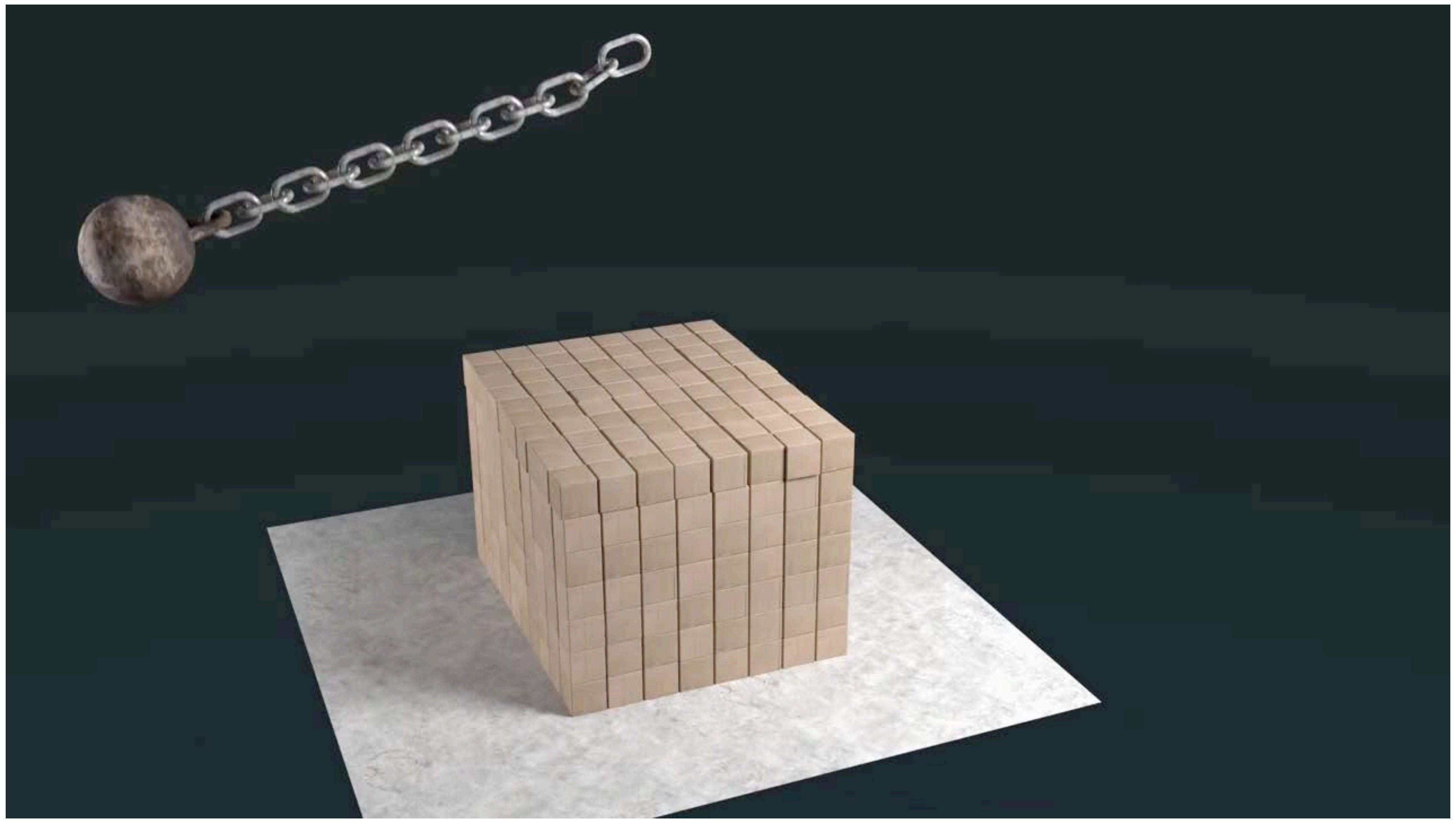
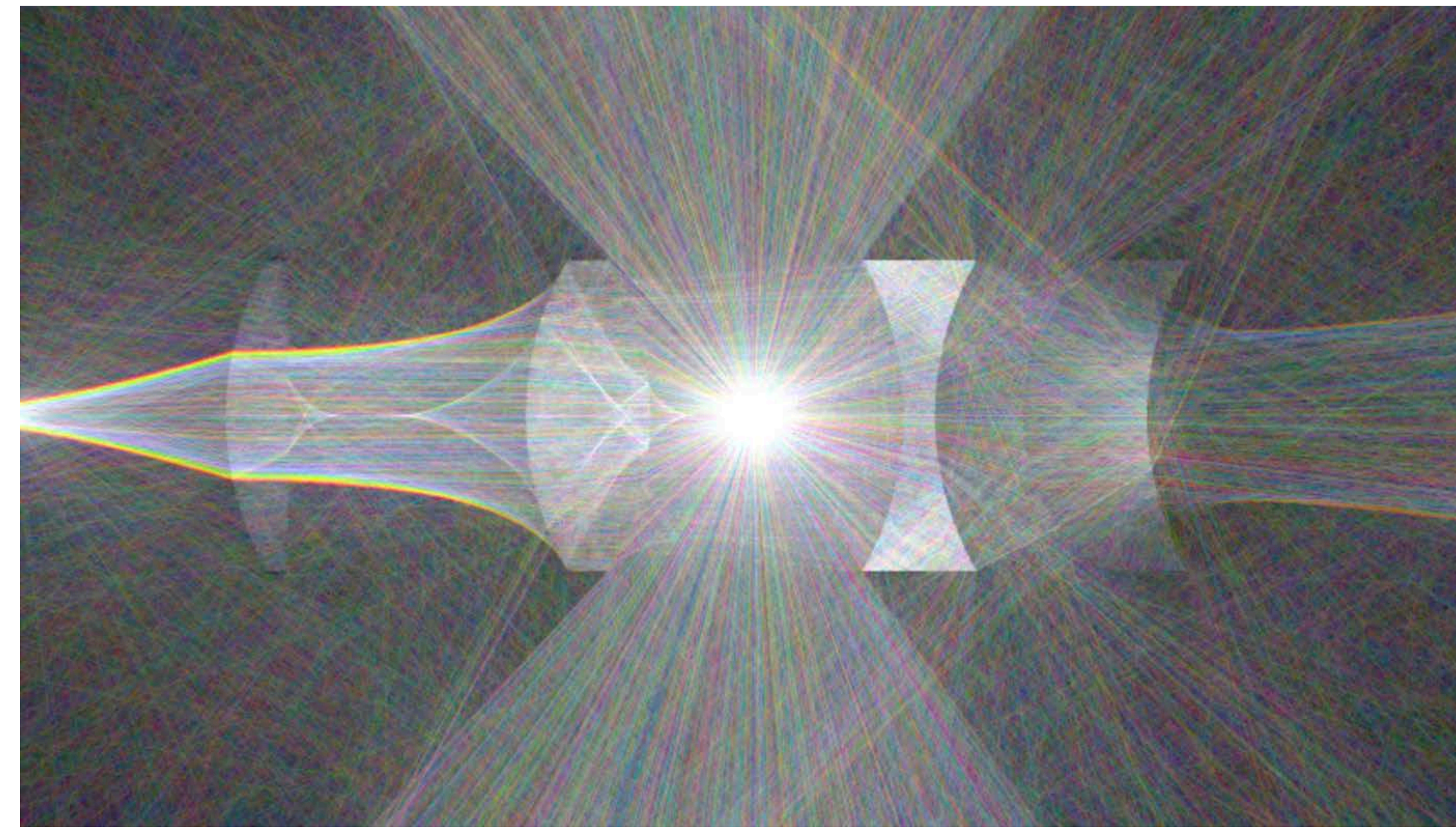
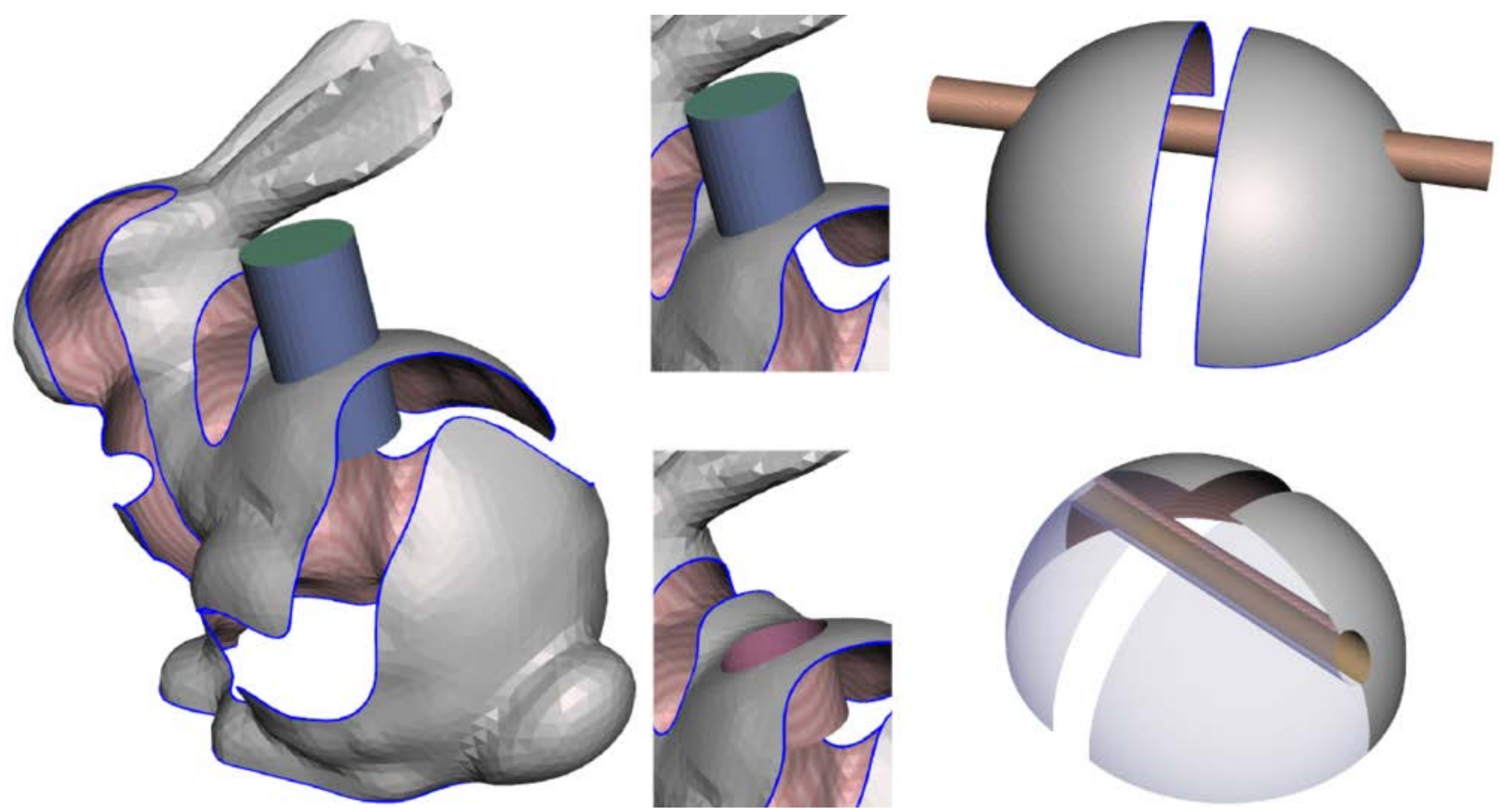
Geometric Queries

