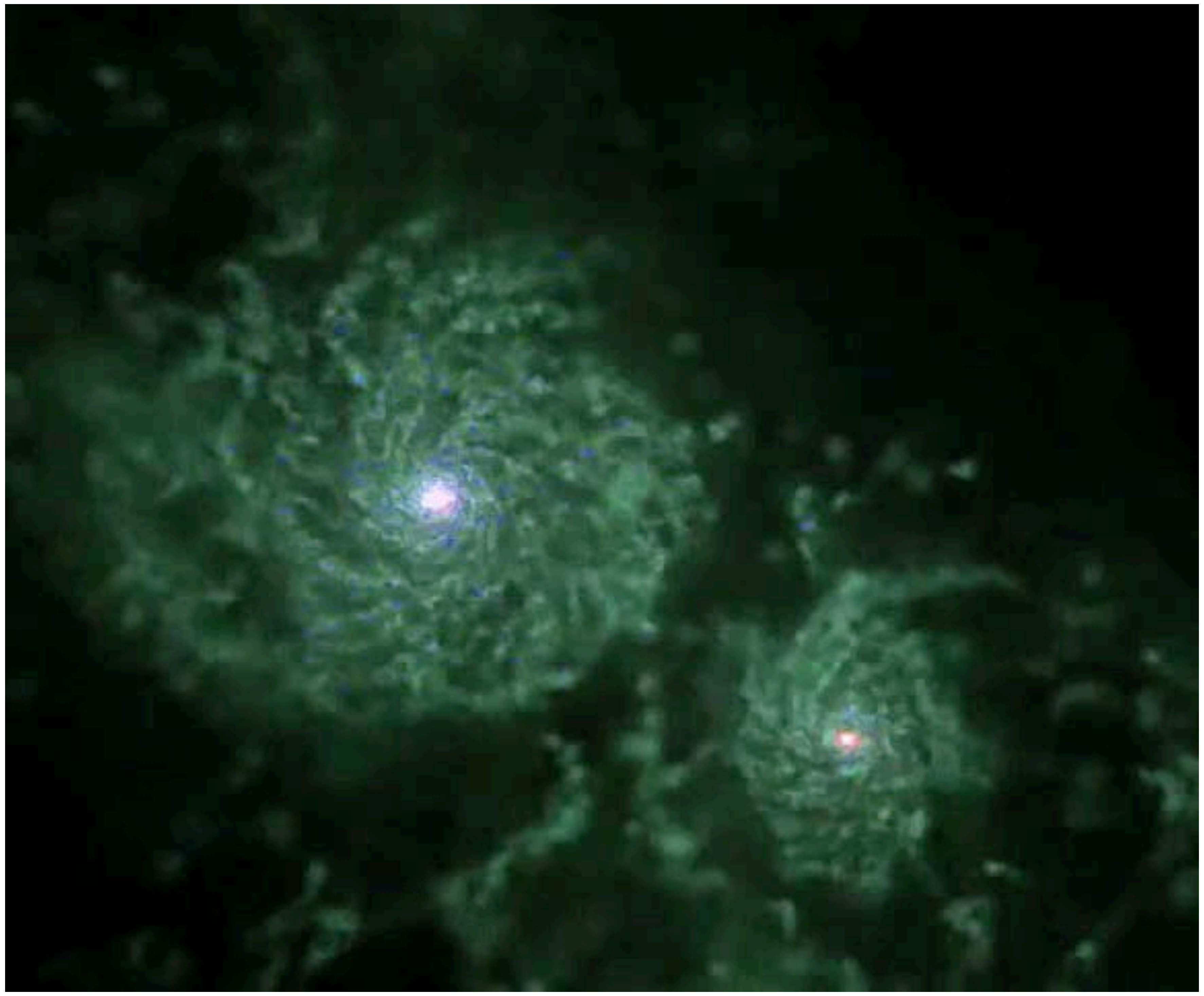
Not-So-Simple Example: Double Pendulum

- Blue ball swings from fixed point; green ball swings from blue one
- Simple system... not-so-simple motion!
- Chaotic: perturb input, wild changes to output
- Must again use numerical approximation

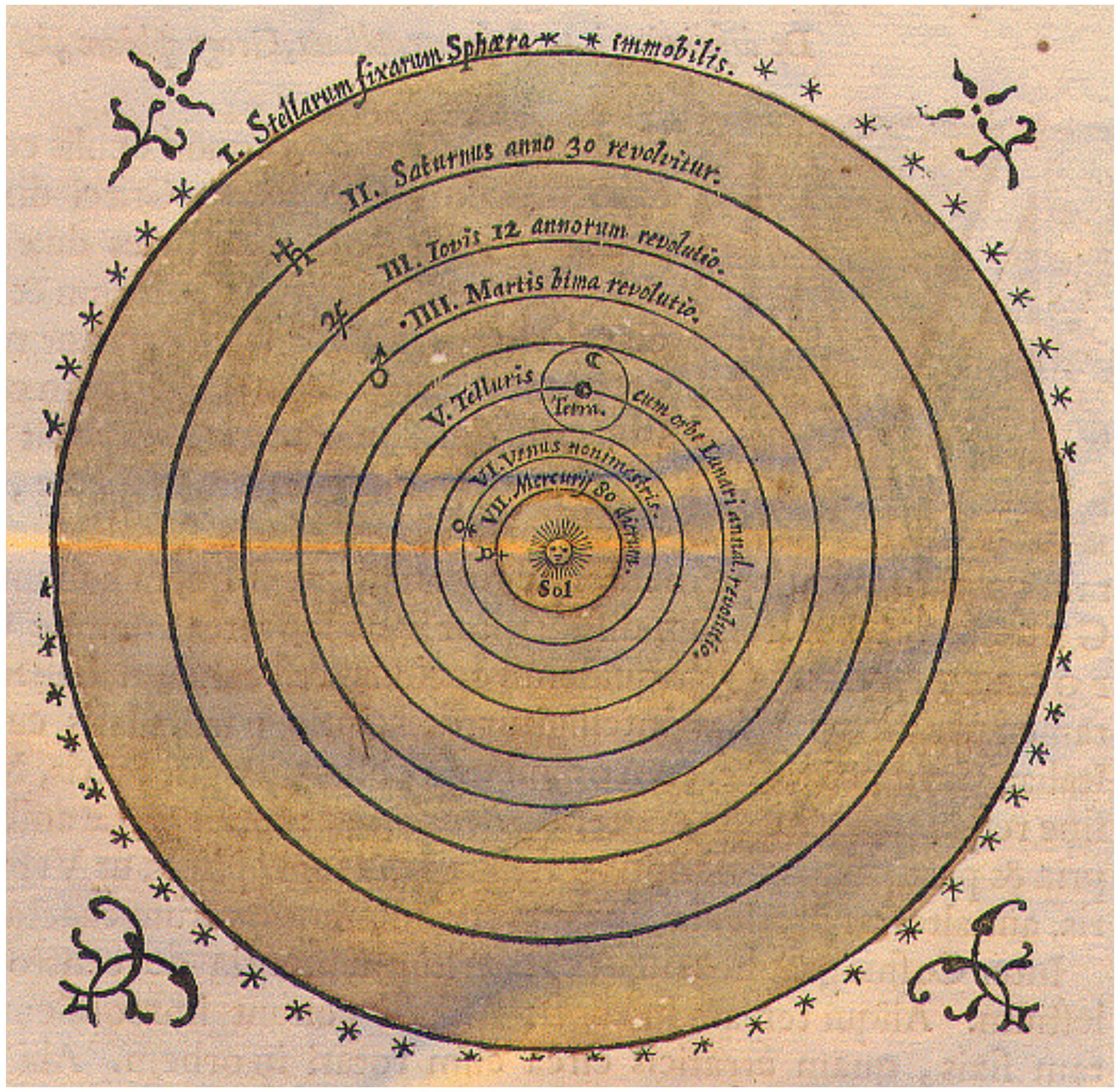


Not-So-Simple Example: *n*-Body Problem

- Consider the Earth, moon, and sun—where do they go?
- Solution is trivial for two bodies (e.g., assume one is fixed)
- As soon as n ≥ 3, again get chaotic solutions (no closed form)
- What if we want to simulate entire *galaxies?*



Credit: Governato et al / NASA

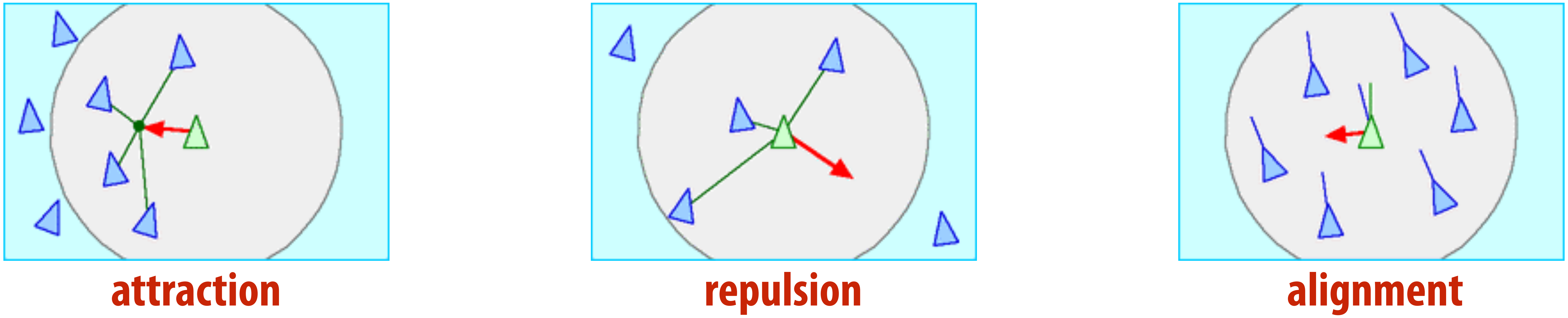


For animation, we *want* to simulate these kinds of phenomena!

