

# **Physically-Based Animation and PDEs**

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**Computer Graphics  
CMU 15-462/15-662**

# Last time: Optimization

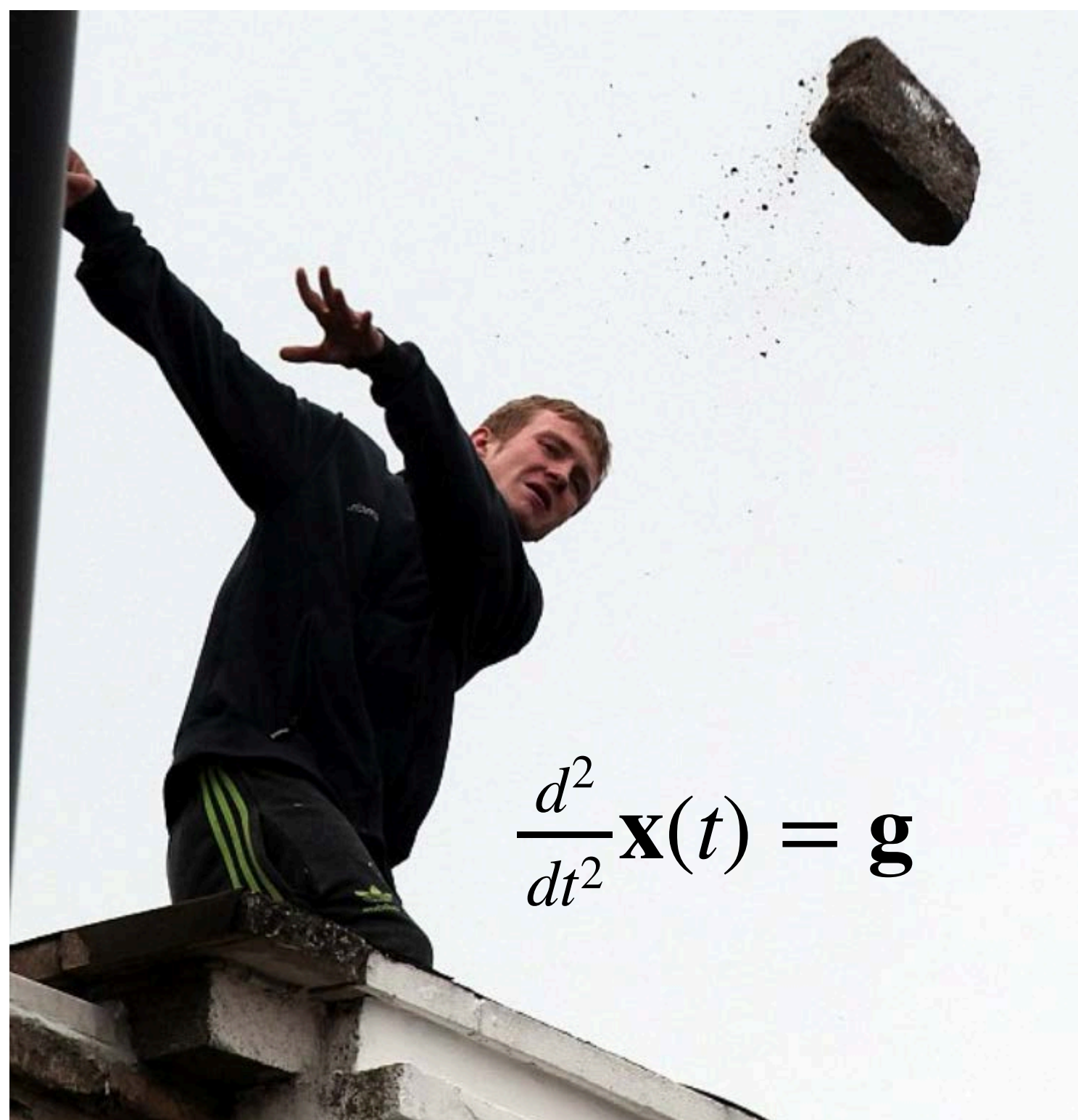
- Modern graphics uses optimization!
- Many complex criteria/constraints
- Basic technique: numerical descent
  - pick initial guess
  - ski downhill
  - keep fingers crossed!
- Gradient descent important example of *ordinary differential equation (ODE)*
- Today: return to differential equations
  - saw ODEs—derivatives in time
  - now PDEs—also have derivatives in space
  - describe many natural phenomena (water, smoke, cloth, ...)
  - recent revolution in CG/visual effects



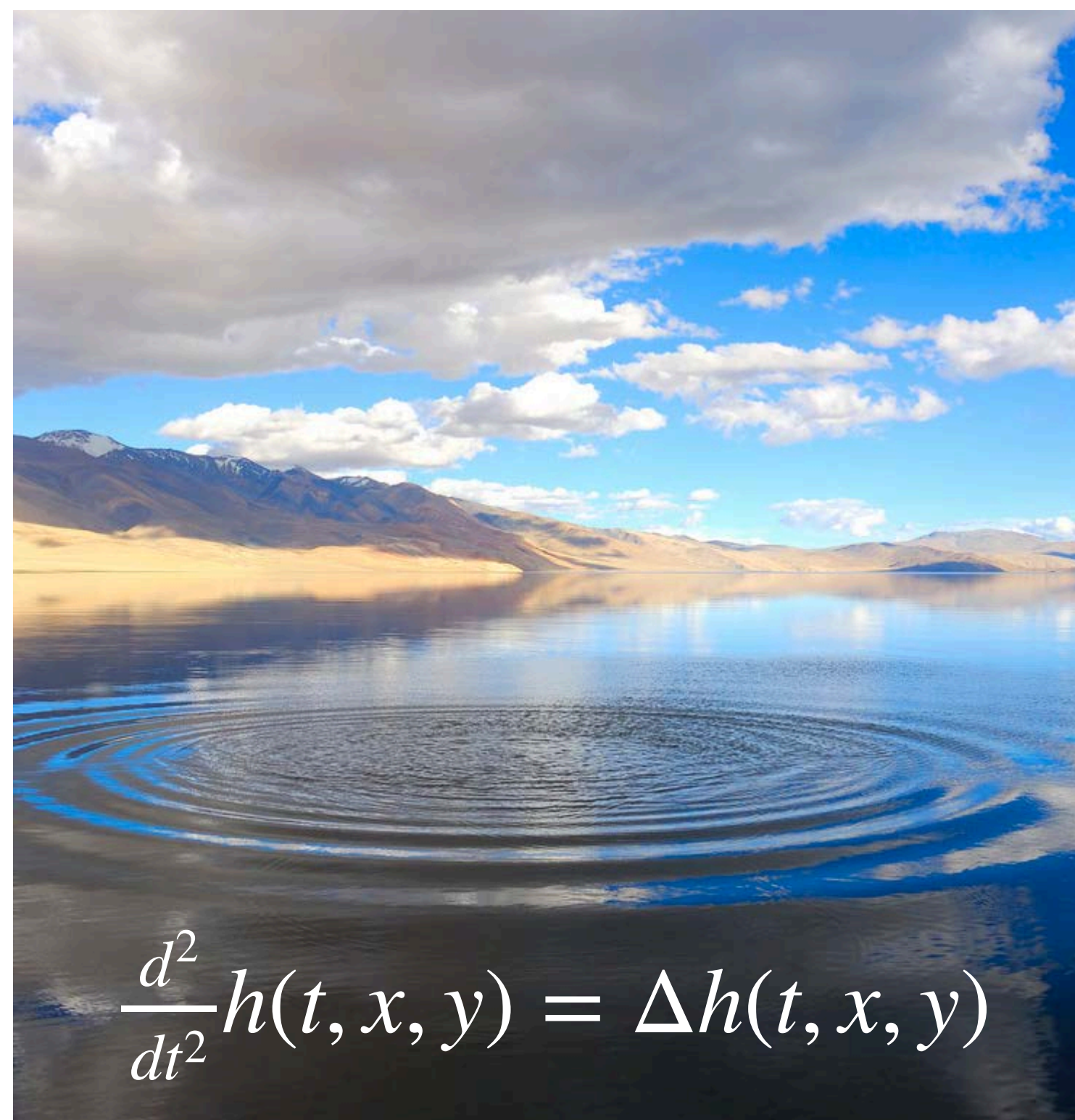
# Partial Differential Equations (PDEs)

- **ODE: *Implicitly* describe function in terms of its time derivatives**
- **PDE: Also include spatial derivatives in implicit description**
- **Like any implicit description, have to solve for actual function**

**ODE—rock flies through air**



**PDE—rock lands in pond**



# To make a long story short...

- Solving ODE looks like *“add a little velocity each time”*

$$q_{k+1} = q_k + \tau f(q)$$

- Solving a PDE looks like *“take weighted combination of neighbors to get velocity (...and add a little velocity each time)”*

	1	
1	-4	1
	1	

$f(q)$

$$q_{k+1} = q_k + \tau f(q)$$

**...obviously there is a lot more to say here!**

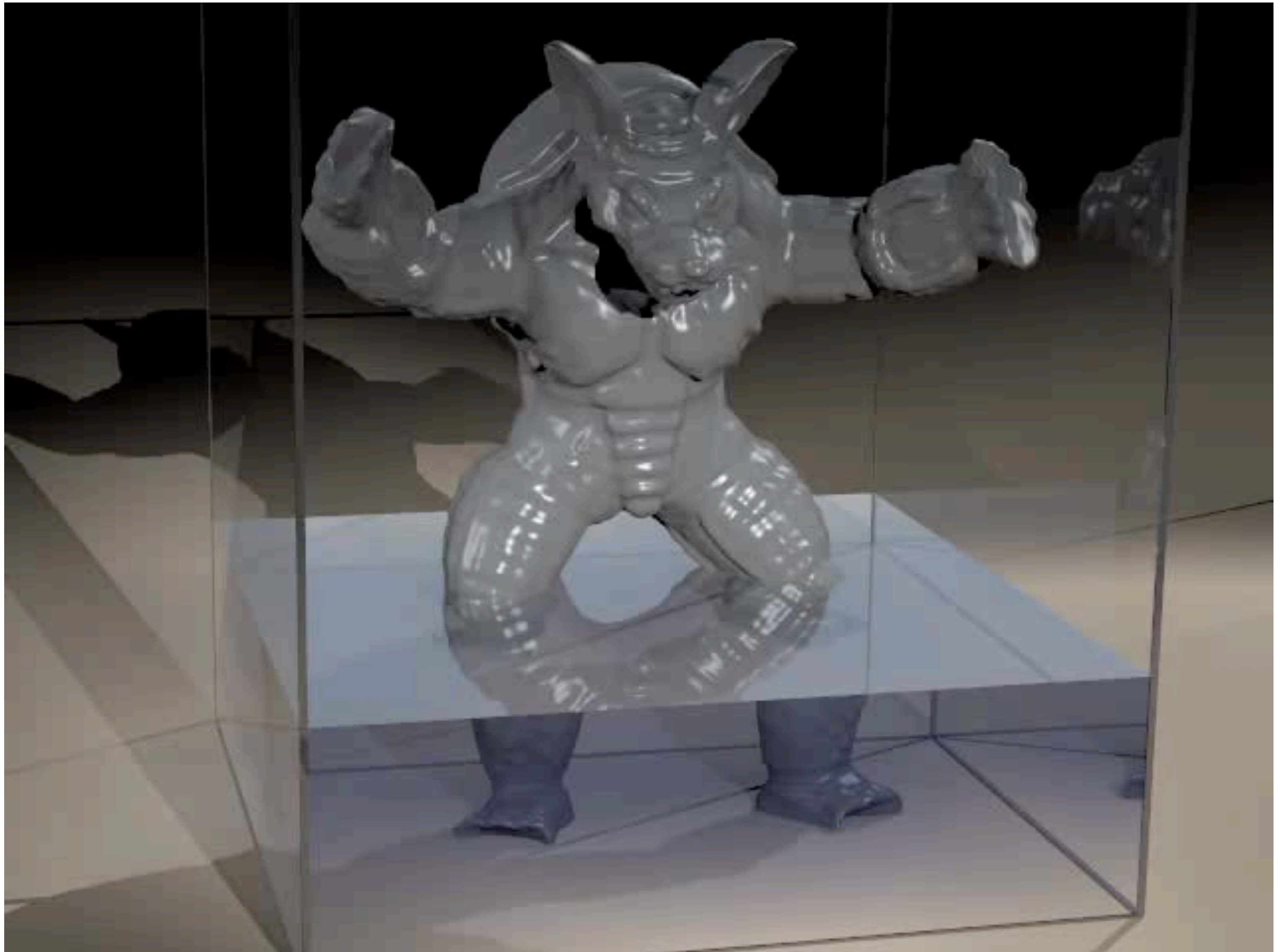
# Solving a PDE in Code

Don't be intimidated—very simple code can give rise to beautiful behavior!

```
void simulateWaves2D() {
    const int N = 128; // grid size
    double u[N][N]; // height
    double v[N][N]; // velocity (time derivative of height)
    const double tau = 0.2; // time step size
    const double alpha = 0.985; // damping factor

    for( int frame = 0; true; frame++ ) { // loop forever
        // drop random "stones"
        if( frame % 100 == 0 ) u[rand()%N][rand()%N] = -1;
        // update velocity
        for( int i = 0; i < N; i++ )
            for( int j = 0; j < N; j++ ) {
                int i0 = (i + N-1) % N; // left
                int i1 = (i + N+1) % N; // right
                int j0 = (j + N-1) % N; // down
                int j1 = (j + N+1) % N; // up
                v[i][j] += tau * (u[i0][j] + u[i1][j] + u[i][j0] + u[i][j1] - 4*u[i][j])
                v[i][j] *= alpha; // damping
            }
        // update height
        for( int i = 0; i < N; i++ )
            for( int j = 0; j < N; j++ ) {
                u[i][j] += tau * v[i][j];
            }
        display( u );
    }
}
```

# Liquid Simulation in Graphics



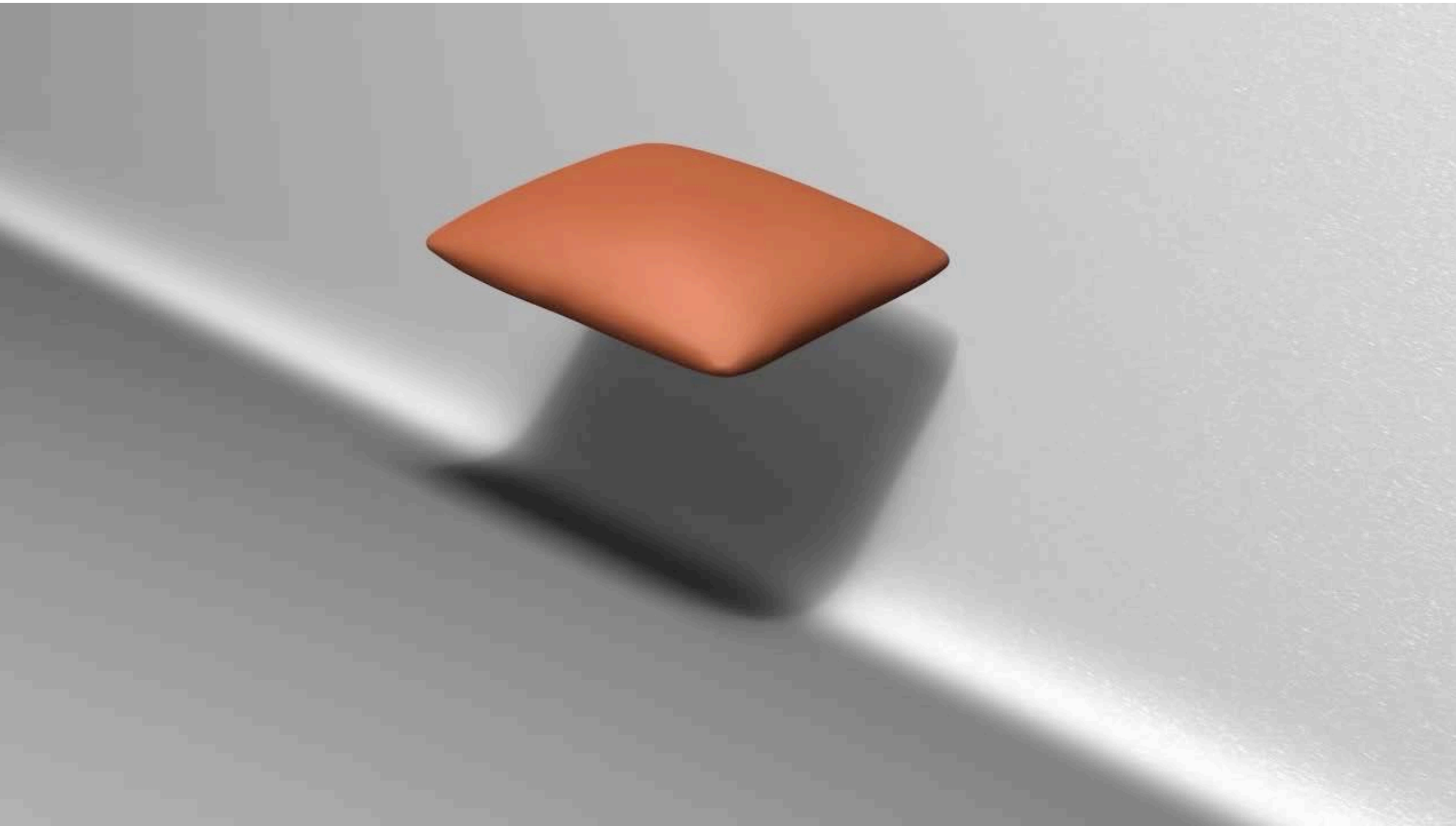
Losasso, F., Shinar, T. Selle, A. and Fedkiw, R., *"Multiple Interacting Liquids"*