Computer Graphics
CMU 15-462/15-662

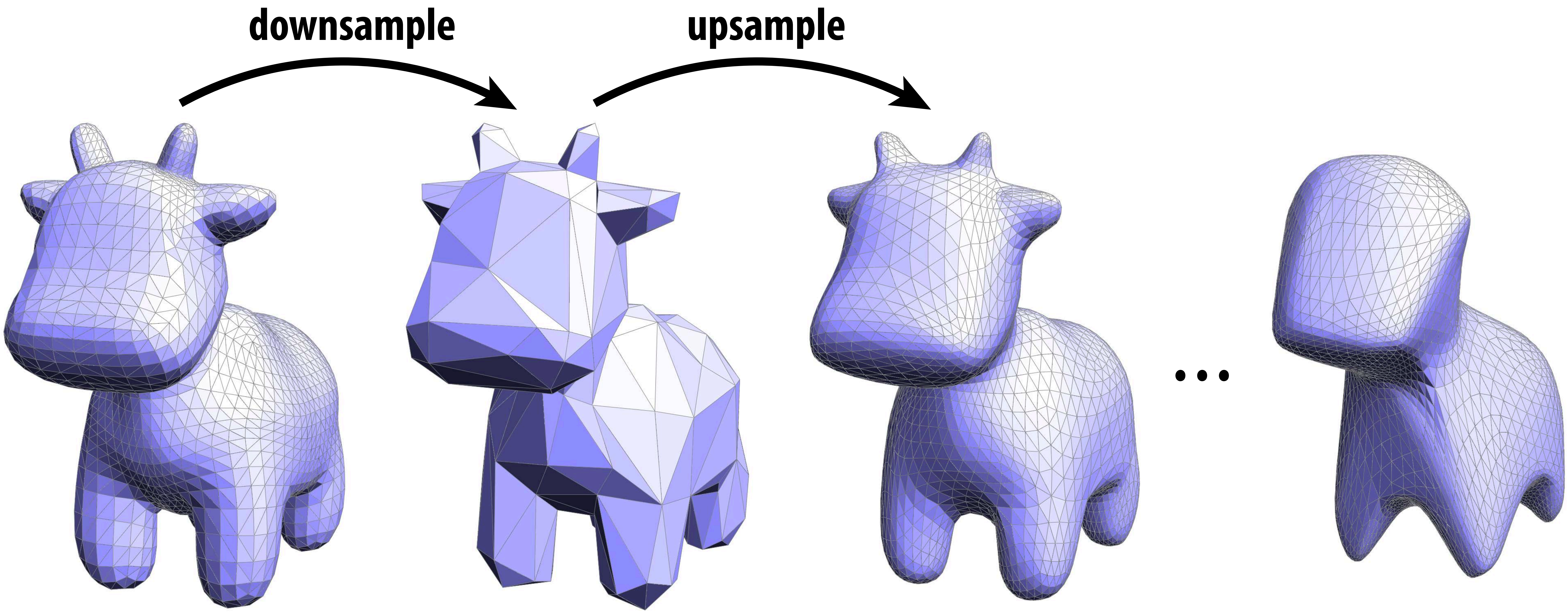
Geometric Queries—Motivation





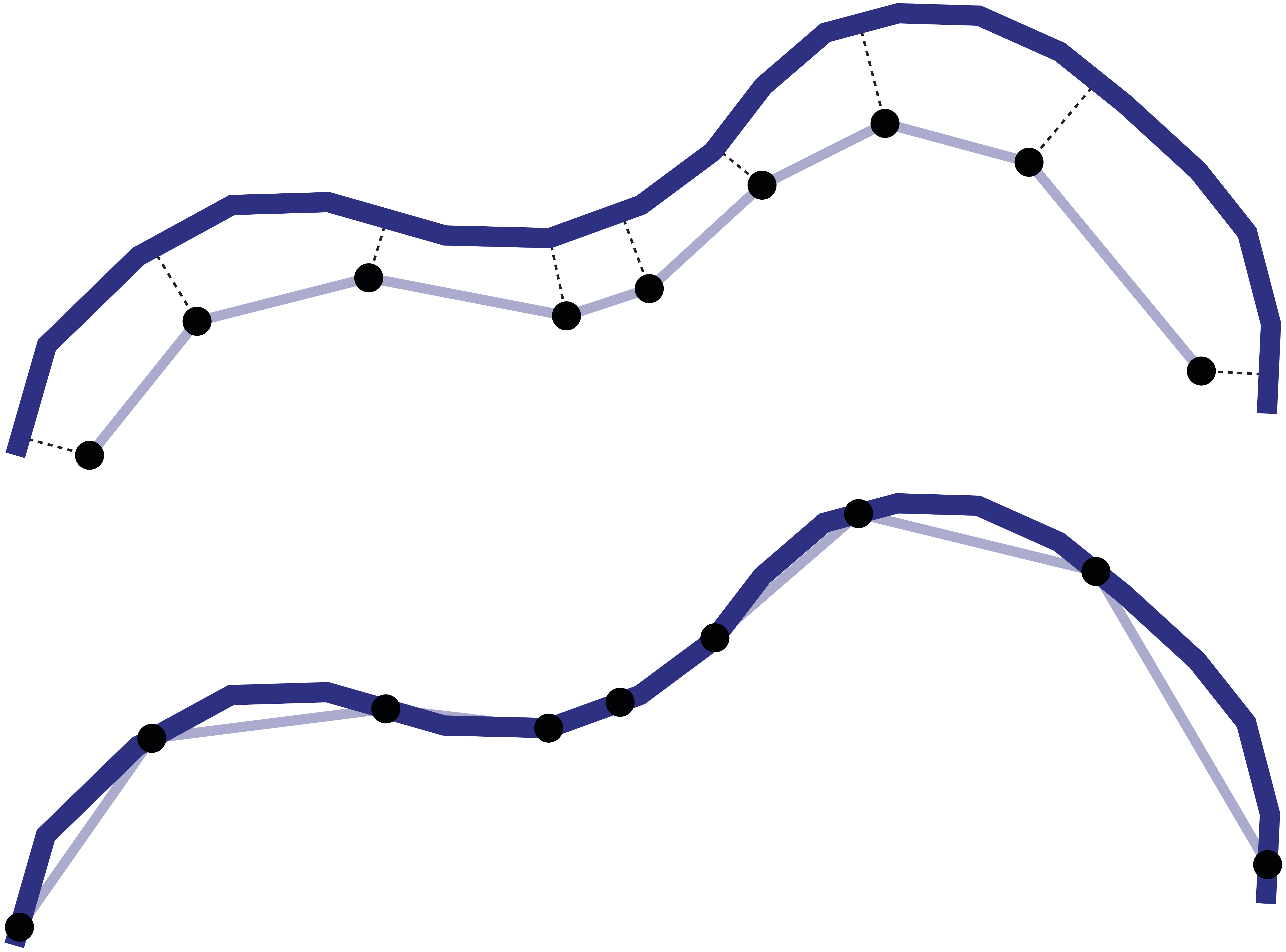


Motivating Example:
Signal Degradation in Geometry Processing



Q: How can we do a better job of preserving the original signal?

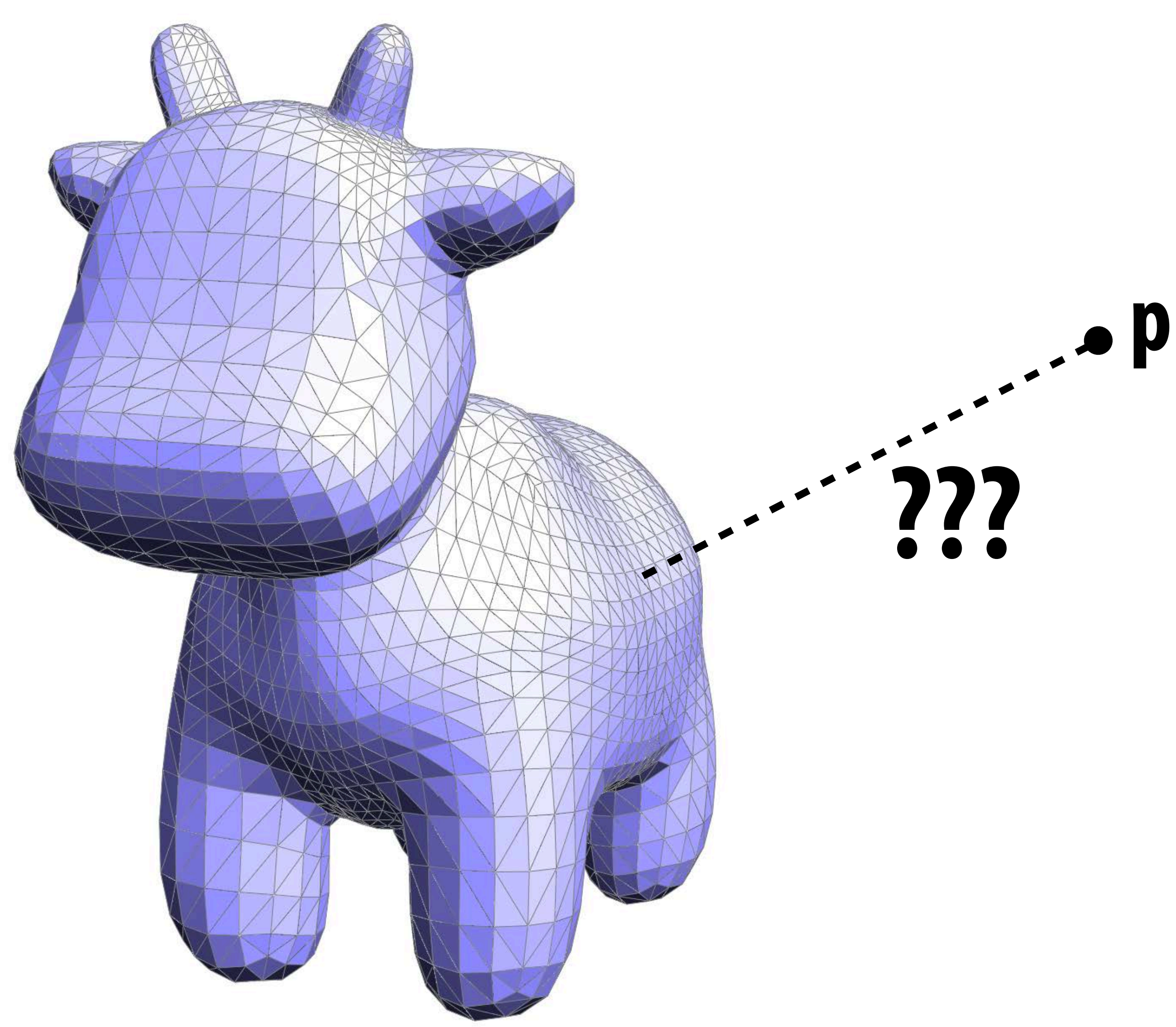
Recovering Fidelity via Closest Point Projection



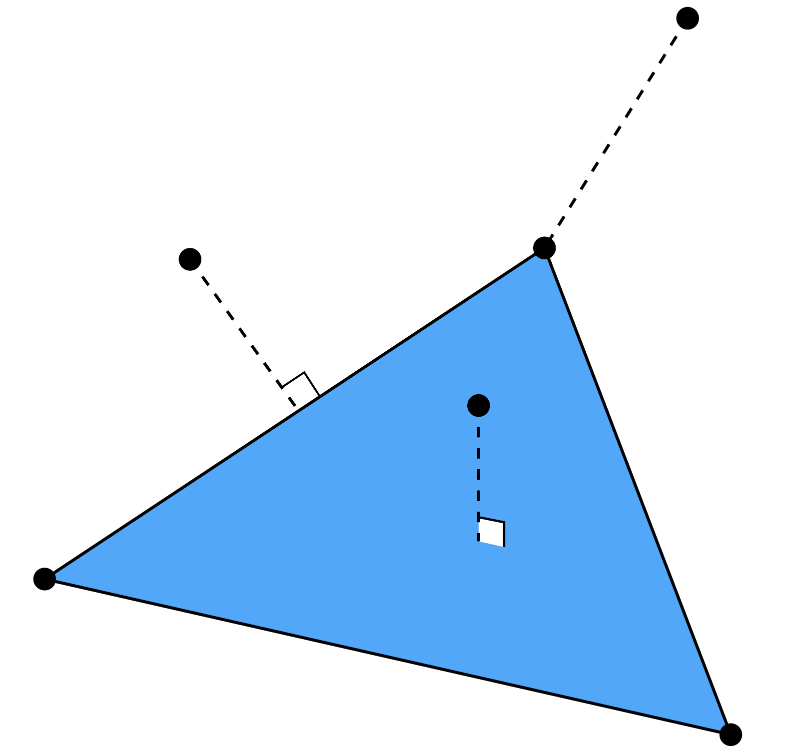
Idea: after resampling, project each vertex onto original mesh