Example: Flocking

Simulated Flocking as an ODE

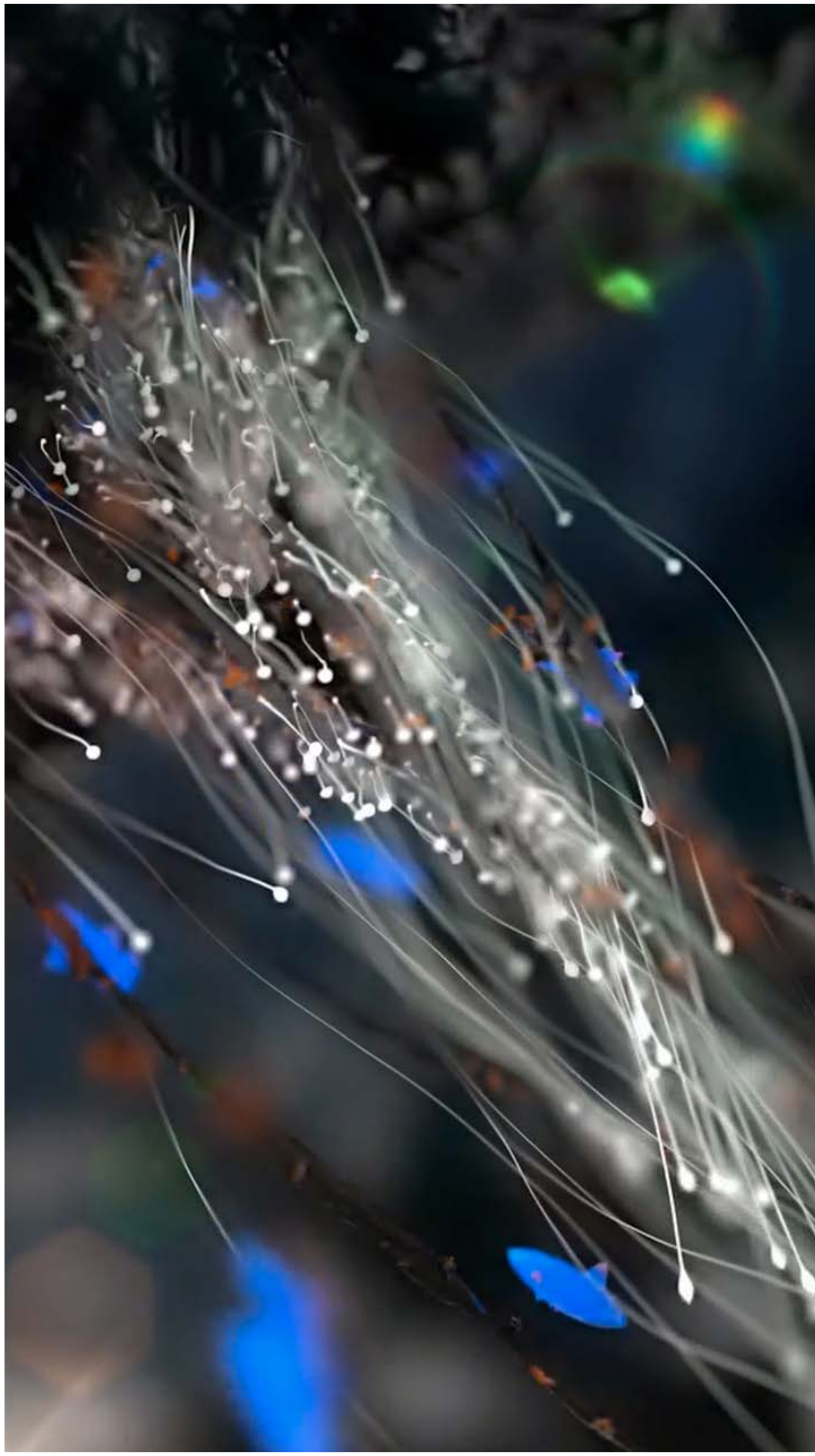
- Each bird is a particle
- Subject to very simple forces:
  - *attraction* to center of neighbors
  - *repulsion* from individual neighbors
  - *alignment* toward average trajectory of neighbors
- Solve large system of ODEs (numerically!)
- Emergent complex behavior (also seen in fish, bees, ...)



Credit: Craig Reynolds (see http://www.red3d.com/cwr/boids/)

Particle Systems

- More generally, model phenomena as large collection of particles
- Each particle has a behavior described by (physical or *non*-physical) forces
- Extremely common in graphics/games
  - easy to understand
  - simple equation for each particle
  - easy to scale up/down
- May need *many* particles to capture certain phenomena (e.g., fluids)
  - may require fast hierarchical data structure (kd-tree, BVH, ...)
  - often better to use continuum model



Example: Crowds

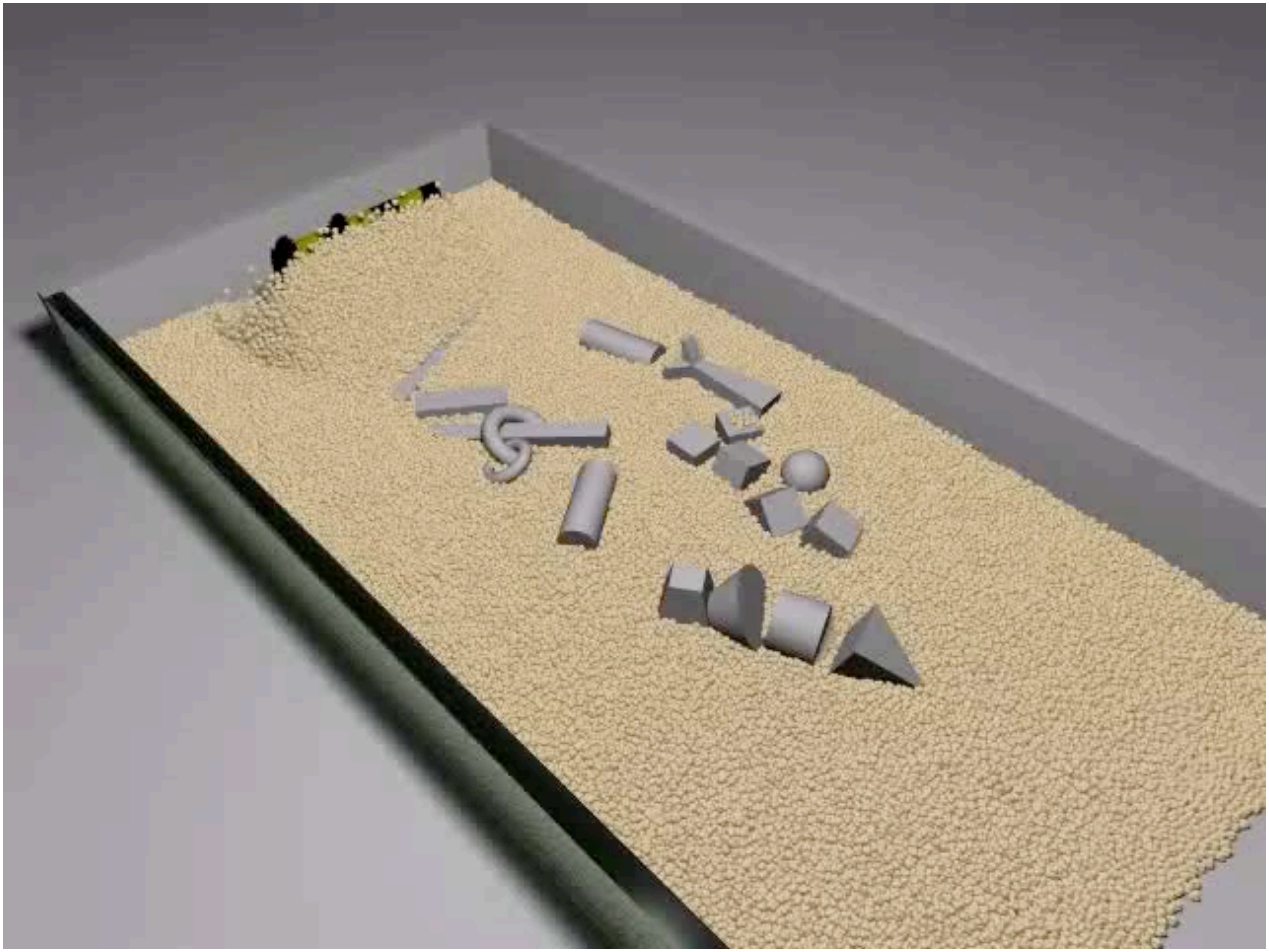
Where are the bottlenecks in a building plan?