# Smoke Simulation in Graphics



S. Weißmann, U. Pinkall. *"Filament-based smoke with vortex shedding and variational reconnection"*

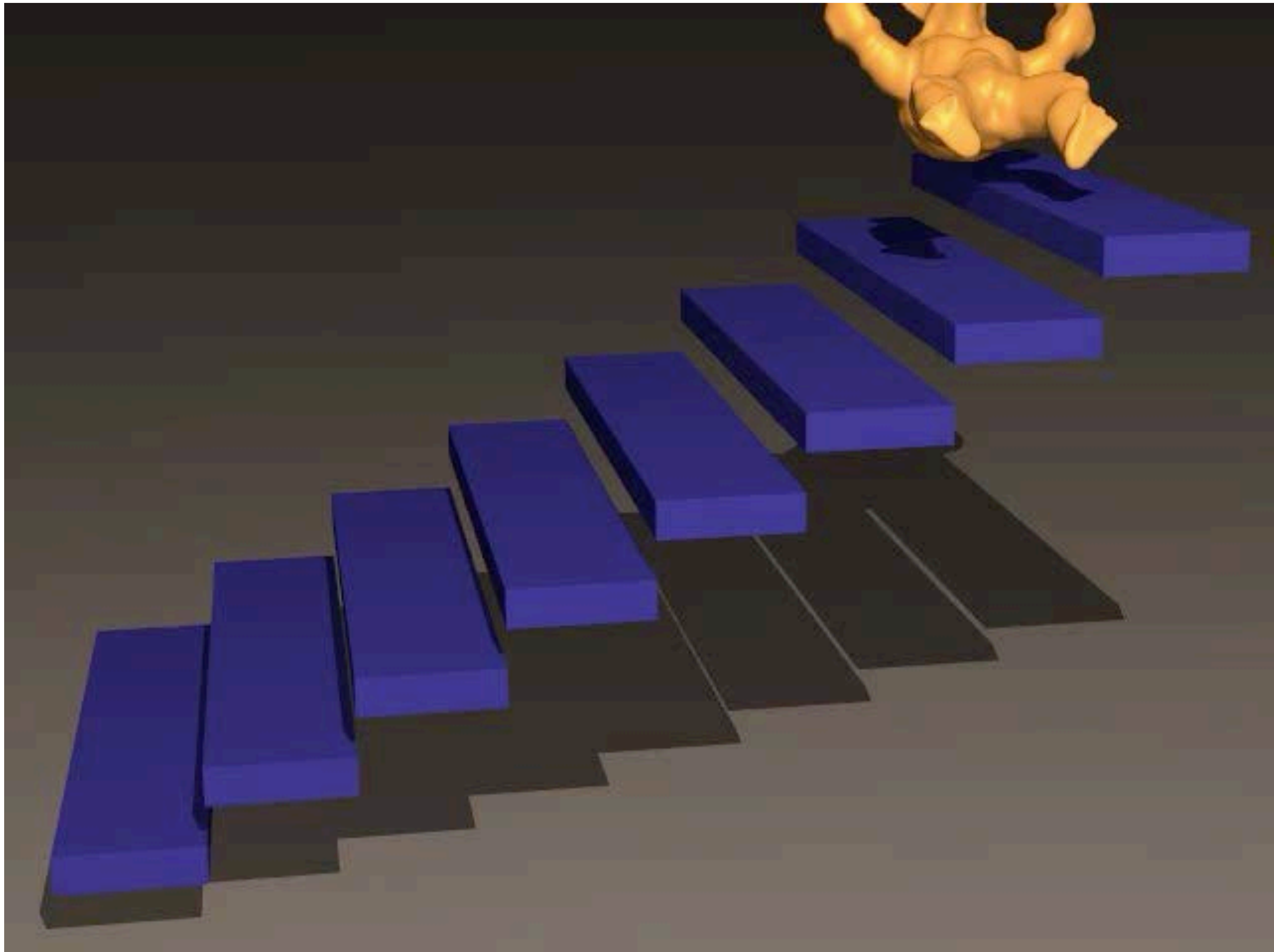
# Cloth Simulation in Graphics



Zhili Chen, Renguo Feng and Huamin Wang, *"Modeling friction and air effects between cloth and deformable bodies"*



# Elasticity in Graphics



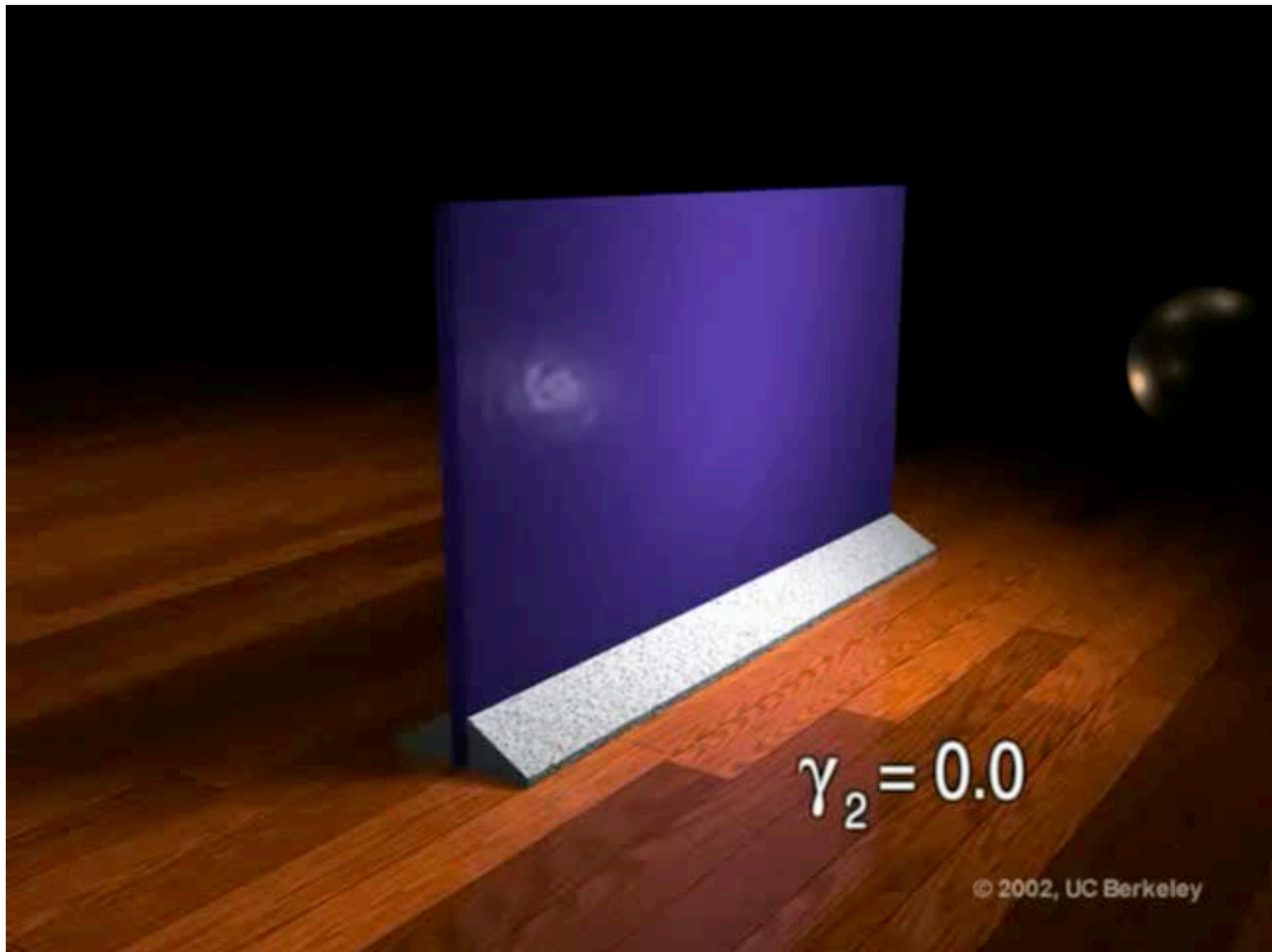
Irving, G., Schroeder, C. and Fedkiw, R., "Volume Conserving Finite Element Simulation of Deformable Models"

# Hair Simulation in Graphics



**Danny M. Kaufman, Rasmus Tamstorf, Breannan Smith, Jean-Marie Aubry, Eitan Grinspun,**  
*“Adaptive Nonlinearity for Collisions in Complex Rod Assemblies”*

# Fracture in Graphics



James F. O'Brien, Adam Bargteil, Jessica Hodgins, *"Graphical Modeling and Animation of Ductile Fracture"*

# Viscoelasticity in Graphics



Chris Wojtan, Greg Turk, *"Fast Viscoelastic Behavior with Thin Features"*

# Snow Simulation in Graphics



©Disney

**Alexey Stomakhin, Craig Schroeder, Lawrence Chai, Joseph Teran, Andrew Selle, "A Material Point Method For Snow Simulation"**

# Definition of a PDE

- Want to solve for a function of time *and* space

$$u(t, x)$$

time      space

- Function given implicitly in terms of derivatives:

$$\dot{u}, \ddot{u}, \frac{d^3 u}{dt^3}, \frac{d^4 u}{dt^4}, \dots$$

any combination of time derivatives

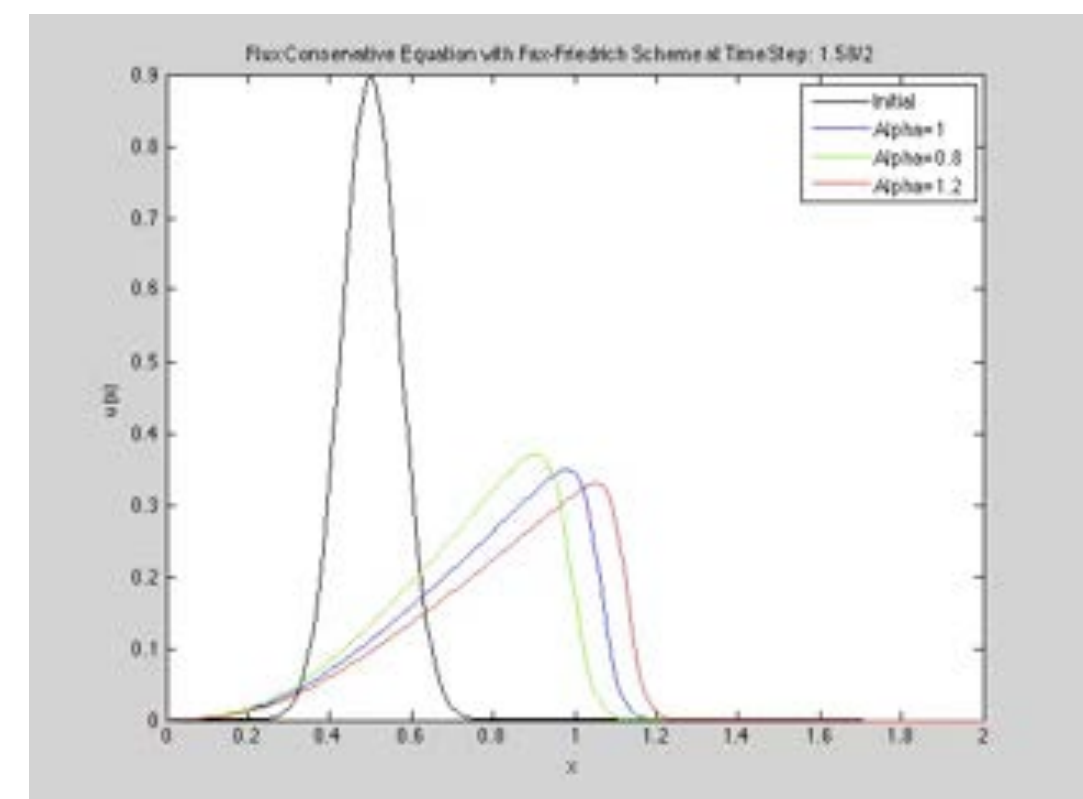
$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^{m+n} u}{\partial x_i^m \partial x_j^n}, \dots$$

plus any combination of space derivatives

- Example:

$$\frac{du}{dt} + u \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

(Burgers' equation)



# Anatomy of a PDE

- Linear vs. nonlinear: how are derivatives combined?

nonlinear!

$$\dot{u} + uu' = au'' \quad \text{(Burgers' equation)}$$

$$\dot{u} = au'' \quad \text{(diffusion equation)}$$

- Order: how many derivatives in space & time?

1st order in time

2nd order in space

$$\dot{u} + uu' = au'' \quad \text{(Burgers' equation)}$$

2nd order in time

2nd order in space

$$\ddot{u} = au'' \quad \text{(wave equation)}$$

**Rule of thumb: nonlinear / higher order  $\Rightarrow$  HARDER TO SOLVE!**

# Model Equations

- Fundamental behavior of many important PDEs is well-captured by three model linear equations:

**LAPLACE EQUATION (“ELLIPTIC”)**  $\Delta u = 0$

“what’s the smoothest function interpolating the given boundary data”

“Laplacian” (more later!)

**HEAT EQUATION (“PARABOLIC”)**  $\dot{u} = \Delta u$

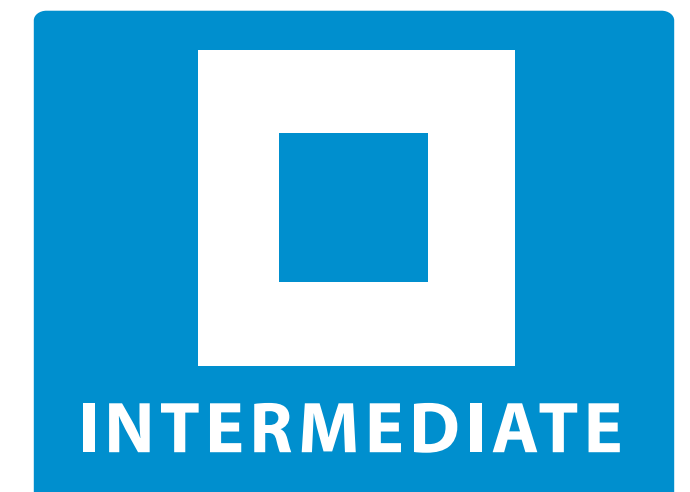
“how does an initial distribution of heat spread out over time?”

**WAVE EQUATION (“HYPERBOLIC”)**  $\ddot{u} = \Delta u$

“if you throw a rock into a pond, how does the wavefront evolve over time?”

[ NONLINEAR + HYPERBOLIC + HIGH-ORDER ]

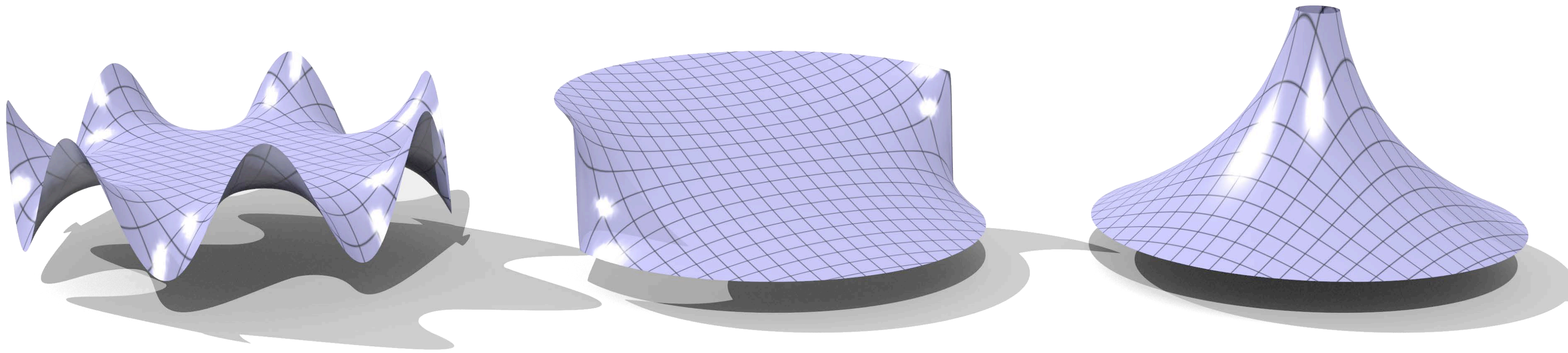
Solve numerically?





