Lecture 12:

Geometric Queries

Computer Graphics
CMU 15-462/15-662

Quiz 10 Review: Geometry Stuff

- **Question 1—Element Ratio**

$$\underline{\textbf{Claim.}} \quad V : E : F \quad \approx \quad 1 : 3 : 2$$

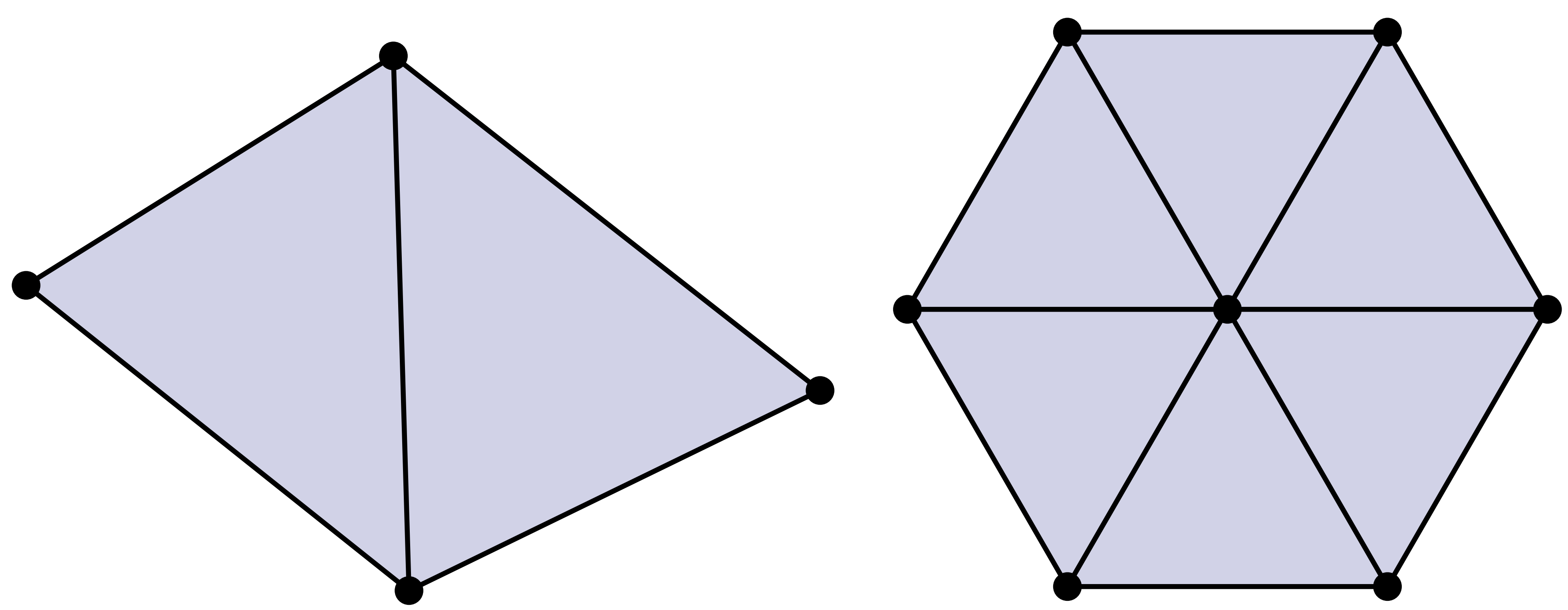
Each face has three edges; each edge is shared by two faces.

$$E = 3F/2 \qquad \Longleftrightarrow \qquad F = 2E/3$$

Each edge has two vertices; each vertex is shared by roughly six edges.

$$V \approx 2E/6 = E/3 \qquad \Longleftrightarrow \qquad E \approx 3V \quad ✓$$

$$\Longrightarrow F \approx 2(3V)/3 = 2V \quad ✓$$



Quiz 10 Review: Geometry Stuff

- **Alternative solution:**

*Euler's formula**: $V - E + F = 2$

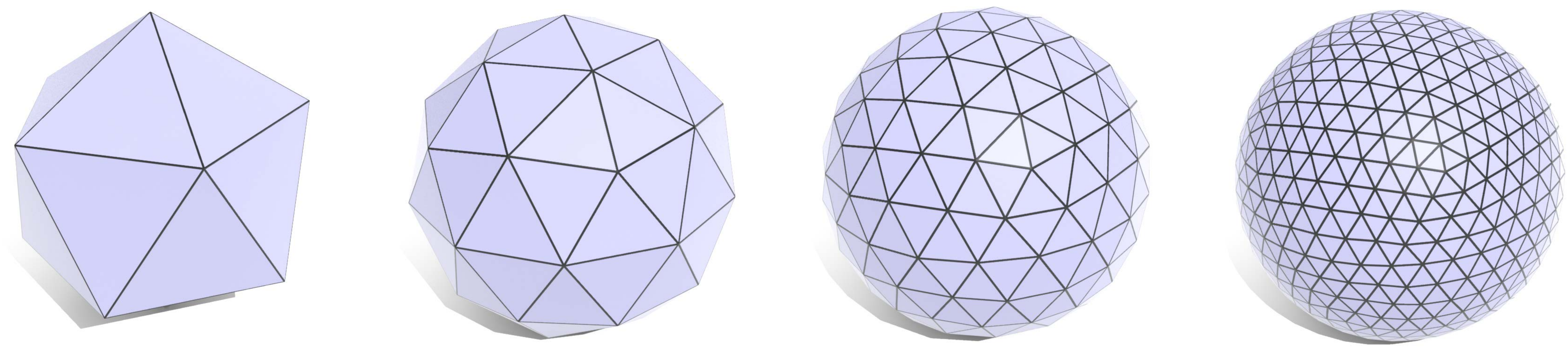
Substitute $2E = 3F$:

$$\implies V - E + 2E/3 = 2$$

$$\implies E = 3V - 6$$

$$\implies F = 2V - 4$$

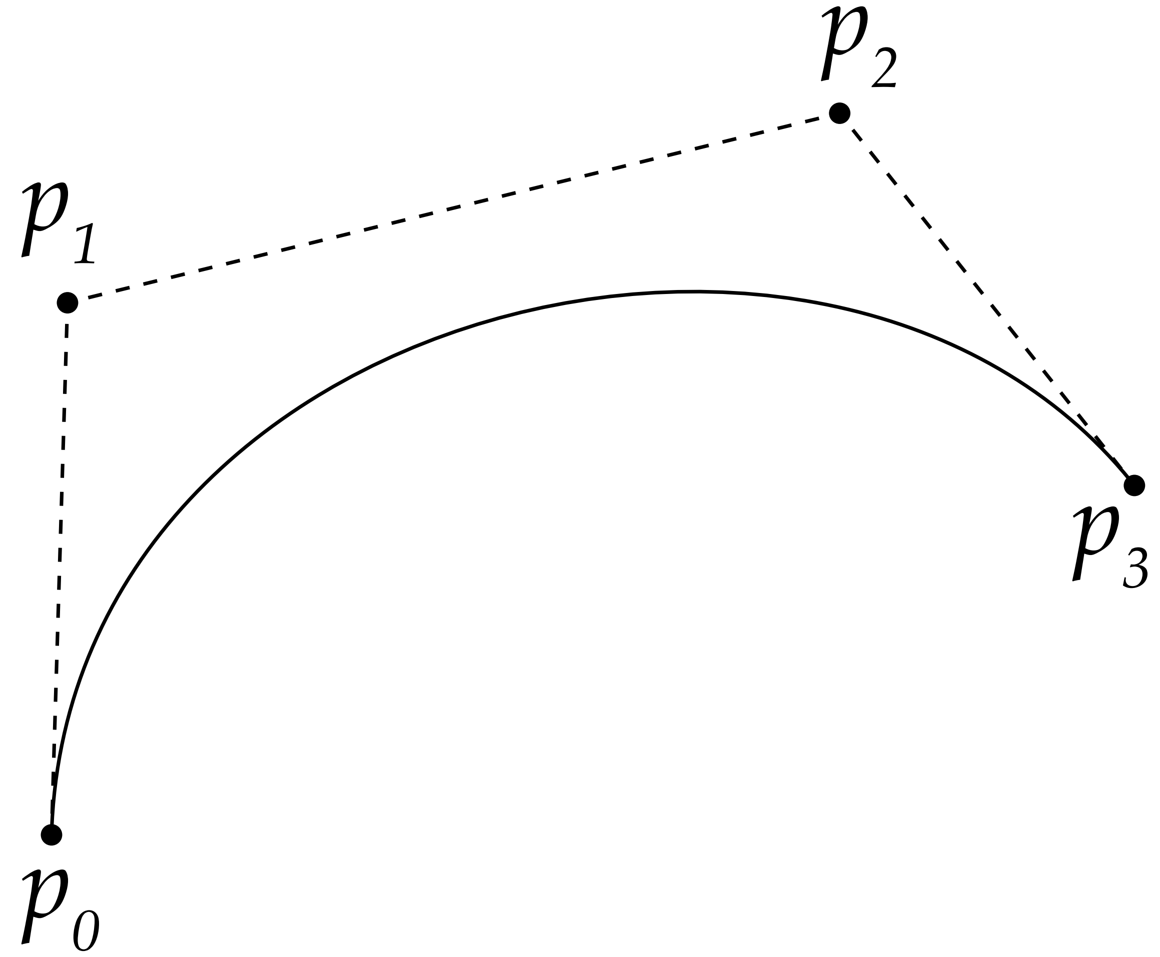
negligible for large number of vertices



***More generally, $V - E + F = 2-2g$; doesn't change story much!**

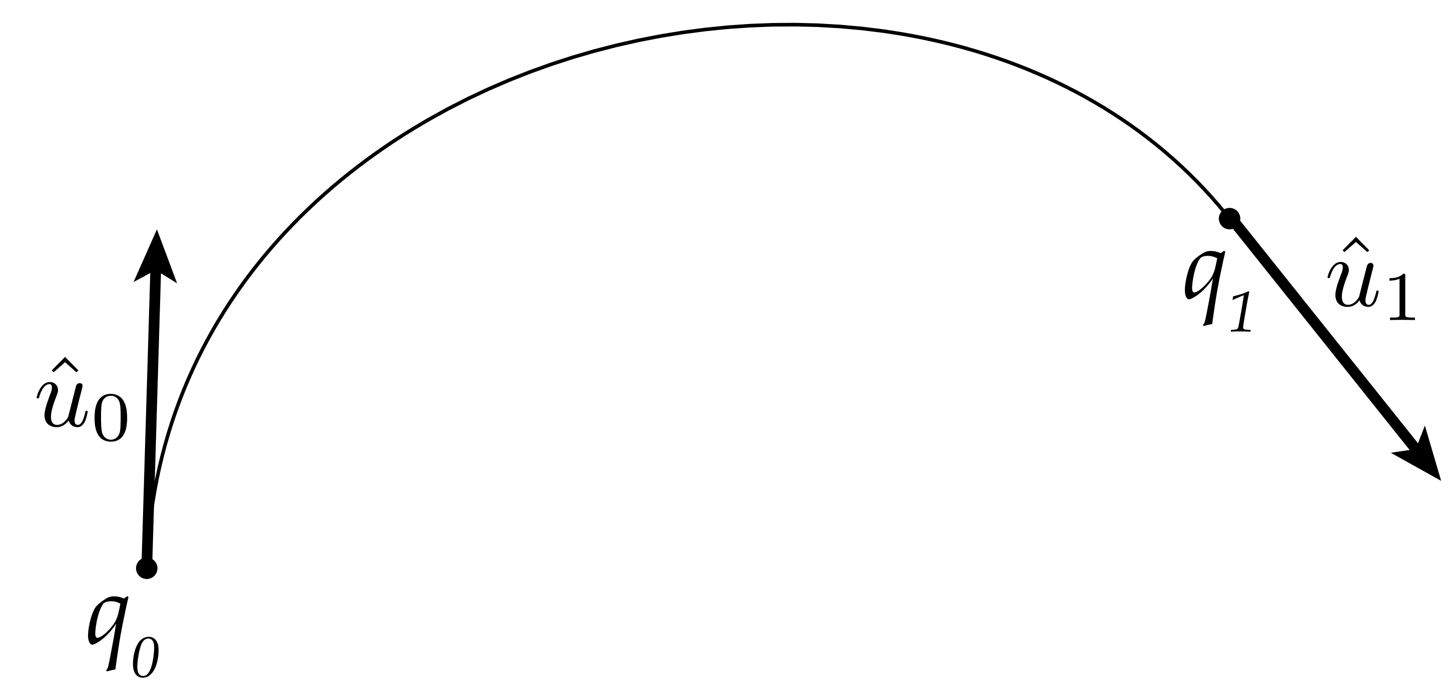
Quiz 10 Review: Geometry Stuff

- **Question 2—Bézier/Hermite Conversion**



cubic Bézier

$$\begin{aligned} B_0(t) &= (1-t)^3, \\ B_1(t) &= 3(1-t)^2 t, \\ B_2(t) &= 3(1-t)t^2, \\ B_3(t) &= t^3. \end{aligned}$$



cubic Hermite

$$\begin{aligned} H_{00}(t) &= (1+2t)(1-t)^2, \\ H_{10}(t) &= t(1-t)^2, \\ H_{01}(t) &= t^2(3-2t), \\ H_{11}(t) &= -t^2(1-t). \end{aligned}$$

$$\mathbf{p}(t) = \mathbf{p}_0 B_0(t) + \mathbf{p}_1 B_1(t) + \mathbf{p}_2 B_2(t) + \mathbf{p}_3 B_3(t), \qquad t \in [0,1].$$

$$\mathbf{q}(t) = \mathbf{q}_0 H_{00}(t) + \mathbf{u}_0 H_{10}(t) + \mathbf{q}_1 H_{01}(t) + \mathbf{u}_1 H_{11}(t), \qquad t \in [0,1]$$

Quiz 10 Review: Geometry Stuff

■ Question 2—Bézier/Hermite Conversion

Step I: Express the Bézier/Hermite basis in terms of monomials:

$$\begin{aligned} B_0(t) &= 1 - 3t + 3t^2 - t^3, \\ B_1(t) &= 3t - 6t^2 + 3t^3, \\ B_2(t) &= 3t^2 - 3t^3, \\ B_3(t) &= t^3. \end{aligned} \qquad \begin{aligned} H_{00}(t) &= 1 - 3t^2 + 2t^3, \\ H_{10}(t) &= t - 2t^2 + t^3, \\ H_{01}(t) &= 3t^2 - 2t^3, \\ H_{11}(t) &= -t^2 + t^3. \end{aligned}$$