Example: Crowds + “Rock” Dynamics

Example: Particle-Based Fluids

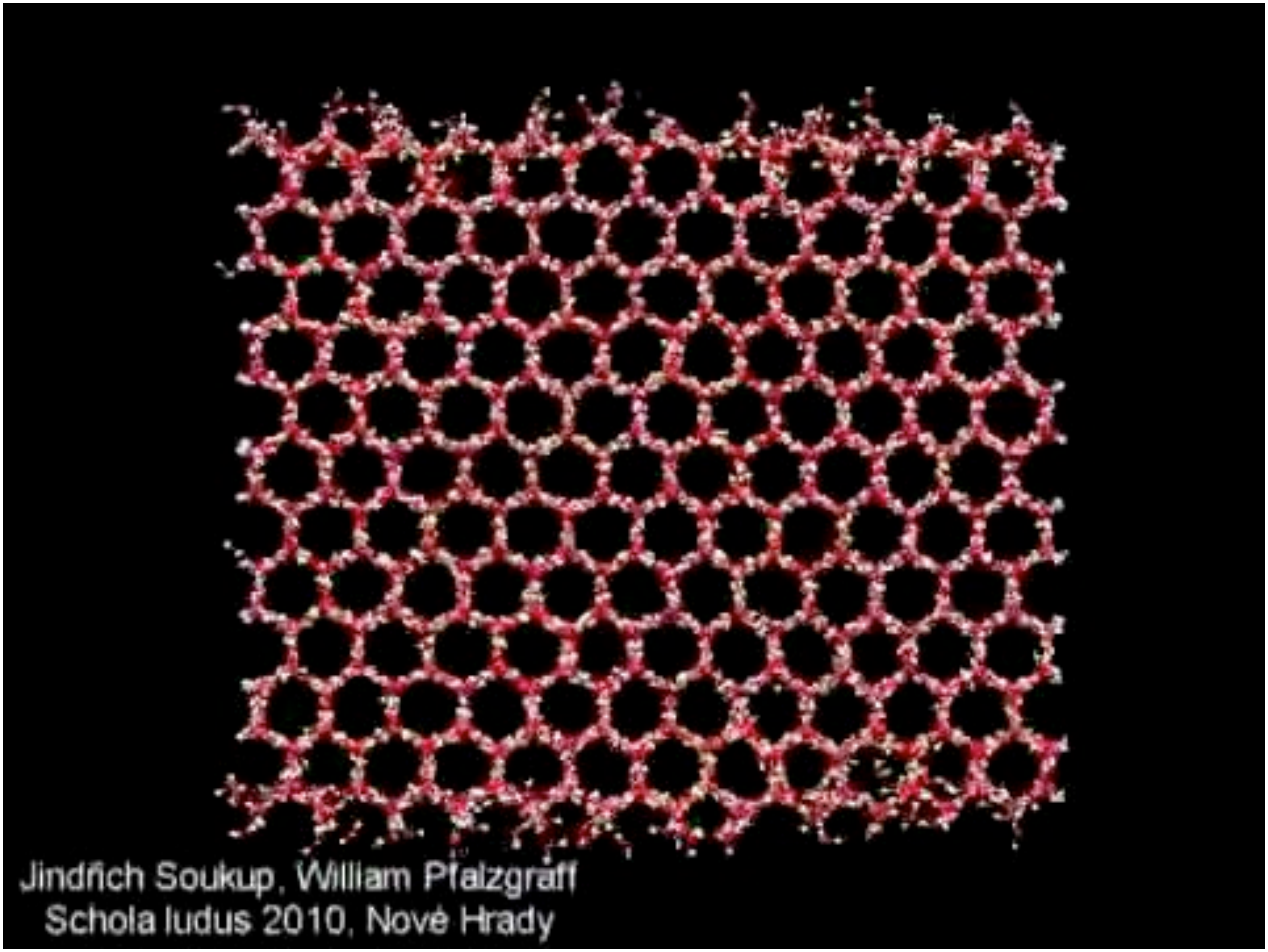
(Fluid: particles or continuum?)

Example: Granular Materials



Bell et al, “Particle-Based Simulation of Granular Materials”

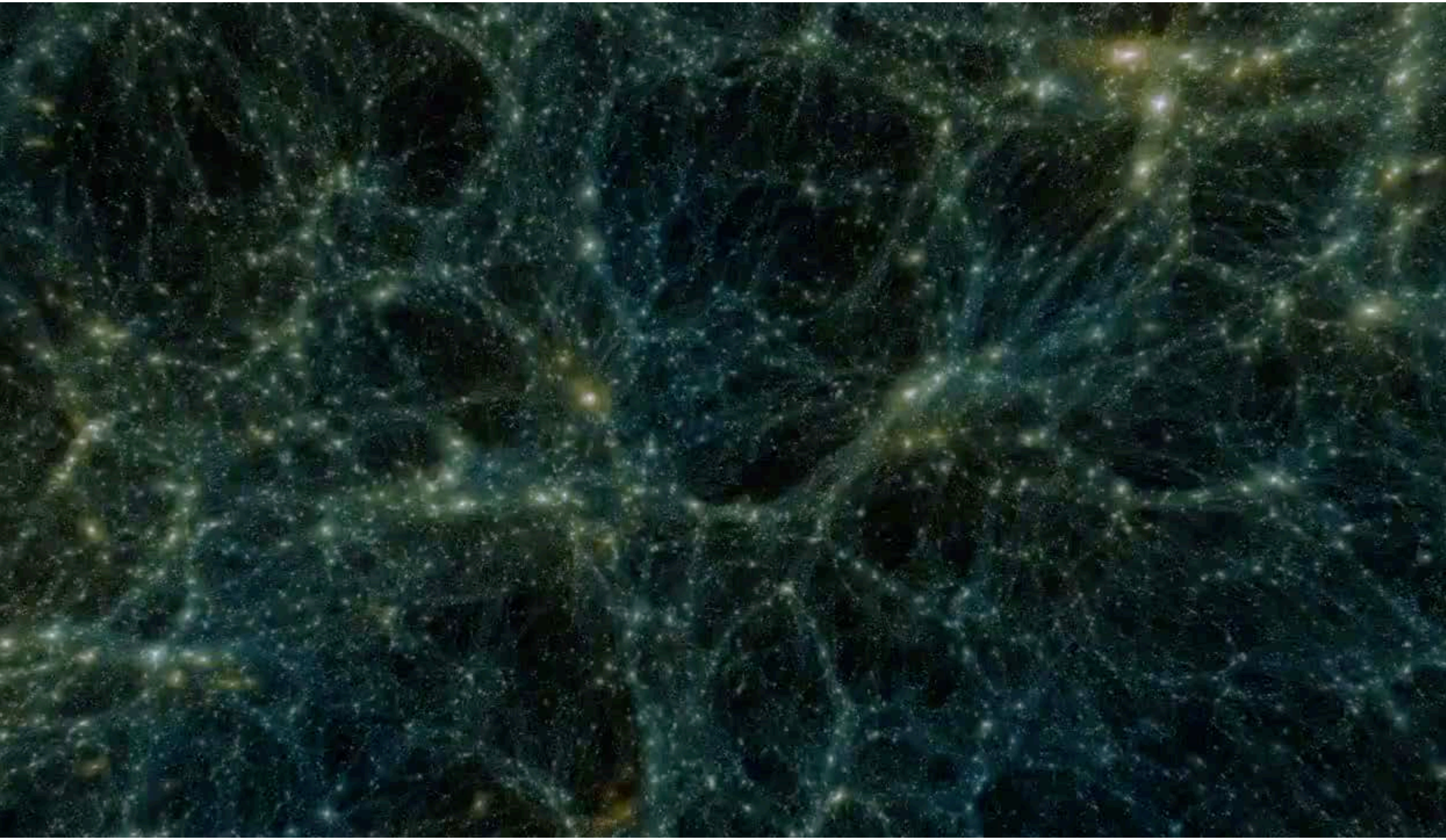
Example: Molecular Dynamics



Jindřich Soukup, William Pfalzgraff
Schola ludus 2010, Nové Hrady

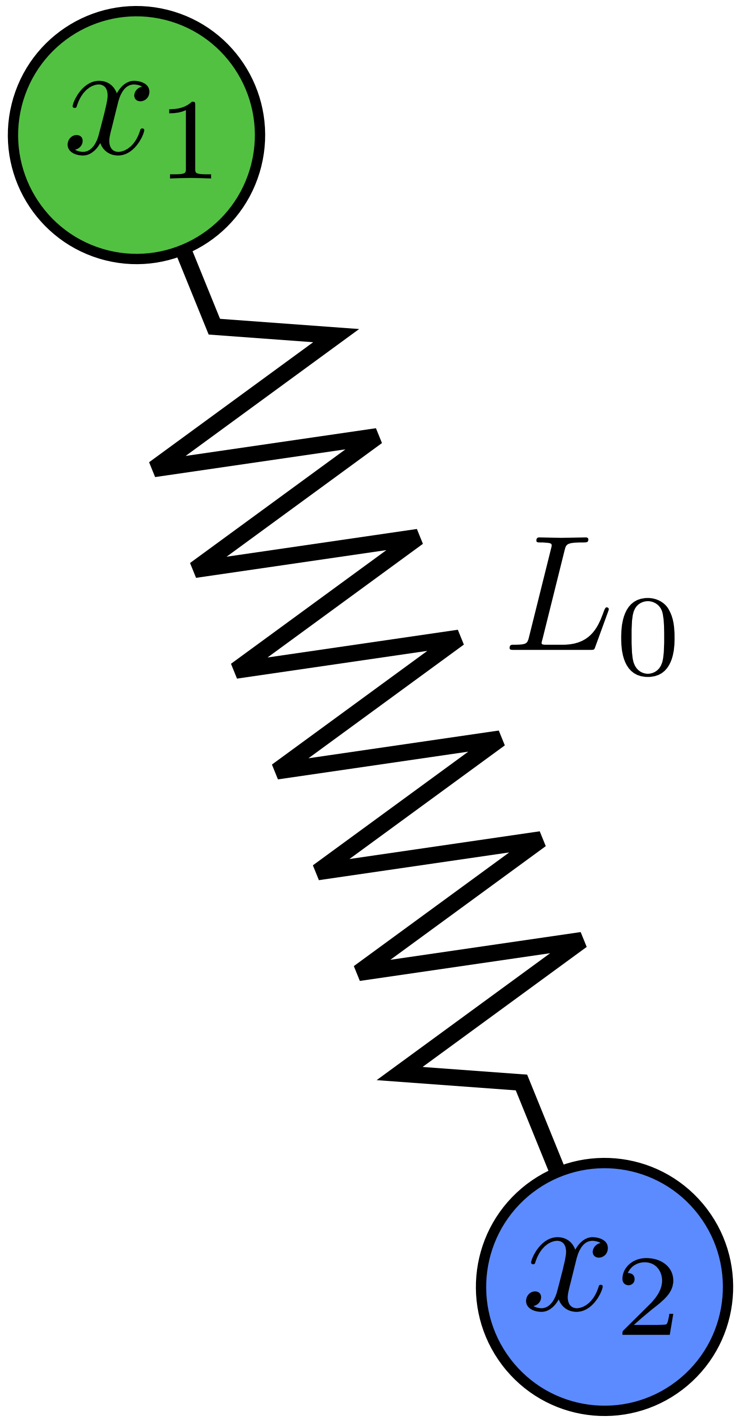
(model of melting ice crystal)

Example: Cosmological Simulation



Tomoaki et al - v^2GC simulation of dark matter (~1 *trillion* particles)