# Elliptic PDEs / Laplace Equation

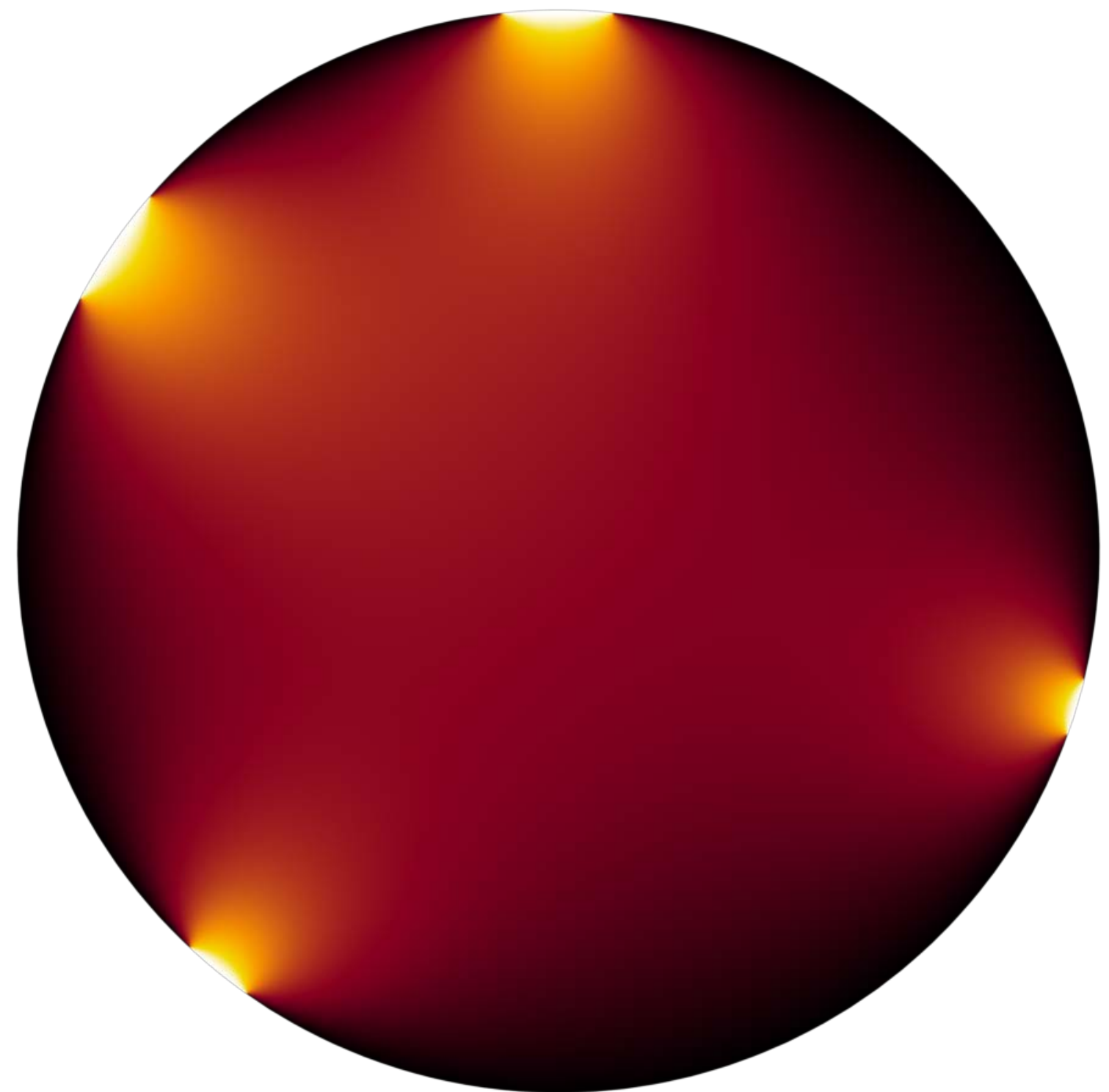
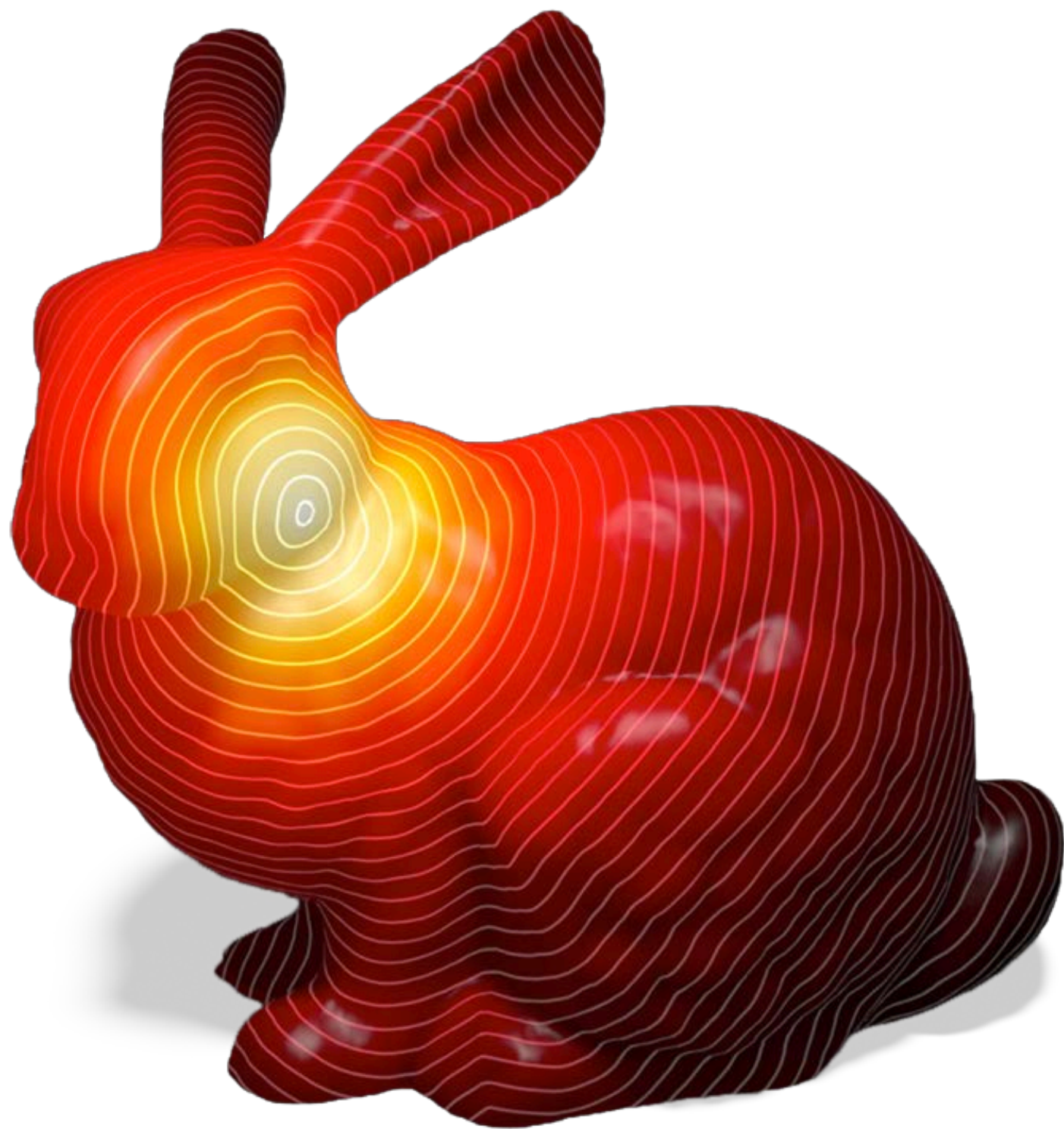
- **“What’s the smoothest function interpolating the given boundary data?”**



- **Conceptually: each value is at the average of its “neighbors”**
- **Roughly speaking, why is it easier to solve?**
- **Very robust to errors: just keep averaging with neighbors!**

# Parabolic PDEs / Heat Equation

- “How does an initial distribution of heat spread out over time?”



- After a long time, solution is same as Laplace equation!
- Models damping / viscosity in many physical systems

# Hyperbolic PDEs / Wave Equation

- **“If you throw a rock into a pond, how does the wavefront evolve over time?”**



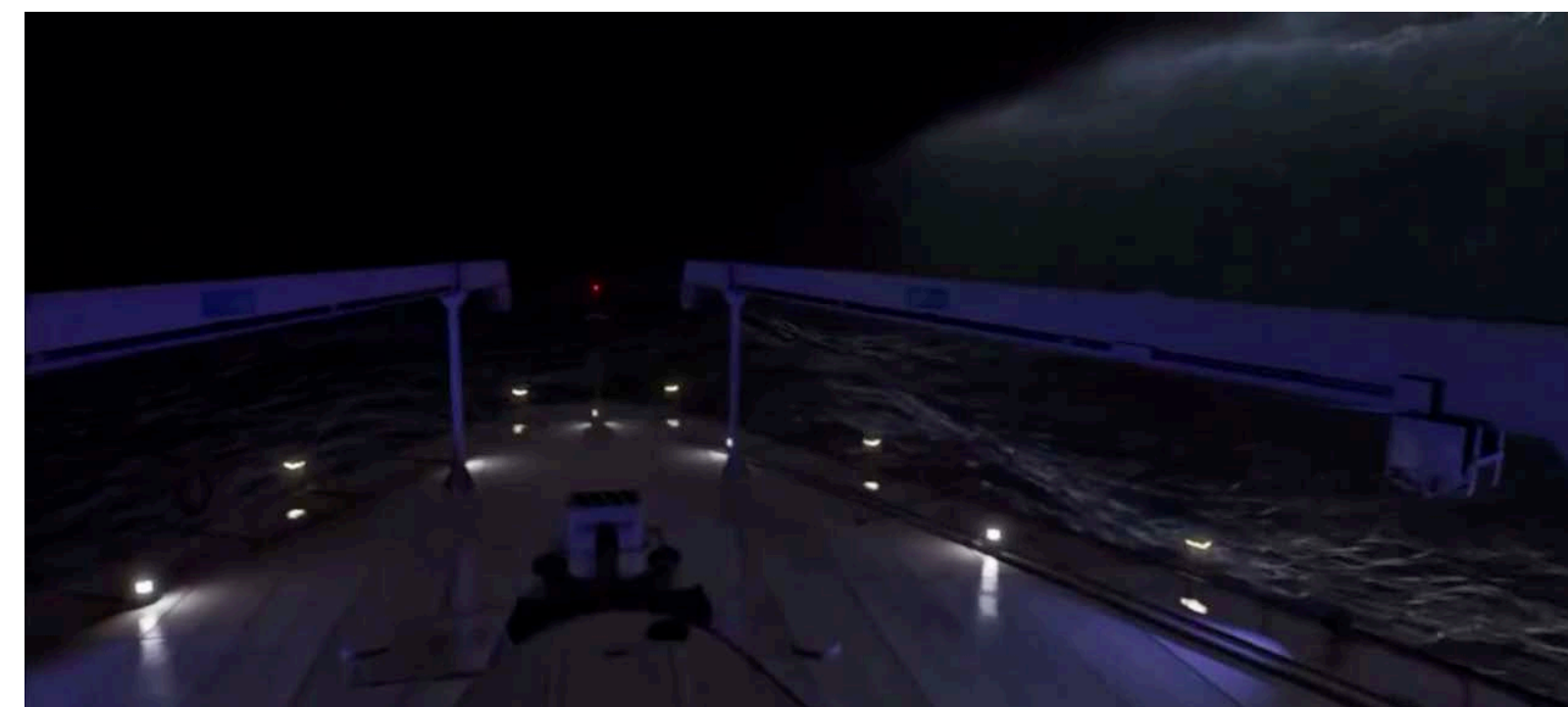
- **Errors made at the beginning will persist for a long time! (hard)**

**PDEs give an implicit description of solution.**

**How do we compute solutions explicitly?**

# Numerical Solution of PDEs—Overview

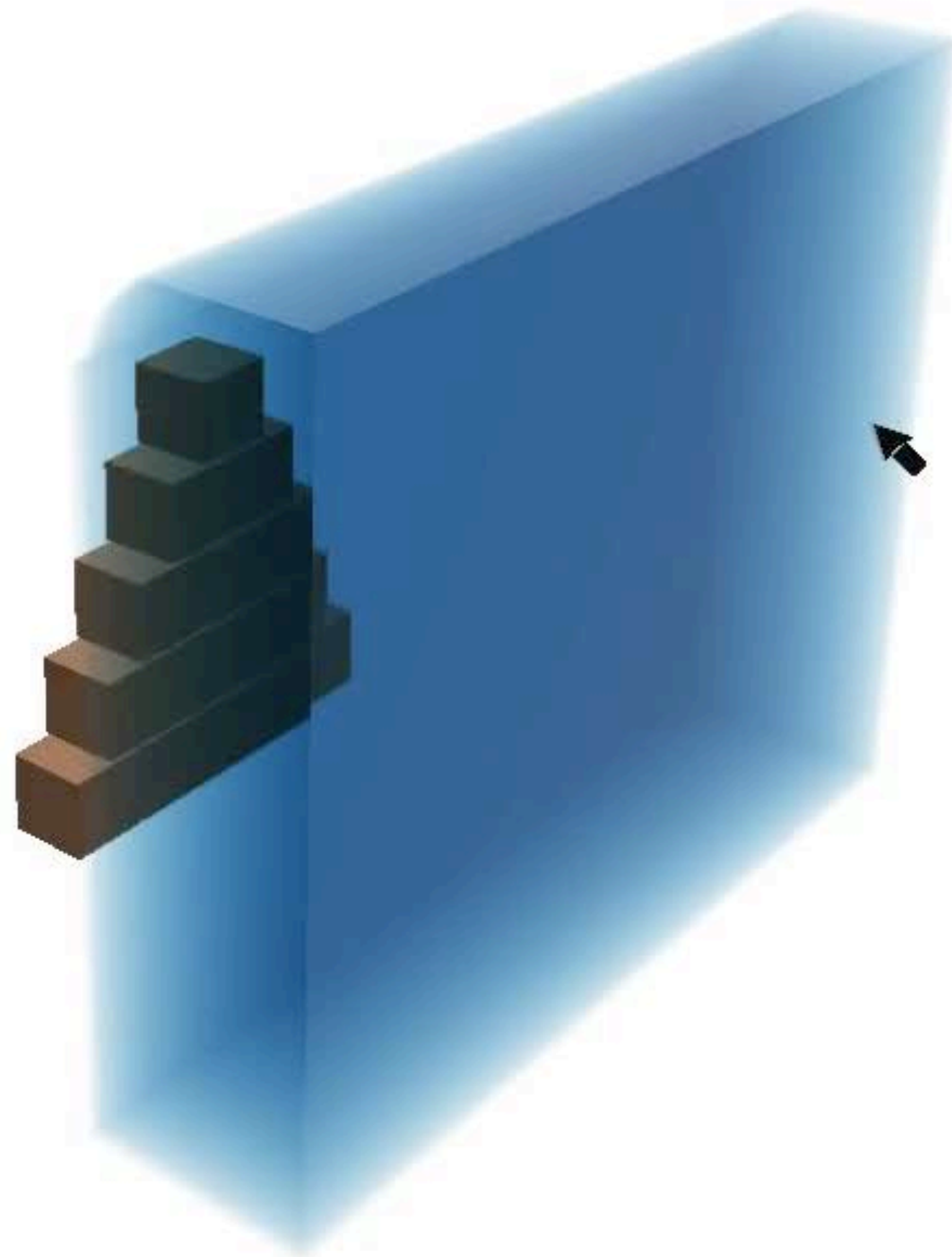
- Like ODEs, most PDEs are difficult/impossible to solve analytically—especially if we want to incorporate data!
- Must instead use numerical time integration
- Basic strategy:
  - pick a time discretization (forward Euler, backward Euler...)
  - pick a spatial discretization (**TODAY**)
  - as with ODEs, perform time-stepping to advance solution
- Historically, very expensive—only for “hero shots” in movies
- Computers are ever faster...
- More & more use of PDEs
  - games, interactive tools, ...