Closest Point Queries

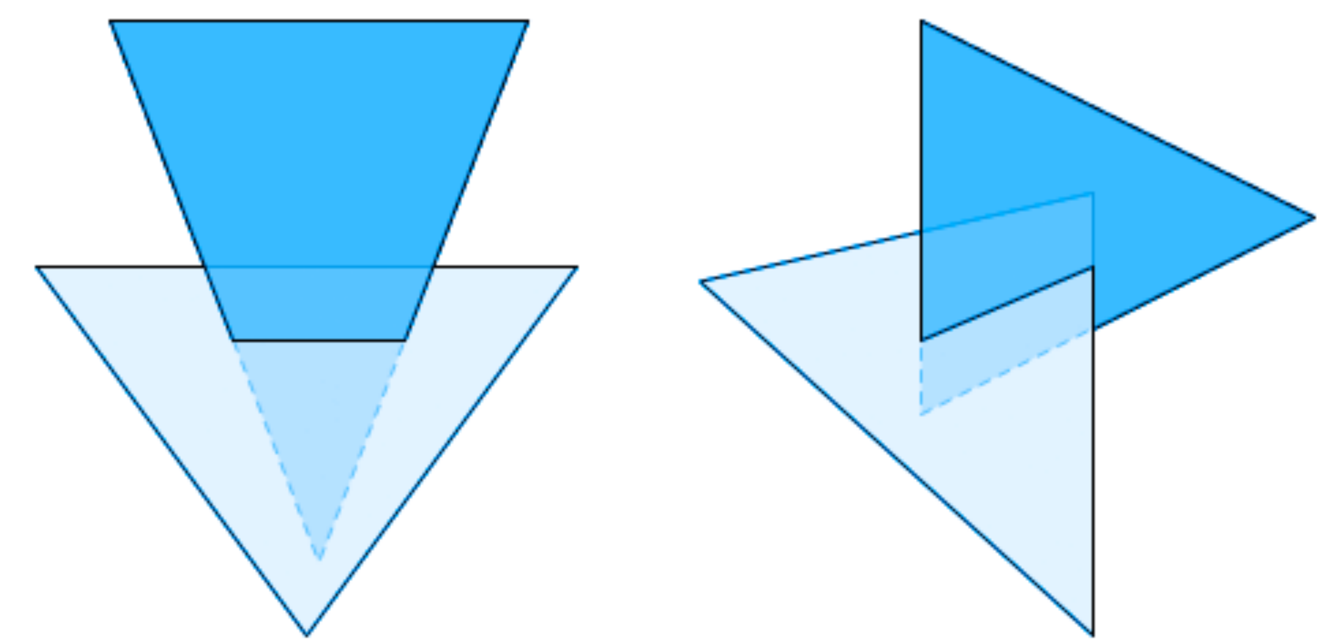
- Q: Given a point, in space (e.g., a new sample point), how do we find the closest point on a given surface?
- Q: Does implicit/explicit representation make this easier?
- Q: Does our choice of mesh data structure make a difference?
- Q: What's the cost of the naïve algorithm?
- Q: How do we find the distance to a single triangle anyway?
- So many questions!



Many types of geometric queries



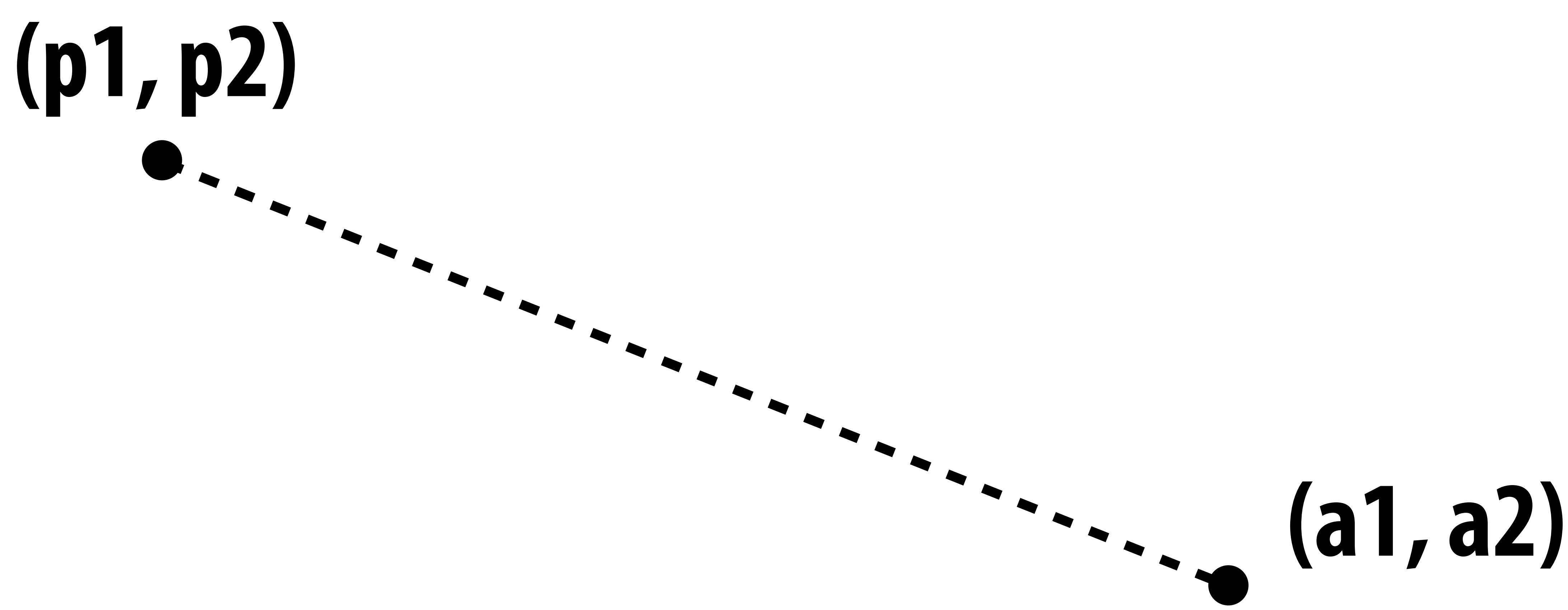
- Already identified need for "closest point" query
- Plenty of other things we might like to know:
  - Do two triangles intersect?
  - Are we inside or outside an object?
  - Does one object contain another?
  - ...



- Data structures we've seen so far not really designed for this...
- Need some new ideas!
- TODAY: come up with simple (read: slow) algorithms.
- NEXT TIME: intelligent ways to accelerate geometric queries.

Warm up: closest point on point

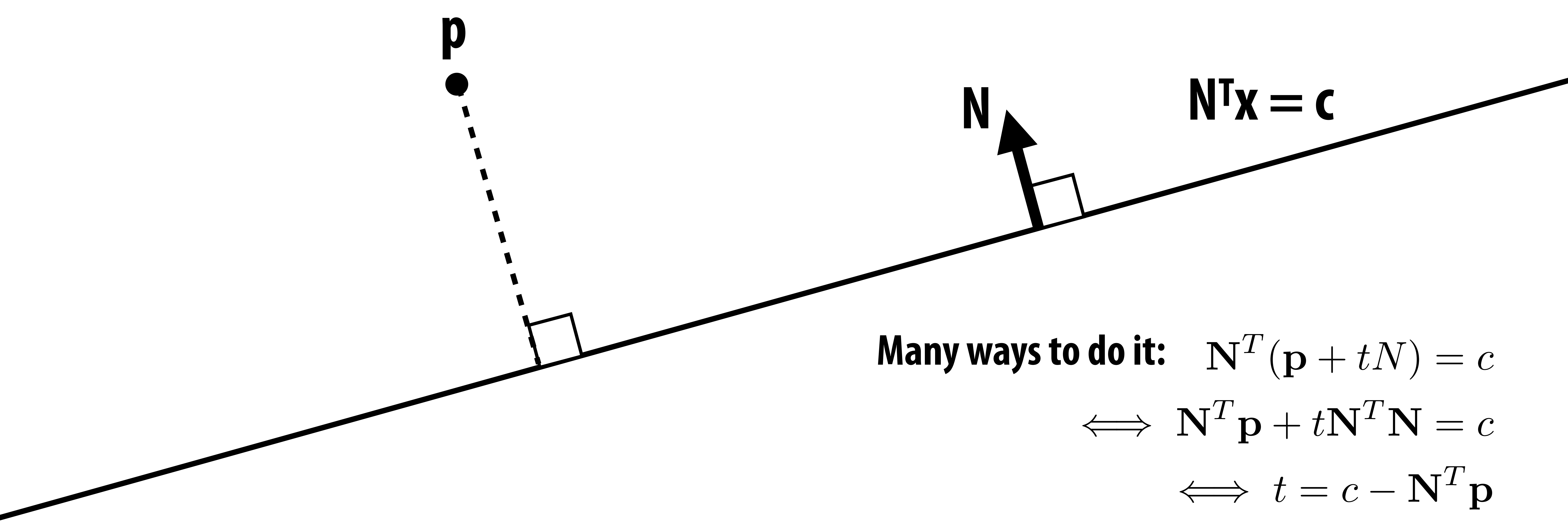
- Goal is to find the point on a mesh closest to a given point.
- *Much* simpler question: given a query point (p1,p2), how do we find the closest point on the point (a1,a2)?



Bonus question: what’s the distance?

Slightly harder: closest point on line

- Now suppose I have a line $\mathbf{N}^T\mathbf{x} = c$, where $\mathbf{N}$ is the unit normal
- How do I find the point closest to my query point $\mathbf{p}$?

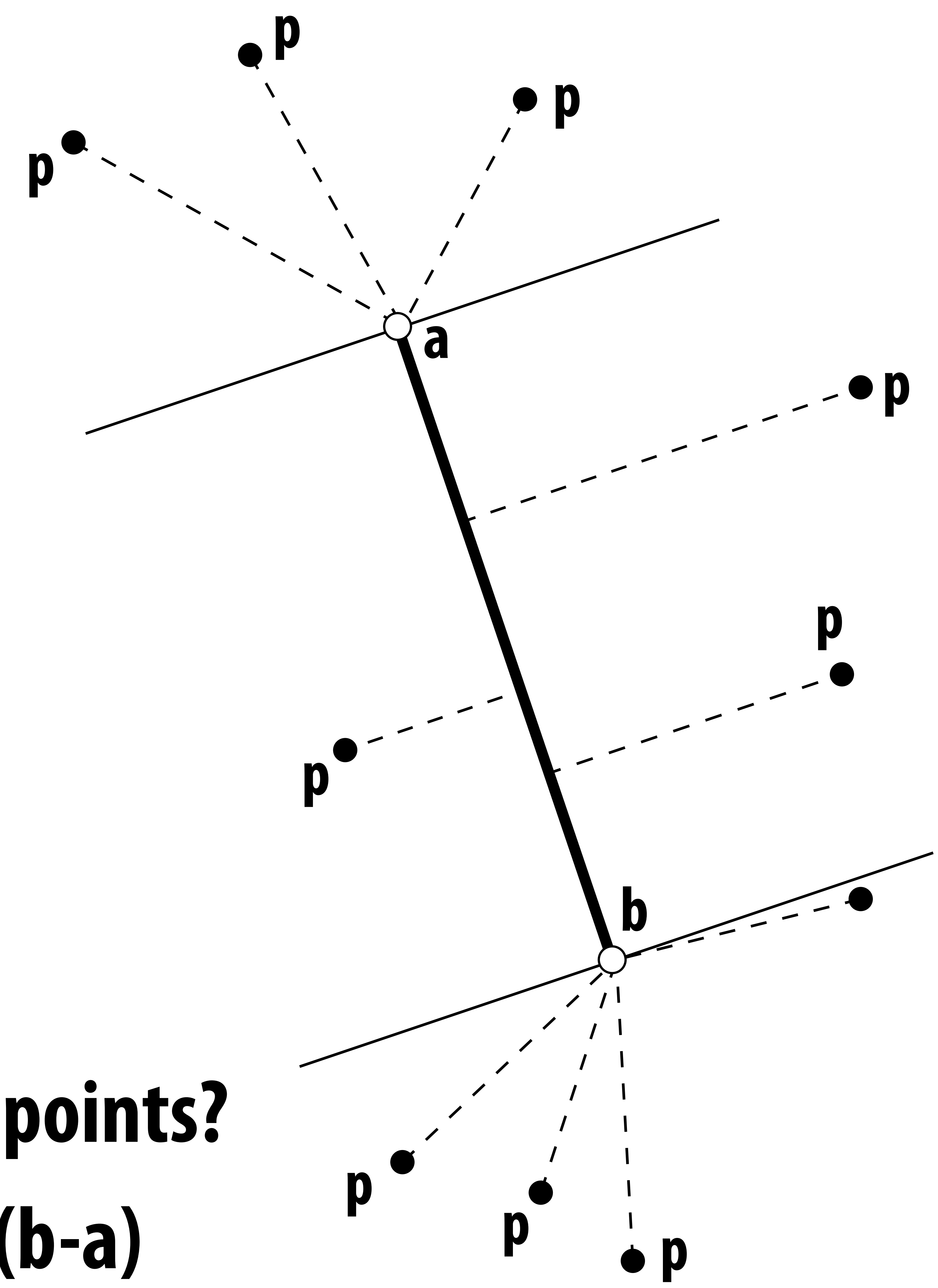


Many ways to do it:

$$
\begin{aligned}
& \mathbf{N}^T(\mathbf{p} + tN) = c \\
\iff\ & \mathbf{N}^T\mathbf{p} + t\mathbf{N}^T\mathbf{N} = c \\
\iff\ & t = c - \mathbf{N}^T\mathbf{p} \\
\Rightarrow\ & \mathbf{p} + t\mathbf{N} = \boxed{\mathbf{p} + (c - \mathbf{N}^T\mathbf{p})\mathbf{N}}
\end{aligned}
$$

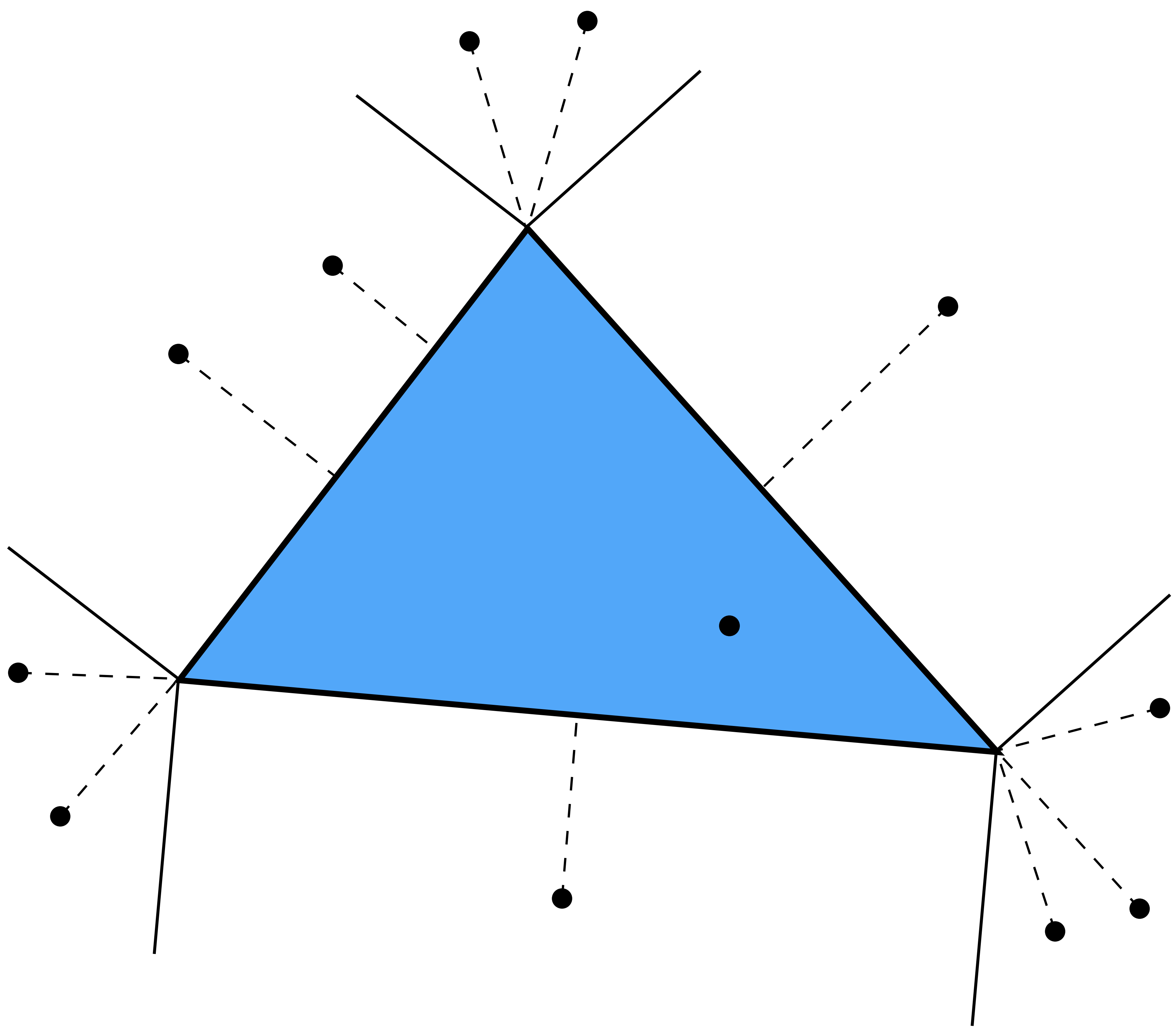
Harder: closest point on line segment

- Two cases: endpoint or interior
- Already have basic components:
  - point-to-point
  - point-to-line
- Algorithm?
  - find closest point on line
  - check if it's between endpoints
  - if not, take closest endpoint
- How do we know if it's between endpoints?
  - write closest point on line as a+t(b-a)
  - if t is between 0 and 1, it's inside the segment!



Even harder: closest point on triangle

- What are all the possibilities for the closest point?
- Almost just minimum distance to three segments:



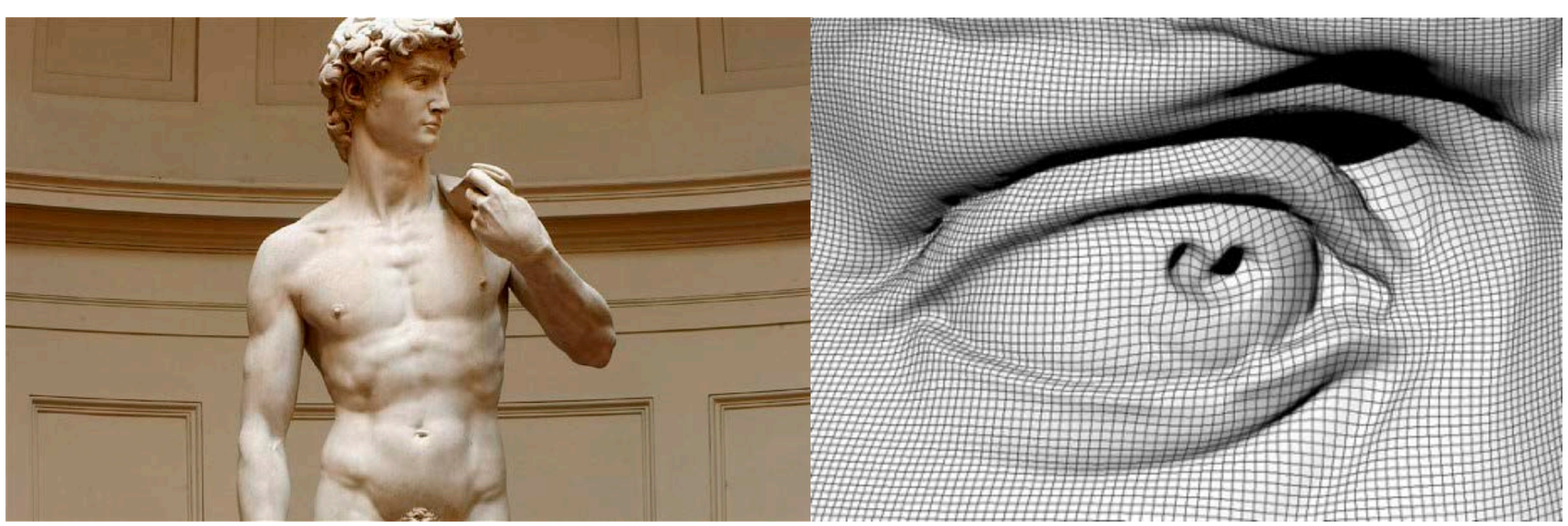
Q: What about a point inside the triangle?

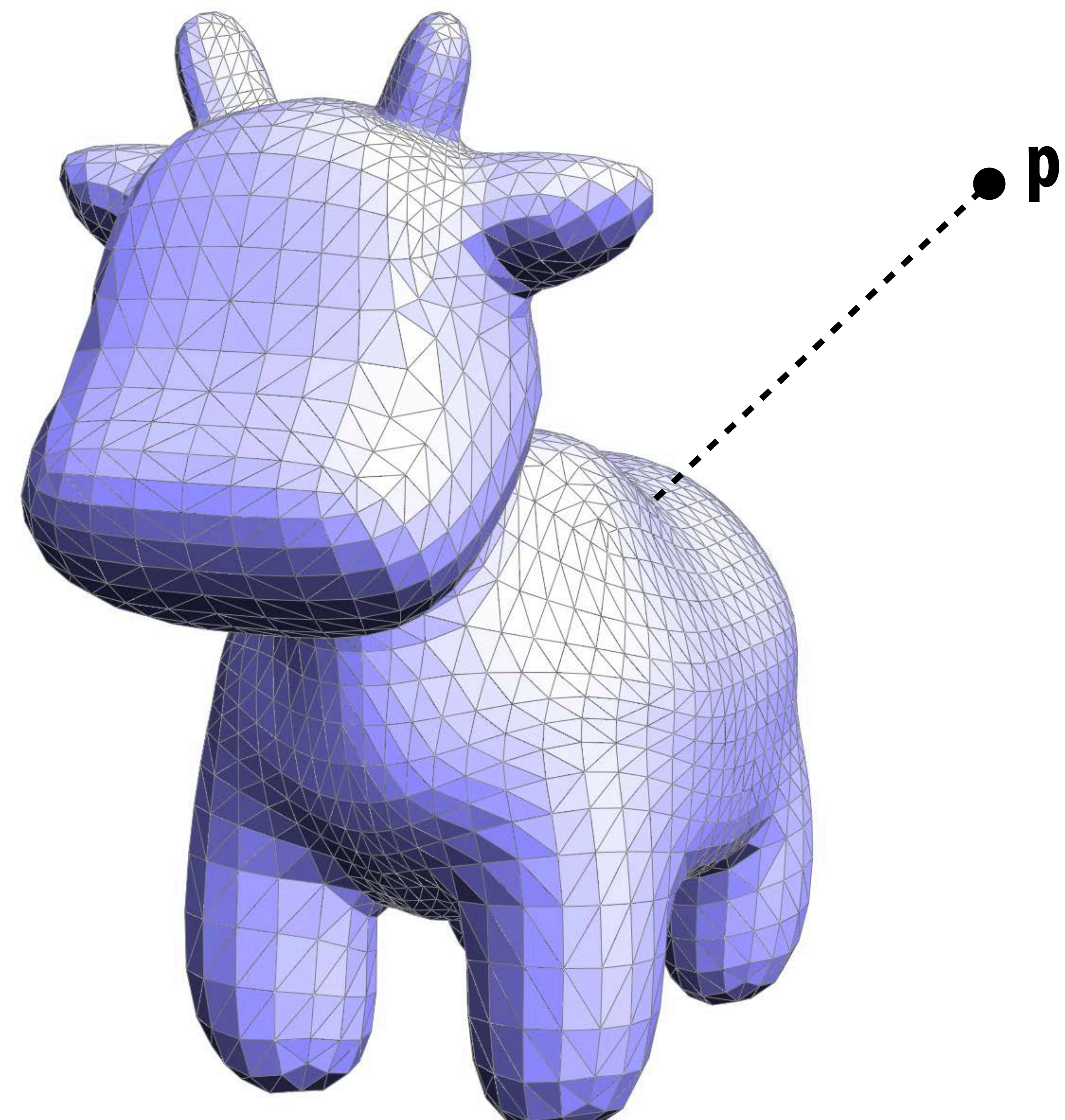
Closest point on triangle in 3D

- Not so different from 2D case
- Algorithm?
  - project onto plane of triangle
  - use half-*space* tests to classify point (vs. half plane)
  - if inside the triangle, we're done!
  - otherwise, find closest point on associated vertex or edge
- By the way, how do we find closest point on plane?
- Same expression as closest point on a line!
- E.g., $p + (c - N^T p) N$

Closest point on triangle *mesh* in 3D?