Step II: Notice that these are just vectors in the basis 1, t, t², t³:

$$\begin{bmatrix} B_0(t) \\ B_1(t) \\ B_2(t) \\ B_3(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} \qquad \mathbf{p}(t) = [\mathbf{p}_0\ \mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]\, U \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$$\begin{bmatrix} H_{00}(t) \\ H_{10}(t) \\ H_{01}(t) \\ H_{11}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{V} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} \qquad \mathbf{q}(t) = [\mathbf{q}_0\ \mathbf{u}_0\ \mathbf{q}_1\ \mathbf{u}_1]\, V \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Quiz 10 Review: Geometry Stuff

■ Question 2—Bézier/Hermite Conversion

Step III: Compute change between two bases:

$$UV^{-1} = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(Bézier to Hermite)

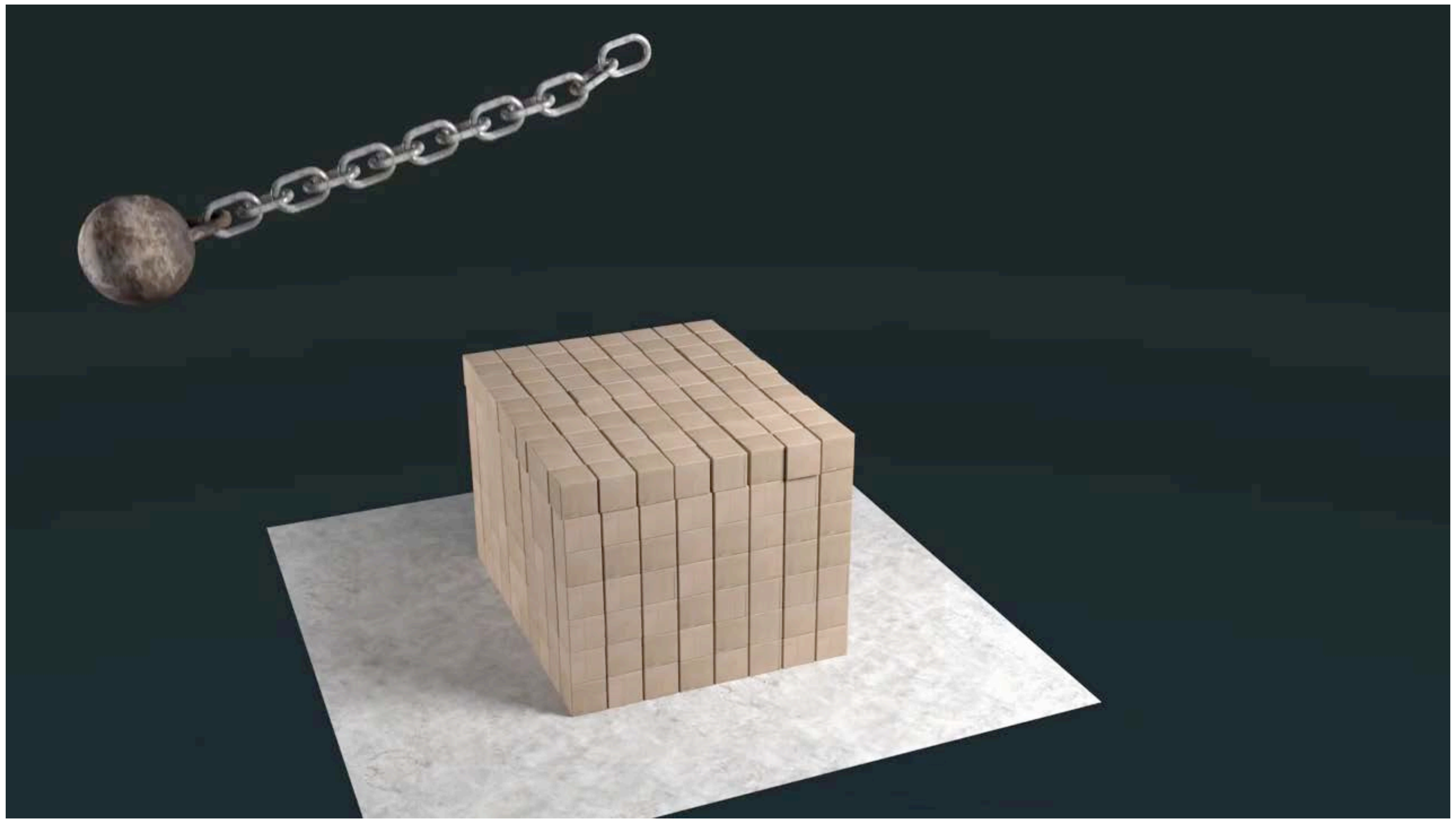
$$VU^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{3} & 0 \end{bmatrix}$$

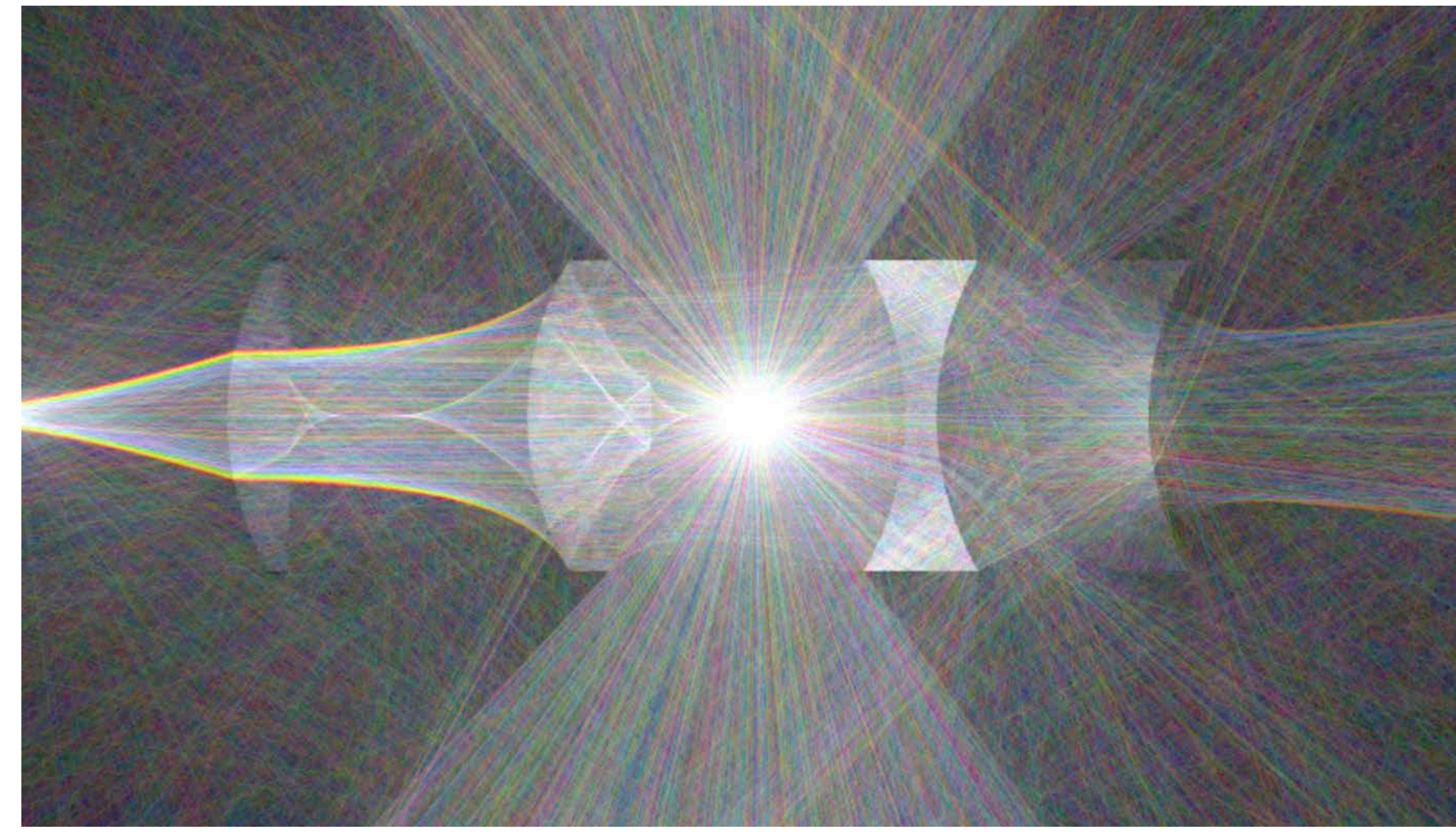
(Hermite to Bézier)

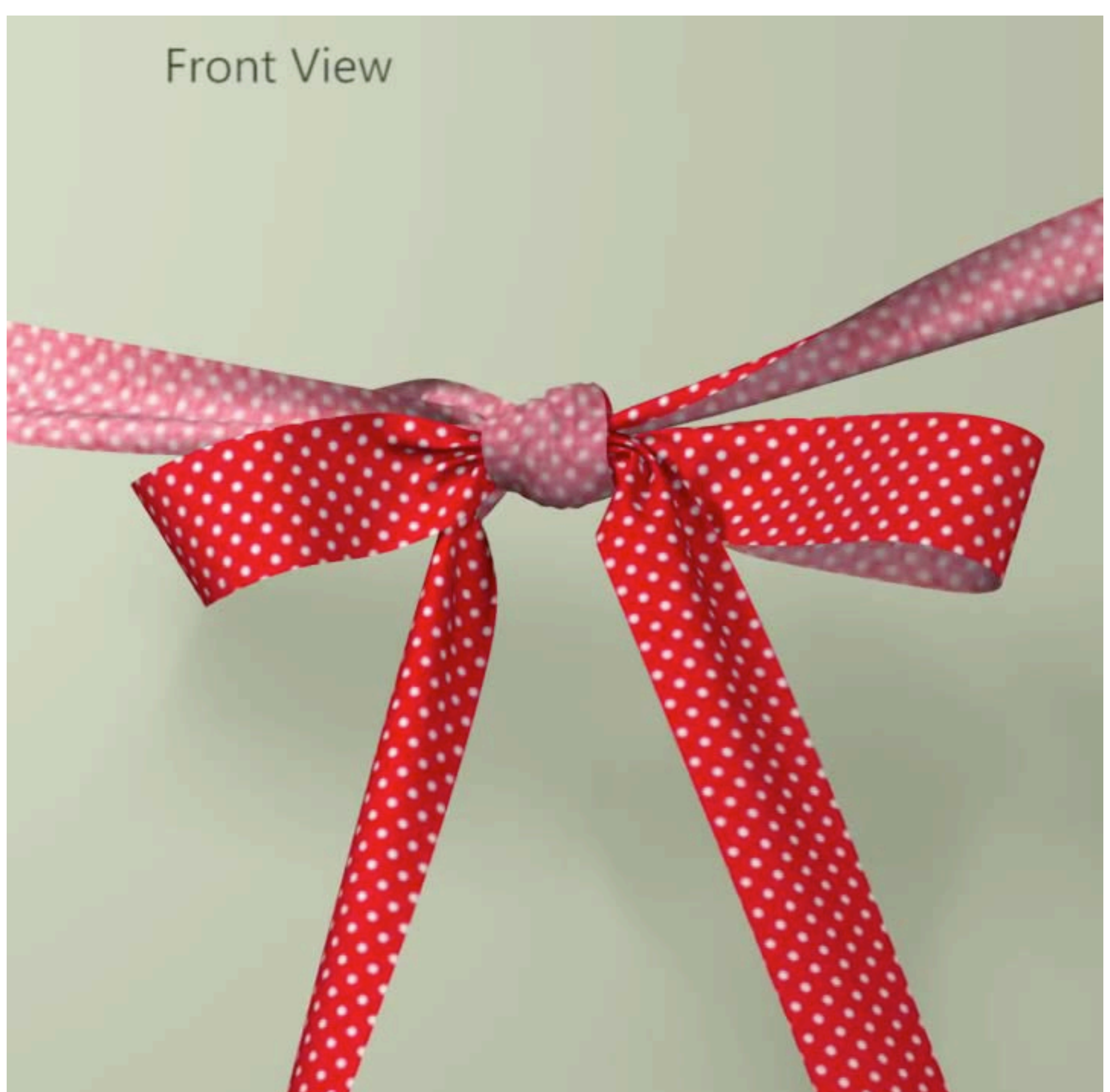
Step IV: Apply change of basis to list of coordinates (being careful about transpose):