Example: Mass-Spring System



- Connect particles x_1, x_2 by a spring of length L_0
- Potential energy is given by

rest length

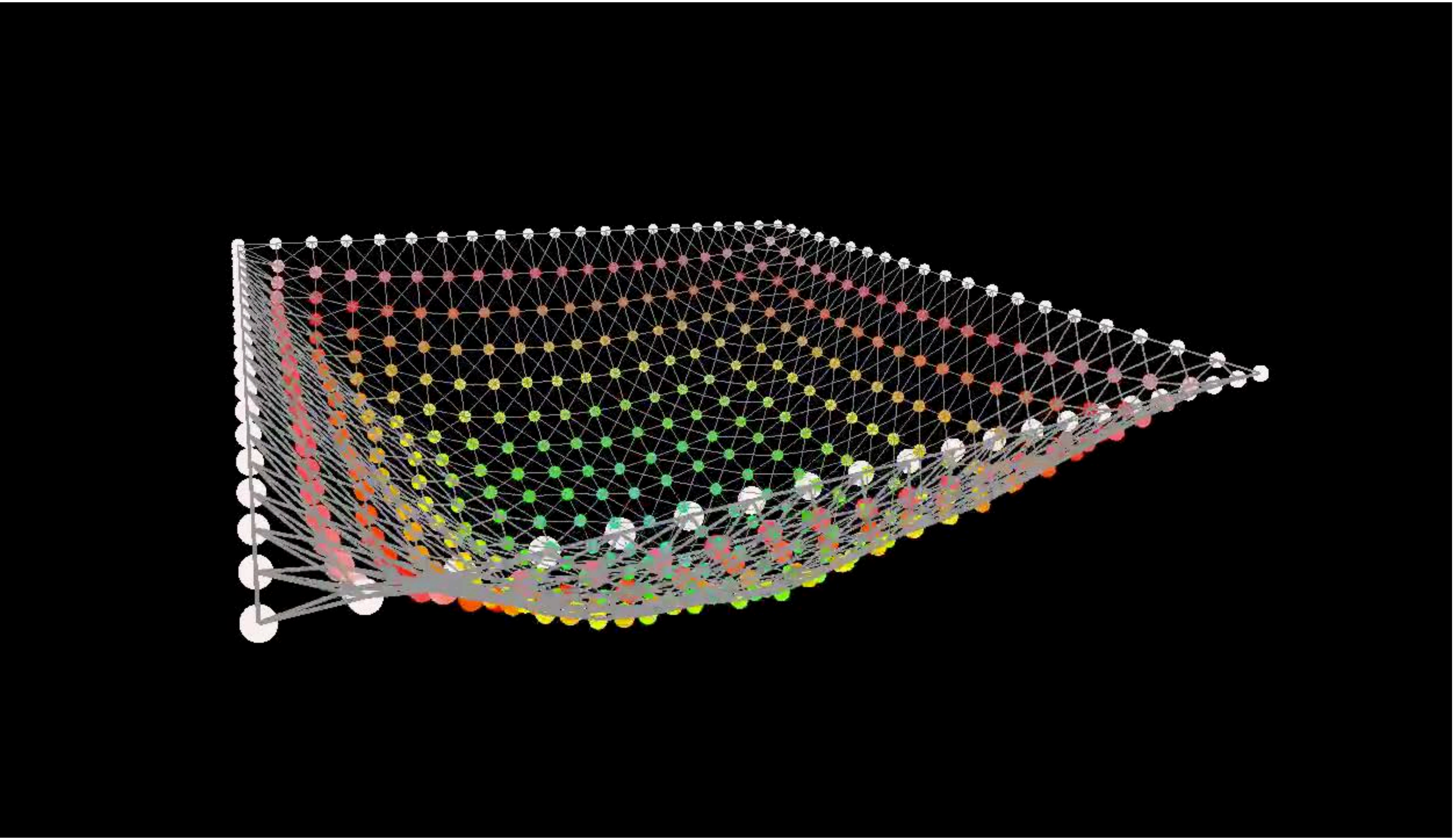
$$U = \tfrac{1}{2}k(L - L_0)^2$$

stiffness current length

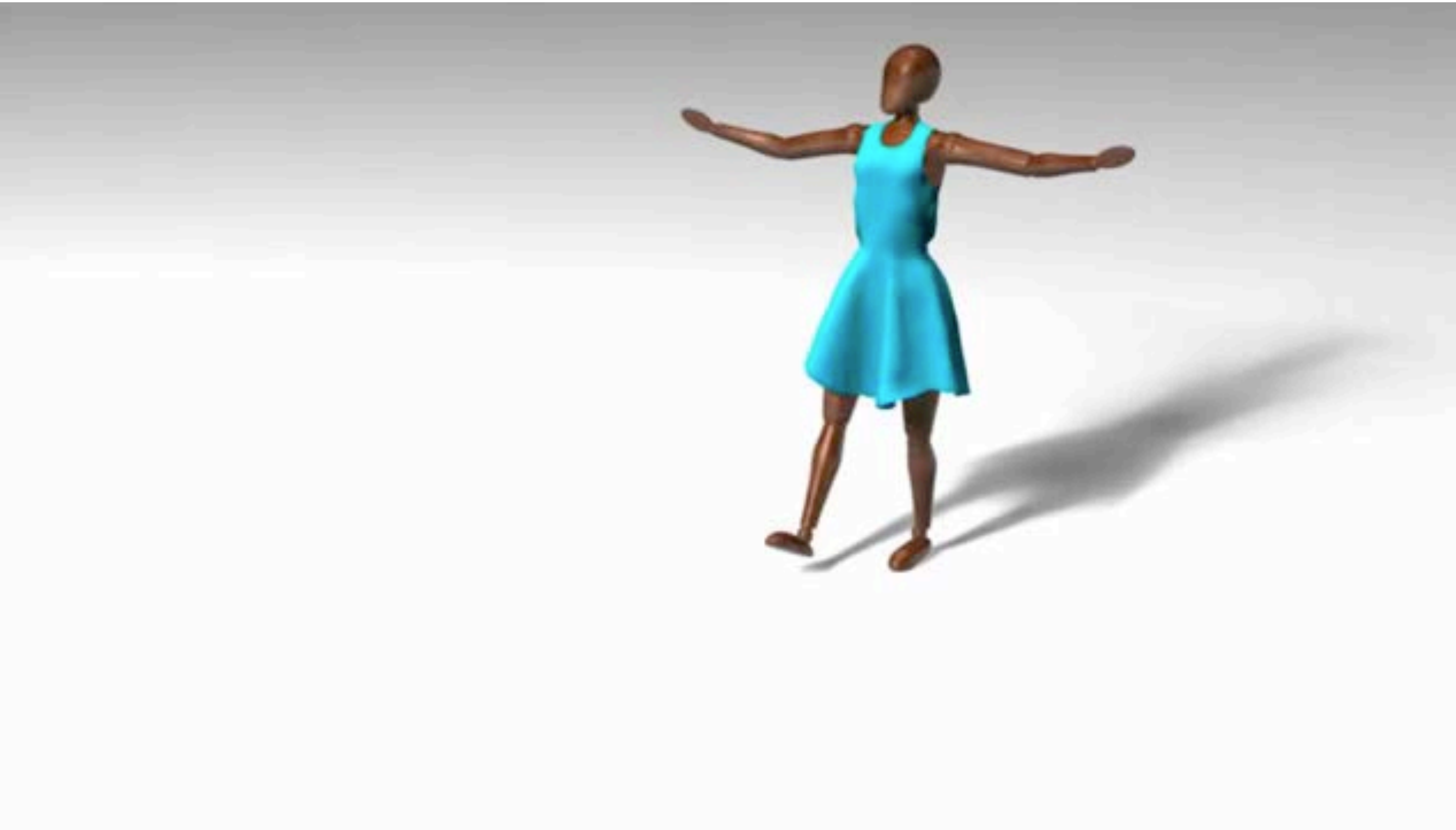
$$= \tfrac{1}{2}k(|x_1 - x_2|^2 - L_0)^2$$

- Connect up many springs to describe interesting phenomena
- Extremely common in graphics/games
  - easy to understand
  - simple equation for each particle
- Often good reasons for using continuum model (PDE)