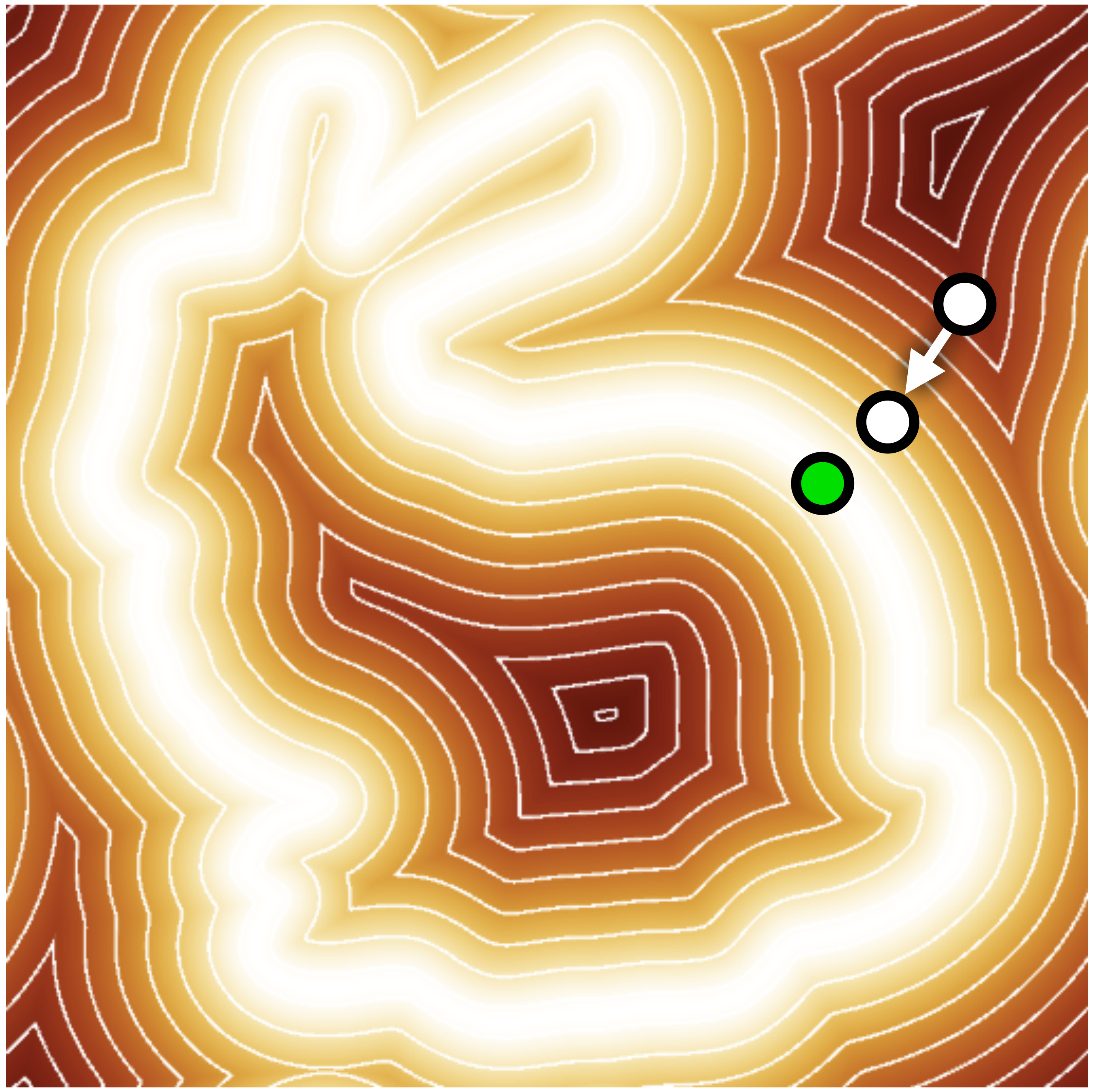
- **Conceptually easy:**
  - loop over all triangles
  - compute closest point to current triangle
  - keep globally closest point
- **Q: What's the cost?**
- **What if we have *billions* of faces?**
- **NEXT TIME: Better data structures!**





Closest point to *implicit* surface?

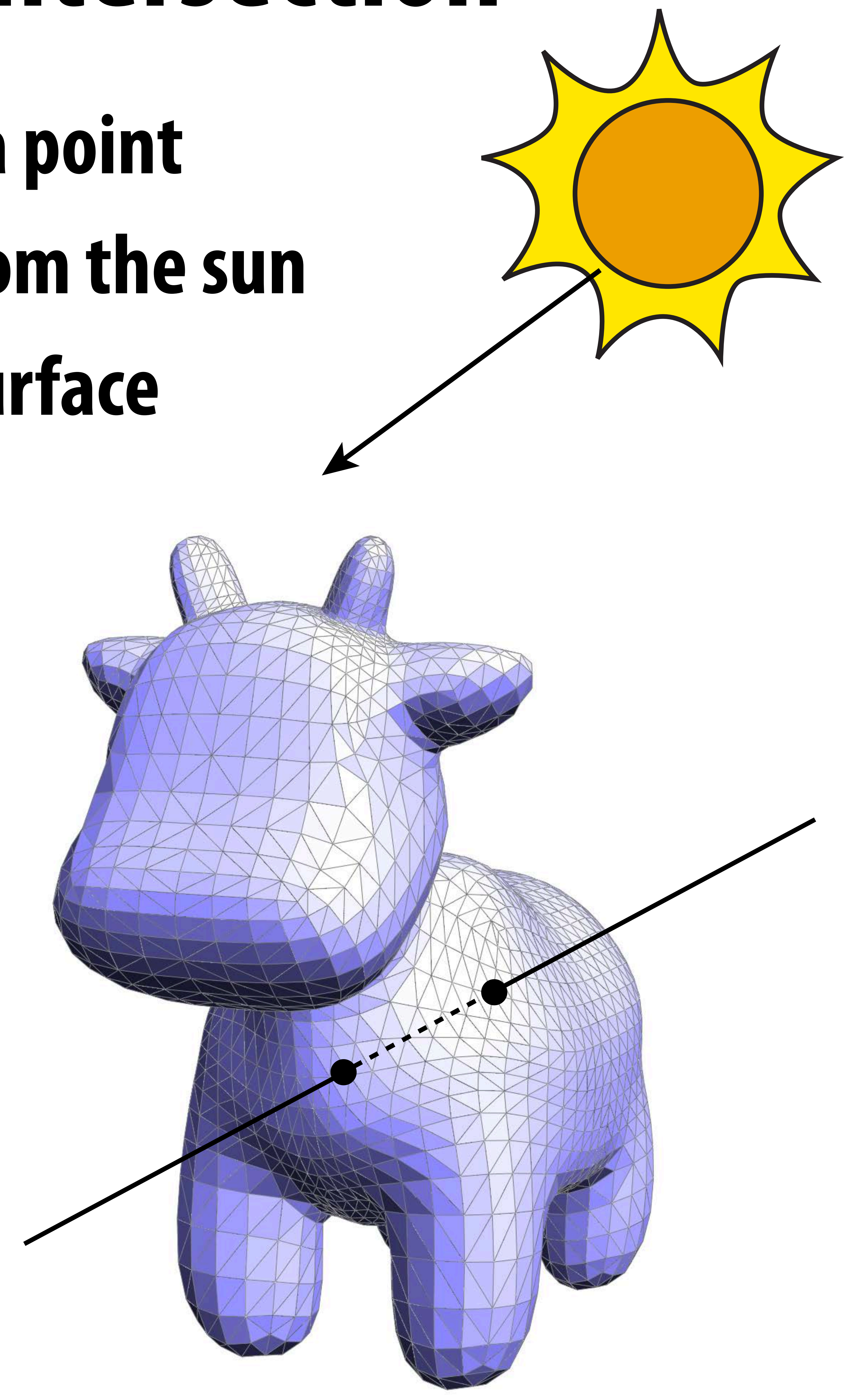
- If we change our representation of geometry, algorithms can change completely
- E.g., how might we compute the closest point on an implicit surface described via its distance function?
- One idea:
  - start at the query point
  - compute gradient of distance (using, e.g., finite differences)
  - take a little step (decrease distance)
  - repeat until we're at the surface (zero distance)



- Better yet: just store closest point for each grid cell! (speed/memory trade off)

Different query: ray-mesh intersection

- A “ray” is an oriented line starting at a point
- Think about a ray of light traveling from the sun
- Want to know where a ray pierces a surface
- Why?
  - GEOMETRY: inside-outside test
  - RENDERING: visibility, ray tracing
  - ANIMATION: collision detection
- Might pierce surface in many places!



Ray equation

- **Can express ray as**

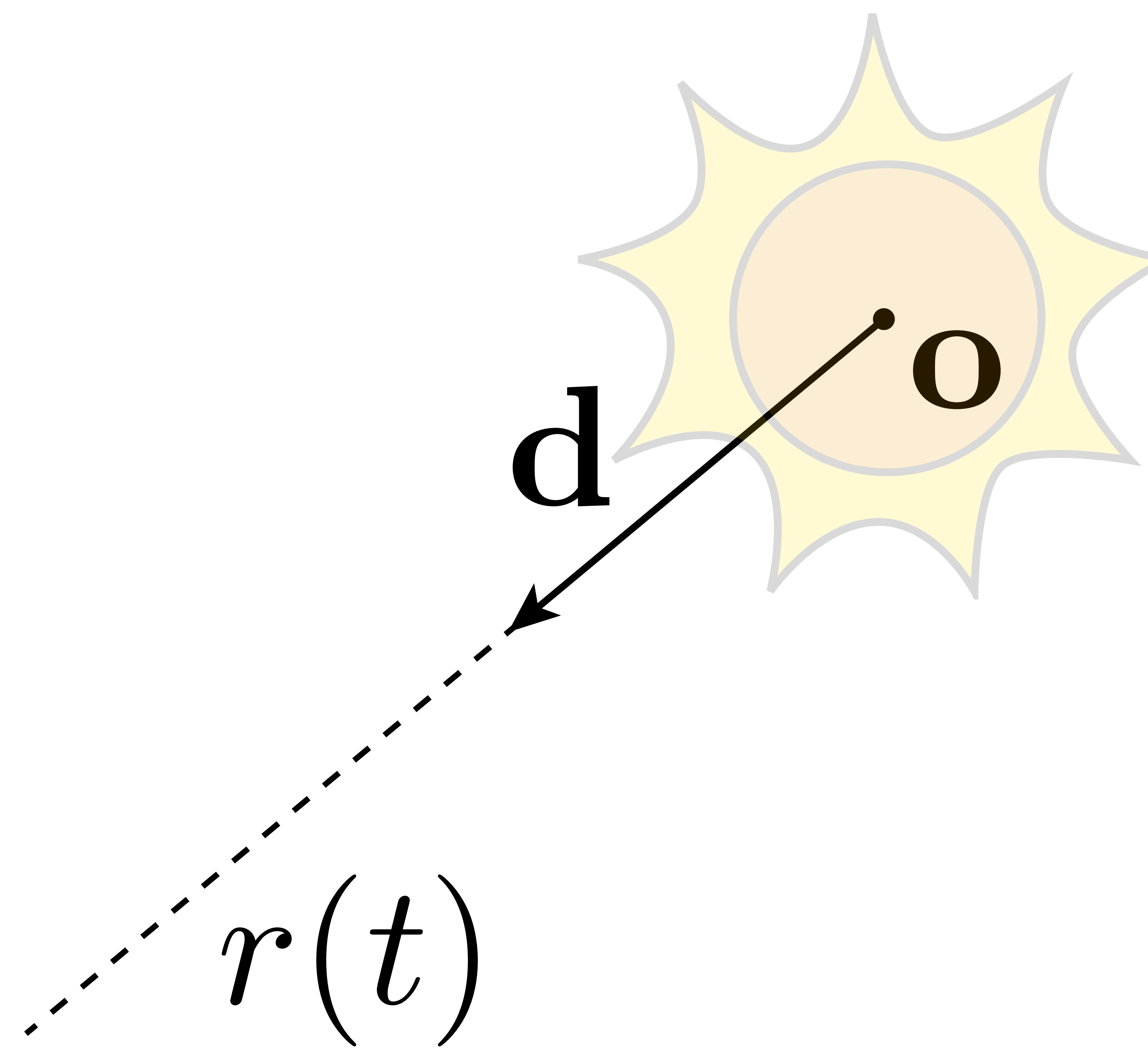
origin

unit direction

$$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$$

point along ray

"time"