# Real Time PDE-Based Simulation (Fire)



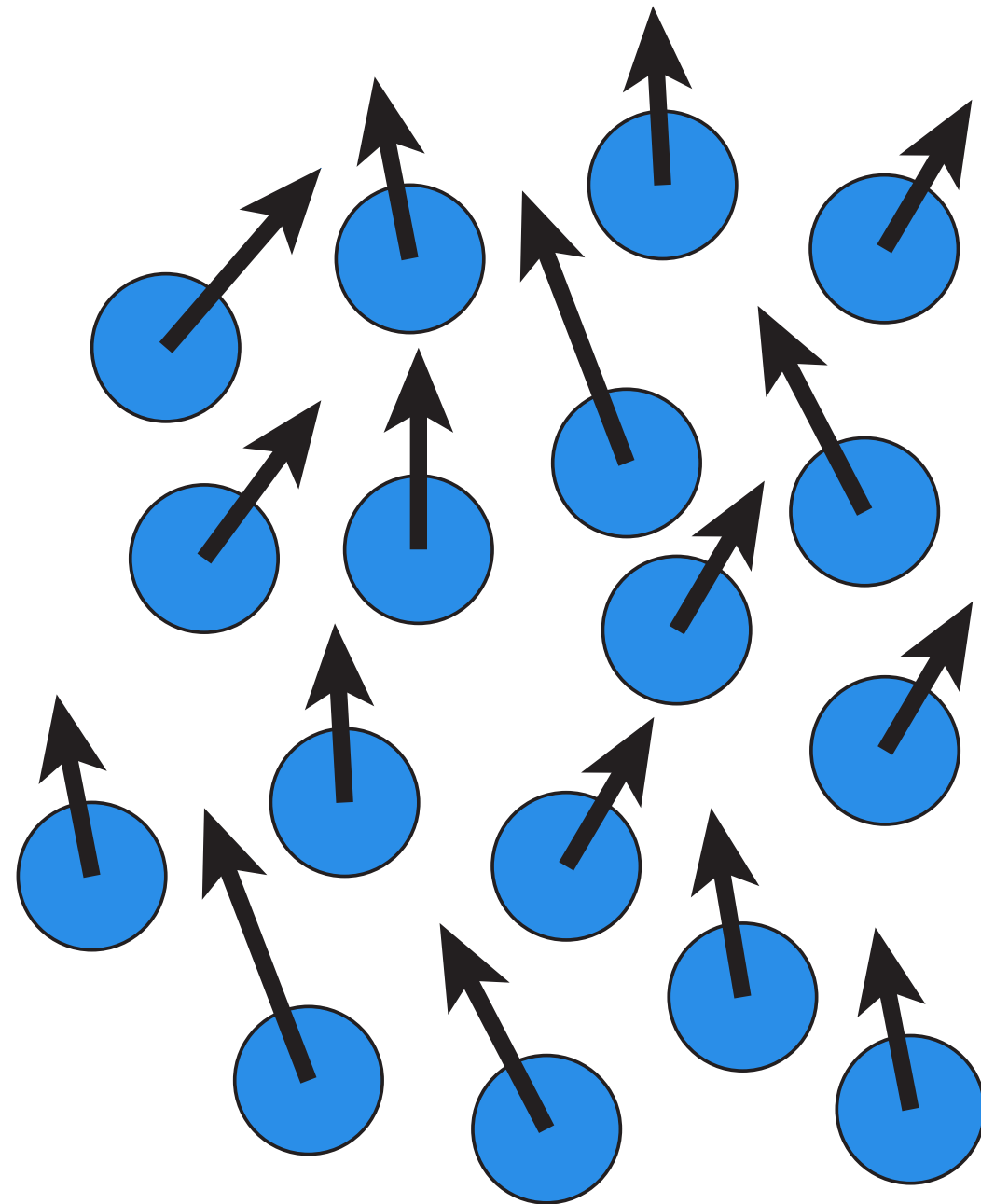
# Real Time PDE-Based Simulation (Water)



# Lagrangian vs. Eulerian

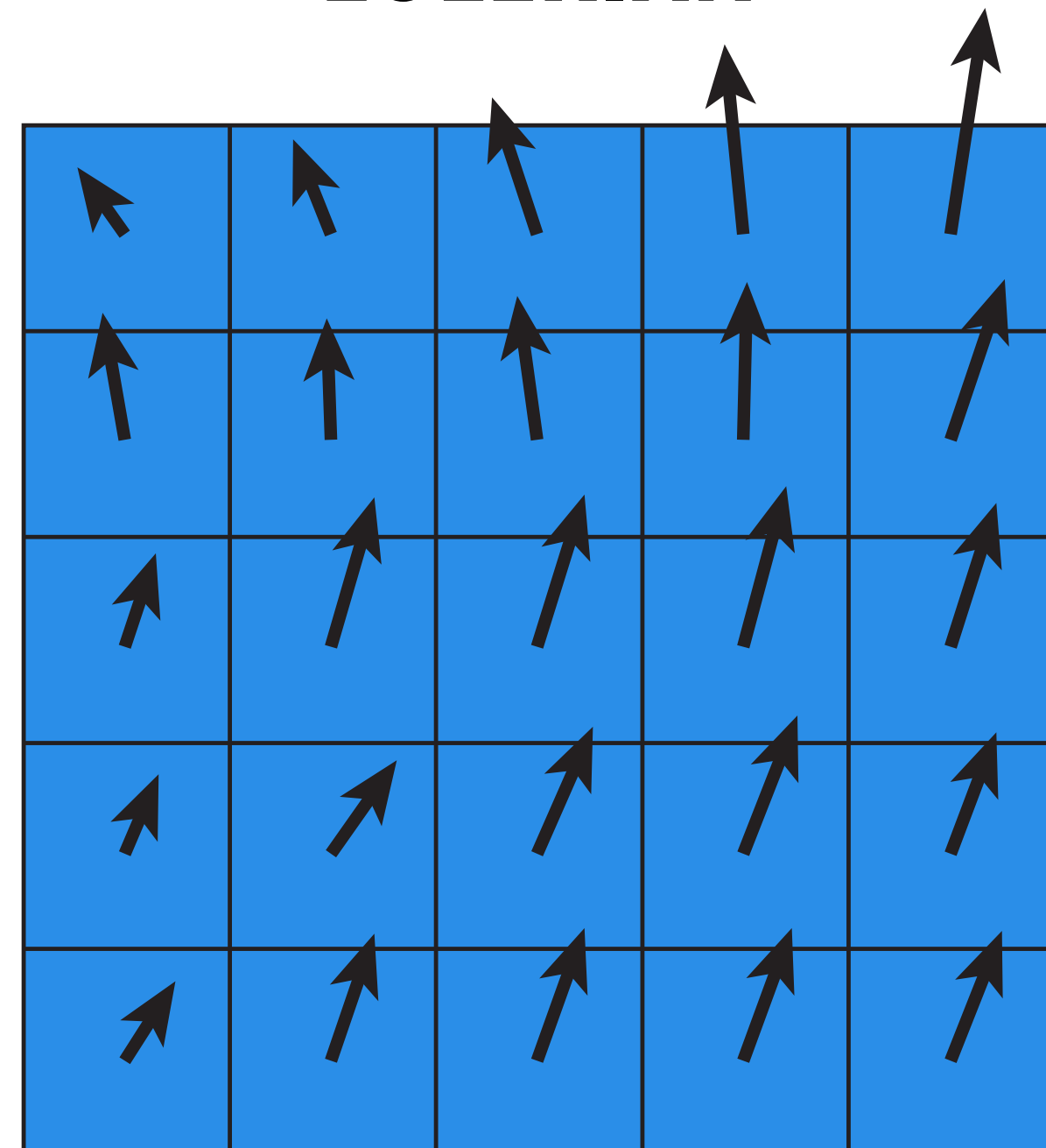
- Two basic ways to discretize space: *Lagrangian* & *Eulerian*
- E.g., suppose we want to encode the motion of a fluid

## LAGRANGIAN



**track position & velocity  
of moving particles**

## EULERIAN



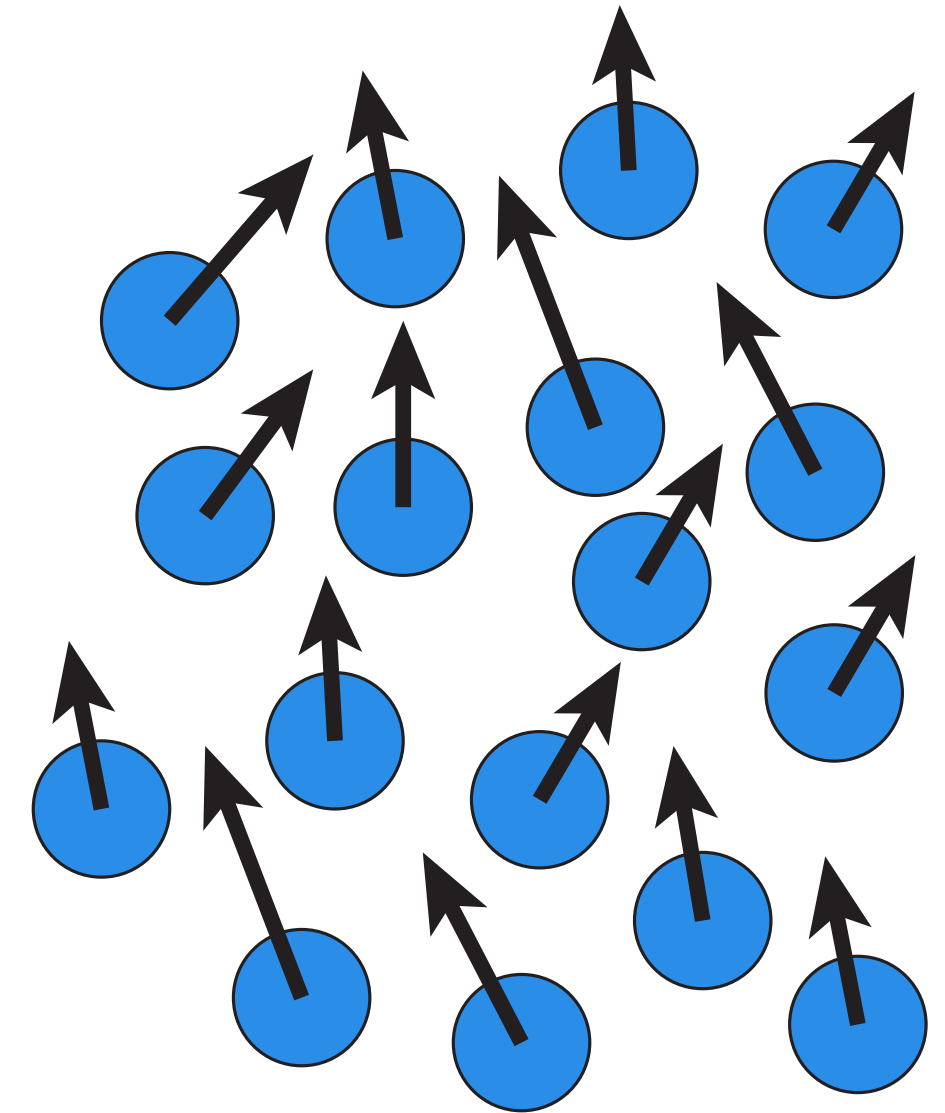
**track velocity (or *flux*)  
at fixed grid locations**



# Lagrangian vs. Eulerian—Trade-Offs

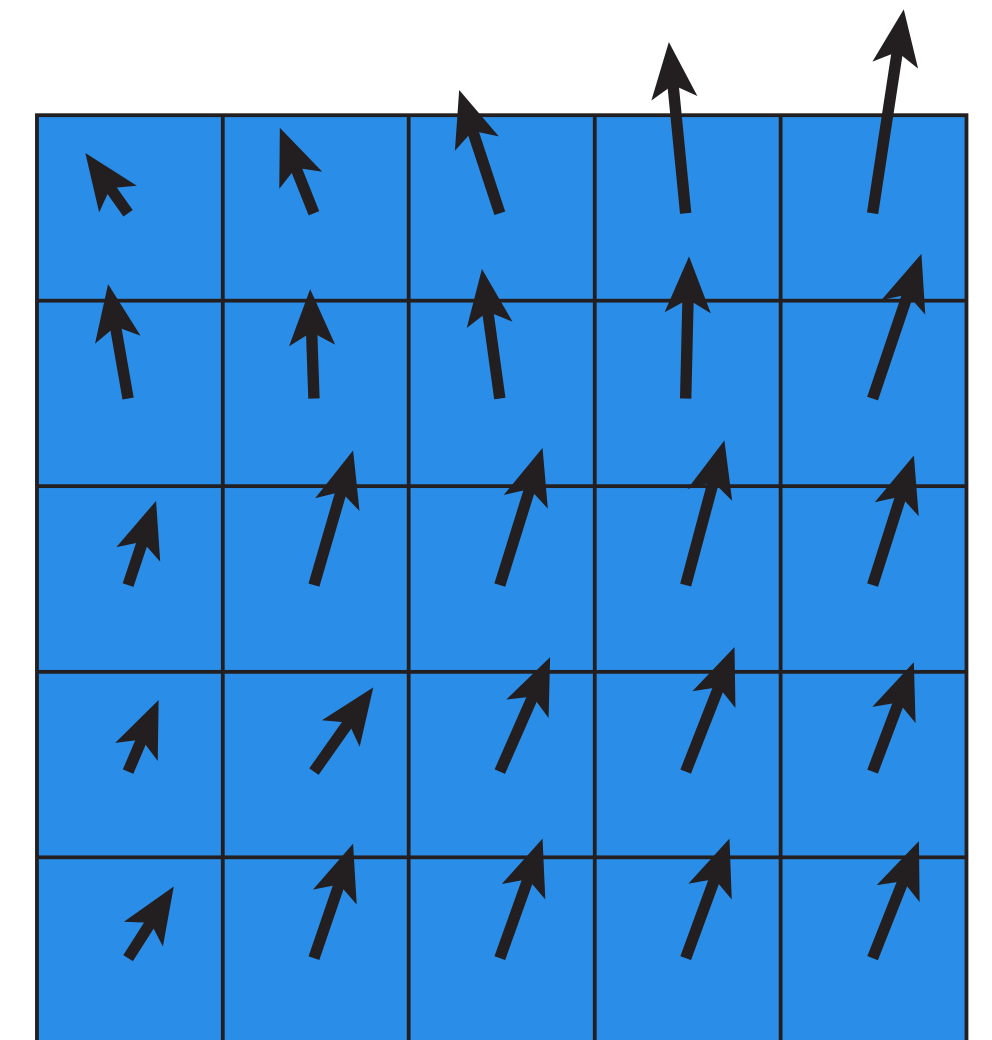
## ■ Lagrangian

- **conceptually easy (like polygon soup!)**
- **resolution/domain not limited by grid**
- **good particle distribution can be tough**
- **finding neighbors can be expensive**



## ■ Eulerian

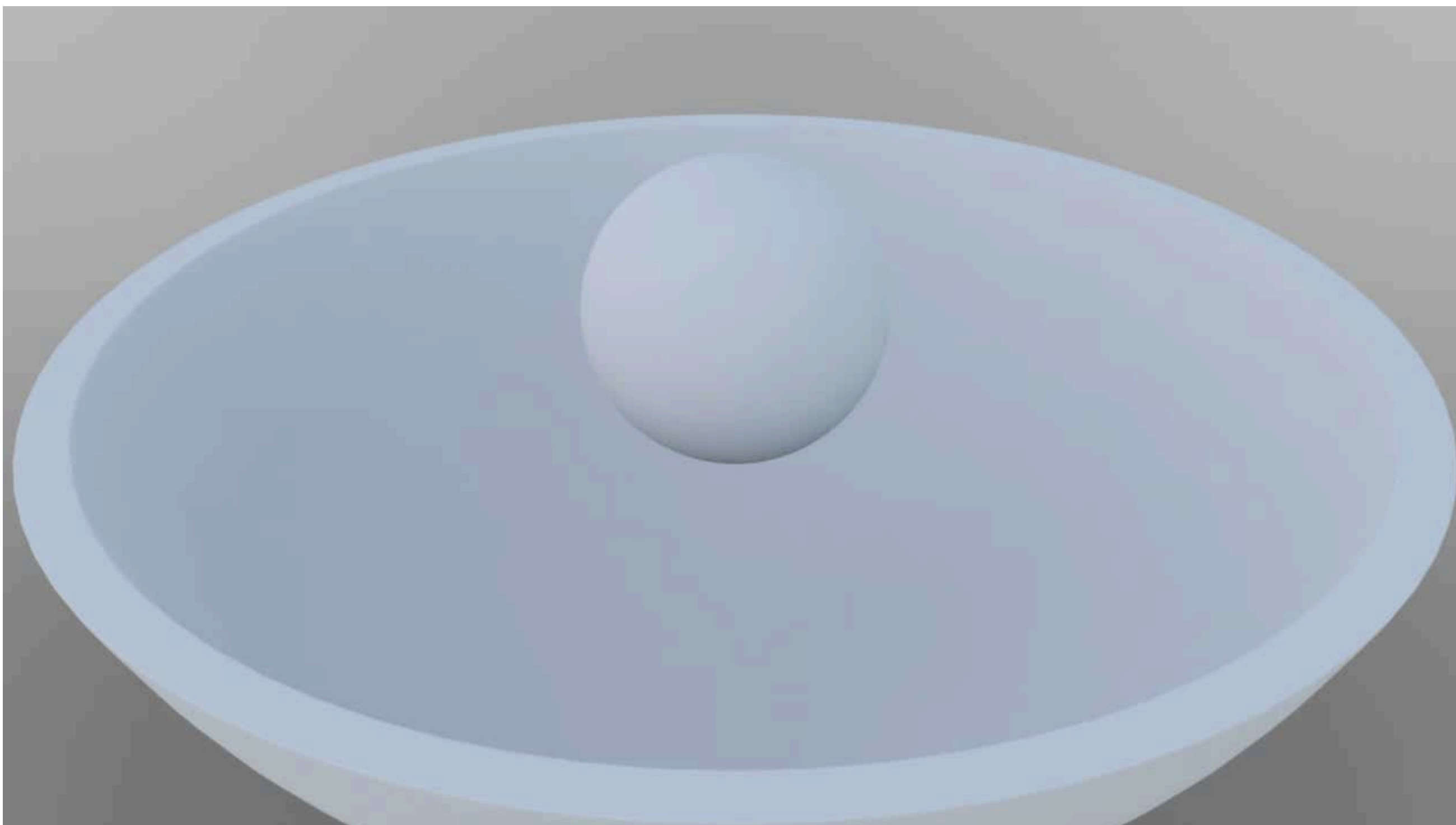
- **fast, regular computation**
- **easy to represent, e.g., smooth surfaces**
- **simulation “trapped” in grid**
- **grid causes “numerical diffusion” (blur)**
- **need to understand PDEs (but you will!)**