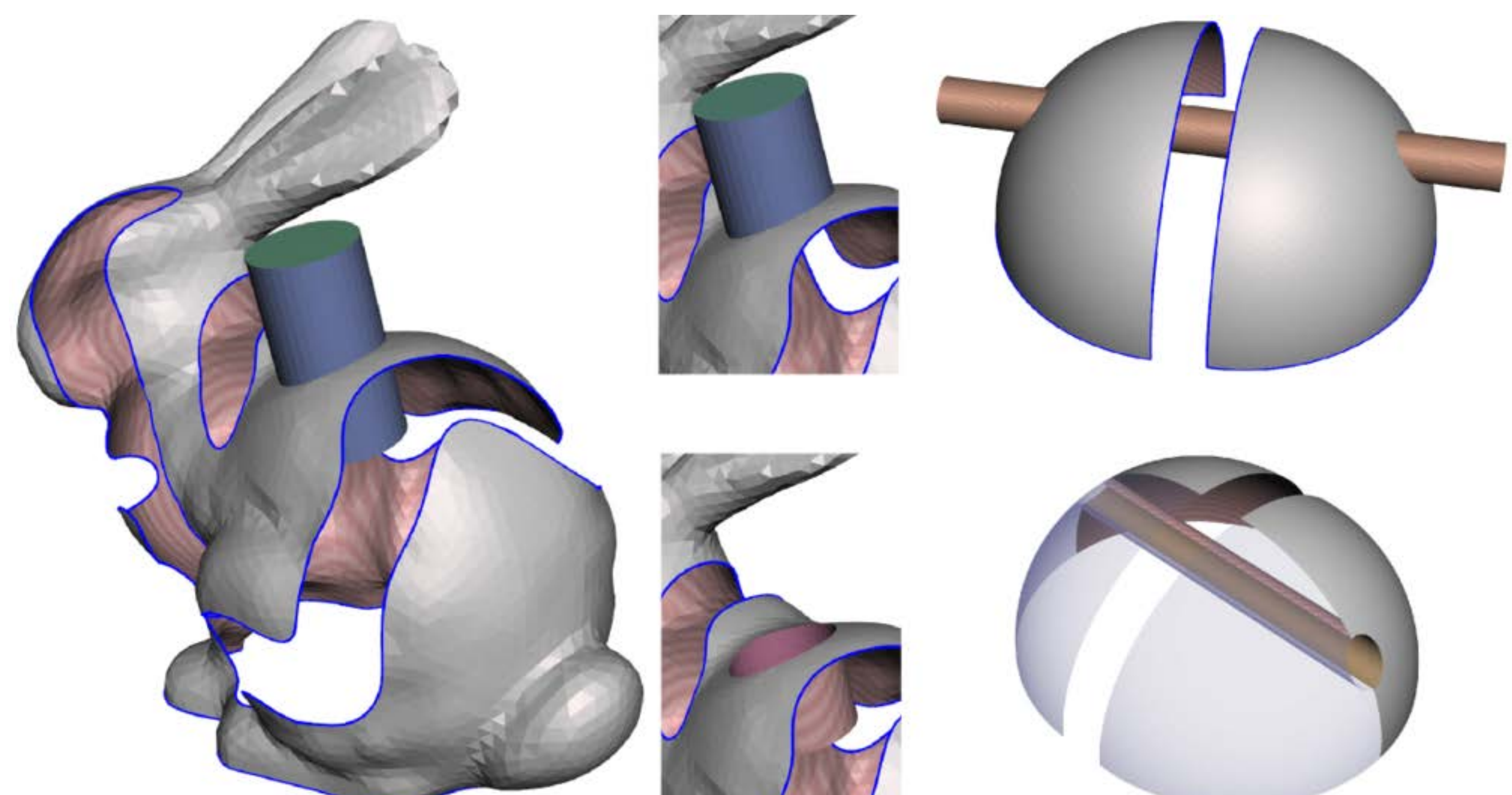
```
void BezierToHermite( Vector  p0, Vector  p1, Vector  p2, Vector  p3,
                      Vector& q0, Vector& q1, Vector& u0, Vector& u1 )
{
   q0 = p0;             // first endpoint equals first control point
   q1 = p3;             // second endpoint equals second control point
   u0 = 3.*(p1-p0);     // tangent is 3x difference of control points
   u1 = 3.*(p3-p2);     // tangent is 3x difference of control points
}
```

Geometric Queries—Motivation

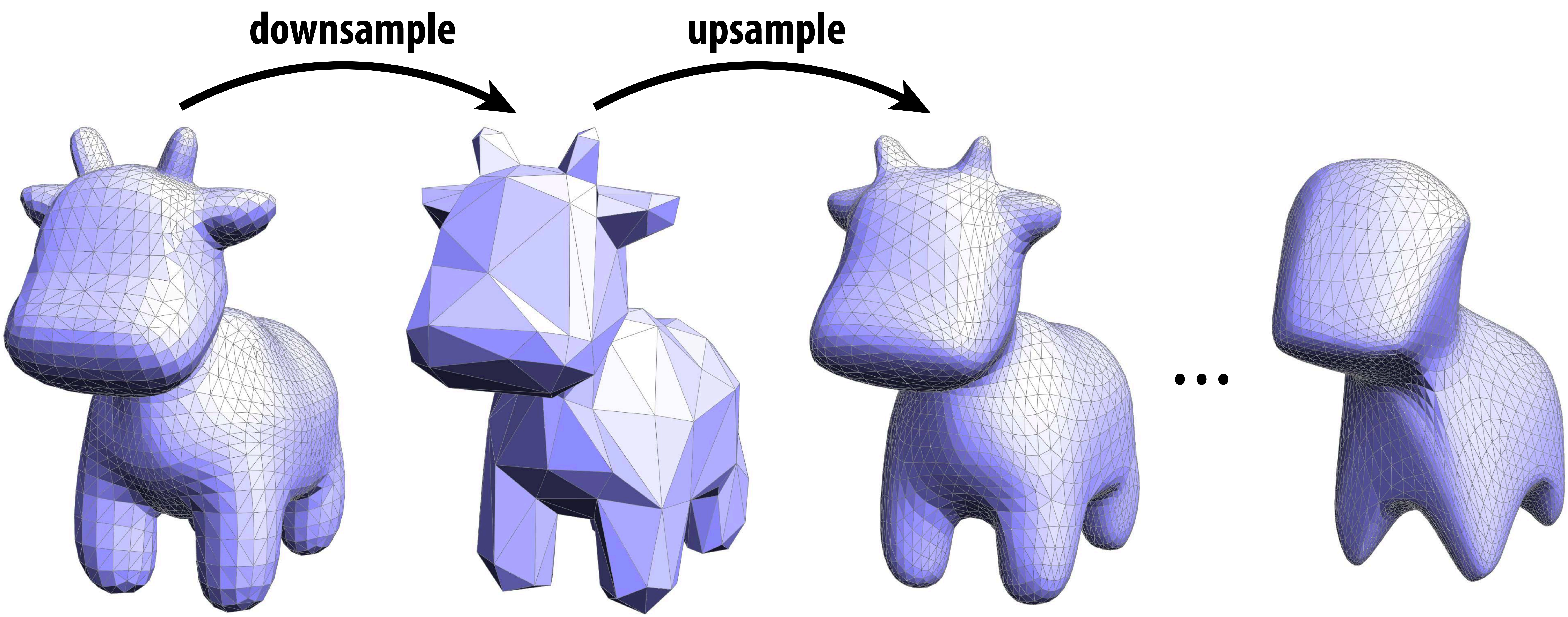








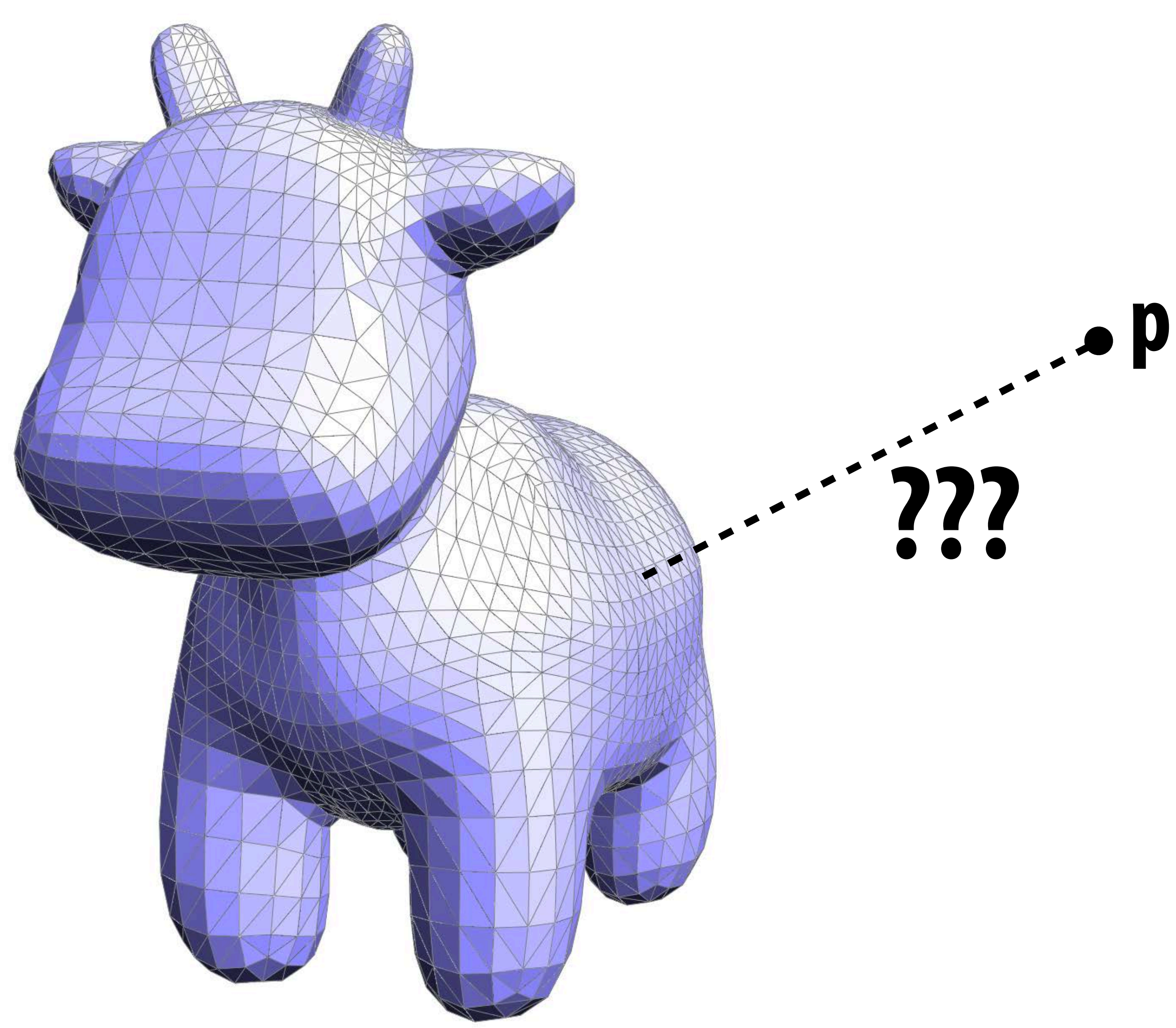
Last Time: Danger of Resampling



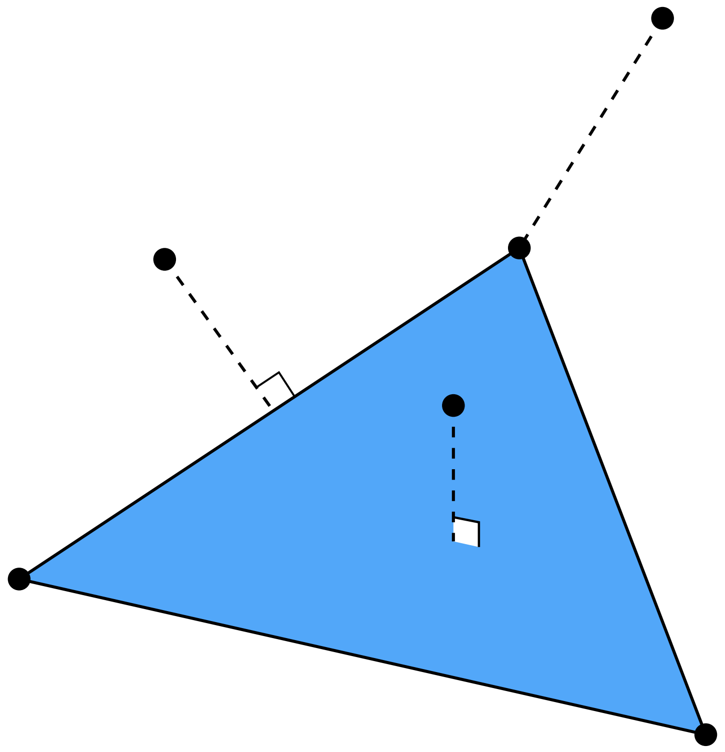
Idea: after resampling, project each vertex onto original mesh

Closest Point Queries

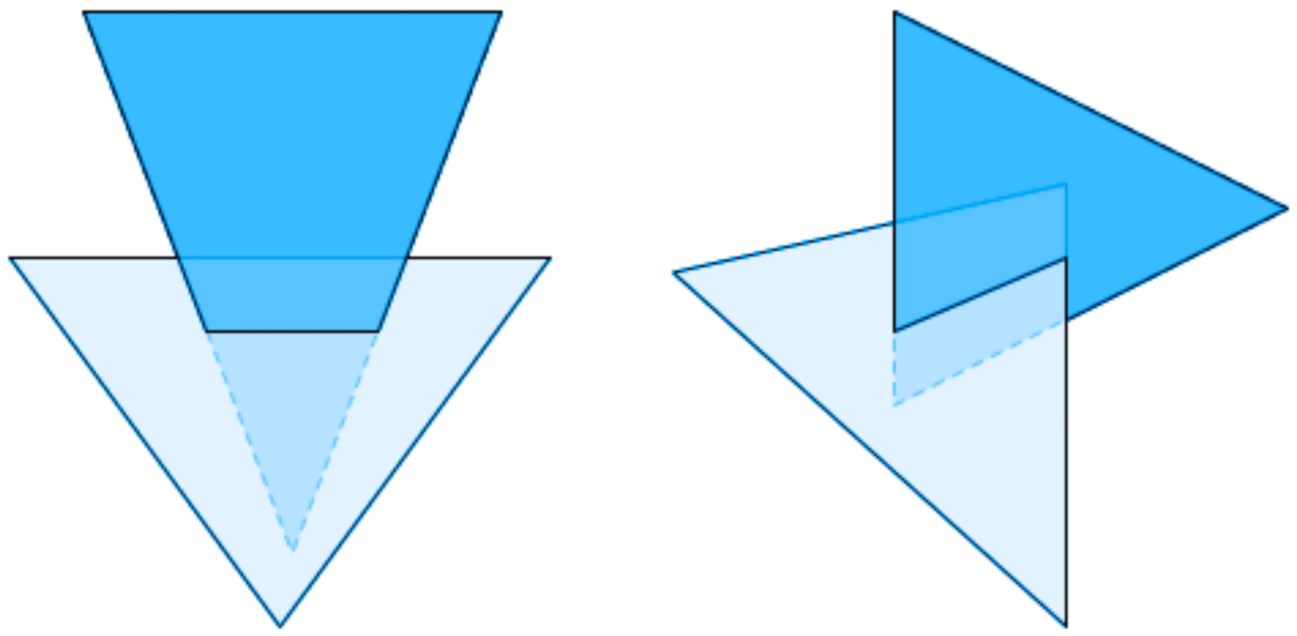
- Q: Given a point, in space (e.g., a new sample point), how do we find the closest point on a given surface?
- Q: Does implicit/explicit representation make this easier?
- Q: Does our halfedge data structure help?
- Q: What's the cost of the naïve algorithm?
- Q: How do we find the distance to a single triangle anyway?
- So many questions!