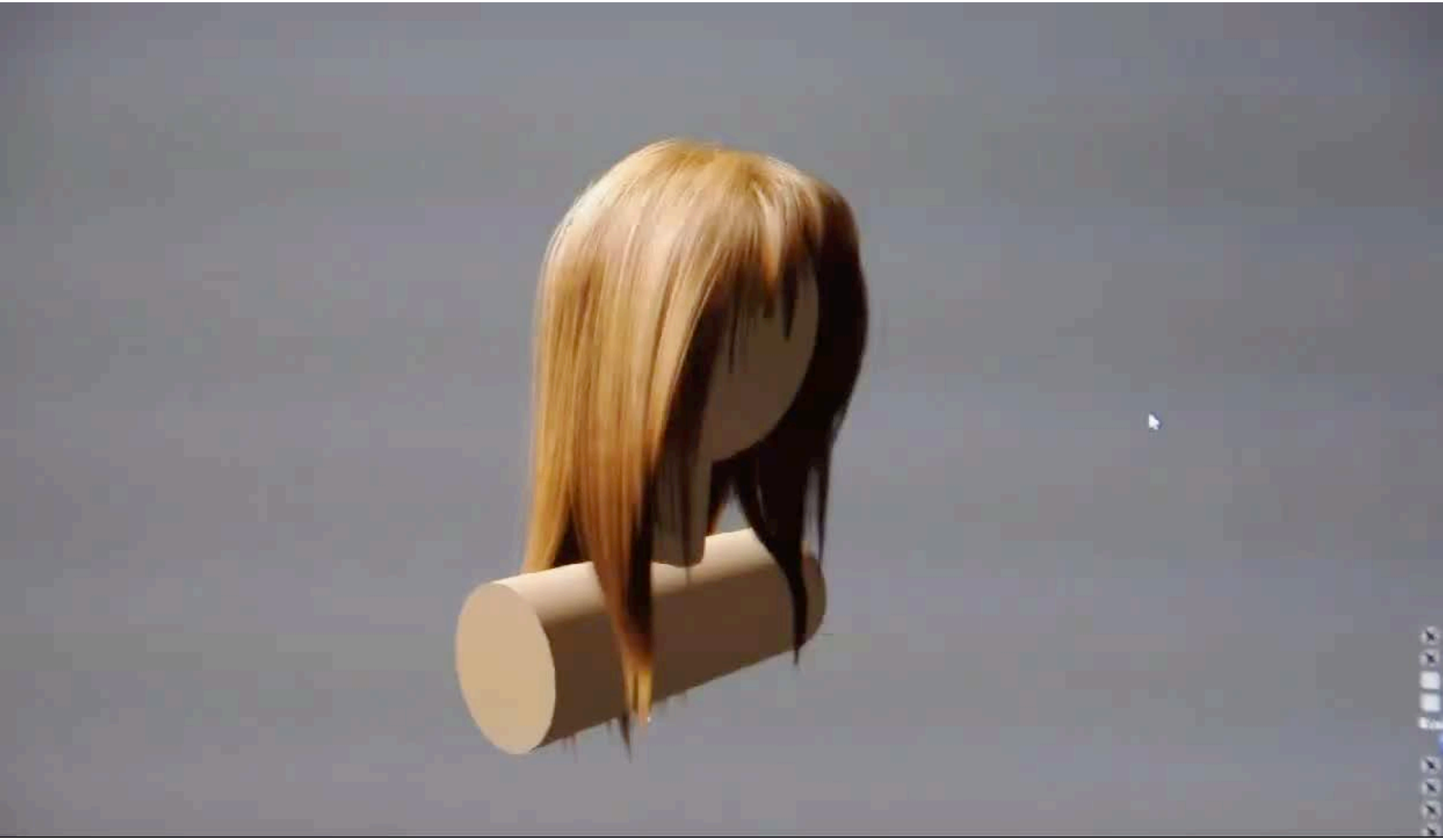
Example: Mass Spring System



Example: Mass Spring + Character



Example: Hair



Ok, I’m convinced. So how do we solve these things numerically?

Numerical Integration

- Key idea: replace *derivatives* with *differences*
- In ODE, only need to worry about derivative in *time*
- Replace time-continuous function q(t) with samples q_k in time

$$\frac{d}{dt}q(t) = f(q(t))$$

$$\Downarrow$$

new configuration
(unknown—want to solve for this!)

current configuration
(known)

$$\frac{q_{k+1} - q_k}{\tau} = f(q)$$

“time step,” i.e., interval of
time between q_k and q_{k+1}

Wait... where do we
evaluate the velocity
function? At the new
or old configuration?

Forward Euler

- **Simplest scheme: evaluate velocity at current configuration**
- **New configuration can then be written *explicitly* in terms of known data:**

new configuration

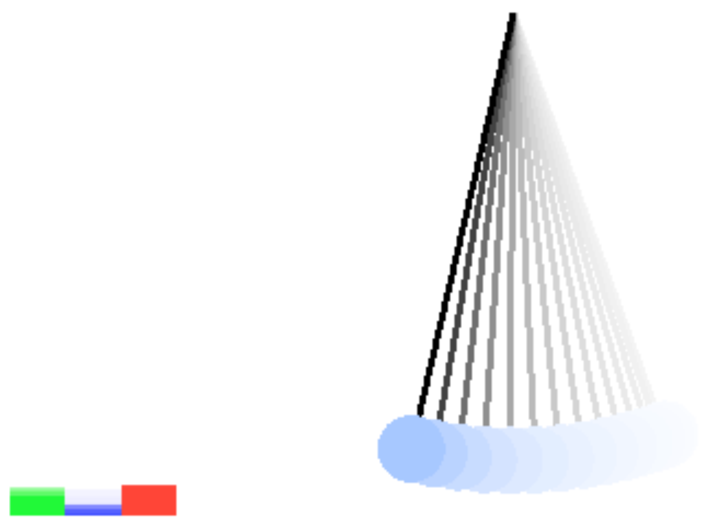
current configuration

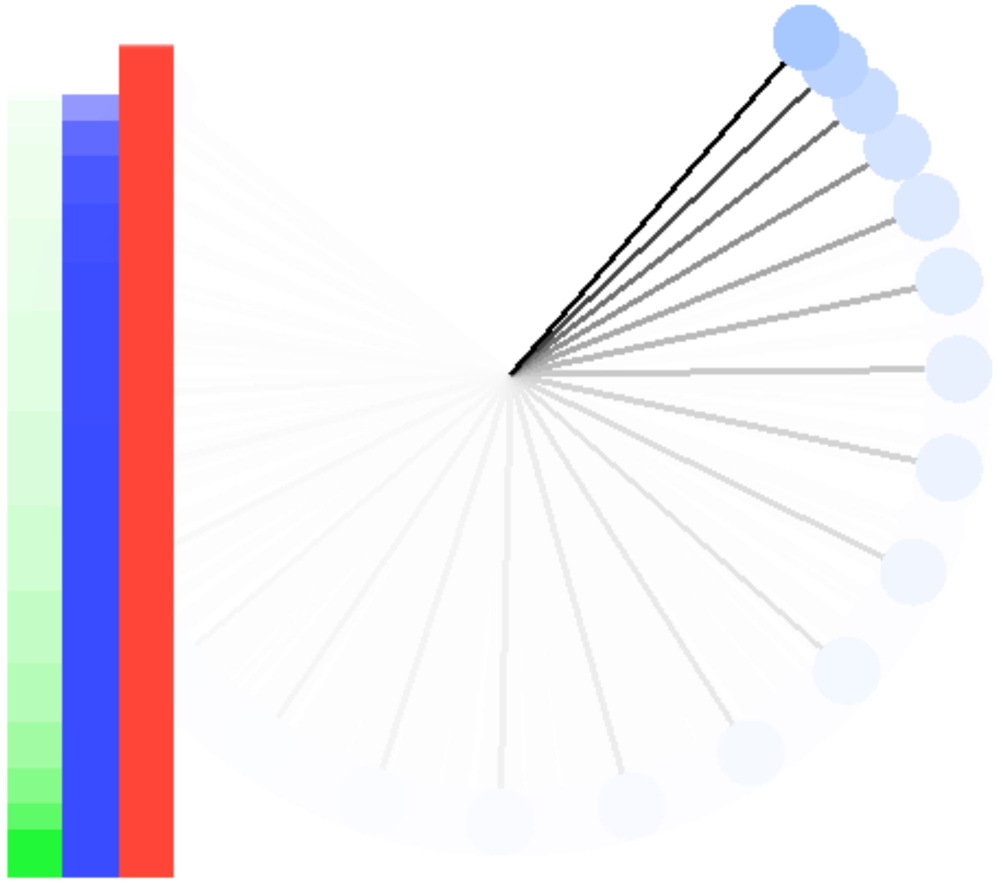
velocity at current time

$$q_{k+1} = q_k + \tau f(q_k)$$

- **Very intuitive: walk a tiny bit in the direction of the velocity**
- **Unfortunately, not very *stable*—consider pendulum:**

starts out slow...





...gradually moves faster & faster!

Where did all this extra energy come from?

Forward Euler - Stability Analysis

- **Let's consider behavior of forward Euler for simple linear ODE:**

$$\dot{u} = -au, \quad a > 0$$

- **Importantly: *u* should *decay* (exact solution is *u*(*t*)=e^{-at})**
- **Forward Euler approximation is**

$$\begin{aligned} u_{k+1} &= u_k - \tau a u_k \\ &= (1 - \tau a) u_k \end{aligned}$$

- **Which means after n steps, we have**

$$u_n = (1 - \tau a)^n u_0$$

- **Decays only if |1-τa| < 1, or equivalently, if τ < 2/a**
- **In practice: need *very small* time steps if a is large ("stiff system")**

Backward Euler

- Let's try something else: evaluate velocity at *next* configuration
- New configuration is then *implicit*, and we must solve for it:

new configuration

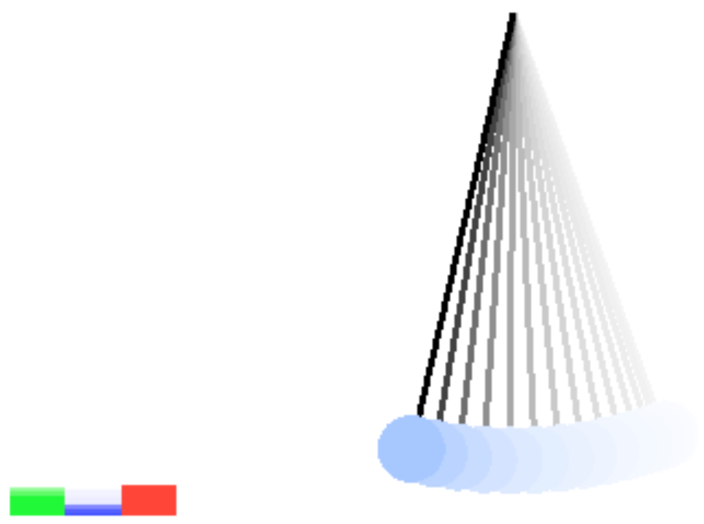
current configuration

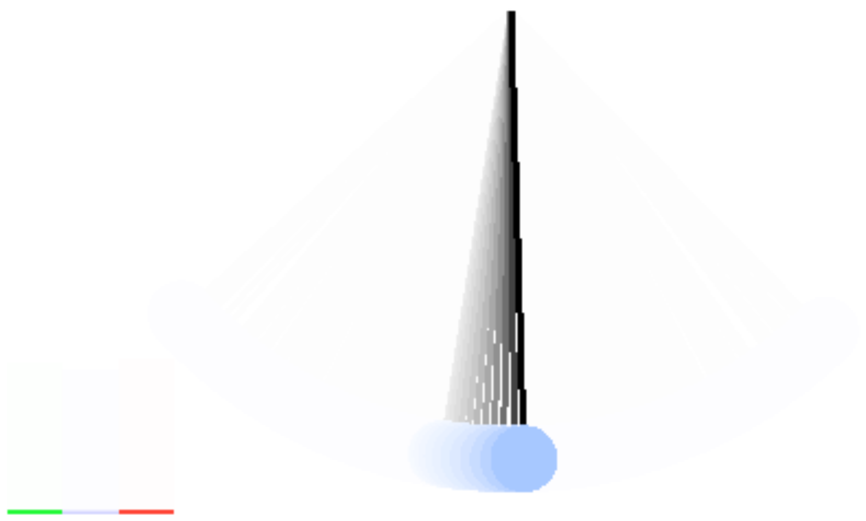
velocity at next time

$$q_{k+1} = q_k + \tau f(q_{k+1})$$

- Much harder to solve, since in general f can be very nonlinear!
- Pendulum is now stable... perhaps *too* stable?

starts out slow...