# Mixing Lagrangian & Eulerian

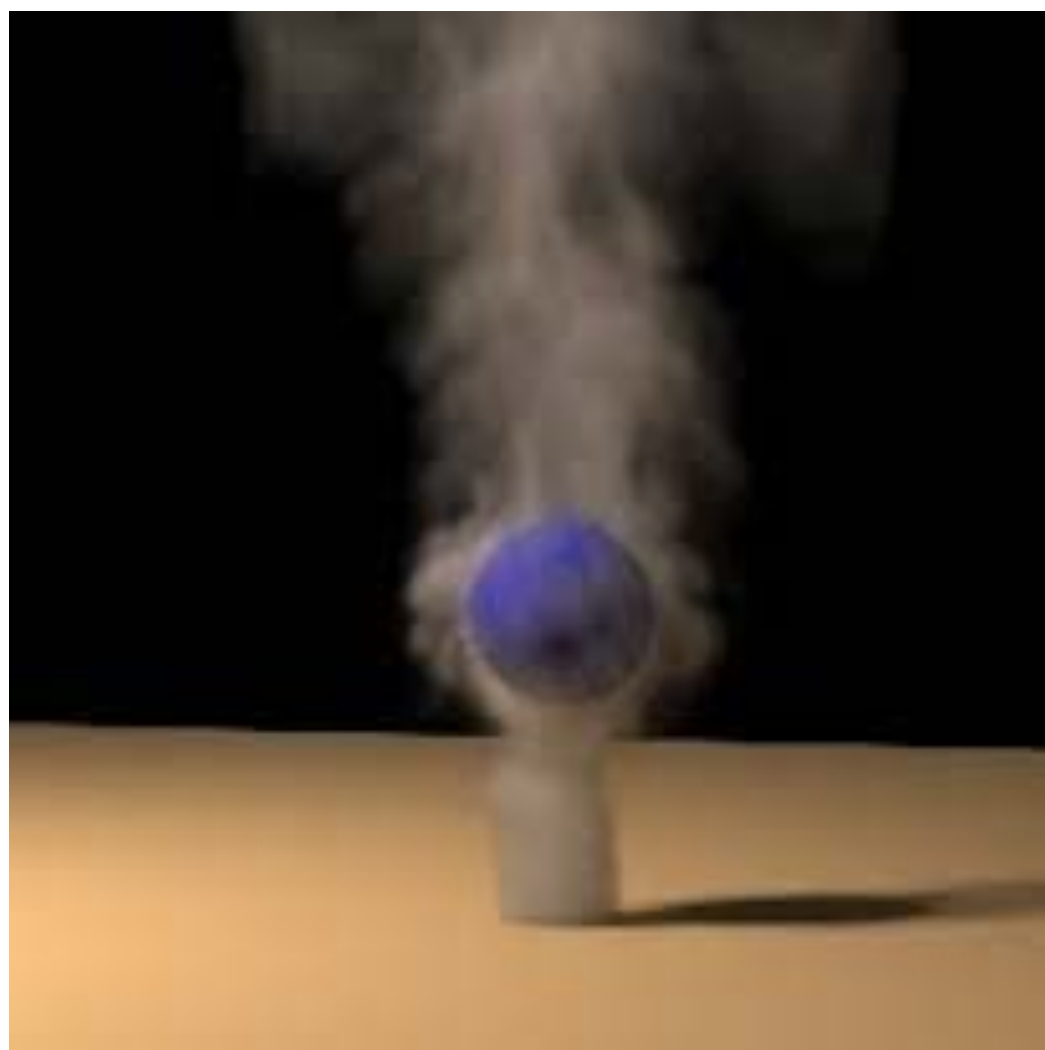
- Of course, no reason you have to choose just one!
- Many modern methods mix Lagrangian & Eulerian:
  - PIC/FLIP, particle level sets, mesh-based surface tracking, Voronoi-based, arbitrary Lagrangian-Eulerian (ALE), ...
- *Pick the right tool for the job!*

*Maya Bifrost*



# Aside: Which Quantity Do We Solve For?

- Many PDEs have mathematically equivalent formulations in terms of different quantities
- E.g., incompressible fluids:
  - *velocity*—how fast is each particle moving?
  - *vorticity*—how fast is fluid “spinning” at each point?
- Computationally, can make a big difference
- *Pick the right tool for the job!*



**Ok, but we're getting *way* ahead of ourselves.  
How do we solve *easy* PDEs?**

# Numerical PDEs—Basic Strategy

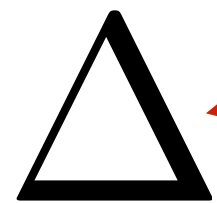
- **Pick PDE formulation**
  - Which quantity do we want to solve for?
  - E.g., velocity or vorticity?
- **Pick spatial discretization**
  - How do we approximate derivatives in space?
- **Pick time discretization**
  - How do we approximate derivatives in time?
  - When do we evaluate forces?
  - Forward Euler, backward Euler, symplectic Euler, ...
- **Finally, we have an *update rule***
- **Repeatedly solve to generate an animation**



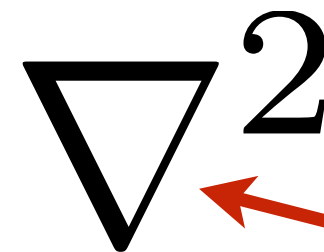
**Richard Courant**

# The Laplace Operator

- All of our model equations used the Laplace operator
- Different conventions for symbol:



← same symbol used for “change”



← same symbol used for Hessian!

- *Unbelievably* important object showing up everywhere across physics, geometry, signal processing, ...
- Ok, but what does it mean?
- *Differential operator*: eats a function, spits out its “2nd derivative”
- What does that mean for a function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ ?

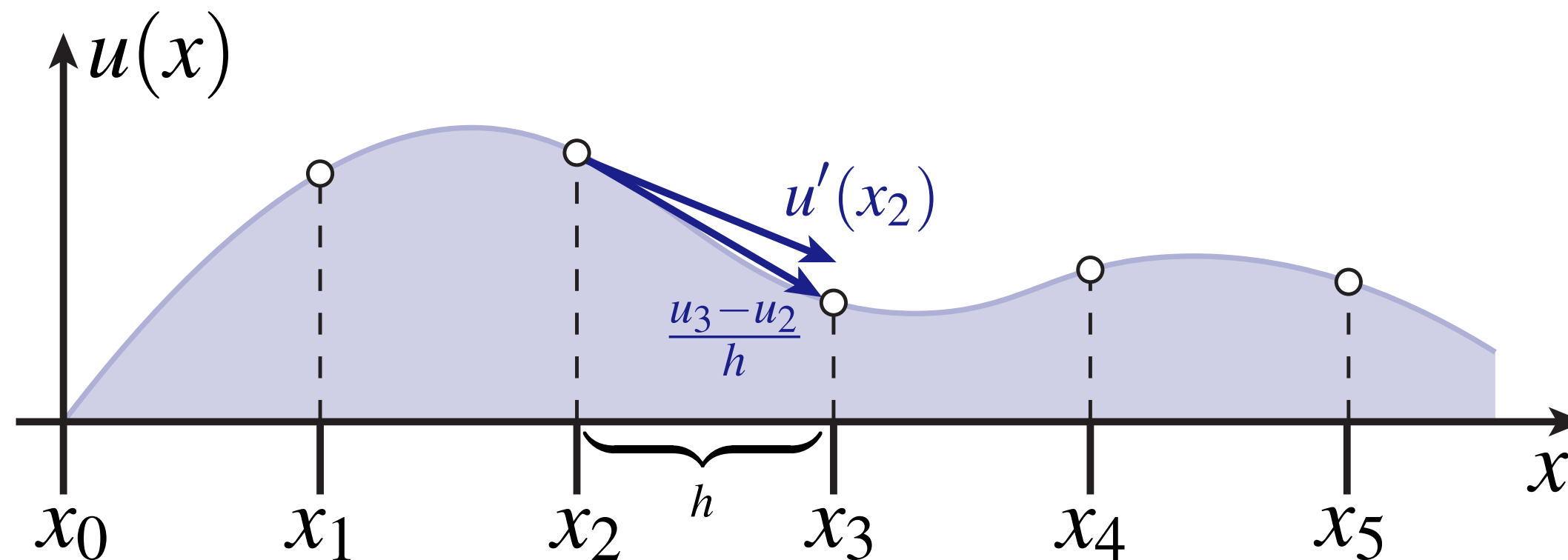
- divergence of gradient
- sum of second derivatives
- deviation from local average
- ...

$$\Delta u = \overset{\text{div}}{\nabla} \cdot \overset{\text{grad}}{\nabla} u$$

$$\Delta u = \frac{\partial u^2}{\partial x_1^2} + \dots + \frac{\partial u^2}{\partial x_n^2}$$

# Discretizing the First Derivative

- To solve any PDE, need to approximate spatial derivatives (e.g., Laplacian)
- Suppose we know a function  $u(x)$  only at regular intervals  $h$



- **Q: How can we approximate the first derivative of  $u$ ?**
- **A: Recall definition of a derivative in terms of limits:**

$$u'(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

- Can hence get an approximation using known values:

$$u'(x_i) \approx \frac{u_{i+1} - u_i}{h}$$

- Approximation gets better for finer grid (smaller  $h$ )

# Discretizing the Second Derivative

- **Q: How can we get an approximation of the second derivative?**
- **A: One idea\*: approximate the first derivative of the approximate first derivative!**

$$u''(x_i) \approx \frac{u'_i - u'_{i-1}}{h} \approx \frac{\left(\frac{u_{i+1} - u_i}{h}\right) - \left(\frac{u_i - u_{i-1}}{h}\right)}{h} =$$

$$\boxed{\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}}$$

- **In general, this approach of approximating derivatives with differences is the “finite difference” approach to PDEs**
- **Not the only way! But works well on regular grids.**

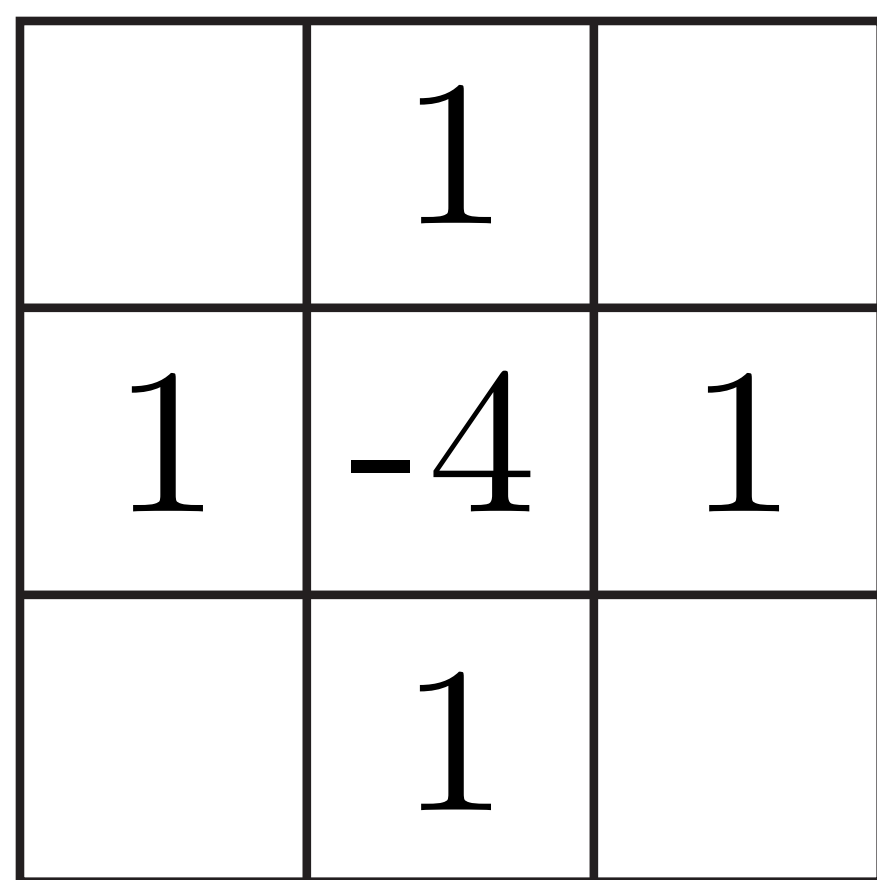
\*Can show this is also a reasonable thing to do, using Taylor series



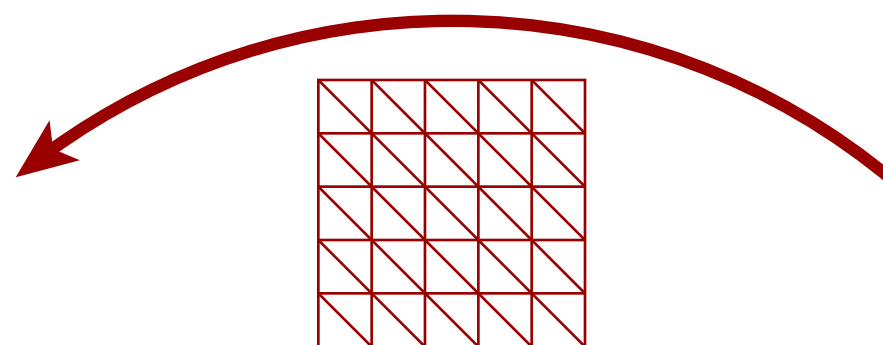
# Discretizing the Laplacian

- How do we approximate the Laplacian?
- Depends on discretization (Eulerian, Lagrangian, grid, mesh, ...)
- Two extremely common ways in graphics:

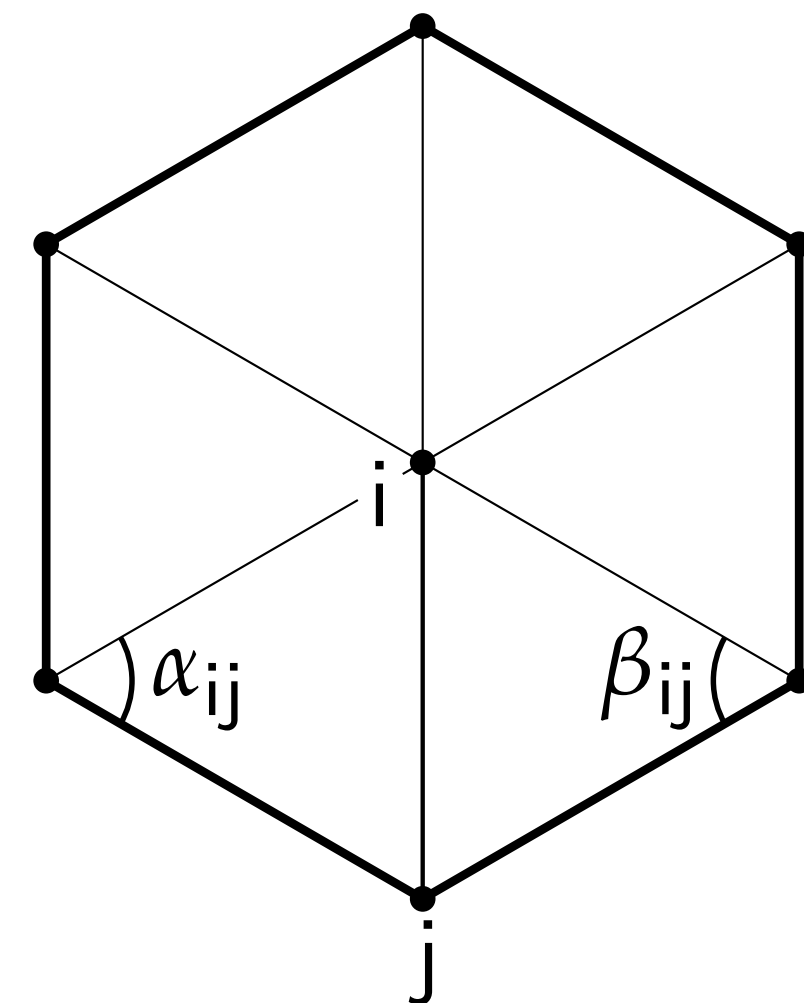
GRID  $h$



(actually, this becomes that)



TRIANGLE MESH



$$\frac{4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1}}{h^2}$$

$$\frac{1}{2} \sum_j (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

- Also not too hard on point clouds, polygon meshes, ...