Many types of geometric queries



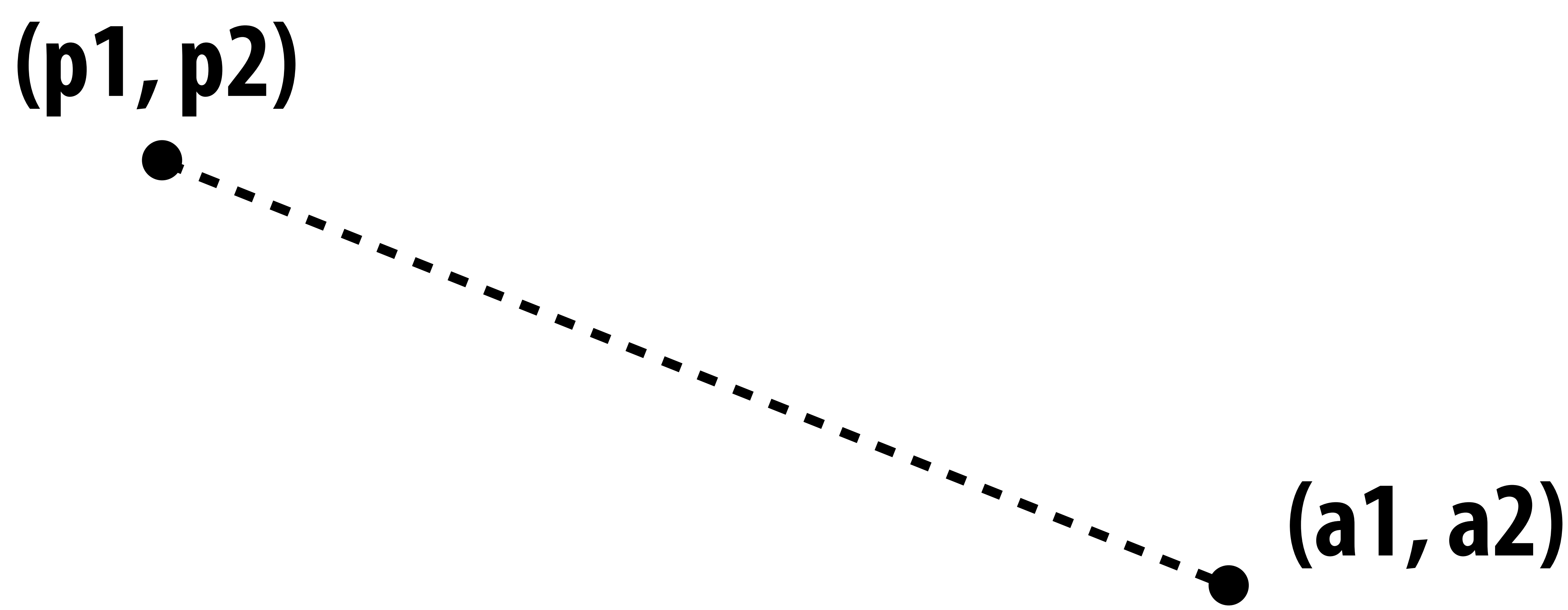
- Already identified need for "closest point" query
- Plenty of other things we might like to know:
  - Do two triangles intersect?
  - Are we inside or outside an object?
  - Does one object contain another?
  - ...



- Data structures we've seen so far not really designed for this...
- Need some new ideas!
- TODAY: come up with simple (read: slow) algorithms.
- NEXT TIME: intelligent ways to accelerate geometric queries.

Warm up: closest point on point

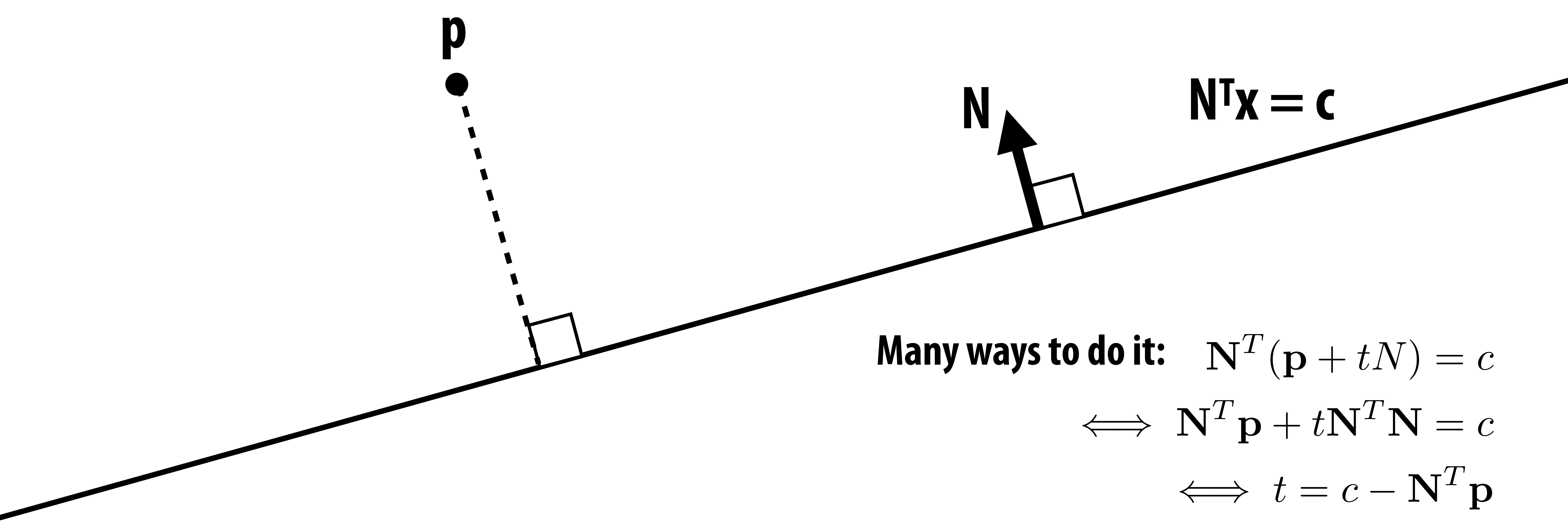
- Goal is to find the point on a mesh closest to a given point.
- *Much* simpler question: given a query point (p1,p2), how do we find the closest point on the point (a1,a2)?



Bonus question: what's the distance?

Slightly harder: closest point on line

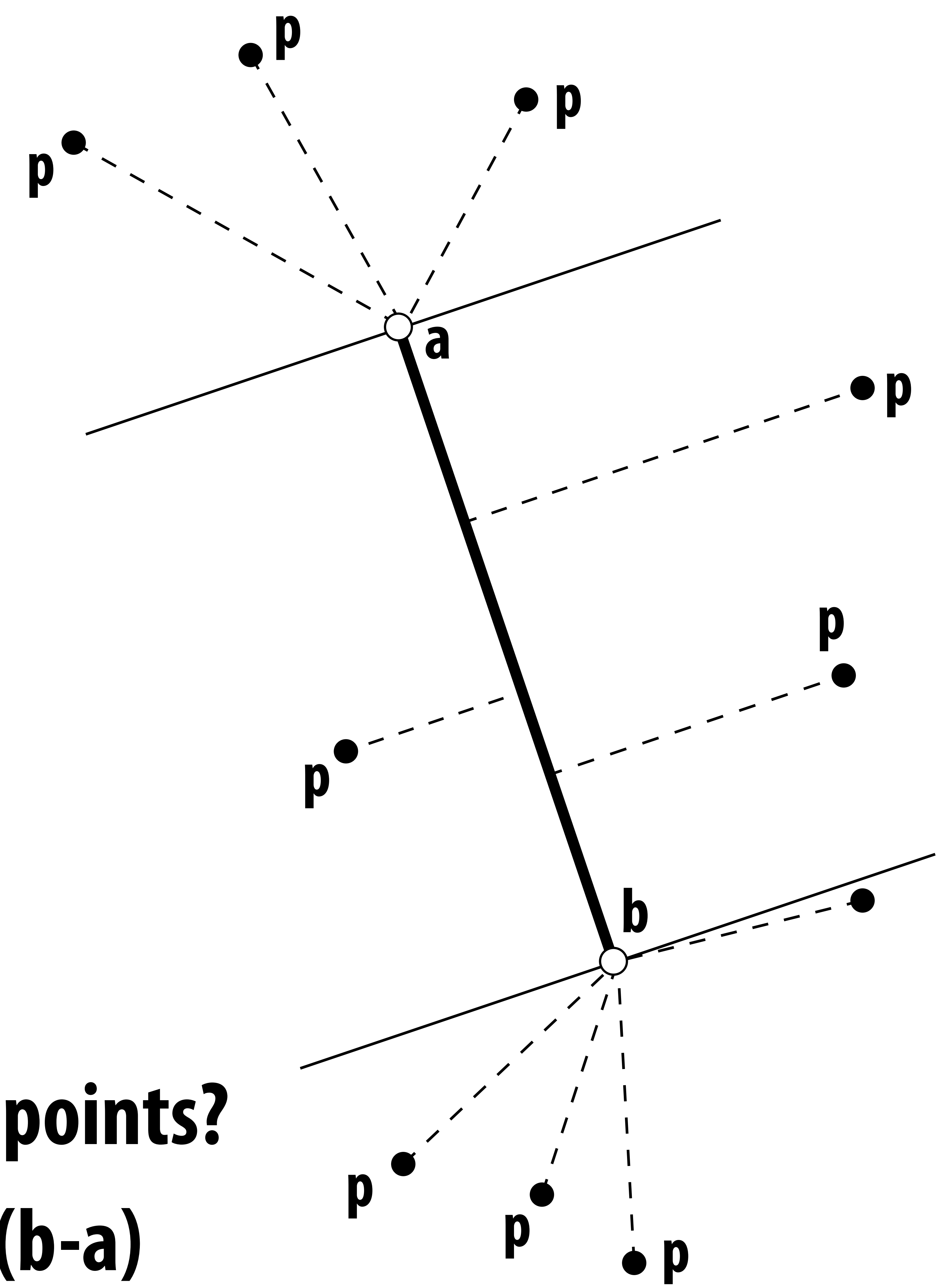
- Now suppose I have a line $N^Tx = c$, where N is the unit normal
- How do I find the point closest to my query point p?



Many ways to do it: $$\mathbf{N}^T(\mathbf{p} + tN) = c$$
$$\iff \mathbf{N}^T\mathbf{p} + t\mathbf{N}^T\mathbf{N} = c$$
$$\iff t = c - \mathbf{N}^T\mathbf{p}$$
$$\Rightarrow \mathbf{p} + t\mathbf{N} = \boxed{\mathbf{p} + (c - \mathbf{N}^T\mathbf{p})\mathbf{N}}$$

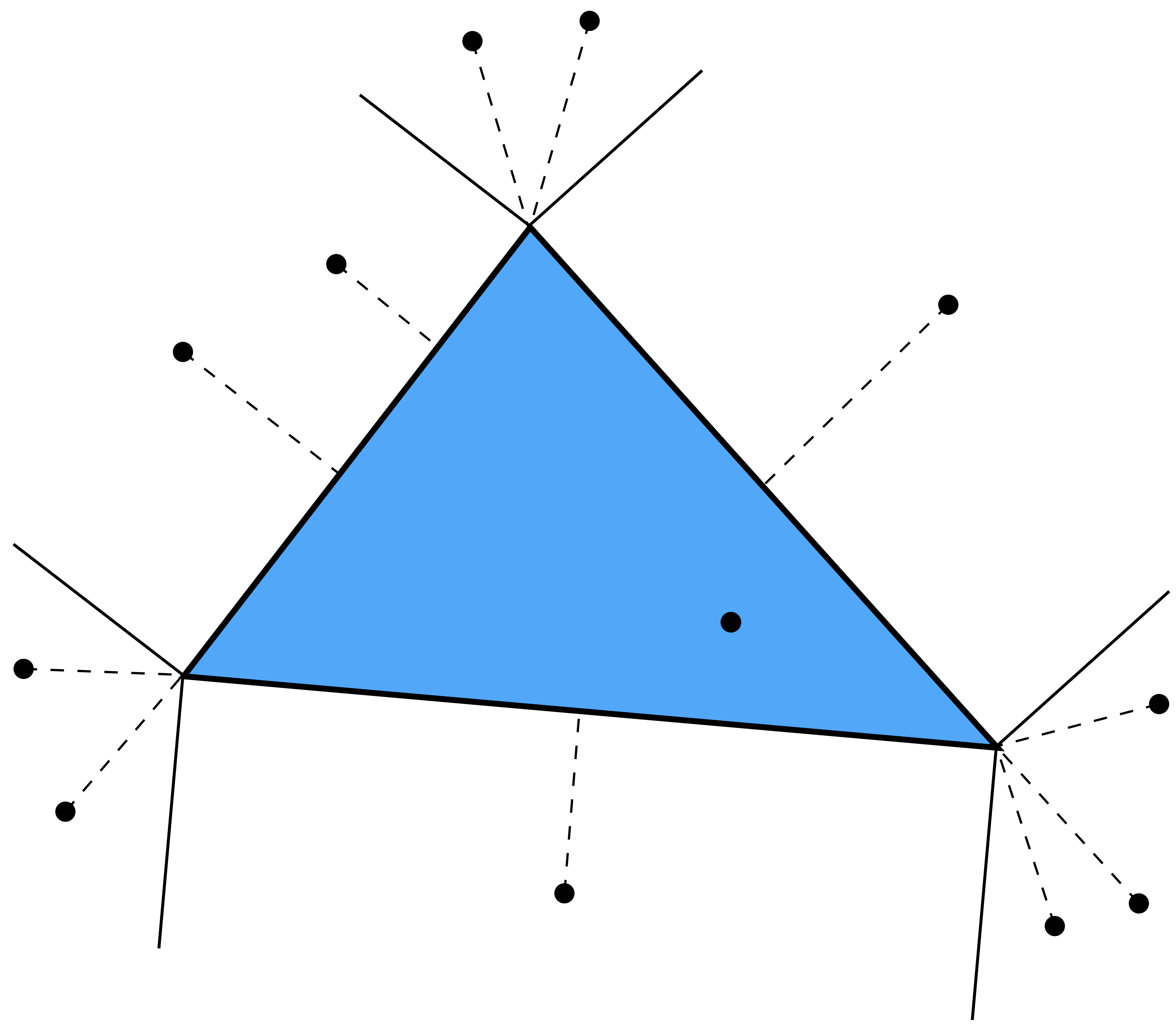
Harder: closest point on line segment

- Two cases: endpoint or interior
- Already have basic components:
  - point-to-point
  - point-to-line
- Algorithm?
  - find closest point on line
  - check if it's between endpoints
  - if not, take closest endpoint
- How do we know if it's between endpoints?
  - write closest point on line as a+t(b-a)
  - if t is between 0 and 1, it's inside the segment!