Intersecting a ray with an implicit surface

- Recall implicit surfaces: all points x such that f(x) = 0
- Q: How do we find points where a ray pierces this surface?
- Well, we know all points along the ray: r(t) = o + td
- Idea: replace "x" with "r" in 1st equation, and solve for t
- Example: unit sphere

$$f(\mathbf{x}) = |\mathbf{x}|^2 - 1$$

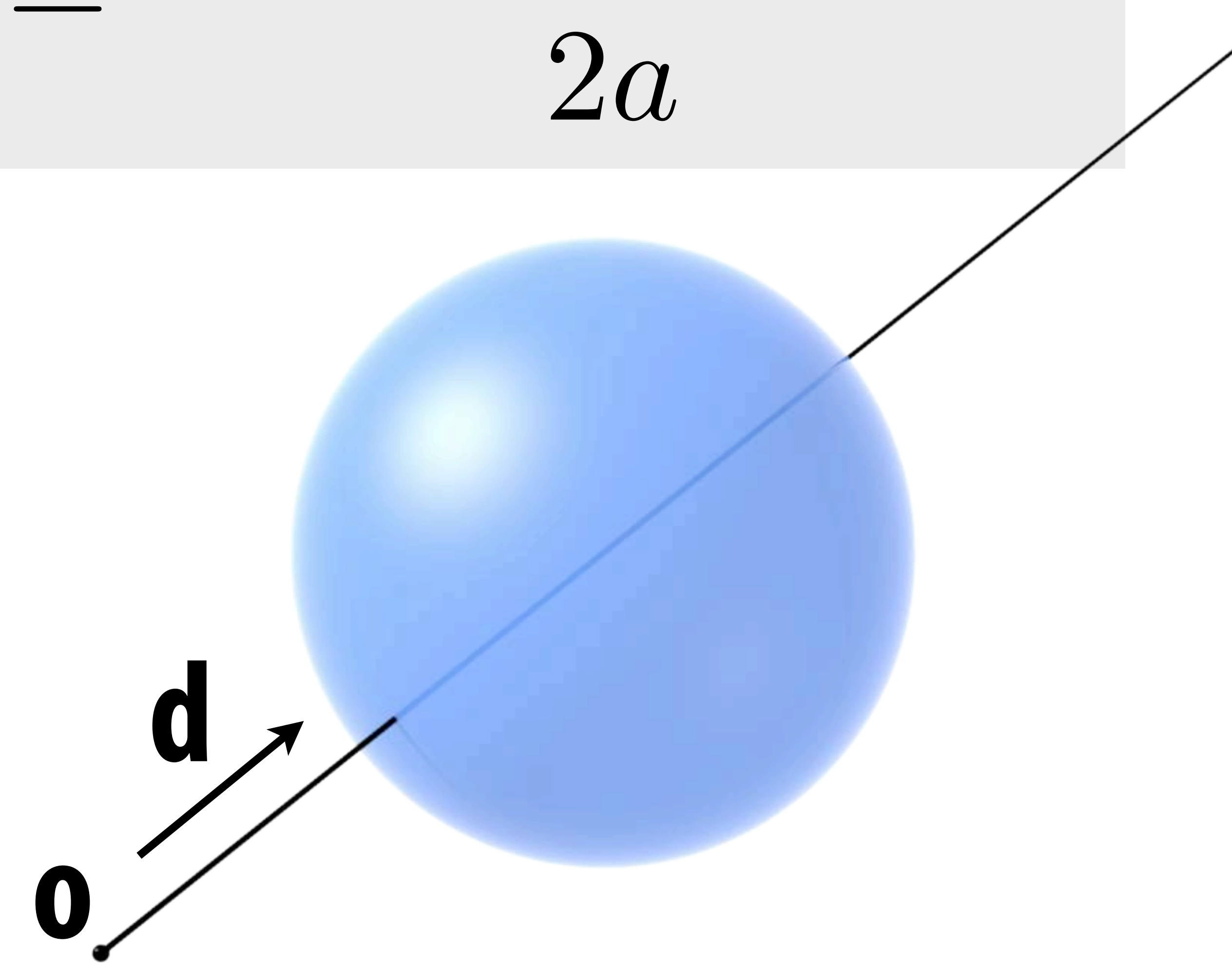
$$\Rightarrow f(\mathbf{r}(t)) = |\mathbf{o} + t\mathbf{d}|^2 - 1$$

$$\underbrace{|\mathbf{d}|^2}_{a} t^2 + \underbrace{2(\mathbf{o} \cdot \mathbf{d})}_{b} t + \underbrace{|\mathbf{o}|^2 - 1}_{c} = 0$$

$$t = \boxed{-\mathbf{o} \cdot \mathbf{d} \pm \sqrt{(\mathbf{o} \cdot \mathbf{d})^2 - |\mathbf{o}|^2 + 1}}$$

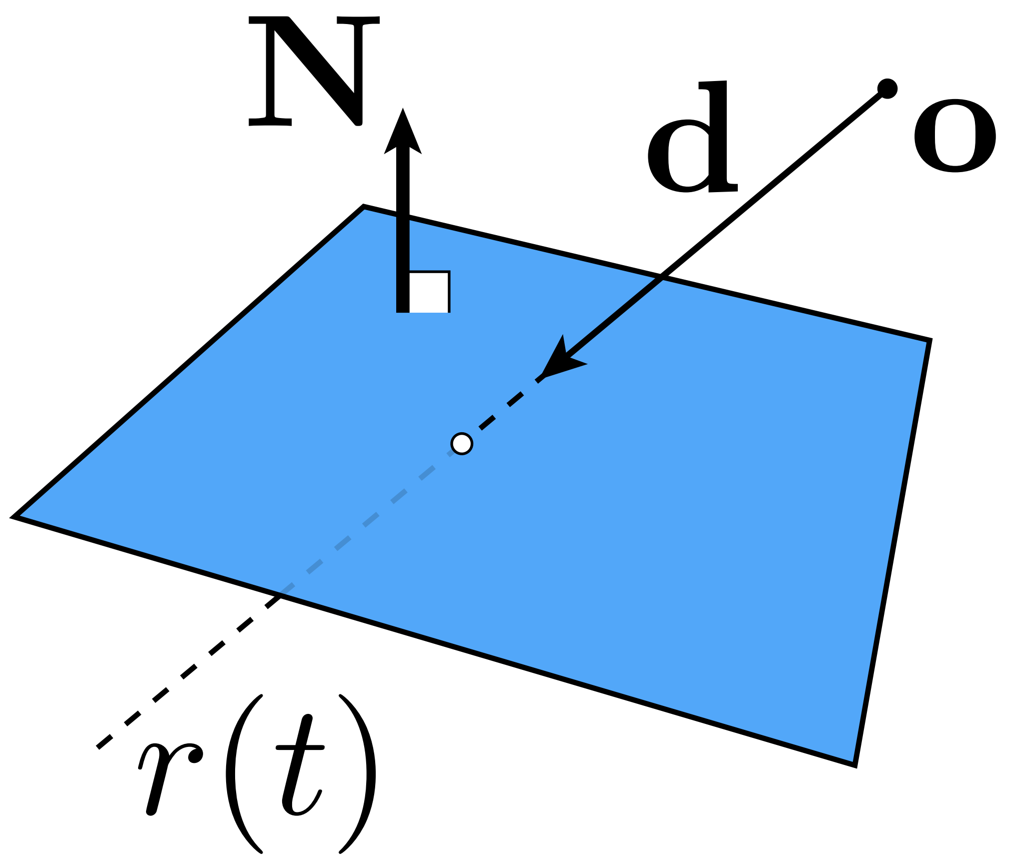
quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Why two solutions?

Ray-plane intersection



- Suppose we have a plane $\mathbf{N}^\mathsf{T}\mathbf{x} = c$
  - N - unit normal
  - c - offset
- How do we find intersection with ray $r(t) = \mathbf{o} + t\mathbf{d}$?
- *Key idea:* again, replace the point x with the ray equation t:

$$\mathbf{N}^\mathsf{T}\mathbf{r}(t) = c$$

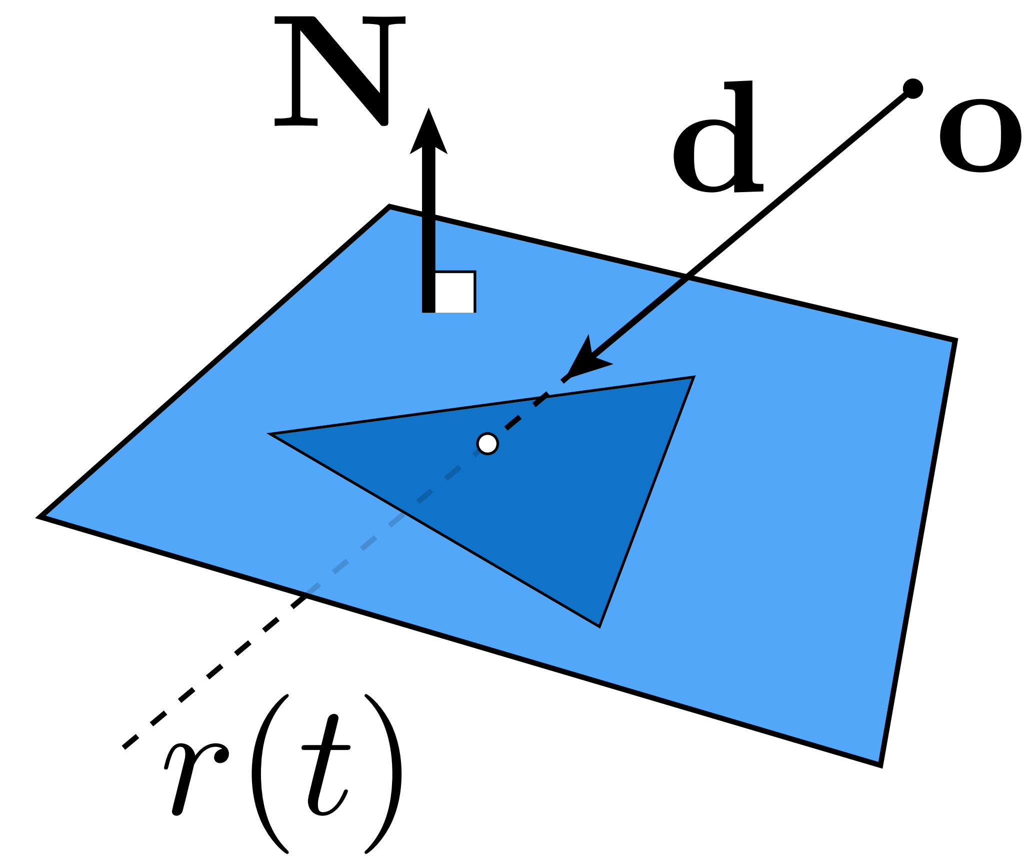
- Now solve for t:

$$\mathbf{N}^\mathsf{T}(\mathbf{o} + t\mathbf{d}) = c \qquad \Rightarrow t = \frac{c - \mathbf{N}^\mathsf{T}\mathbf{o}}{\mathbf{N}^\mathsf{T}\mathbf{d}}$$