# Numerically Solving the Laplace Equation

- Want to solve  $\Delta u = 0$

- Plug in one of our discretizations, e.g.,

	$u_{i,j+1}$	
$u_{i-1,j}$	$u_{i,j}$	$u_{i+1,j}$
	$u_{i,j-1}$	

$$\frac{4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}}{h^2} = 0$$

$$\iff u_{i,j} = \frac{1}{4} \left( u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} \right)$$

- If  $u$  is a solution, then each value must be the average of the neighboring values ( $u$  is a “harmonic function”)
- How do we solve this?
- One idea: keep averaging with neighbors! (“Jacobi method”)
- Correct, but *slow*. Much better to use modern linear solver

# Aside: PDEs and Linear Equations

- How can we turn our Laplace equation into a linear solve?

- Have a bunch of equations of the form

$$4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

- On a 4x4 grid, assign each cell  $u_{i,j}$  a unique index  $1, \dots, 16$

- Can then write equations as a 16x16 matrix equation\*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$$\begin{bmatrix}
 -4 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 1 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 1 & 0 & 0 & 0 & -4 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -4 & 0
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4 \\
 u_5 \\
 u_6 \\
 u_7 \\
 u_8 \\
 u_9 \\
 u_{10} \\
 u_{11} \\
 u_{12} \\
 u_{13} \\
 u_{14} \\
 u_{15} \\
 u_{16}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
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 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

- Compute solution by calling sparse linear solver (SuiteSparse, Eigen, ...)

- **Q: By the way, what's wrong with our problem setup here? :-)**

\*assuming neighbors wrap around left/right and top/bottom

# Boundary Conditions for Discrete Laplace

- What values do we use to compute averages near the boundary?

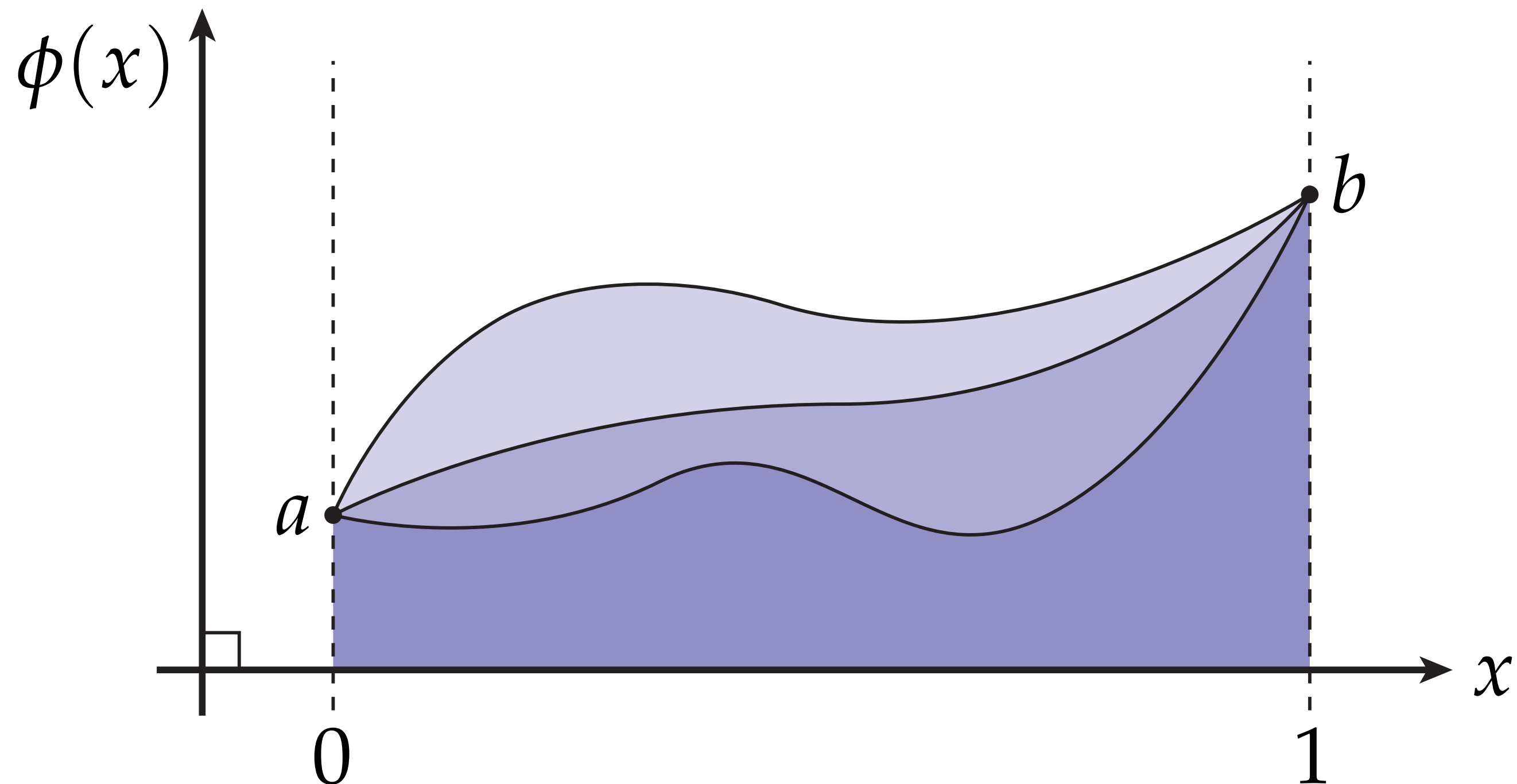
	$c$	
?	$a$	$b$
	$e$	

$$a = \frac{1}{4} (b + c + ? + e)$$

- **A: We get to choose**—this is the data we want to interpolate!
- **Two basic boundary conditions:**
  1. ***Dirichlet***—boundary data always set to fixed values
  2. ***Neumann***—specify derivative (difference) across boundary
- **Also mixed (*Robin*) boundary conditions (and more, in general)**

# Dirichlet Boundary Conditions

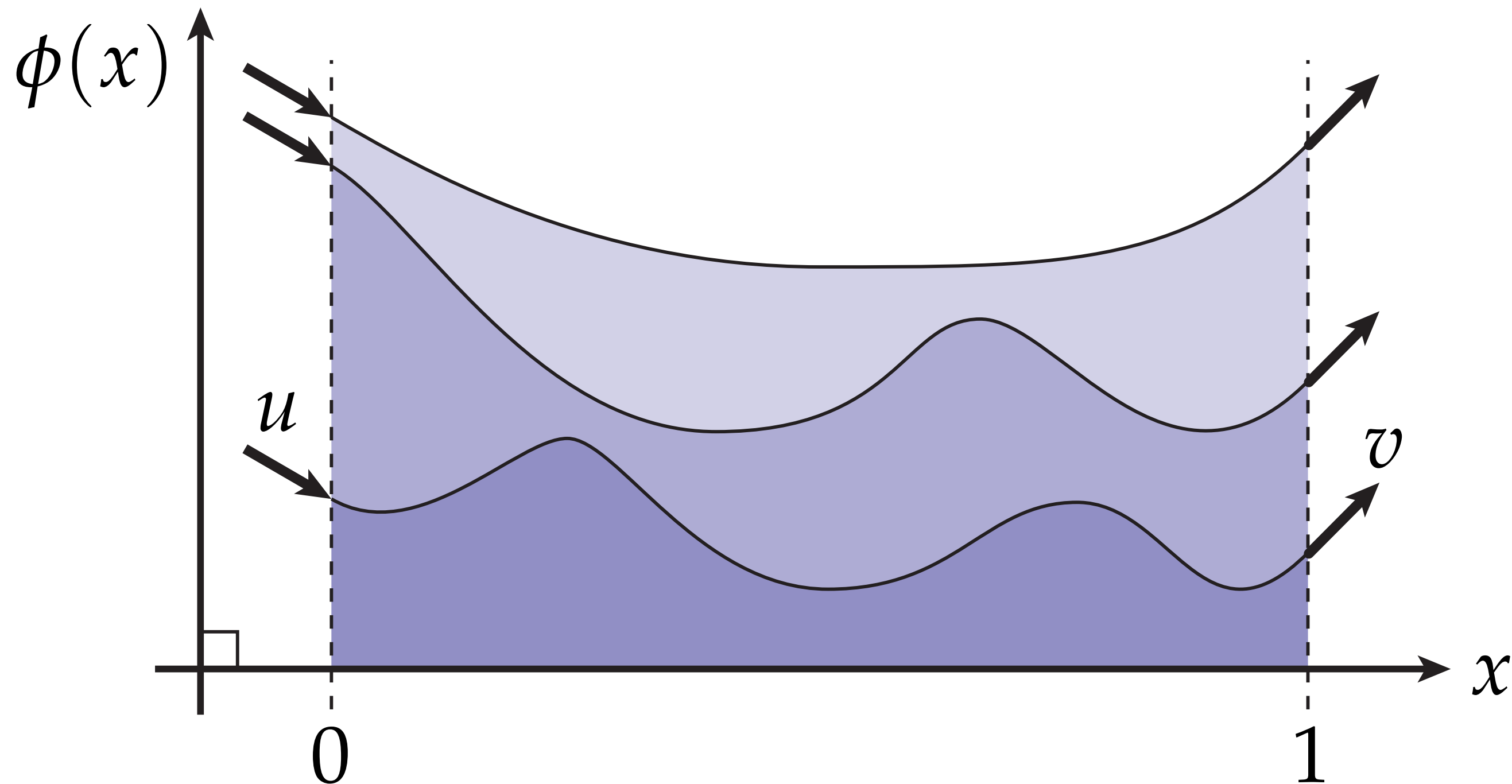
- Let's go back to smooth setting, function on real line
- *Dirichlet* means “prescribe values”
- E.g.,  $\phi(0) = a, \phi(1) = b$



- Many possible functions “in between”!

# Neumann Boundary Conditions

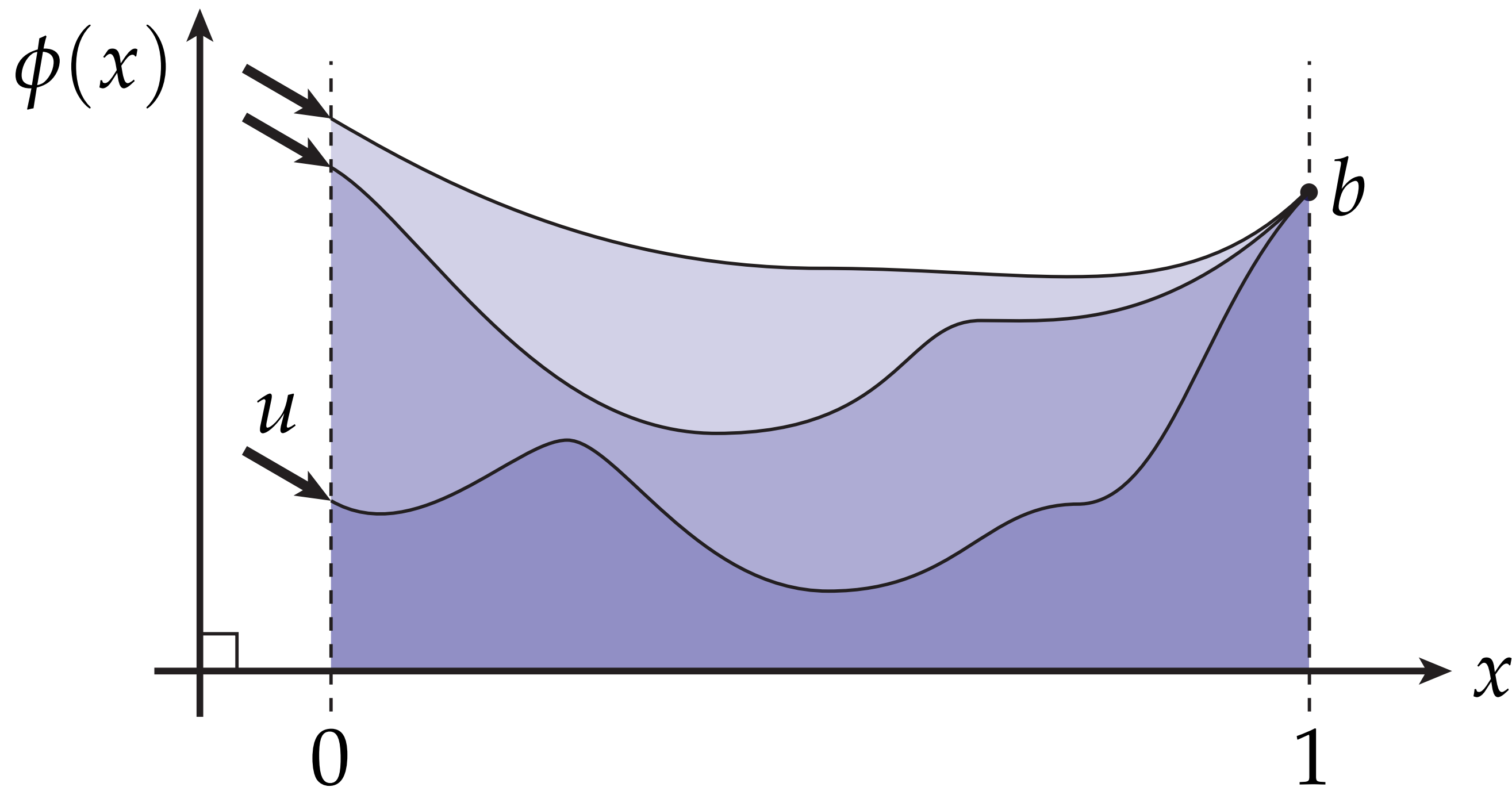
- *Neumann* means “prescribe derivatives”
- E.g.,  $\phi'(0) = u$ ,  $\phi'(1) = v$



- Again, many possible functions!