Even harder: closest point on triangle

- What are all the possibilities for the closest point?
- Almost just minimum distance to three segments:



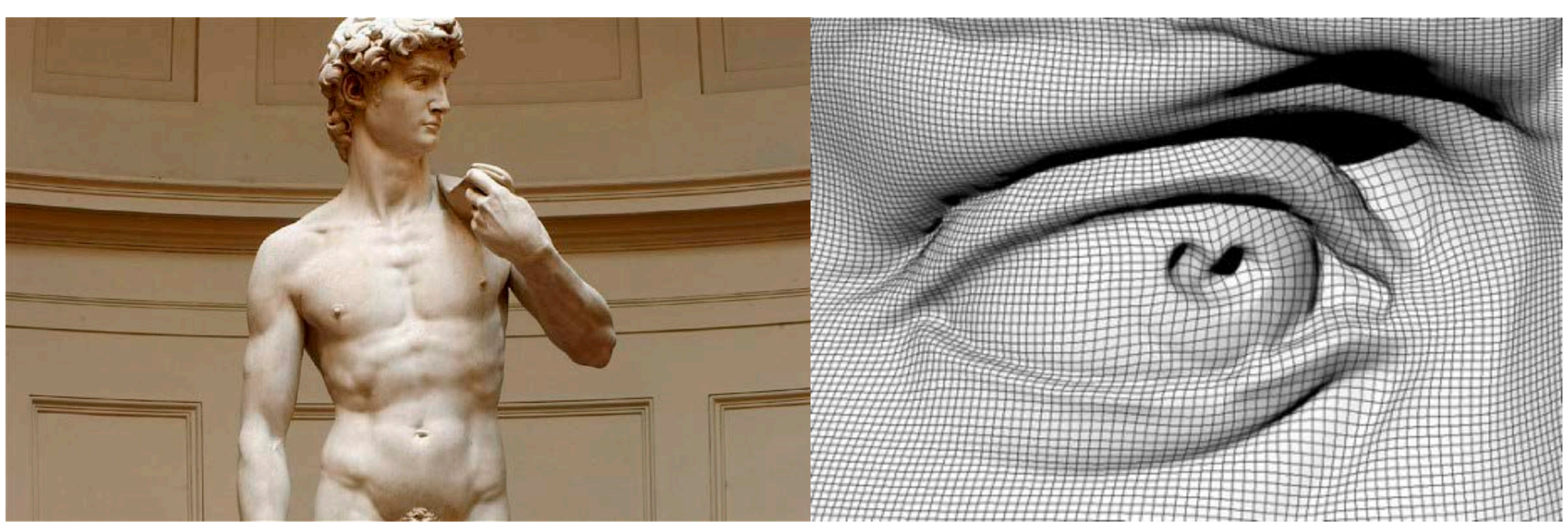
Q: What about a point inside the triangle?

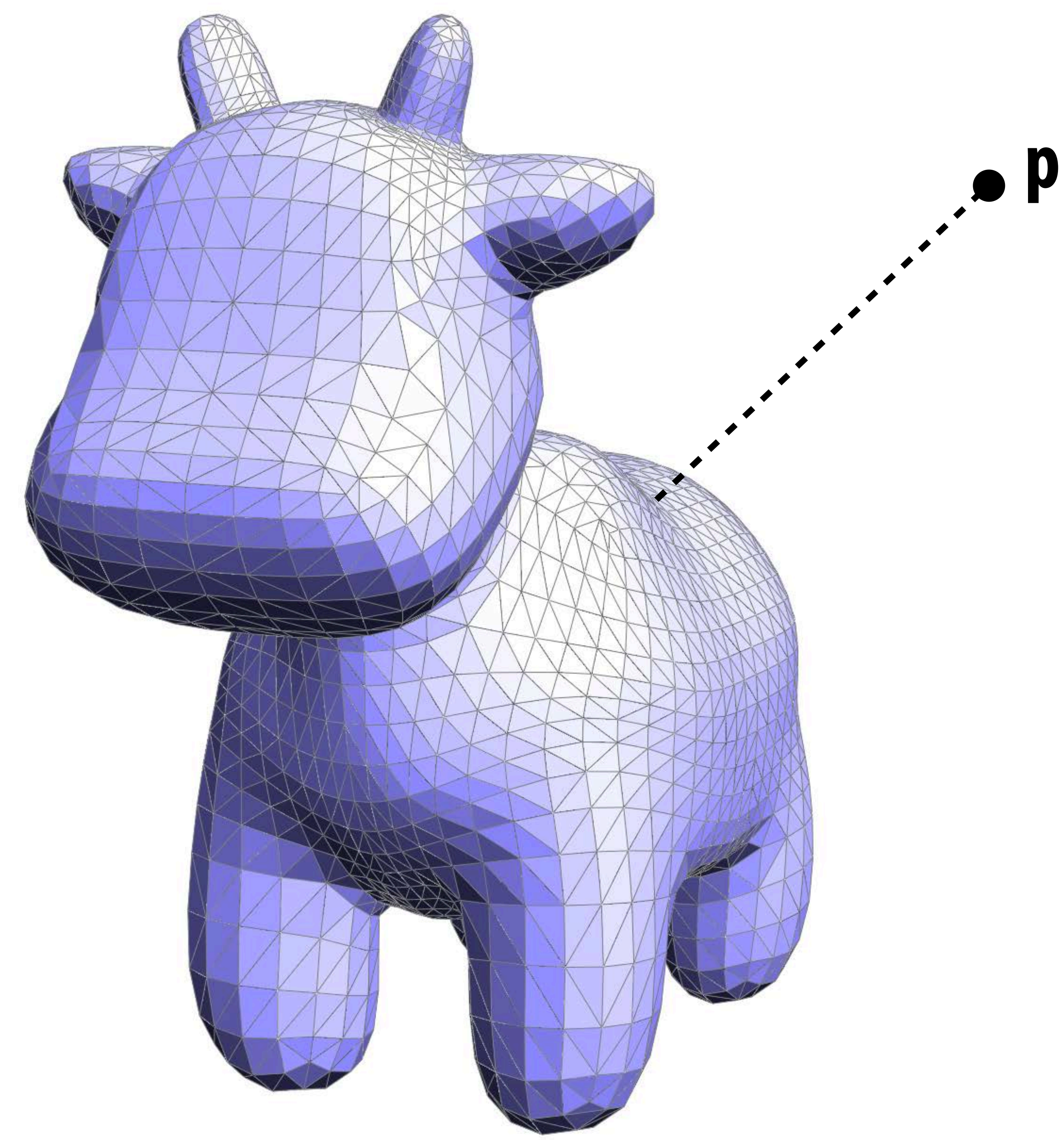
Closest point on triangle in 3D

- Not so different from 2D case
- Algorithm?
  - project onto plane of triangle
  - use half-*space* tests to classify point (vs. half plane)
  - if inside the triangle, we're done!
  - otherwise, find closest point on associated vertex or edge
- By the way, how do we find closest point on plane?
- Same expression as closest point on a line!
- E.g., $p + (c - N^T p) N$

Closest point on triangle *mesh* in 3D?

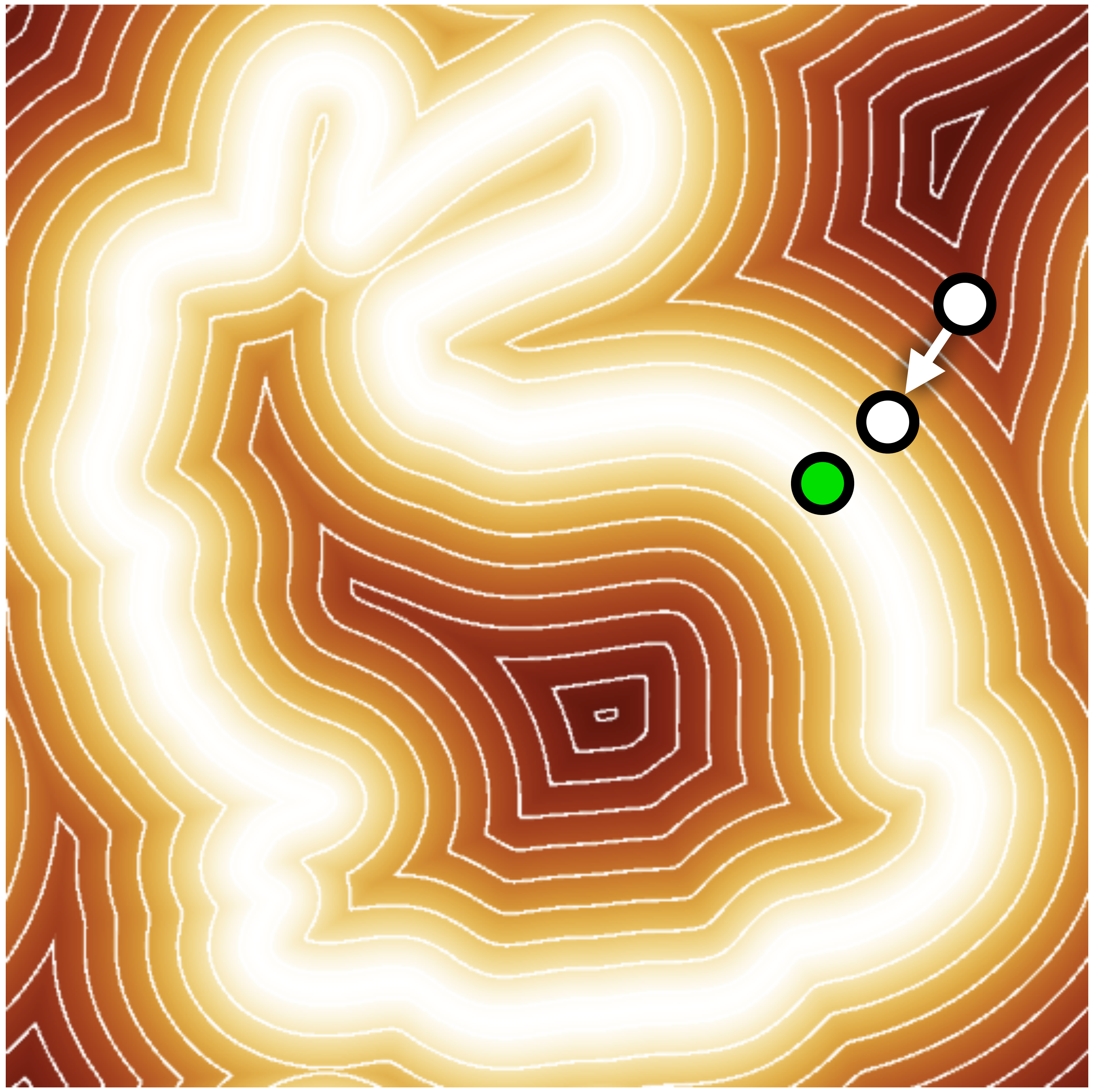
- **Conceptually easy:**
  - **loop over all triangles**
  - **compute closest point to current triangle**
  - **keep globally closest point**
- **Q: What's the cost?**
- **What if we have *billions* of faces?**
- **NEXT TIME: Better data structures!**





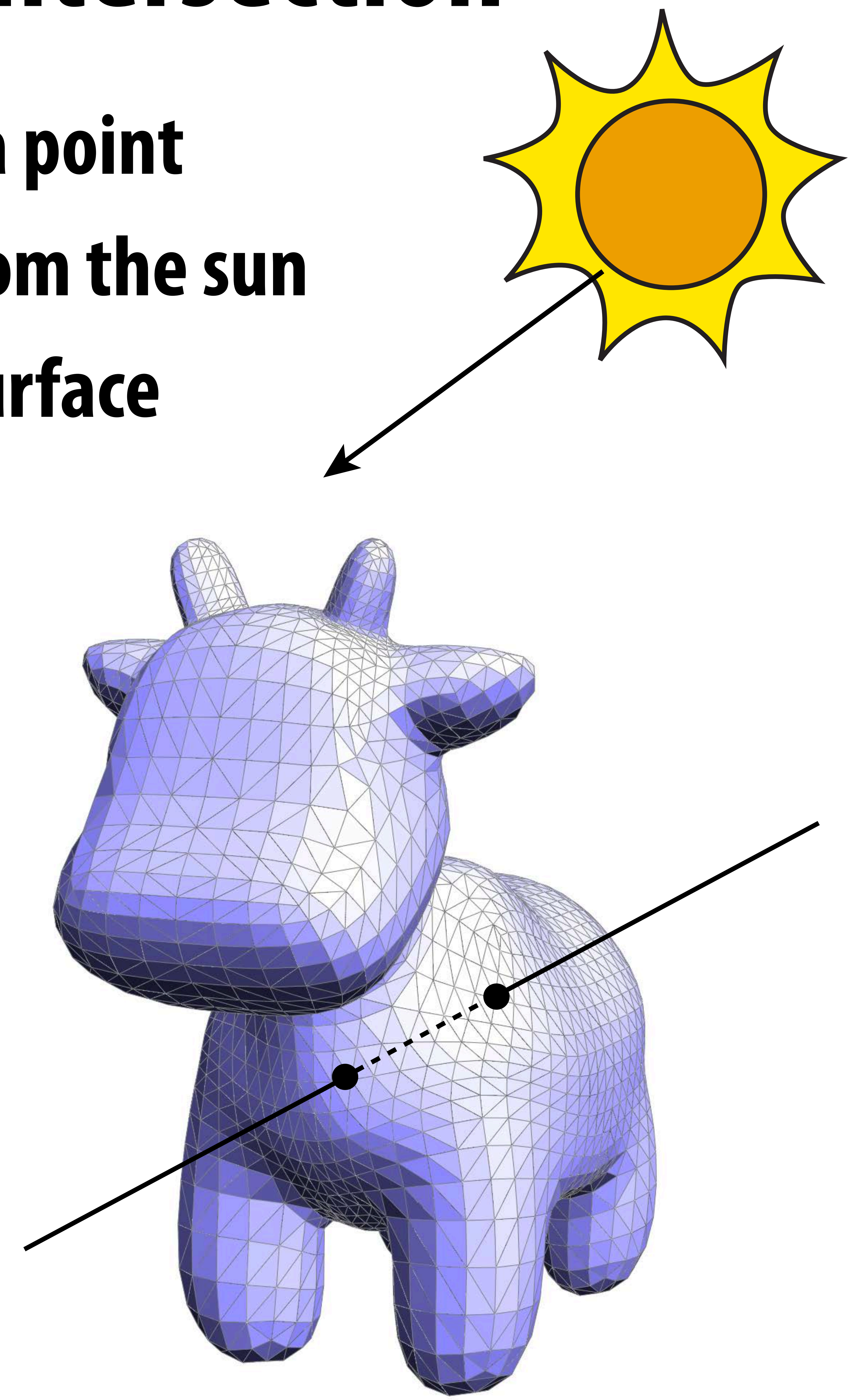
Closest point to *implicit* surface?