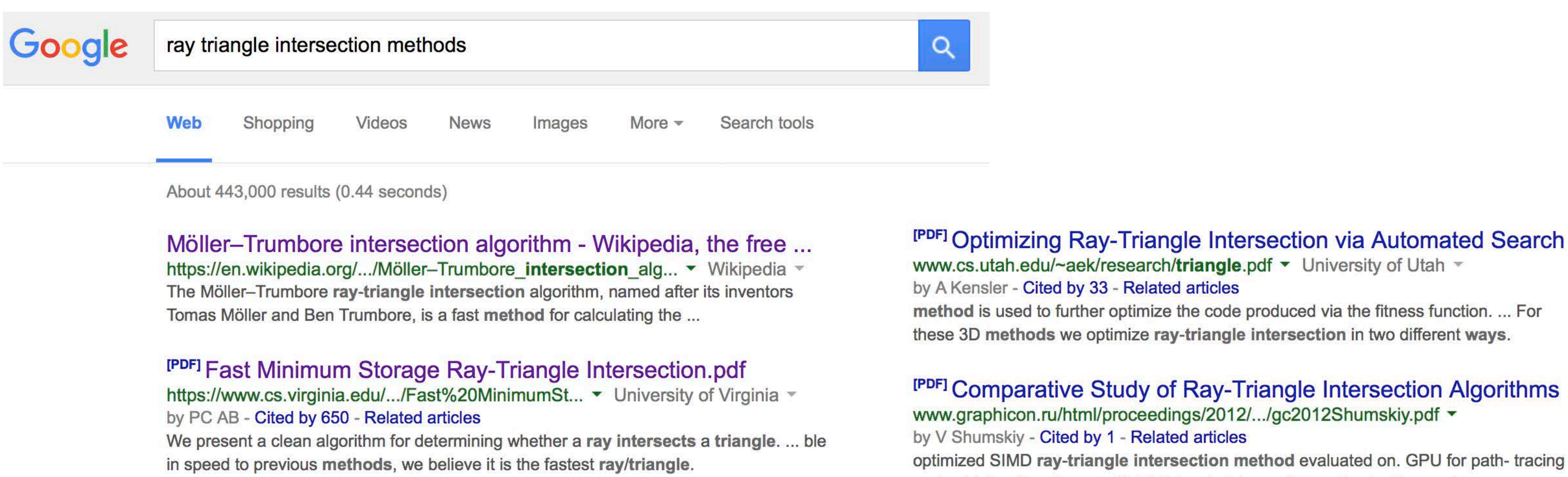
- And plug t back into ray equation:

$$r(t) = \mathbf{o} + \frac{c - \mathbf{N}^\mathsf{T}\mathbf{o}}{\mathbf{N}^\mathsf{T}\mathbf{d}}\mathbf{d}$$

Ray-triangle intersection



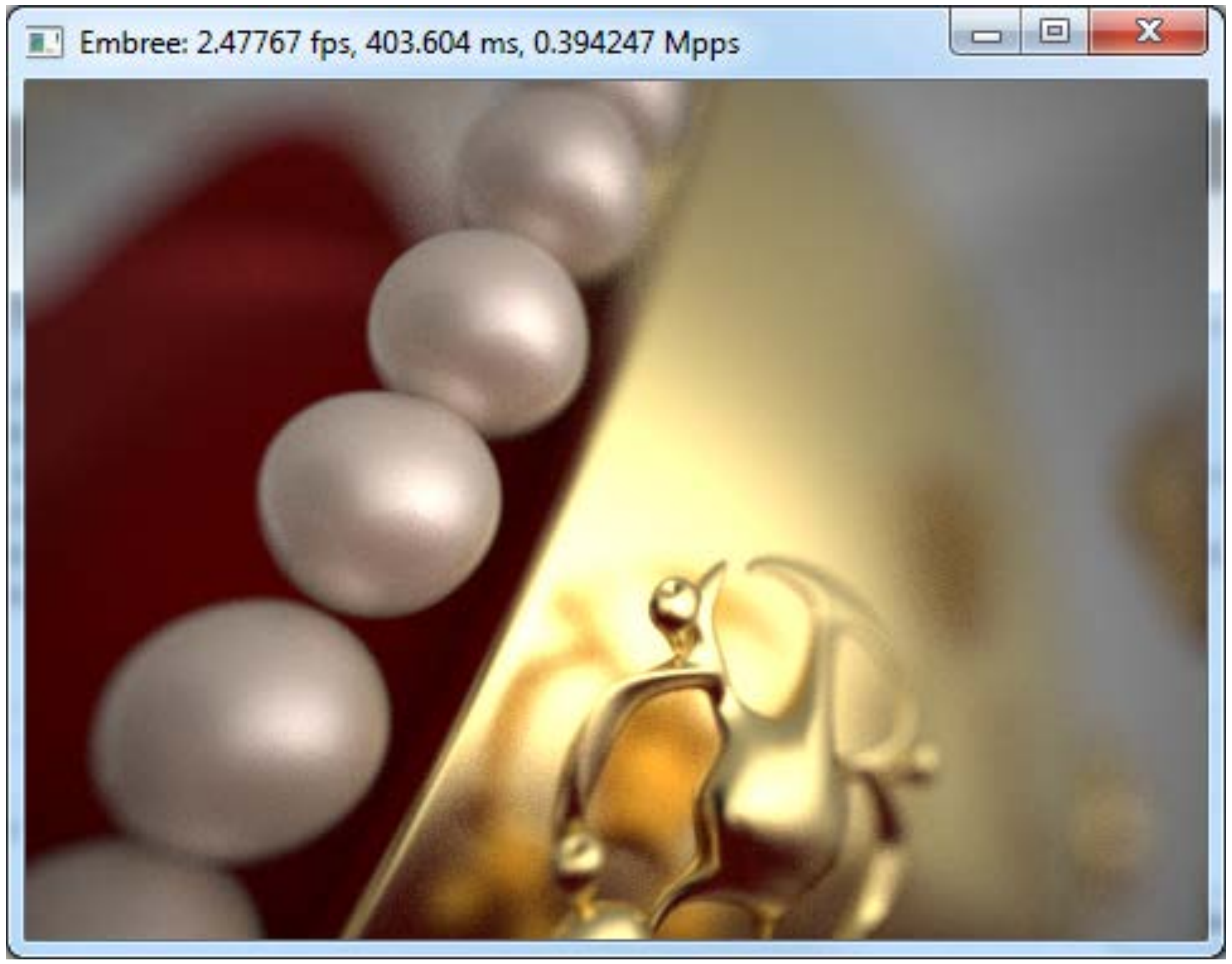
- Triangle is in a plane...
- Not much more to say!
  - Compute ray-plane intersection
  - Q: What do we do now?
  - A: Why not compute barycentric coordinates of hit point?
  - If barycentric coordinates are all positive, point in triangle
- Actually, a *lot* more to say... if you care about performance!



Why care about performance?

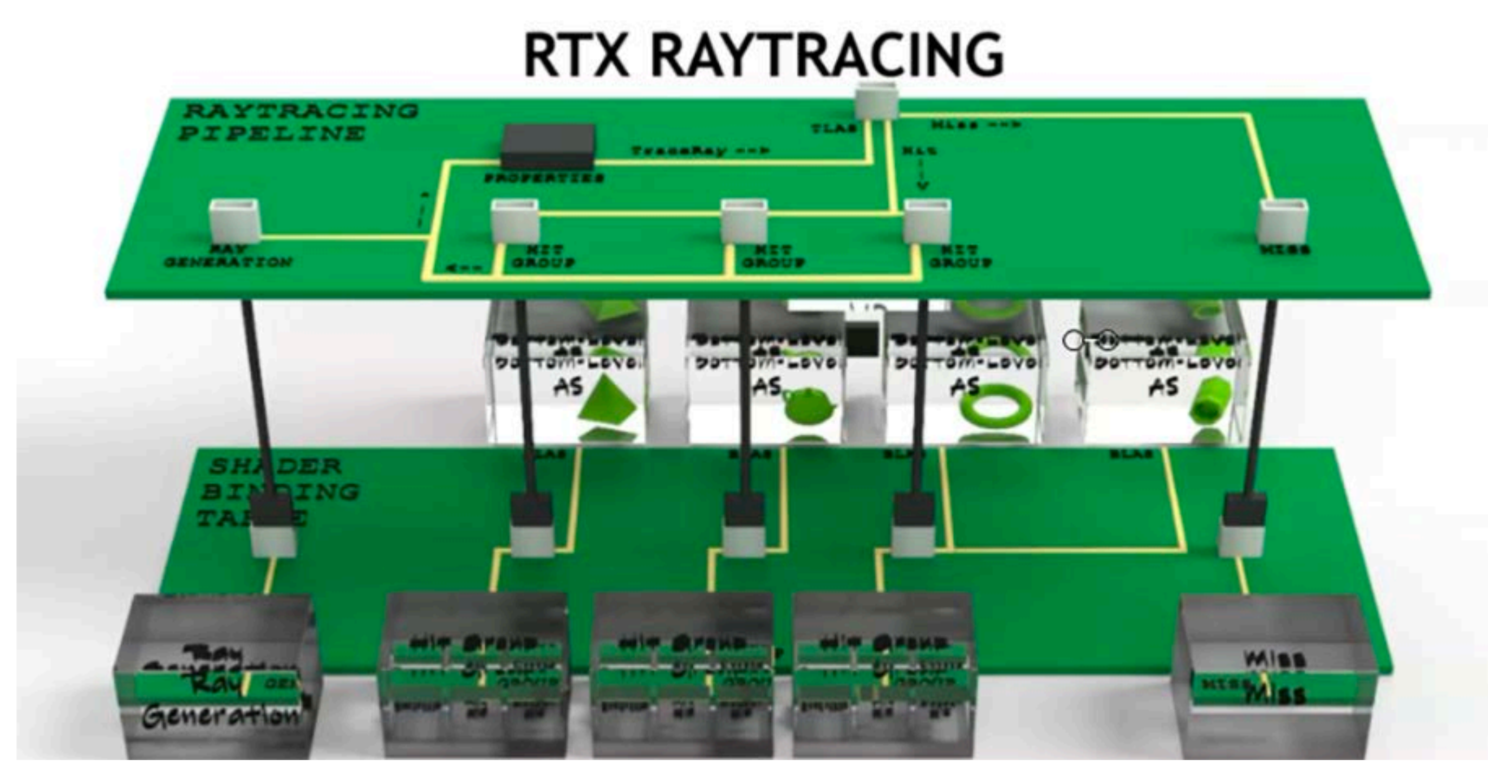
Pixar's "Coco" — about 50 <u>hours</u> per frame (@24 frames/sec)