# Both Neumann & Dirichlet

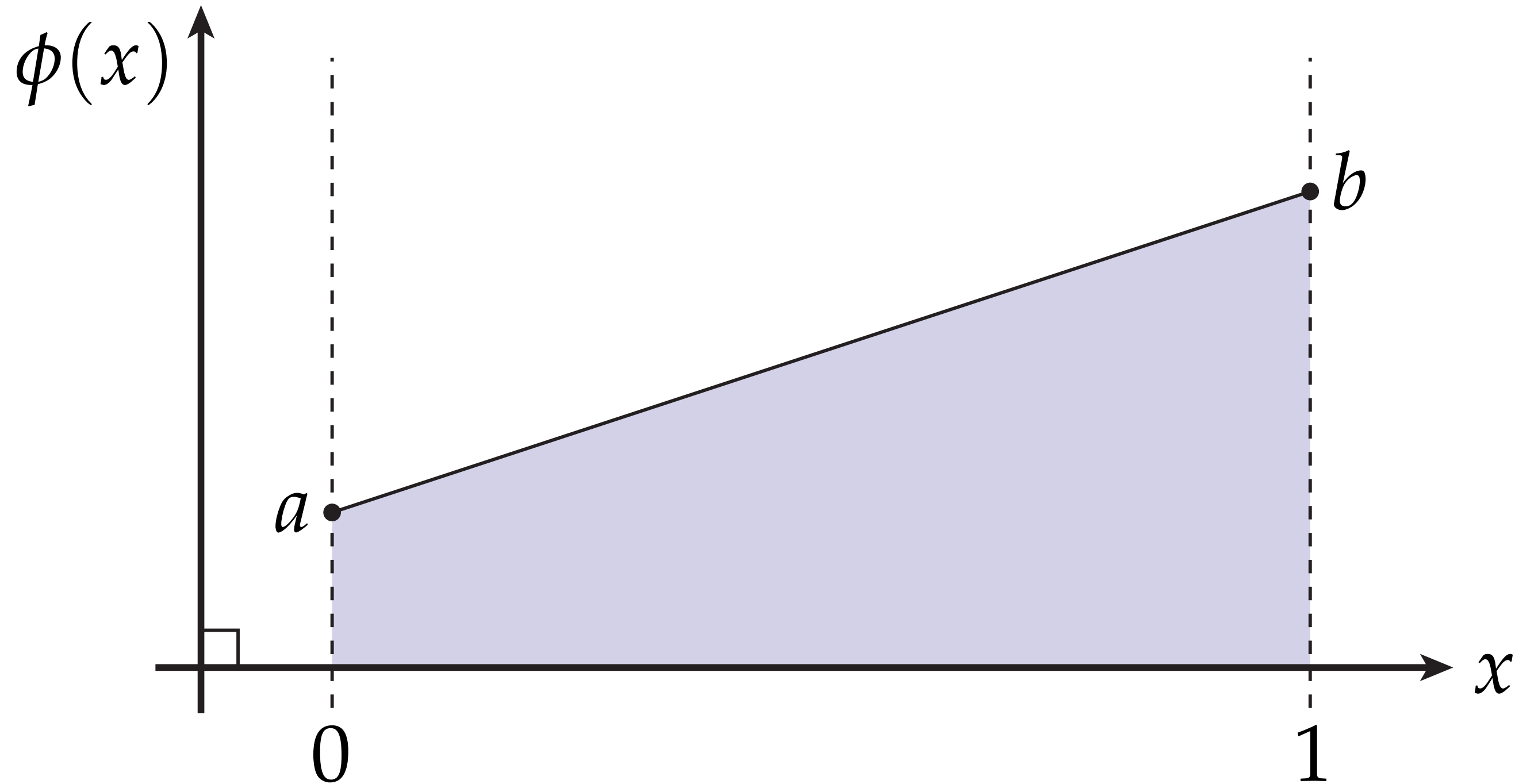
- Or: prescribe some values, some derivatives
- E.g.,  $\phi'(0) = u, \phi(1) = b$



- Q: What about  $\phi'(1) = v, \phi(1) = b$ ? Does that work?
- Q: What about  $\phi'(0) + \phi(0) = p, \phi'(1) + \phi(1) = q$ ? (*Robin*)

# 1D Laplace w/ Dirichlet BCs

- **1D Laplace:**  $\partial^2 \phi / \partial x^2 = 0$
- **Solutions:**  $\phi(x) = cx + d$
- **Q: Can we *always* satisfy given Dirichlet boundary conditions?**

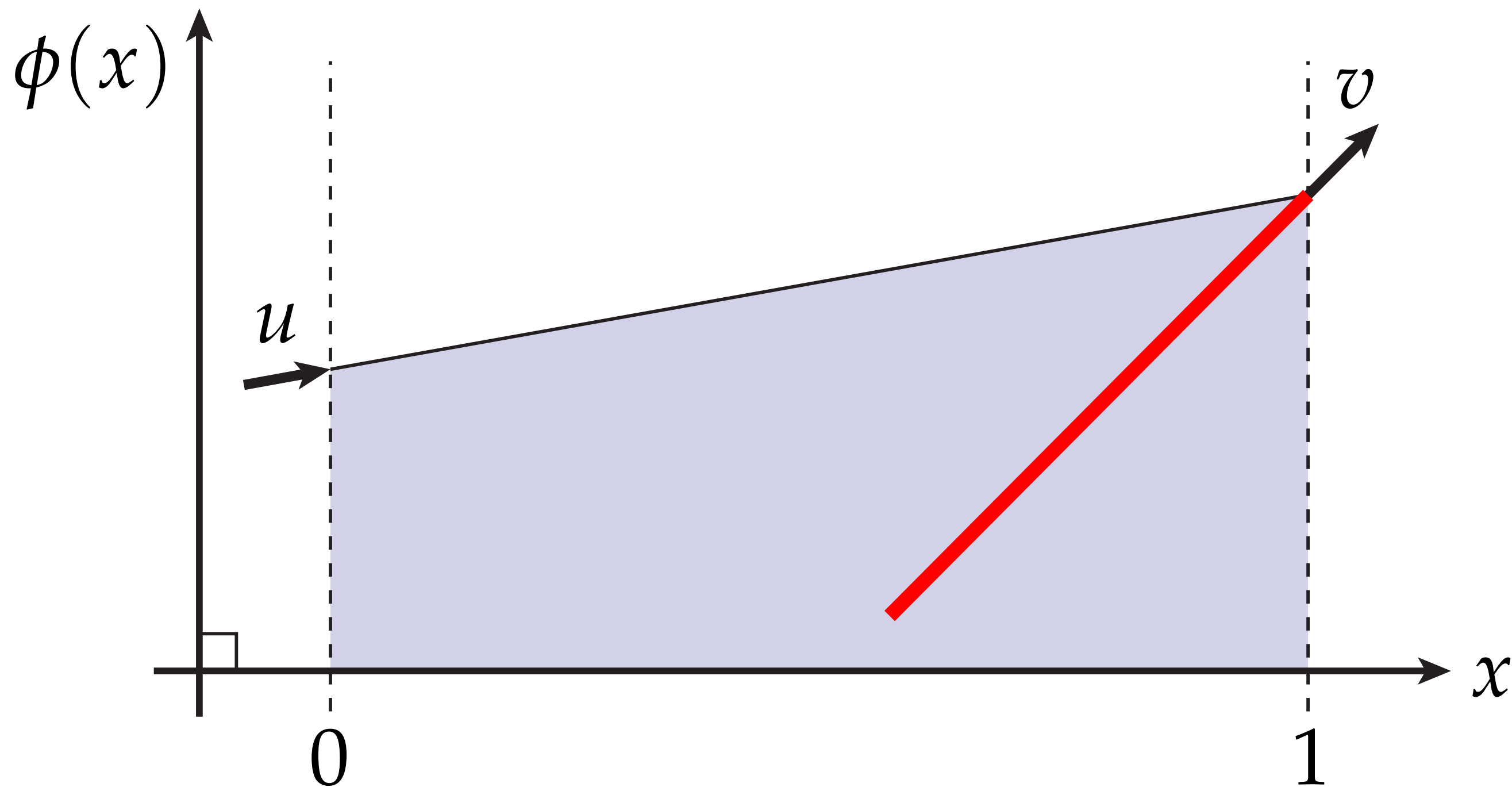


- **Yes: a line can interpolate any two points.**



# 1D Laplace w/ Neumann BCs

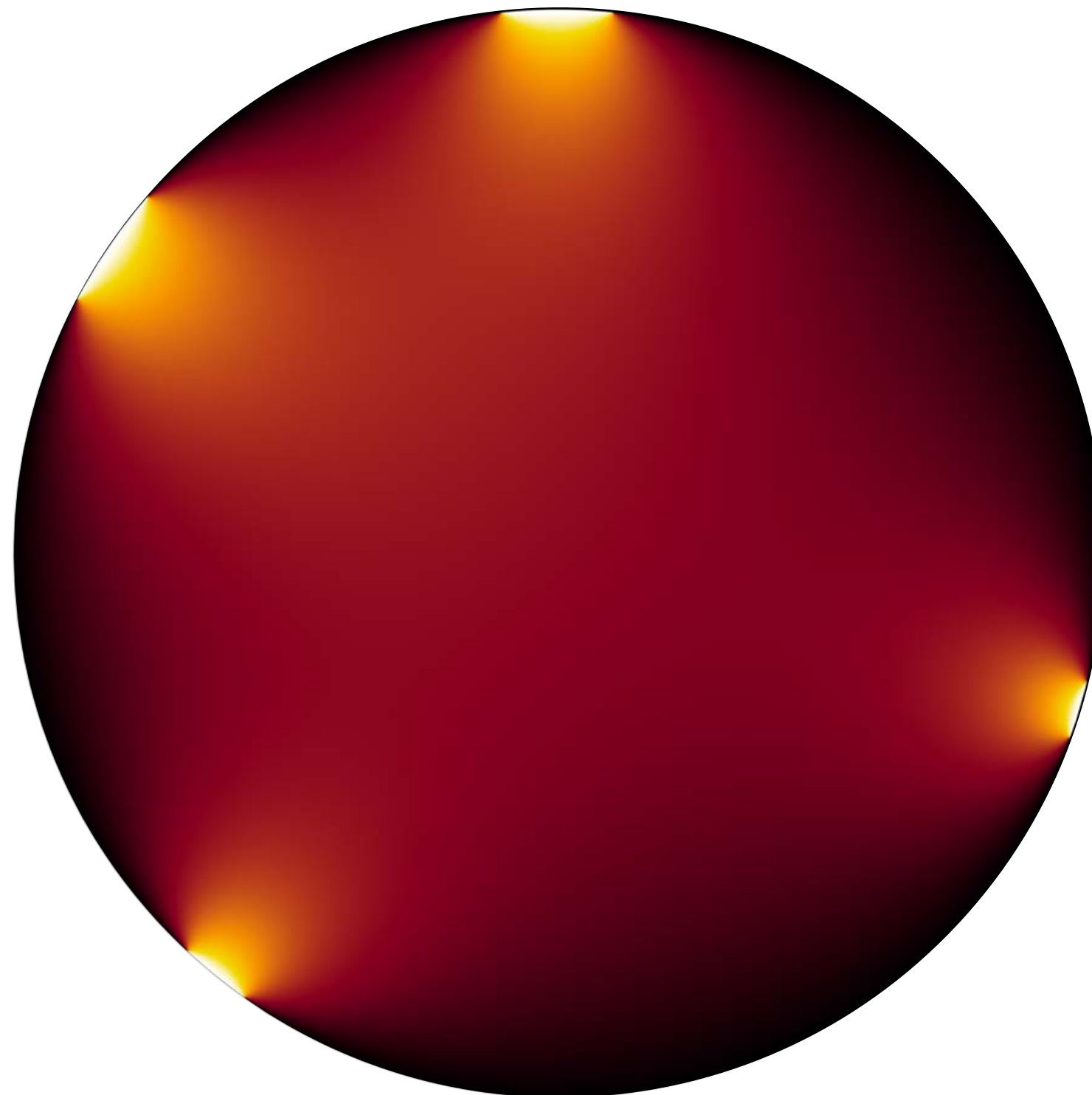
- What about Neumann BCs?
- Q: Can we prescribe the derivative at both ends?



- No! A line has only one slope.
- In general, solution to a PDE may not exist for given BCs.

# 2D Laplace w/ Dirichlet BCs

- 2D Laplace:  $\Delta\phi = 0$
- Q: Can satisfy any Dirichlet BCs? (given data along boundary)



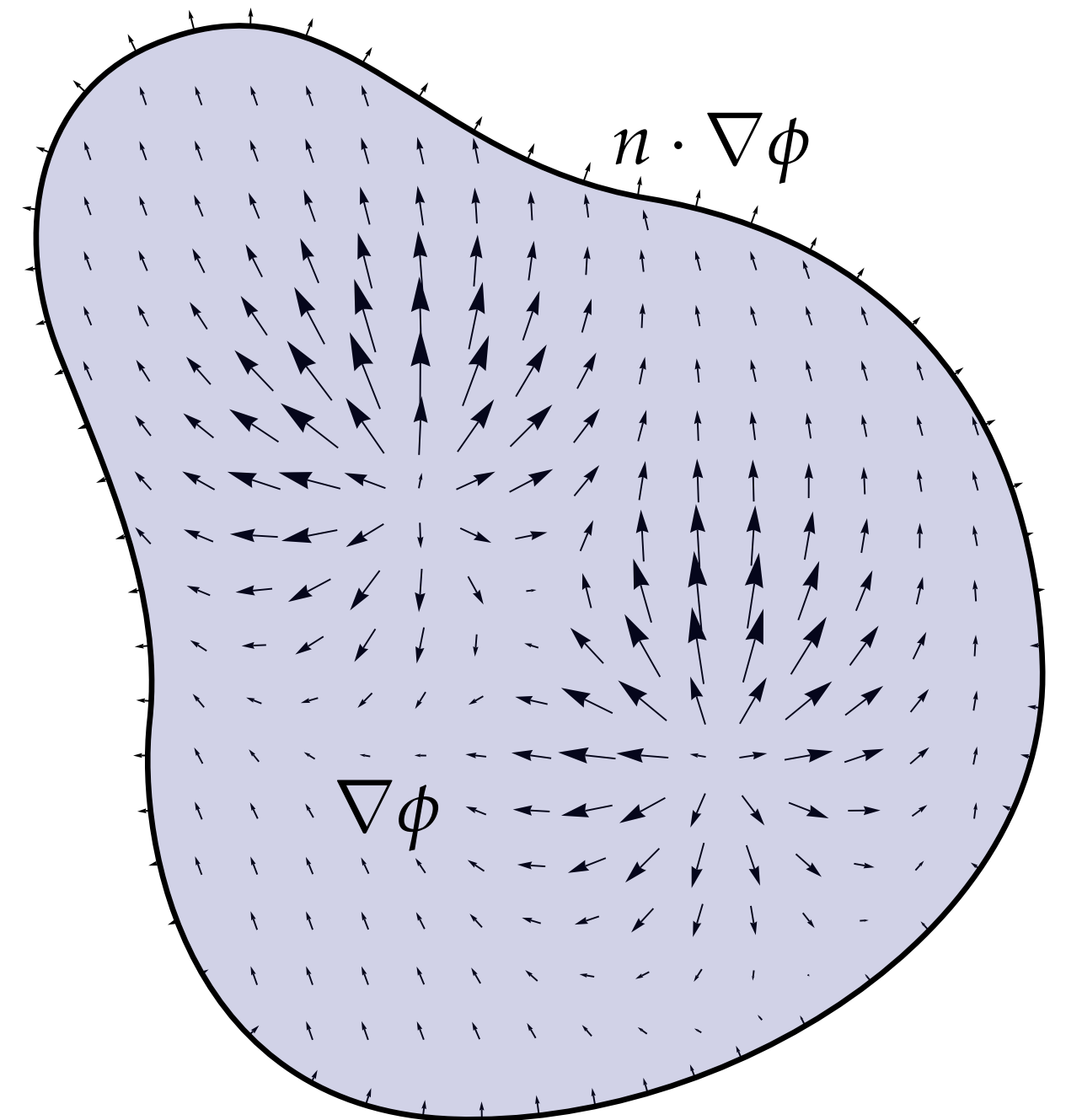
- Yes: Laplace is long-time solution to heat flow
- Data is “heat” at boundary. Then just let it flow...

# 2D Laplace w/ Neumann BCs

- What about Neumann BCs for  $\Delta\phi = 0$ ?
- Neumann BCs prescribe derivative in normal direction:  $n \cdot \nabla\phi$
- Q: Can it always be done? (Wasn't possible in 1D...)
- In 2D, we have the *divergence theorem*:

$$\int_{\partial\Omega} n \cdot \nabla\phi = \int_{\Omega} \nabla \cdot \nabla\phi = \int_{\Omega} \Delta\phi \stackrel{!}{=} 0$$

- Should be called, "*what goes in must come out theorem!*"
- Can't have a solution unless the net flux through the boundary is zero.
- Numerical libraries will *not* always tell you if there's a problem!
- Trust, but verify (e.g., after solving  $Ax = b$ , compute  $\|b - Ax\|$ )



# Solving the Heat Equation

- Back to our three model equations, want to solve *heat eqn.*

$$\dot{u} = \Delta u$$

- Just saw how to discretize Laplacian
- Also know how to do time (forward Euler, backward Euler, ...)
- E.g., forward Euler:

$$u^{k+1} = u^k + \Delta u^k$$

- Q: On a grid, what's our overall update now at  $u_{i,j}$ ?

$$u_{i,j}^{k+1} = u^k + \frac{\tau}{h^2} (4u_{i,j}^k - u_{i+1,j}^k - u_{i-1,j}^k - u_{i,j+1}^k - u_{i,j-1}^k)$$

- *Not* hard to implement! Loop over grid, add up some neighbors.