- If we change our representation of geometry, algorithms can change completely
- E.g., how might we compute the closest point on an implicit surface described via its distance function?
- One idea:
  - start at the query point
  - compute gradient of distance (using, e.g., finite differences)
  - take a little step (decrease distance)
  - repeat until we're at the surface (zero distance)
- Better yet: just store closest point for each grid cell! (speed/memory trade off)



Different query: ray-mesh intersection

- A "ray" is an oriented line starting at a point
- Think about a ray of light traveling from the sun
- Want to know where a ray pierces a surface
- Why?
  - GEOMETRY: inside-outside test
  - RENDERING: visibility, ray tracing
  - ANIMATION: collision detection
- Might pierce surface in many places!



Ray equation

- Can express ray as

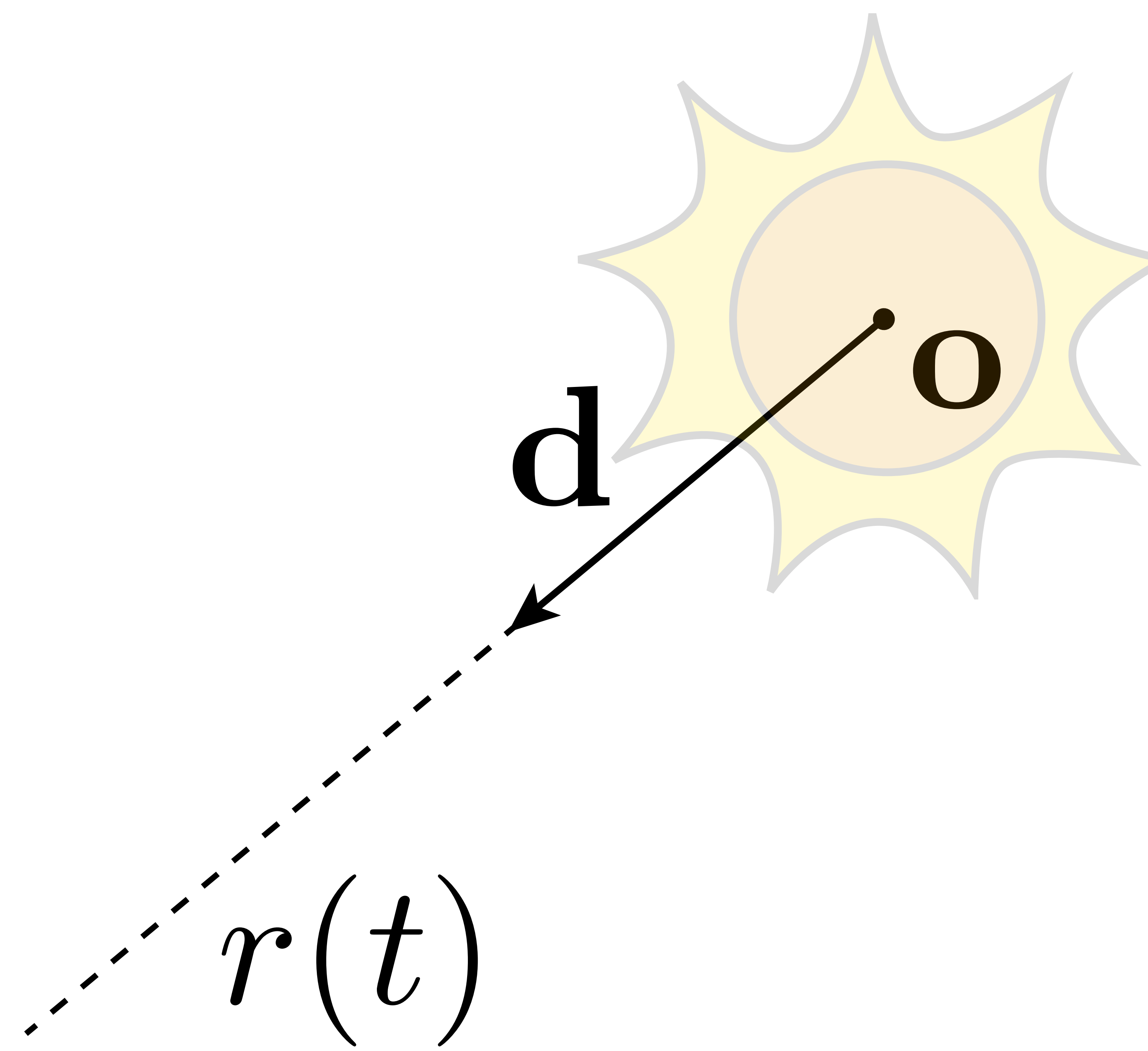
origin

unit direction

$$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$$

point along ray

"time"



Intersecting a ray with an implicit surface

- **Recall implicit surfaces: all points x such that f(x) = 0**
- **Q: How do we find points where a ray pierces this surface?**
- **Well, we know all points along the ray: r(t) = o + td**
- **Idea: replace "x" with "r" in 1st equation, and solve for t**
- **Example: unit sphere**

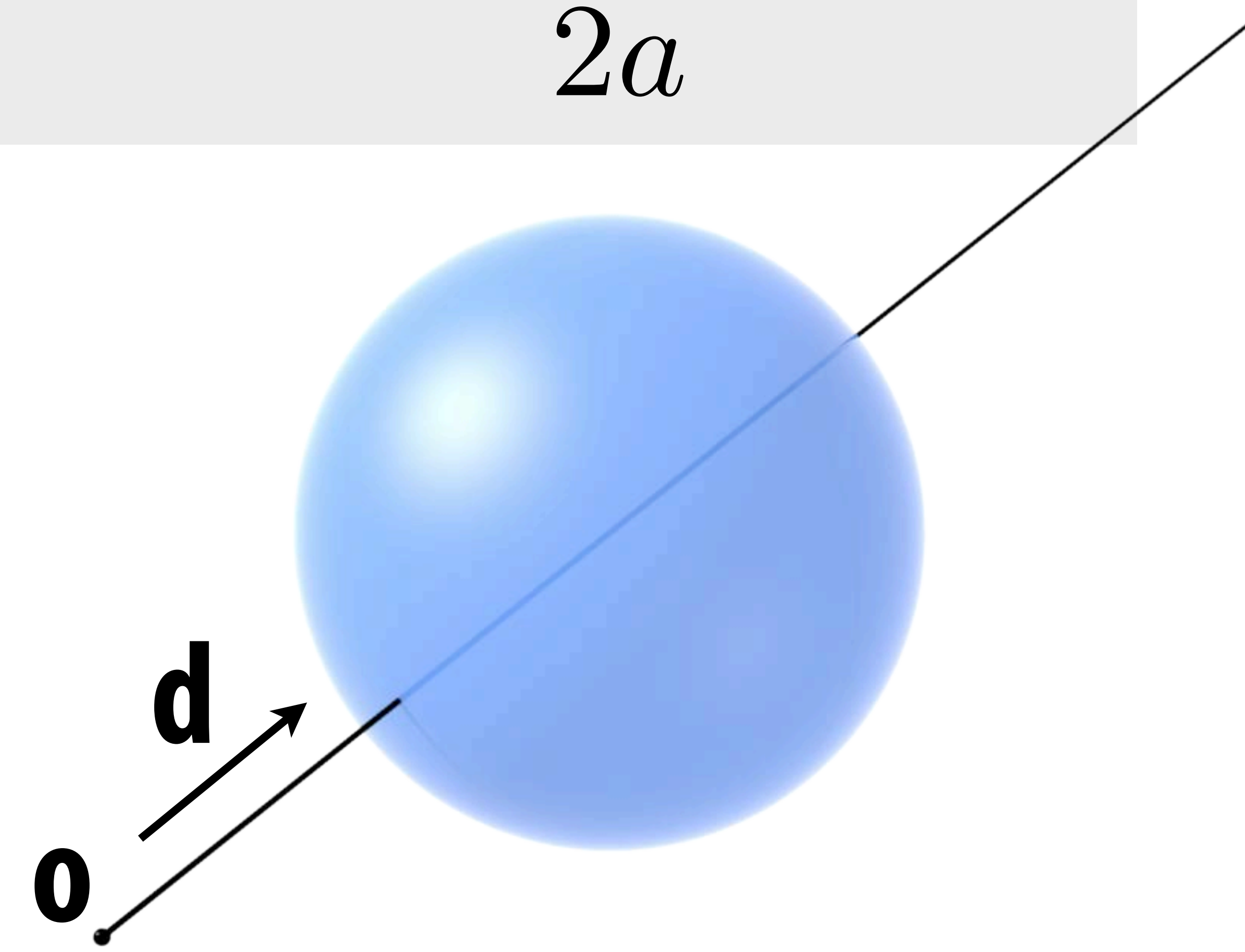
quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$f(\mathbf{x}) = |\mathbf{x}|^2 - 1$$

$$\Rightarrow f(\mathbf{r}(t)) = |\mathbf{o} + t\mathbf{d}|^2 - 1$$

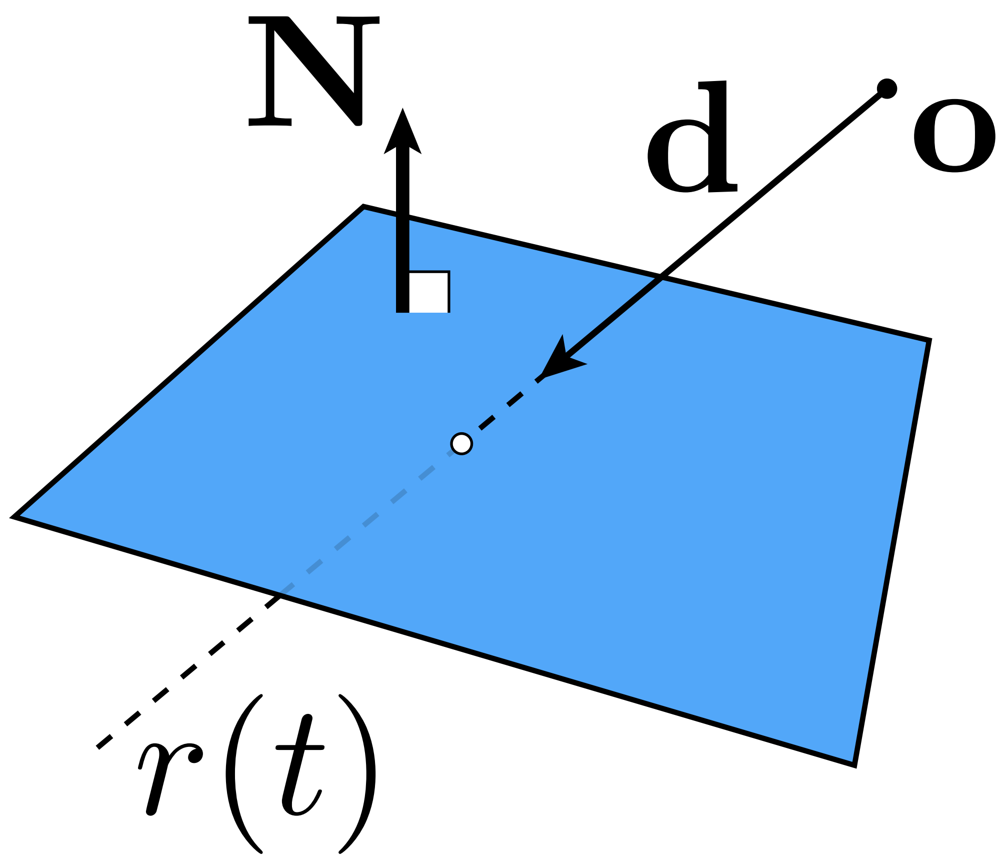
$$\underbrace{|\mathbf{d}|^2}_{a} t^2 + \underbrace{2(\mathbf{o} \cdot \mathbf{d})}_{b} t + \underbrace{|\mathbf{o}|^2 - 1}_{c} = 0$$



$$t = \boxed{-\mathbf{o} \cdot \mathbf{d} \pm \sqrt{(\mathbf{o} \cdot \mathbf{d})^2 - |\mathbf{o}|^2 + 1}}$$

Why two solutions?