Where did all the energy go?

...and eventually stops moving completely.

Backward Euler - Stability Analysis

- **Again consider a simple linear ODE:**

$$\dot{u} = -au, \quad a > 0$$

- **Remember: *u* should *decay* (exact solution is *u*(*t*)=e^{-at})**
- **Backward Euler approximation is**

$$(u_{k+1} - u_k)/\tau = -au_{k+1}$$

$$\iff \quad u_{k+1} = \tfrac{1}{1+\tau a} u_k$$

- **Which means after n steps, we have**

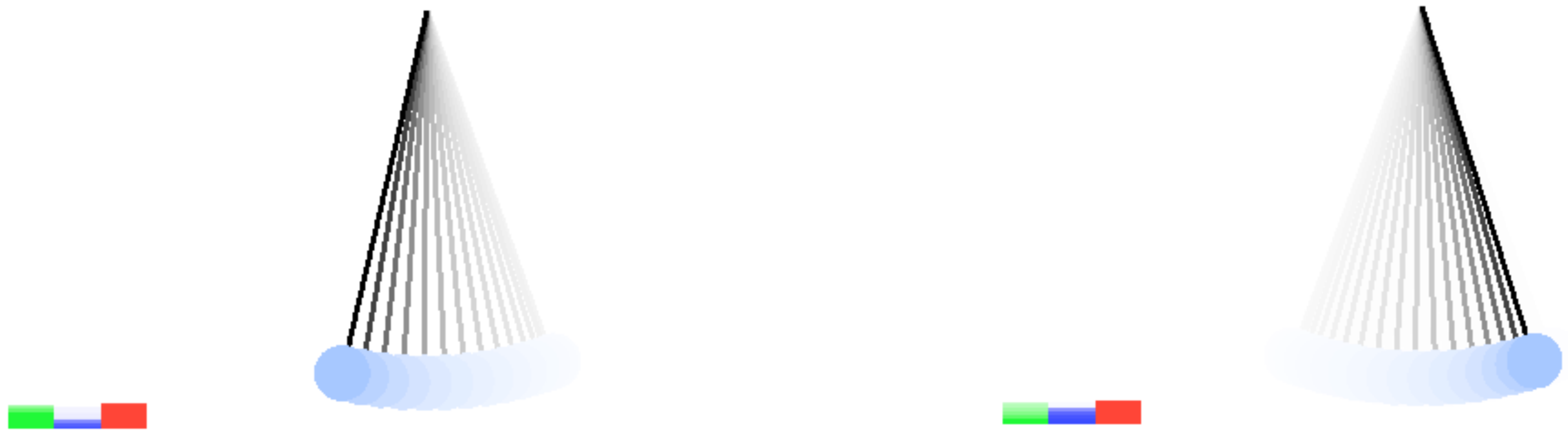
$$u_n = \left(\frac{1}{1+\tau a}\right)^n u_0$$

- **Decays if |1+τa| > 1, which is always true!**
- **⇒Backward Euler is *unconditionally stable* for linear ODEs**

Symplectic Euler

- Backward Euler was stable, but we also saw (empirically) that it exibits *numerical damping* (damping not found in original eqn.)
- Nice alternative is symplectic Euler
  - update velocity using current configuration
  - update configuration using *new* velocity
- Easy to implement; used often in practice (or leapfrog, Verlet, ...)
- Pendulum now conserves energy *almost exactly*, forever:

starts out slow...

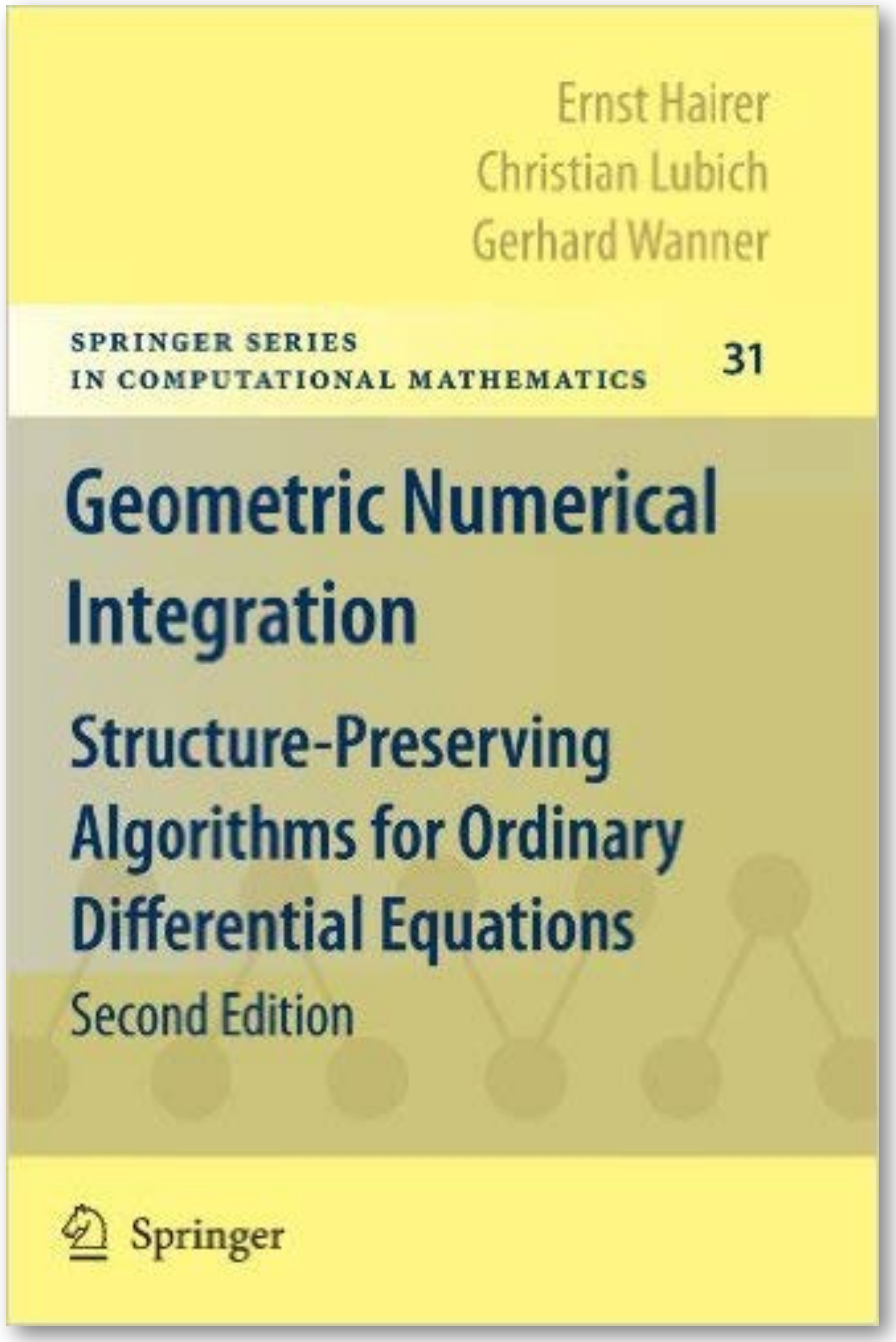


...and keeps on ticking.

(Proof? The analysis is not quite as easy...)

Numerical Integrators

- Barely scratched the surface
- *Many* different integrators
- Why? Because many notions of “good”:
    - stability
    - accuracy
    - consistency/convergence
    - conservation, symmetry, ...
    - computational efficiency (!)
- No one “best” integrator—*pick the right tool for the job!*
- Could do (at least) an entire course on time integration...
- Great book: Hairer, Lubich, Wanner



Ernst Hairer
Christian Lubich
Gerhard Wanner

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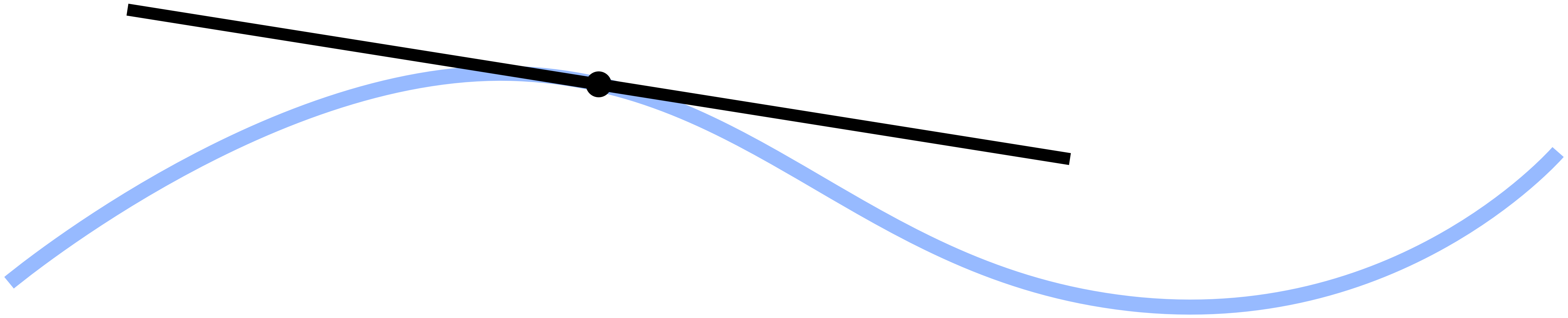
Second Edition

Springer

Computational Differentiation

- So far, we've been taking derivatives by hand
- Very often in simulation, need to differentiate *extremely complicated* functions (e.g., potential energy, to get forces)
- Several different techniques:
  - **keep doing it by hand!** (laborious & error prone, but potentially fast)
  - **numerical differentiation** (simple to code, but usually poor accuracy)
  - **automatic differentiation** (bigger code investment, better accuracy)
  - **symbolic differentiation** (can help w/ "by-hand", often messy results)
  - **geometric differentiation** (sometimes simplifies "by hand" expressions)



Review: Derivatives

- **Suppose I have a function** $f : \mathbb{R} \to \mathbb{R}; x \mapsto f(x)$
- **Q: How do I define its first derivative with respect to *x*, at *x*$_0$?**

$$f'(x_0) := \lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon) - f(x_0)}{\epsilon}$$

- **In dynamical simulation, often need to consider functions**

$$f : \mathbb{R}^n \to \mathbb{R}; q \mapsto f(q)$$

(e.g., potential)

- ***Directional derivative* looks a lot like ordinary derivative:**

$$D_X f(q_0) := \lim_{\epsilon \to 0} \frac{f(q_0 + \epsilon X) - f(q_0)}{\epsilon}$$

(Q: is D$_X$f vector or scalar?)