# Solving the Wave Equation

- Finally, wave equation:

$$\ddot{u} = \Delta u$$

- Not much different; now have 2nd derivative in time

- By now we've learned two different techniques:

- Convert to two 1st order (in time) equations:

$$\dot{u} = v, \quad \dot{v} = \Delta u$$

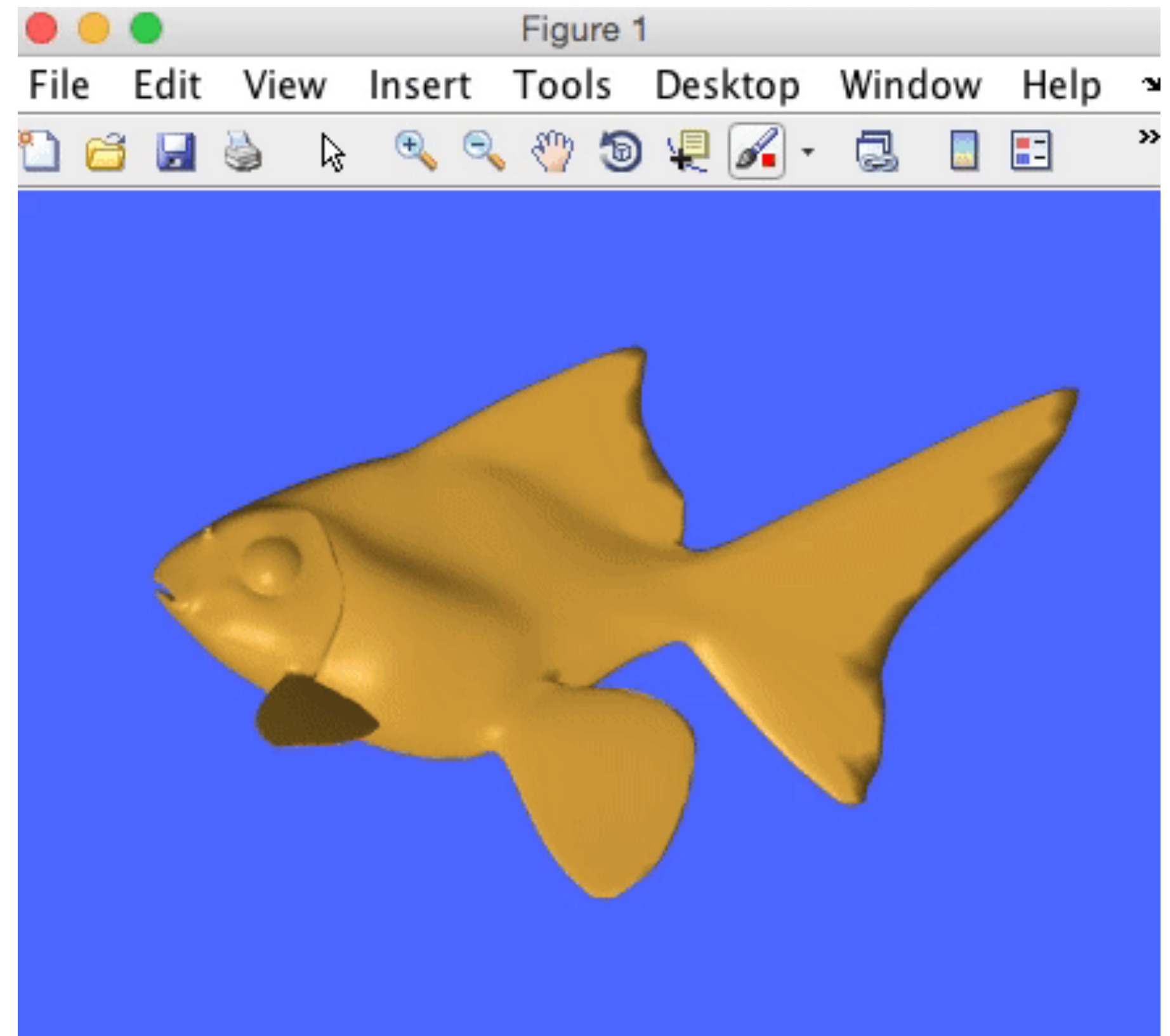
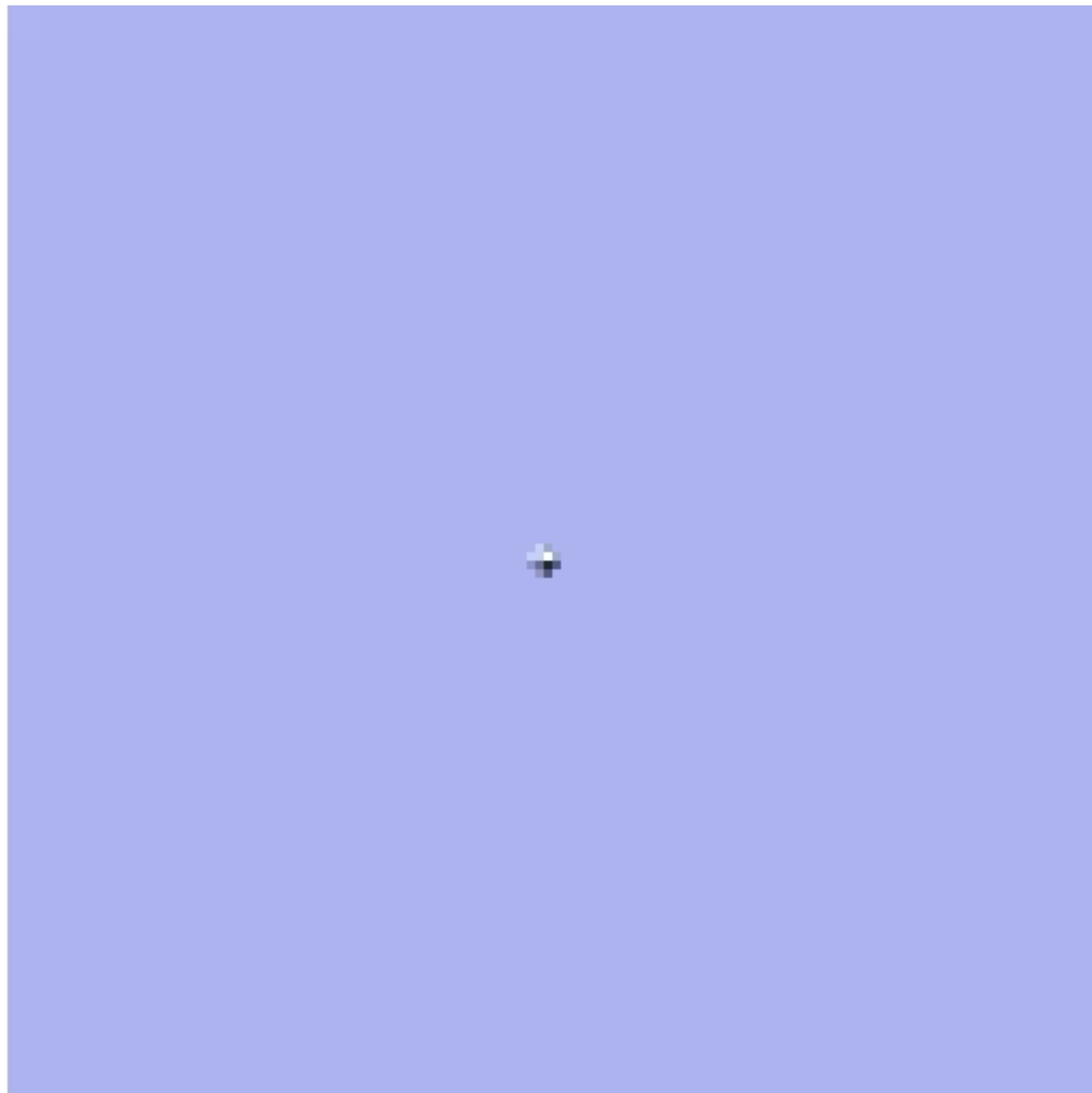
- Or, use centered difference (like Laplace) in time:

$$\frac{u^{k+1} - 2u^k + u^{k-1}}{\tau^2} = \Delta u^k$$

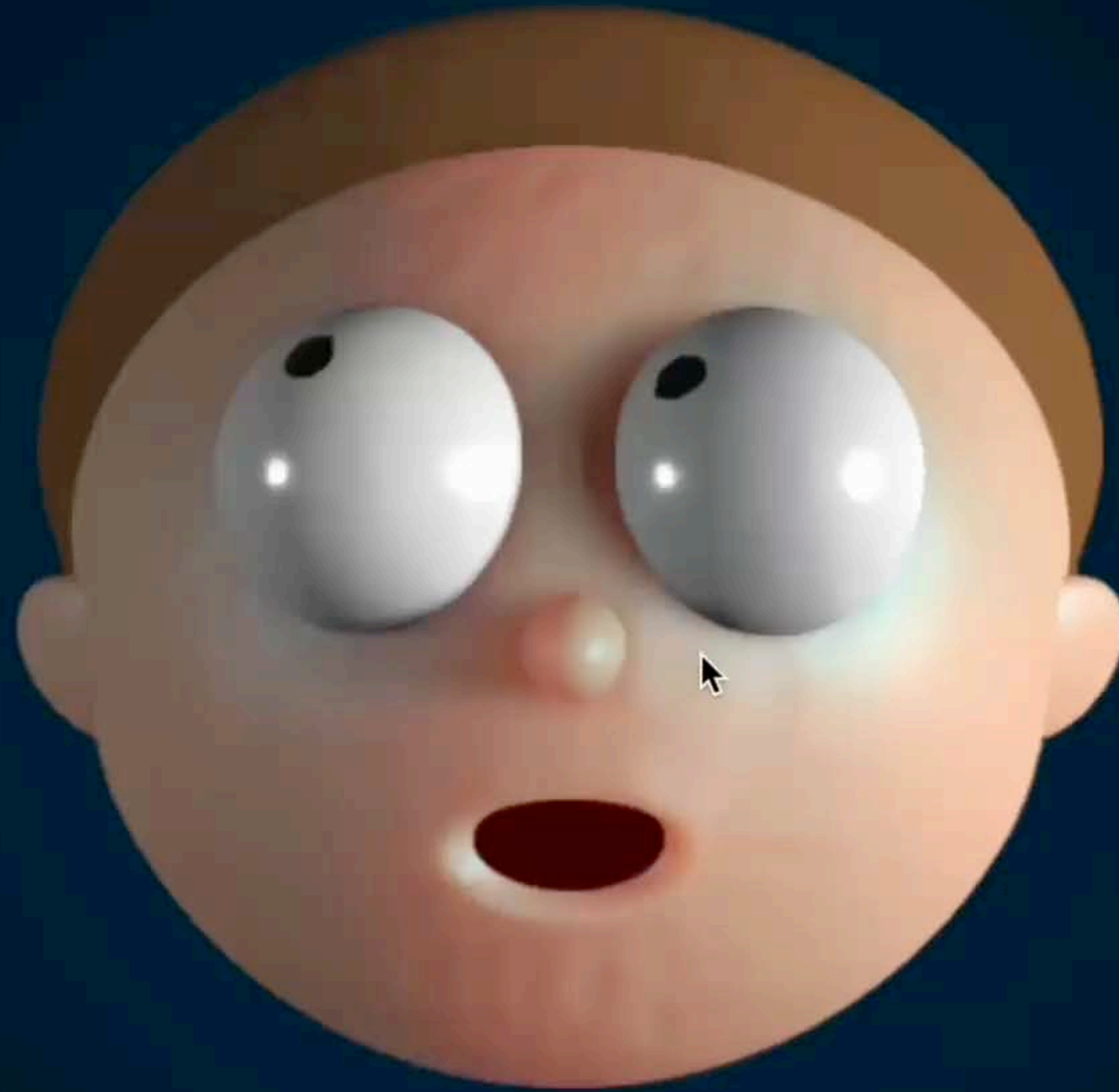
- Plus all our choices about how to discretize Laplacian.

- *So many choices!* And many, many (*many*) more we didn't discuss.

# Wave Equation on a Grid, Triangle Mesh



# Fun with wave-like equations...



<https://www.adultswim.com/etcetera/elastic-man/>

author: David Li

Technique: low-res thin shell simulation (via "position-based dynamics") + Loop subdivision

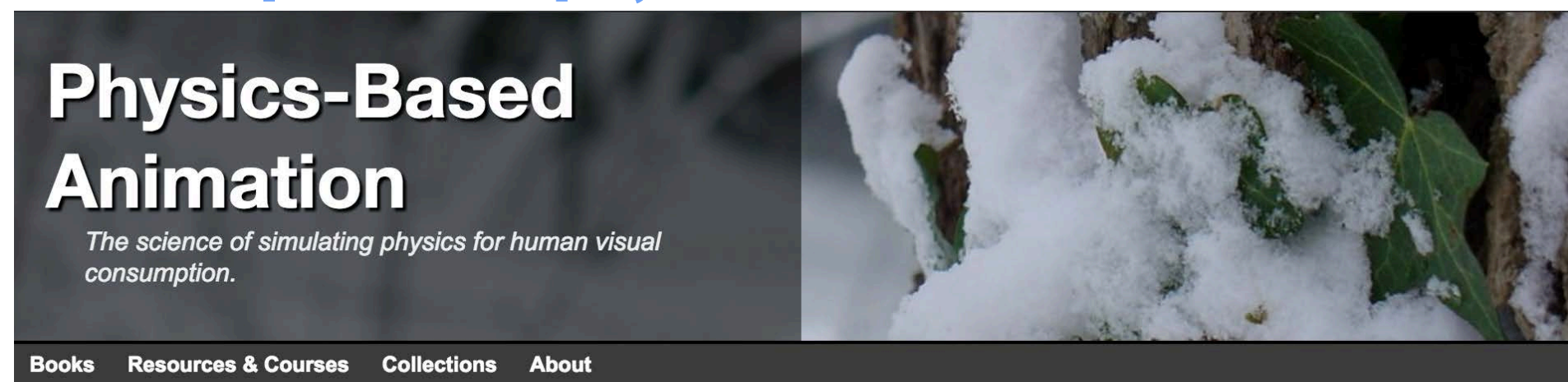
**Wait, what about all that other cool stuff?  
(Fluids, hair, cloth, ...)**



# Want to Know More?

- There are some good books:
- And papers:

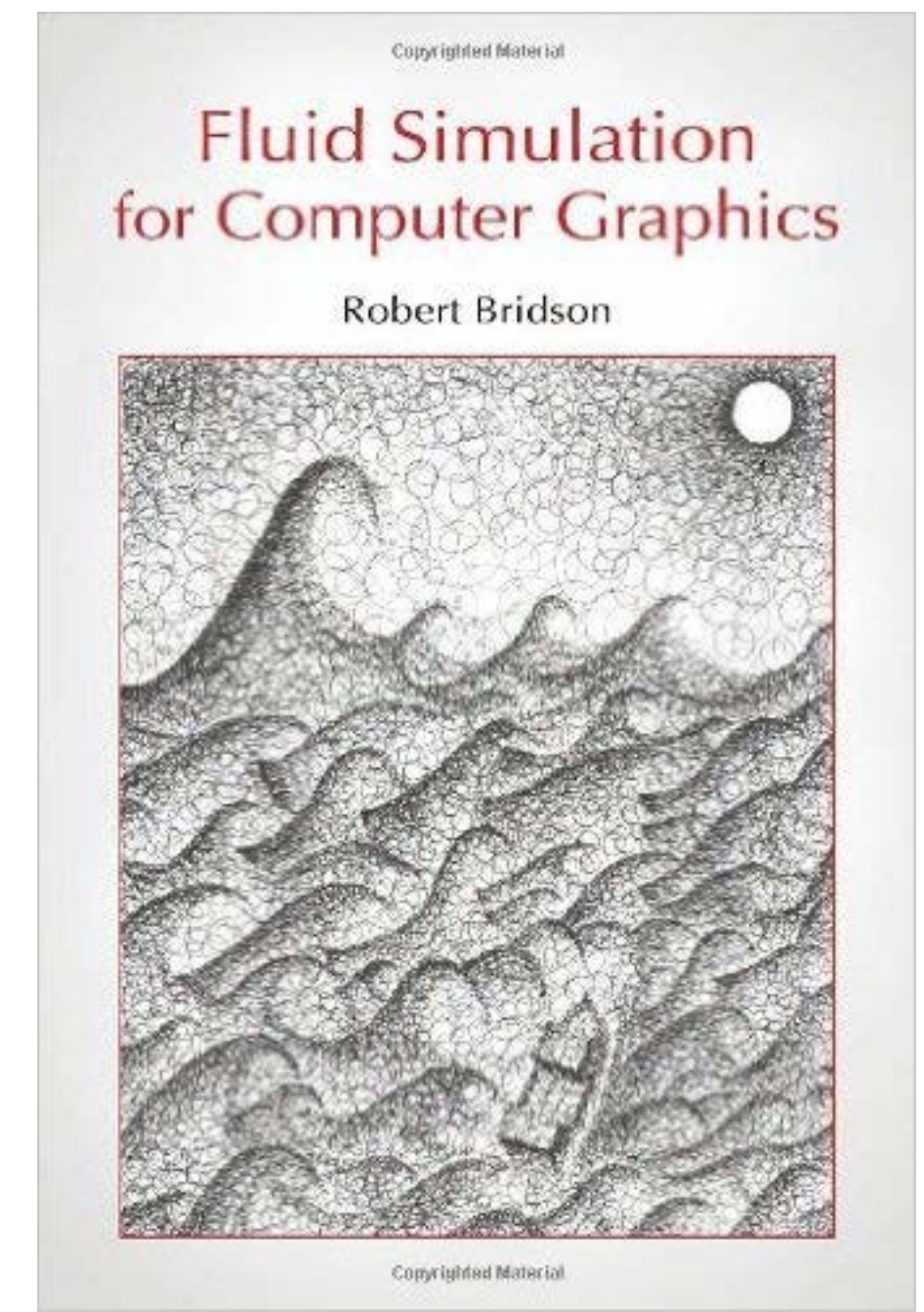
<http://www.physicsbasedanimation.com/>



**Biomechanical Simulation and Control of Hands and Tendinous Systems**

Prashant Sachdeva, Shinjiro Sueda, Susanne Bradley, Mikhail Fain, Dinesh K. Pai

Search...  
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This site is managed by Christopher Batty from the University of Waterloo



- Also, what did the folks who *wrote* these books & papers read?