Ray-plane intersection



- Suppose we have a plane $N^T x = c$
  - N - unit normal
  - c - offset
- How do we find intersection with ray $r(t) = o + td$?
- *Key idea:* again, replace the point x with the ray equation t:

$$\mathbf{N}^{\mathsf{T}}\mathbf{r}(t) = c$$

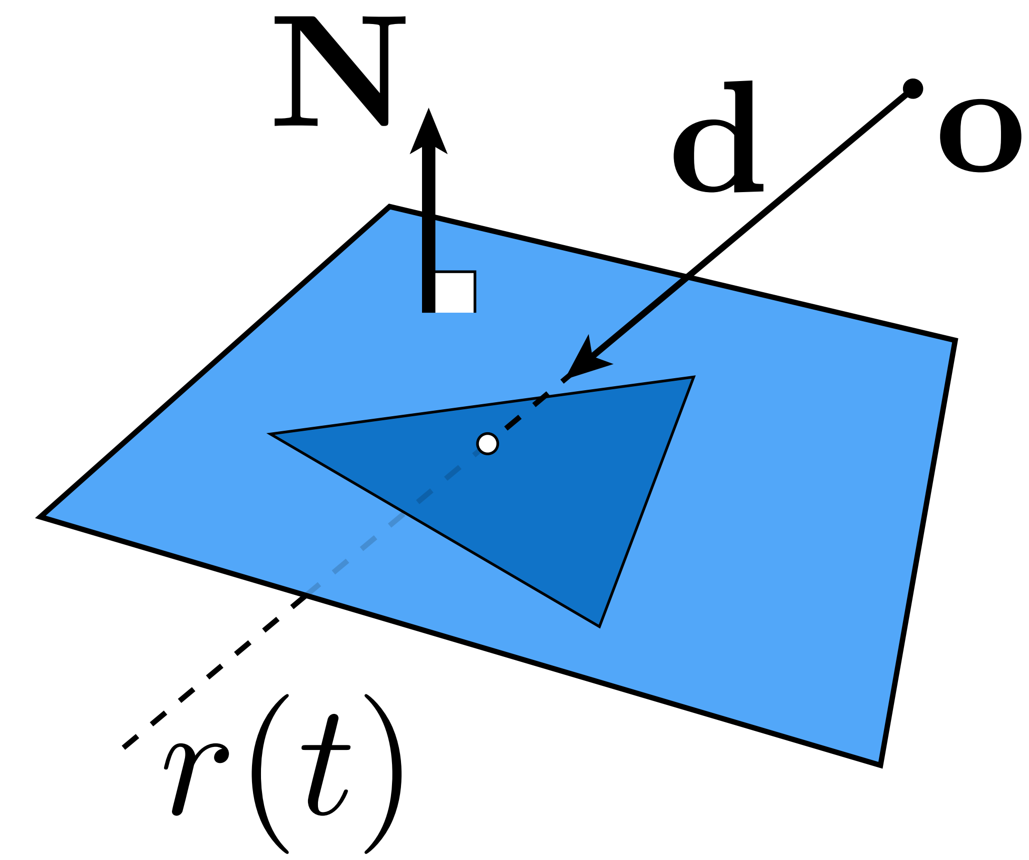
- Now solve for t:

$$\mathbf{N}^{\mathsf{T}}(\mathbf{o} + t\mathbf{d}) = c \qquad \Rightarrow t = \frac{c - \mathbf{N}^{\mathsf{T}}\mathbf{o}}{\mathbf{N}^{\mathsf{T}}\mathbf{d}}$$

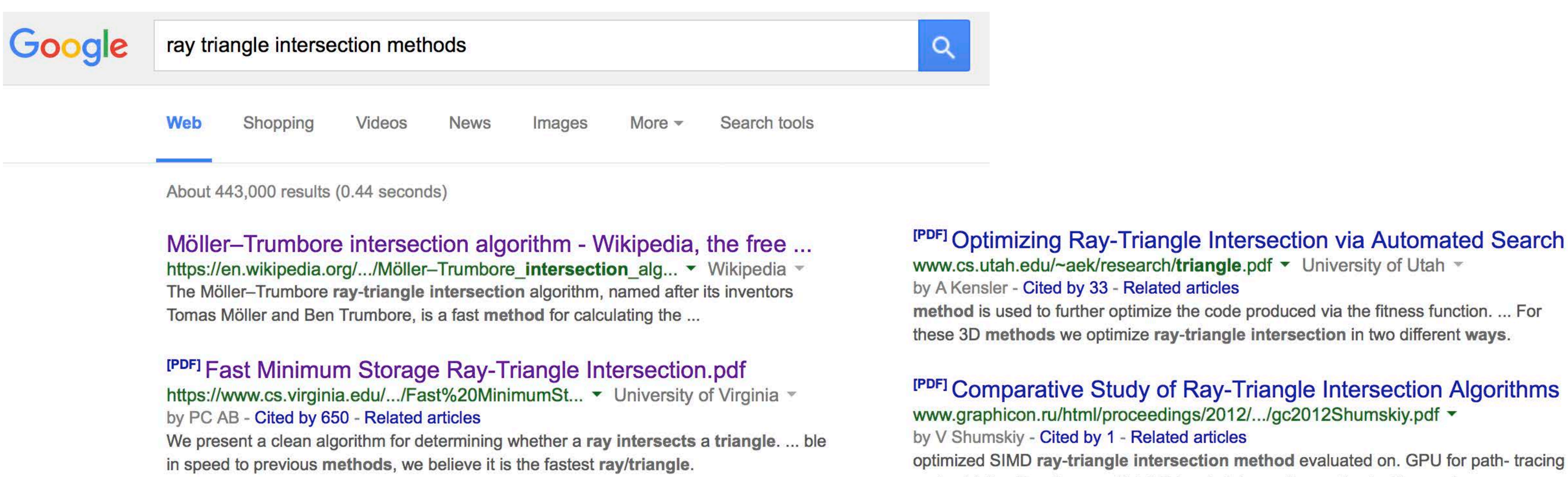
- And plug t back into ray equation:

$$r(t) = \mathbf{o} + \frac{c - \mathbf{N}^{\mathsf{T}}\mathbf{o}}{\mathbf{N}^{\mathsf{T}}\mathbf{d}}\mathbf{d}$$

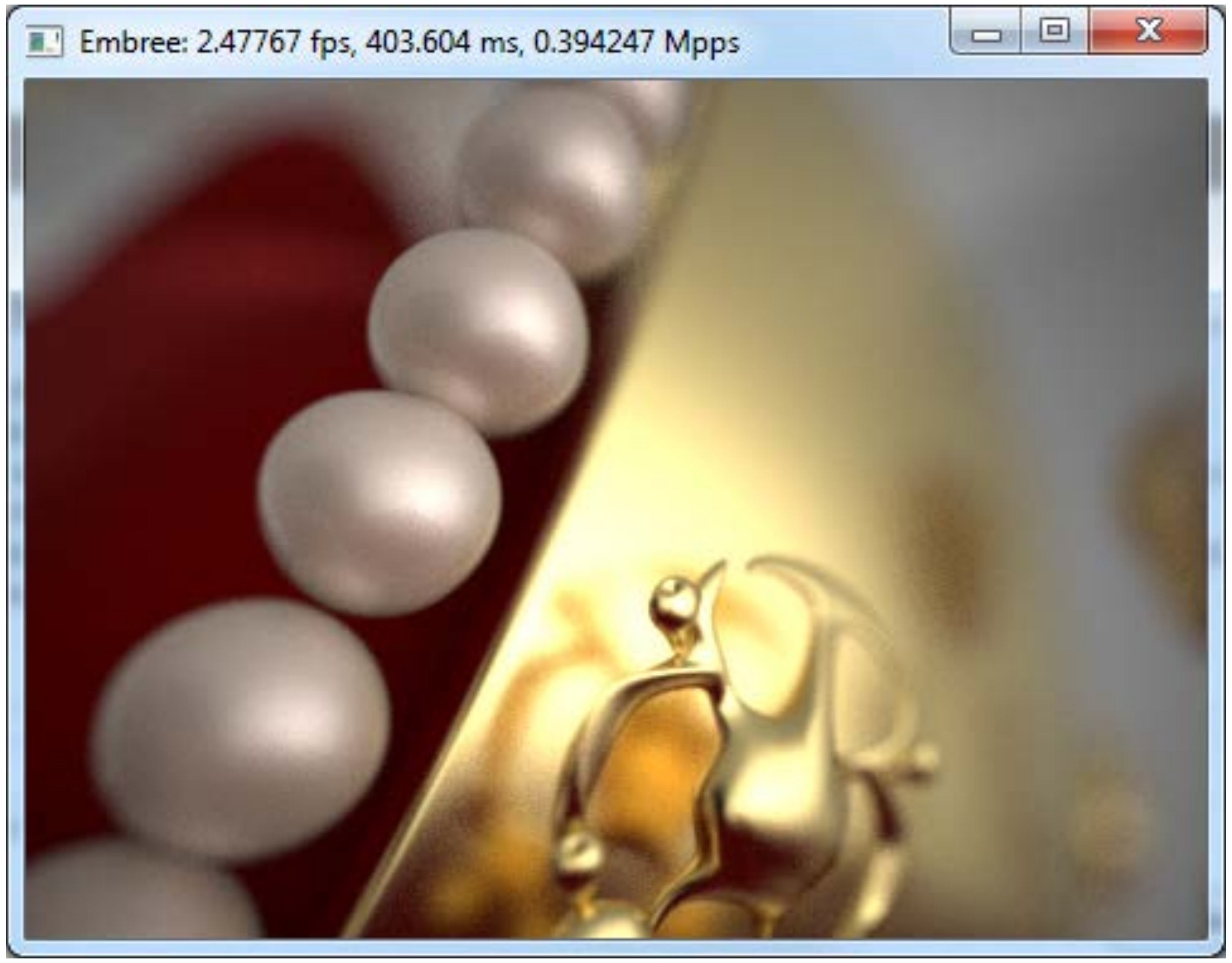
Ray-triangle intersection



- Triangle is in a plane...
- Not much more to say!
    - Compute ray-plane intersection
    - Q: What do we do now?
    - A: Why not compute barycentric coordinates of hit point?
    - If barycentric coordinates are all positive, point in triangle
- Actually, a *lot* more to say... if you care about performance!



Why care about performance?



Intel Embree

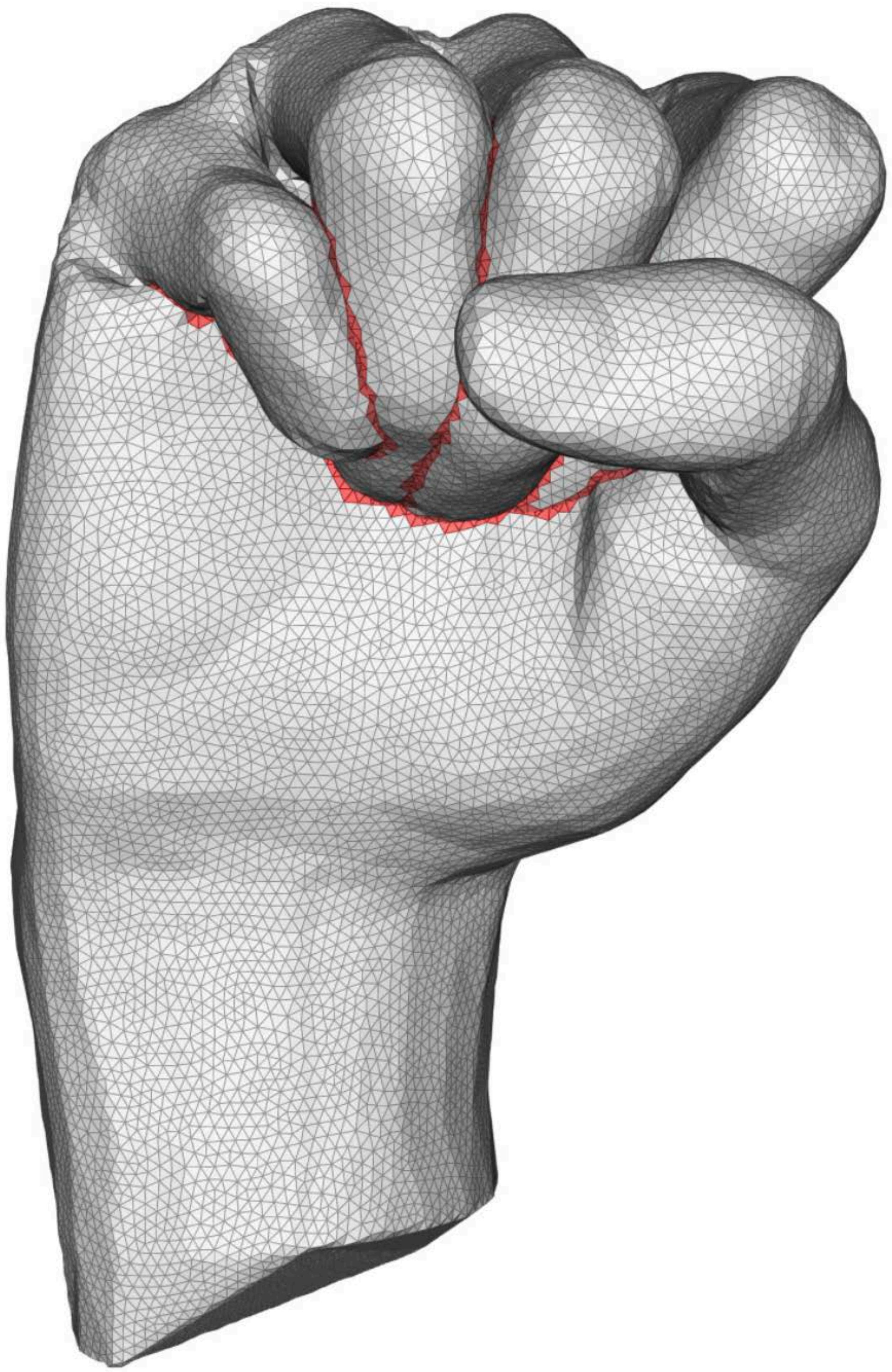
NVIDIA OptiX

Why care about performance?

"Brigade 3" real time path tracing demo

One more query: mesh-mesh intersection

- GEOMETRY: How do we know if a mesh intersects itself?
- ANIMATION: How do we know if a collision occurred?