- ***Gradient* is vector ∇*f* that yields D$_X$f when you take inner product:**

$$\langle \nabla f(q_0), X \rangle = D_X f(q_0)$$

(e.g., gradient of potential is force)

Numerical Differentiation

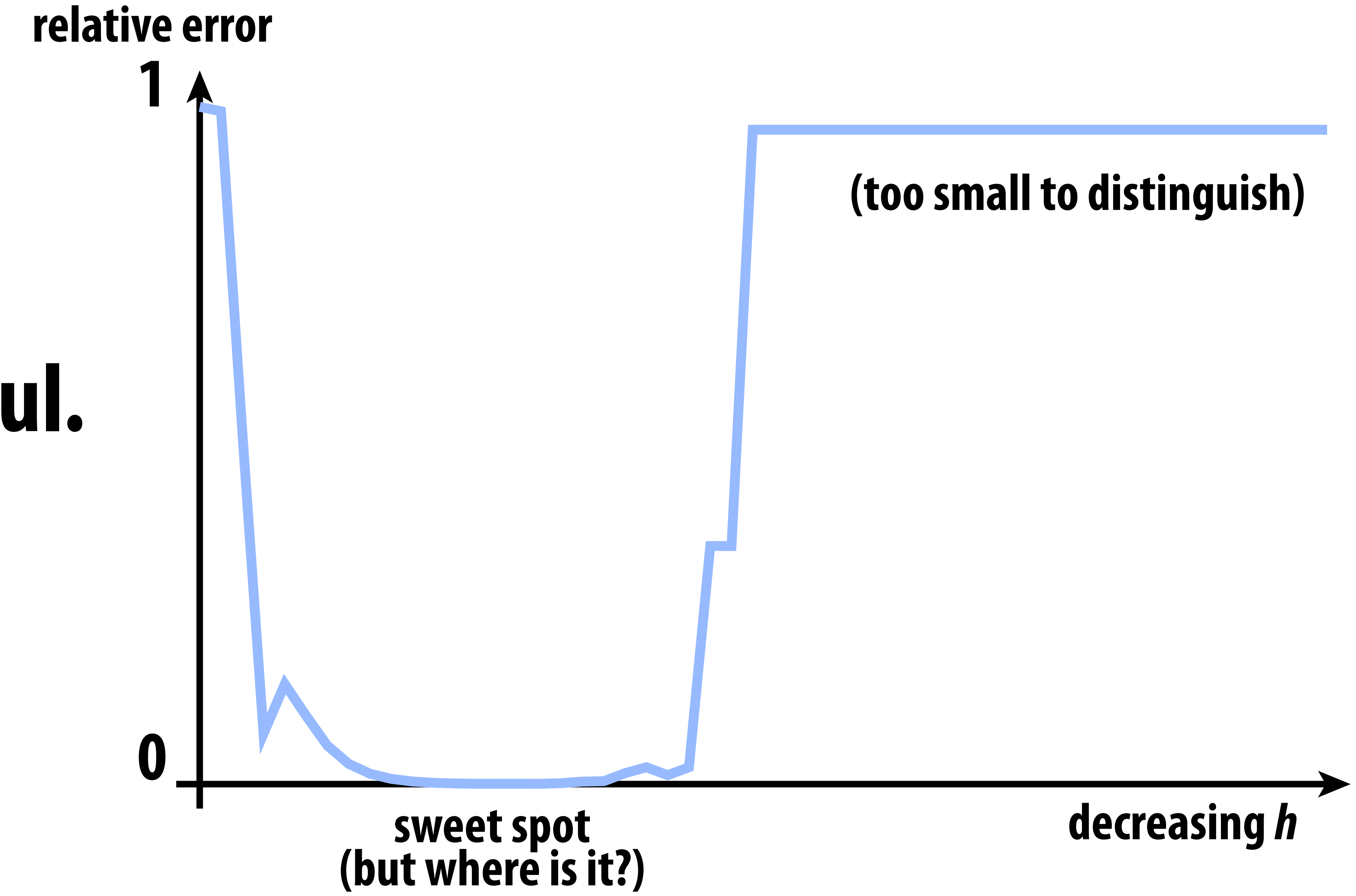
- Taking all those derivatives by hand is a lot of work! (Especially if you're just developing/debugging)
- Idea: replace *derivatives* with *differences* (as we did w/ time):

$$f'(x_0) \quad \Rightarrow \quad \frac{f(x_0+h)-f(x_0)}{h}$$

now has *fixed* size

- But how do we pick *h*?
- Smaller is better… right?
- Not always! Must be careful.
- Can also be *expensive!*

e.g., what if *f* were some kind of radiance integral?



Automatic Differentiation

- Completely different idea: do arithmetic simultaneously on a function *and* its derivative.
- I.e., rather than work with values *f*, work with tuples (*f*,*f′*)
- Use *chain rule* to determine rules for manipulating tuples
- Example function: $f(x) = ax^2$
- Suppose we want the value and derivative at *x*=2
- Start with the tuple $(x, \frac{\partial}{\partial x}x)\,|_{x=2} = (2, 1)$
- How do we *multiply* tuples? $(u, u') * (v, v') = (\underset{\text{values just get multiplied}}{uv}, \underset{\text{for derivatives, we apply the chain rule}}{uv' + vu'})$
- So, squaring our tuple yields $(2, 1) * (2, 1) = (4, 4)$
- And multiplying by *a* scales the value *and* derivative: $(4a, 4a)$

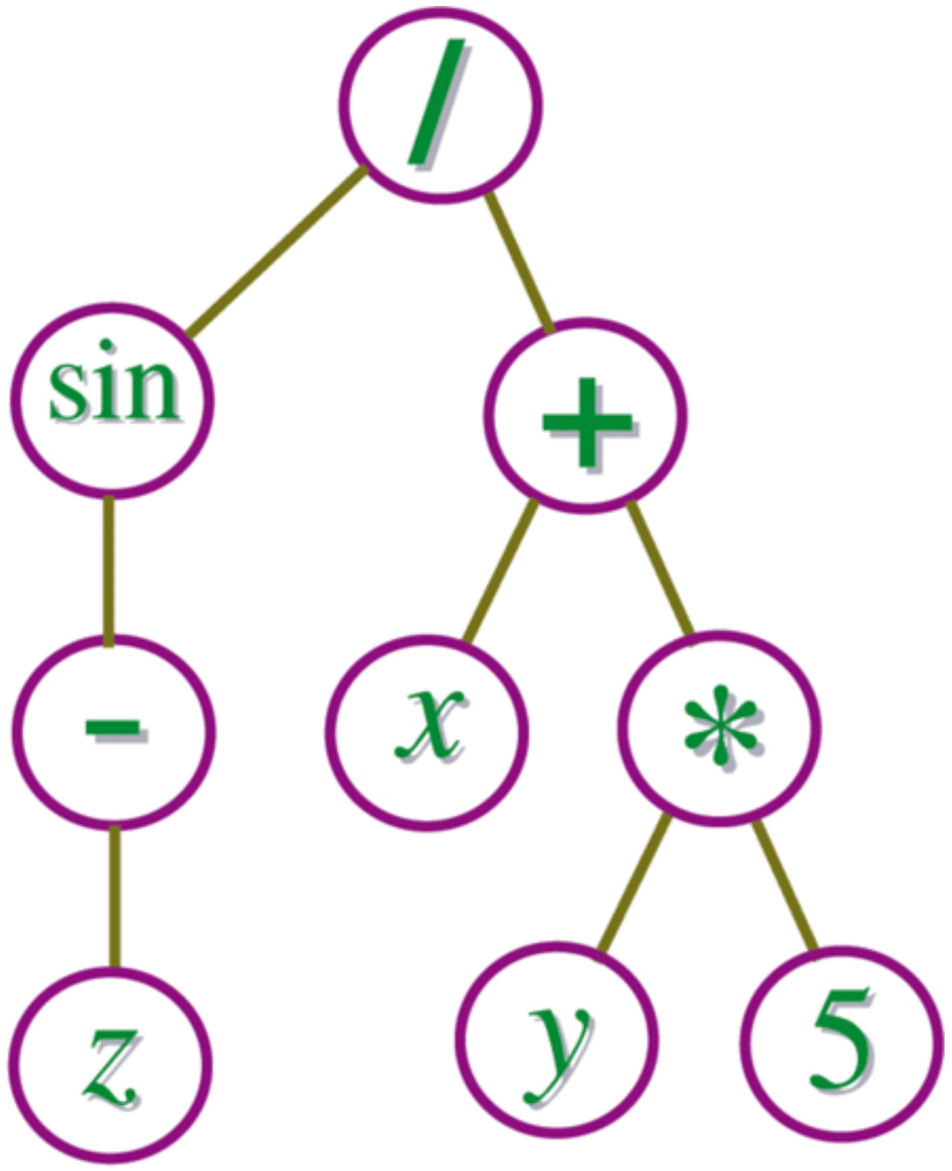
(did we get it right?)

- Pros: good accuracy, *reasonably* fast
- Cons: have to redefine all our arithmetic operators!

(must have access to code!)

Symbolic Differentiation

- Yet another approach (though related to automatic one...)
- Build explicit tree representing expression
- Apply transformations to obtain derivative
- Pros: only needs to happen once!
- Cons: *serious* development investment
- But, can often use existing tools
  - *Mathematica*, *Maple*, etc.
- Current systems not *great* with vectors, 3D
- Often produce unnecessarily complex formulae...



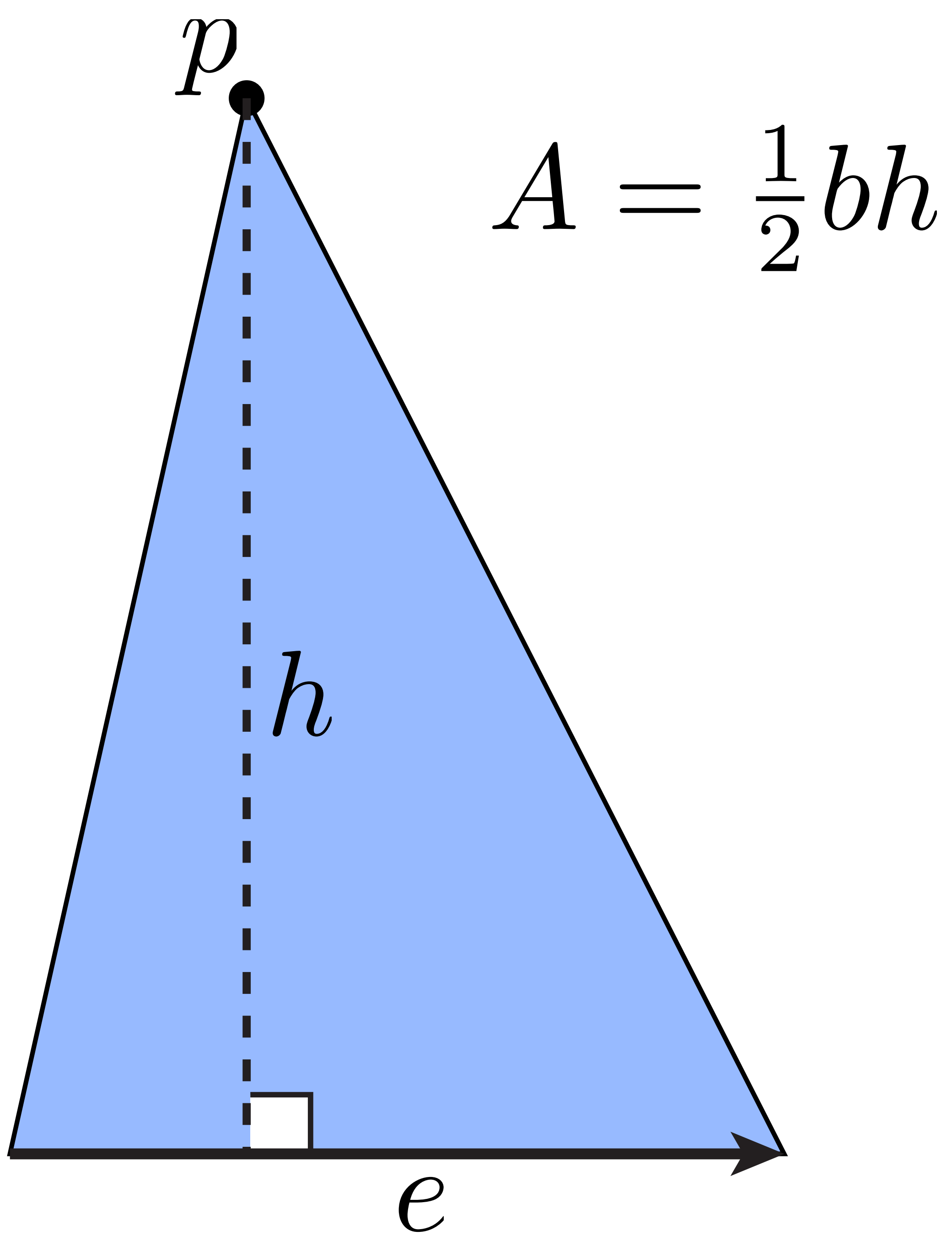
$$\frac{\sin(-z)}{x+5y}$$

Geometric Differentiation

- Sometimes symbolic differentiation misses the "big picture"
- E.g., gradient of triangle area w.r.t. vertex position p

***Mathematica* output:**

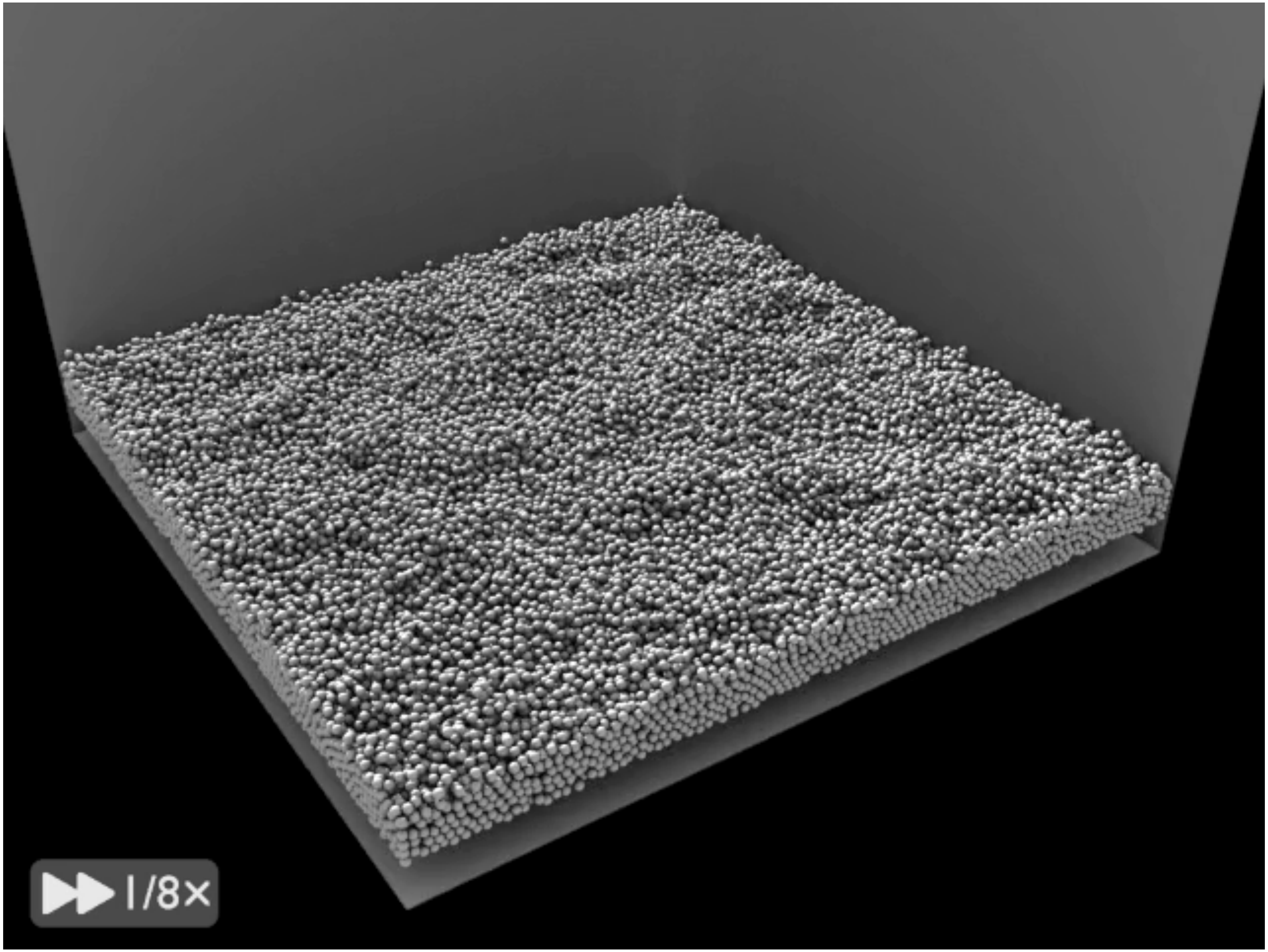
```
(2 (b2 - c2) (-b2 c1 + a2 (-b1 + c1) + a1 (b2 - c2) +
b1 c2) + 2 (b3 - c3) (-b3 c1 + a3 (-b1 + c1) + a1 (b3
- c3) + b1 c3))/(4 Sqrt((a2 b1 - a1 b2 - a2 c1 + b2 c1
+ a1 c2 - p b1 c2)^2 + (a3 b1 - a1 b3 - a3 c1 + b3 c1
+ a1 c3 - b1 c3)^2 + (a3 b2 - a2 b3 - a3 c2 + b3 c2 +
a2 c3 - b2 c3)^2)), (2 (b1 - c1) (a2 (b1 - c1) + b2 c1
- b1 c2 + a1 (-b2 + c2)) + 2 (b3 - c3) (-b3 c2 + a3 (-
b2 + c2) + a2 (b3 - c3) + b2 c3))/(4 Sqrt((a2 b1 - a1
b2 - a2 c1 + b2 c1 + a1 c2 - b1 c2)^2 + (a3 b1 - a1 b3
- a3 c1 + b3 c1 + a1 c3 - b1 c3)^2 + (a3 b2 - a2 b3 -
a3 c2 + b3 c2 + a2 c3 - b2 c3)^2)), (2 (b1 - c1) (a3
(b1 - c1) + b3 c1 - b1 c3 + a1 (-b3 + c3)) + 2 (b2 -
c2) (a3 (b2 - c2) + b3 c2 - b2 c3 + a2 (-b3 + c3)))/(4
Sqrt((a2 b1 - a1 b2 - a2 c1 + b2 c1 + a1 c2 - b1 c2)^2
+ (a3 b1 - a1 b3 - a3 c1 + b3 c1 + a1 c3 - b1 c3)^2 +
(a3 b2 - a2 b3 - a3 c2 + b3 c2 + a2 c3 - b2 c3)^2))
```



"Geometric" derivative:

$$\nabla_p A = \frac{1}{2} N \times e$$

Not Covered: Contact Mechanics



Smith et al, ***"Reflections on Simultaneous Impact"***

Coming up next: Optimization