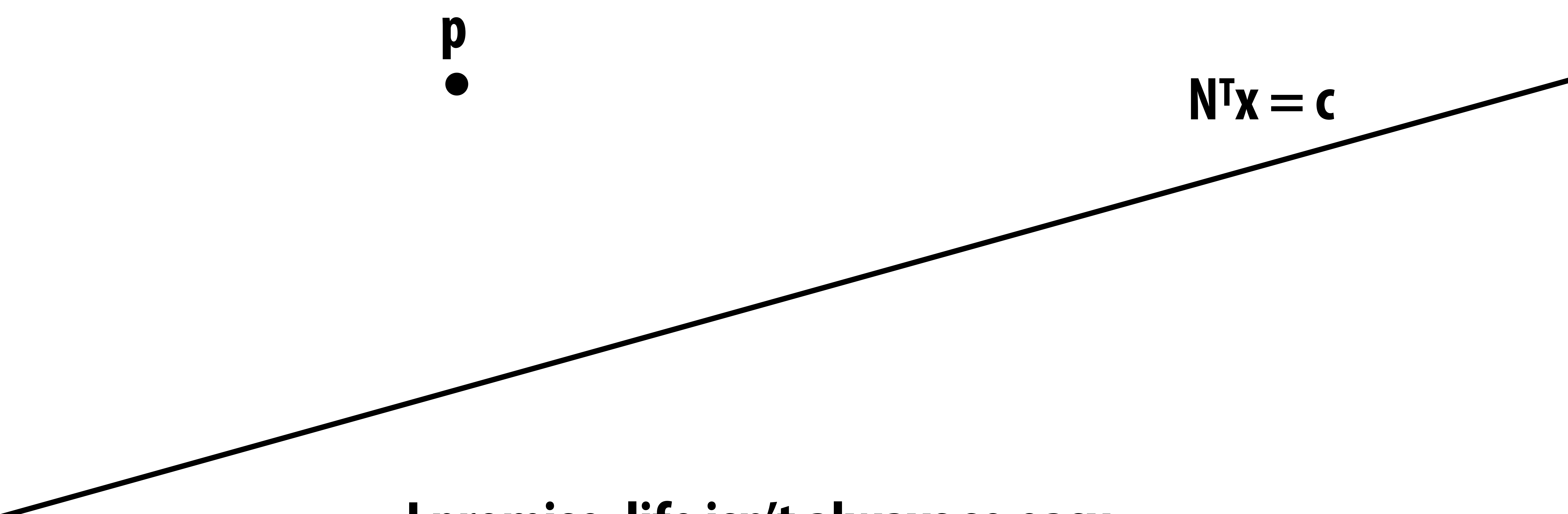
Warm up: point-point intersection

- **Q: How do we know if p intersects a?**
- **A: ...check if they're the same point!**

Sadly, life is not always so easy.

Slightly harder: point-line intersection

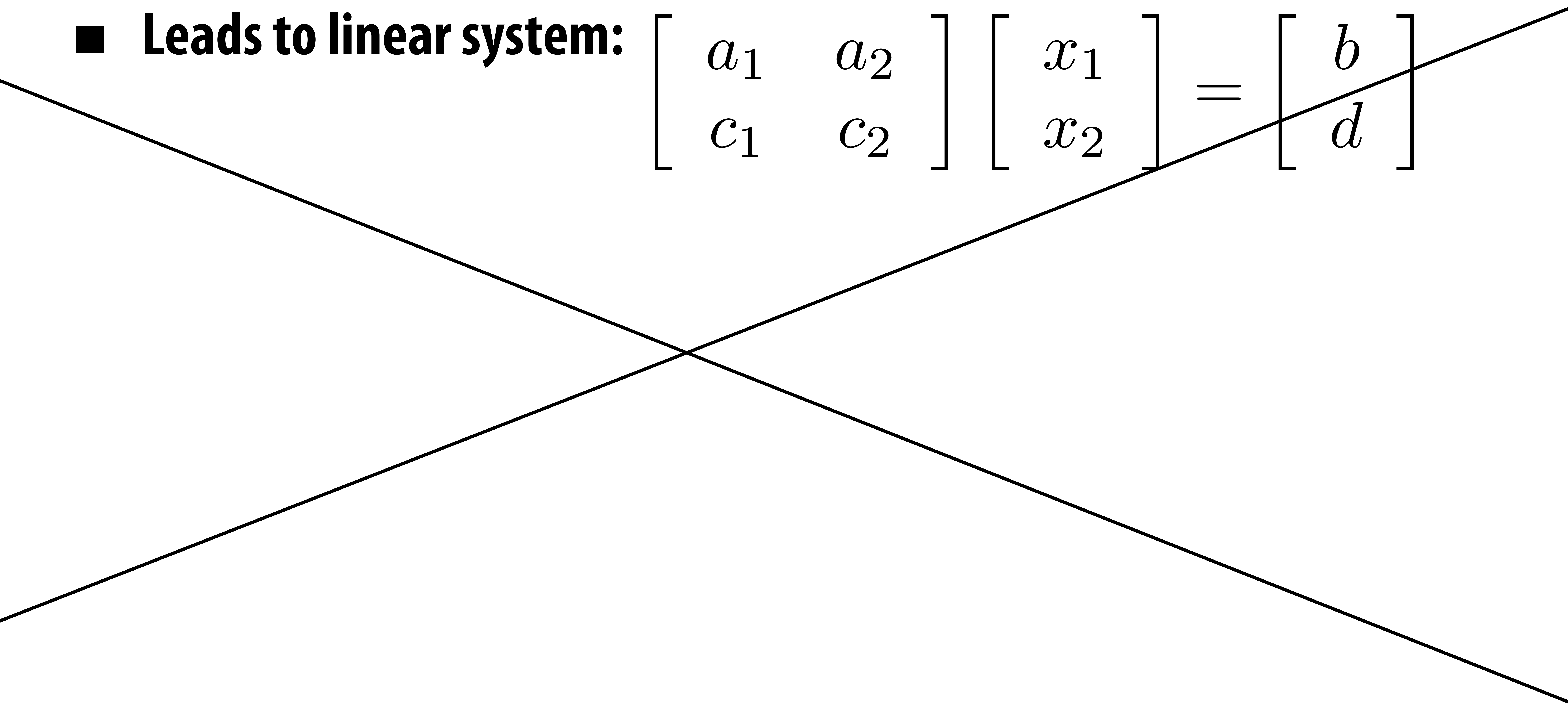
- **Q: How do we know if a point intersects a given line?**
- **A: ...plug it into the line equation!**



I promise, life isn't always so easy.

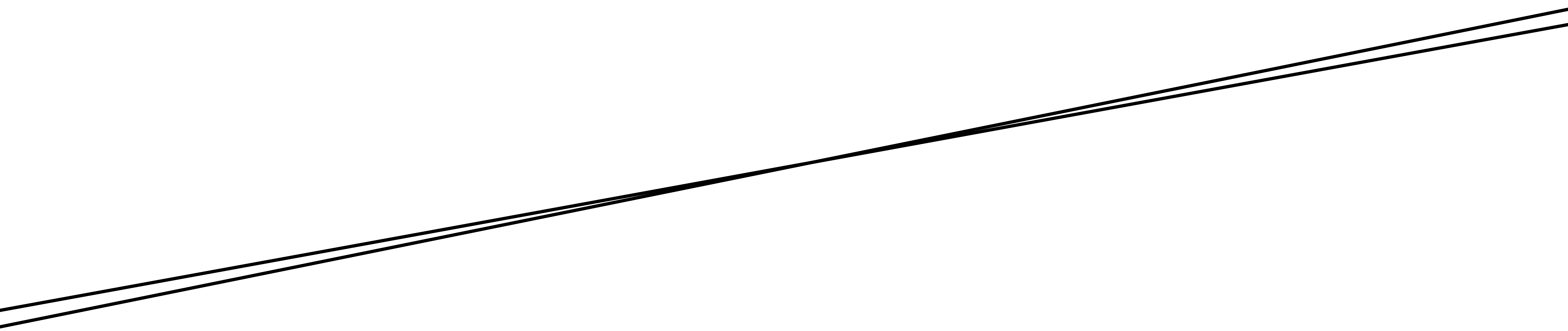
Finally interesting: line-line intersection

- **Two lines: ax=b and cx=d**
- **Q: How do we find the intersection?**
- **A: See if there is a simultaneous solution**
- **Leads to linear system:** $$\begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



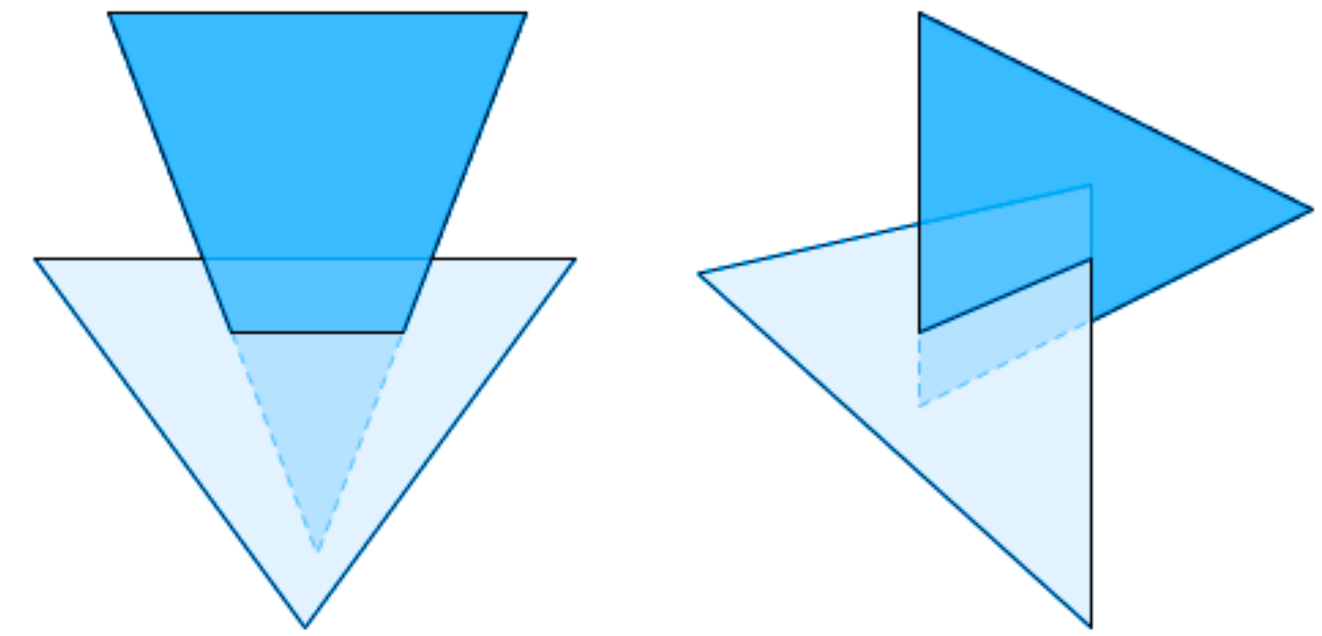
Degenerate line-line intersection?

- What if lines are almost parallel?
- Small change in normal can lead to big change in intersection!
- Instability very common, very important with geometric predicates. Demands special care (e.g., analysis of matrix).

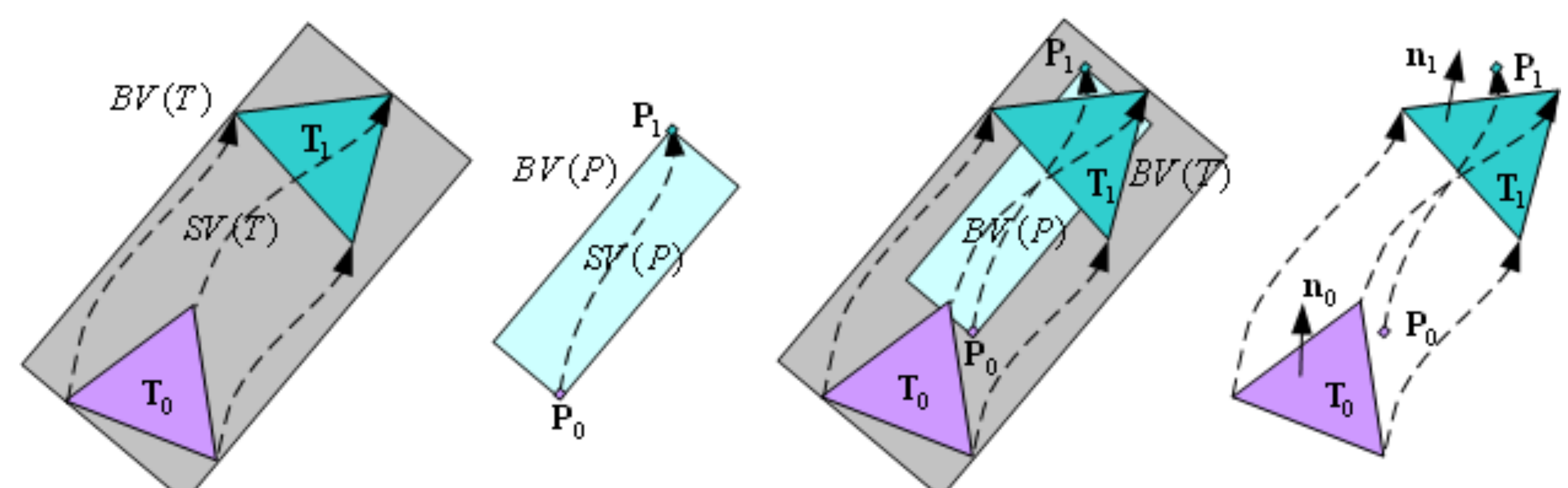


See for example Shewchuk, "*Adaptive Precision Floating-Point Arithmetic and Fast Robust Geometric Predicates*"

Triangle-Triangle Intersection?



- Lots of ways to do it
- Basic idea:
  - Q: Any ideas?
  - One way: reduce to edge-triangle intersection
  - Check if each line passes through plane
  - Then do interval test
- What if triangle is *moving*?
  - Important case for animation
  - Can think of triangles as *prisms* in time
  - Turns dynamic problem (nD + time) into purely geometric problem in (n+1)-dimensions



(a) Bounding volume of a deforming triangle (b) Bounding volume of a deforming vertex (c) Bounding volume test (d) Coplanarity test

Up Next: Spatial Acceleration Data Strucutres

- Testing every element is *slow!*
- E.g., linearly scanning through a list vs. binary search
- Can apply this same kind of thinking to geometric queries