High-performance ray tracing



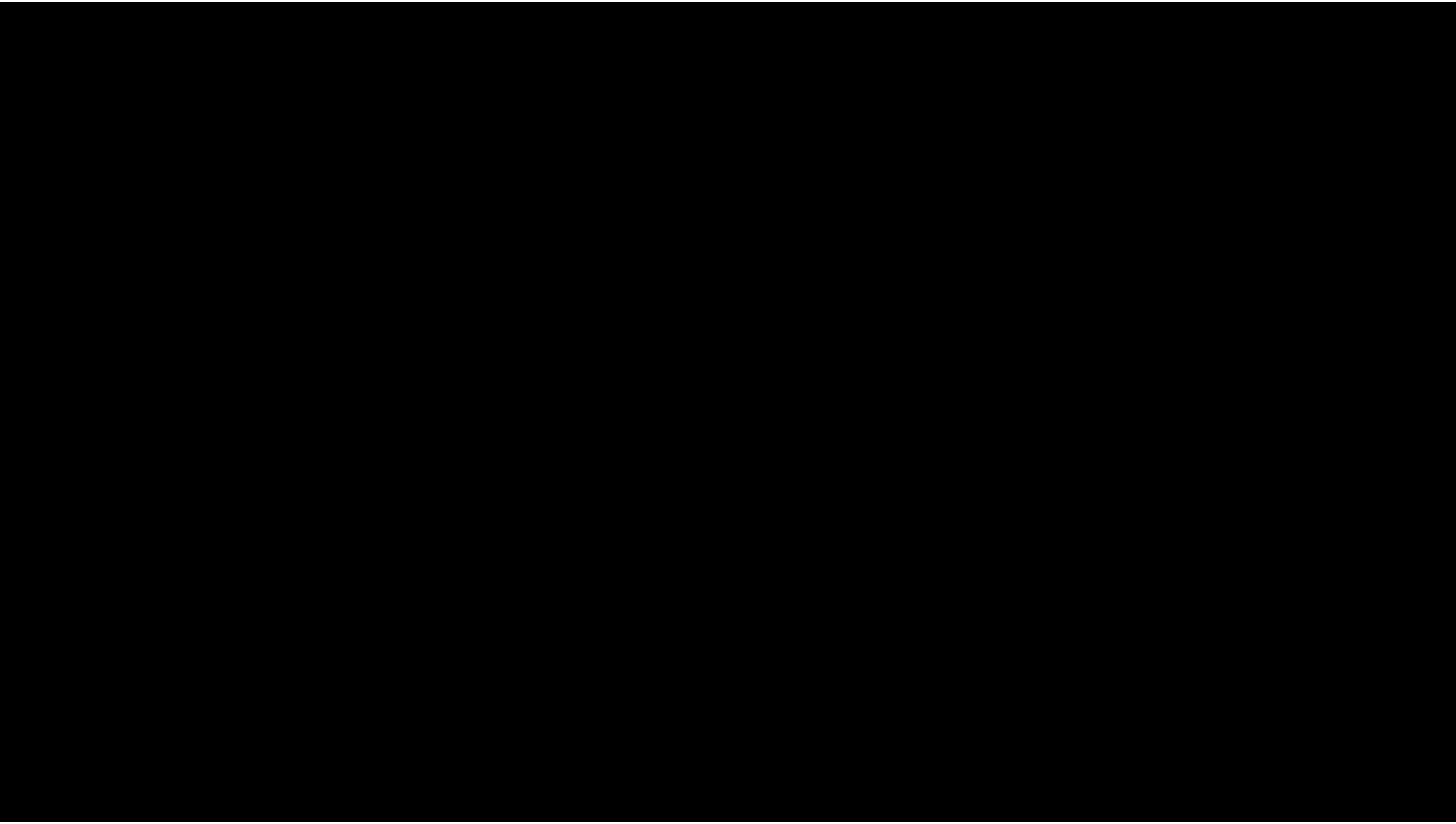
Intel Embree

NVIDIA OptiX



Hardware-accelerated ray tracing (RTX)

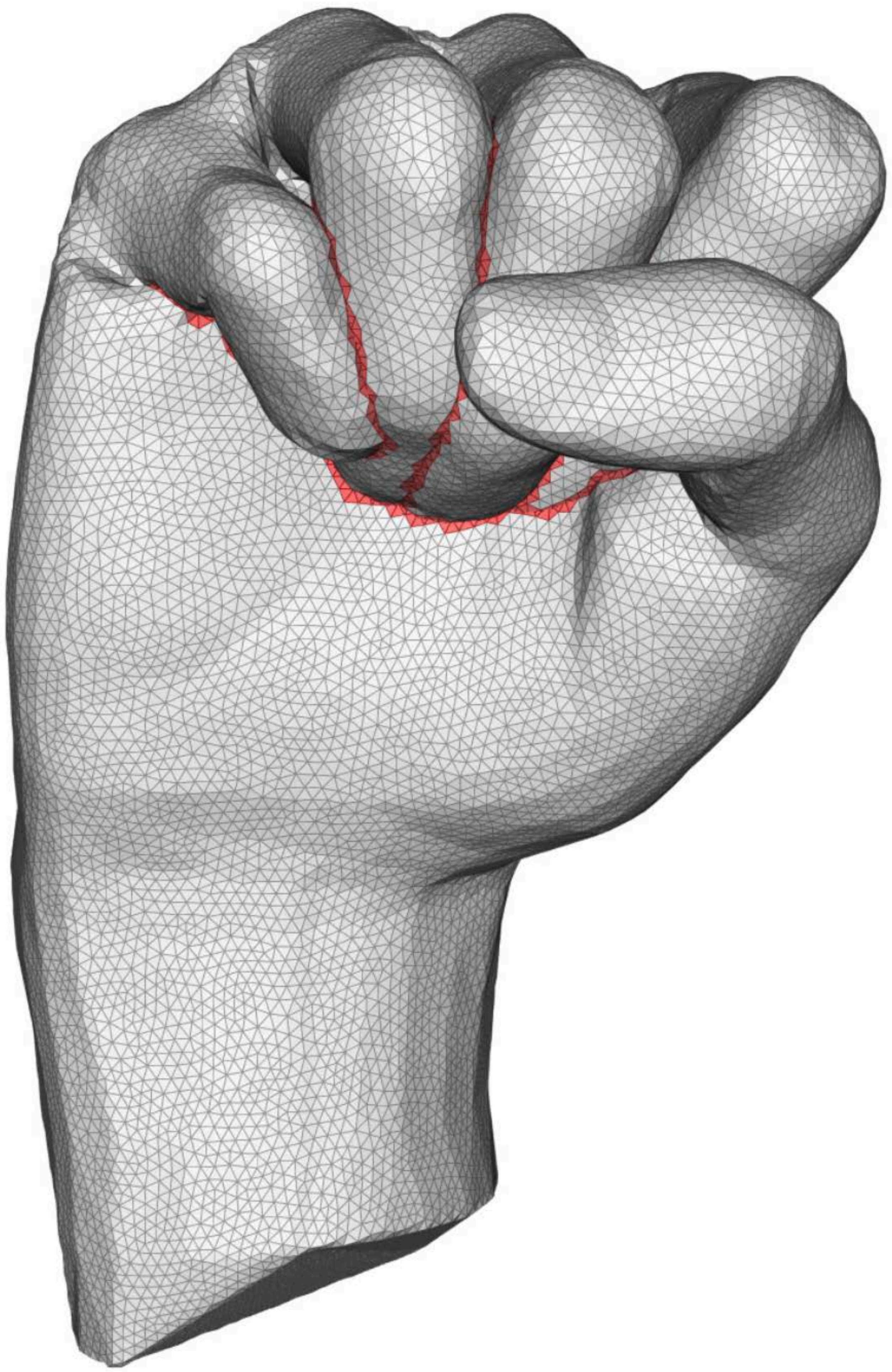
High-performance ray tracing



NVIDIA's "Project Sol" — real time ray tracing demo (RTX)

One more query: mesh-mesh intersection

- **GEOMETRY: How do we know if a mesh intersects itself?**
- **ANIMATION: How do we know if a collision occurred?**



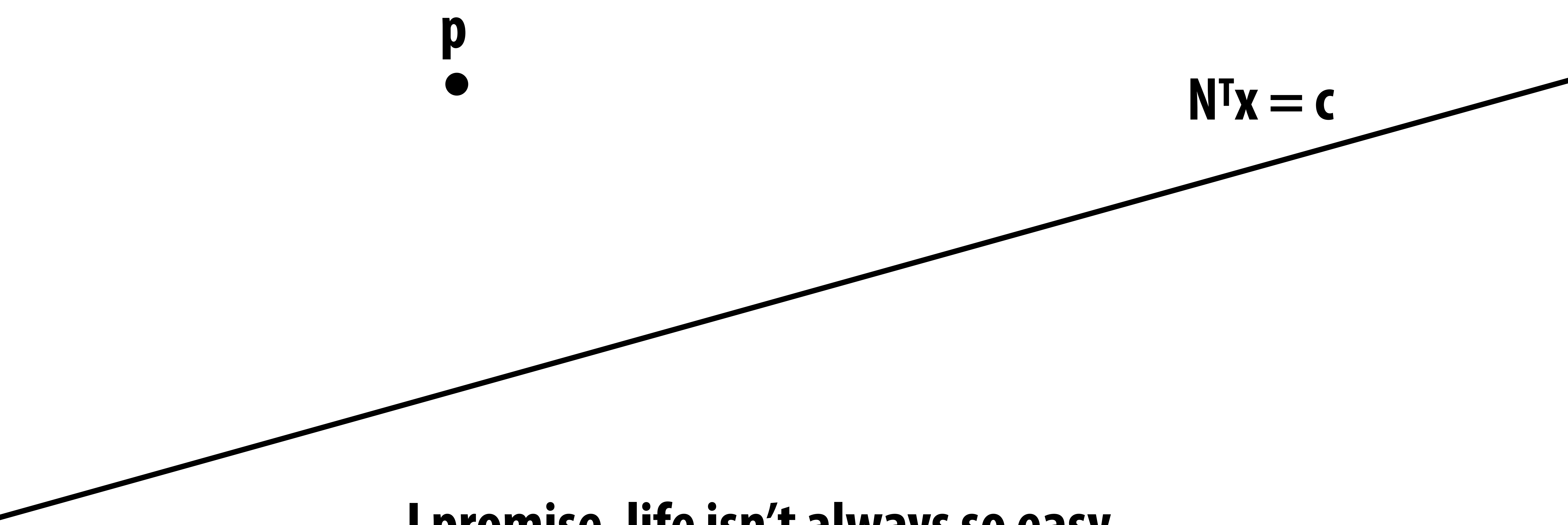
Warm up: point-point intersection

- Q: How do we know if p intersects a?
- A: ...check if they're the same point!

Sadly, life is not always so easy.

Slightly harder: point-line intersection

- Q: How do we know if a point intersects a given line?
- A: ...plug it into the line equation!



I promise, life isn't always so easy.

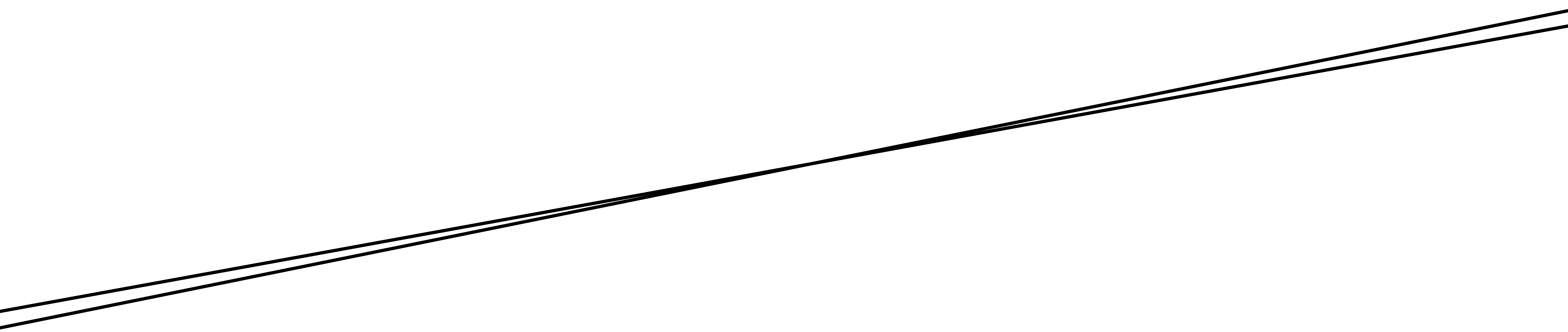
Finally interesting: line-line intersection

- Two lines: ax=b and cx=d
- Q: How do we find the intersection?
- A: See if there is a simultaneous solution
- Leads to linear system:

$$\begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

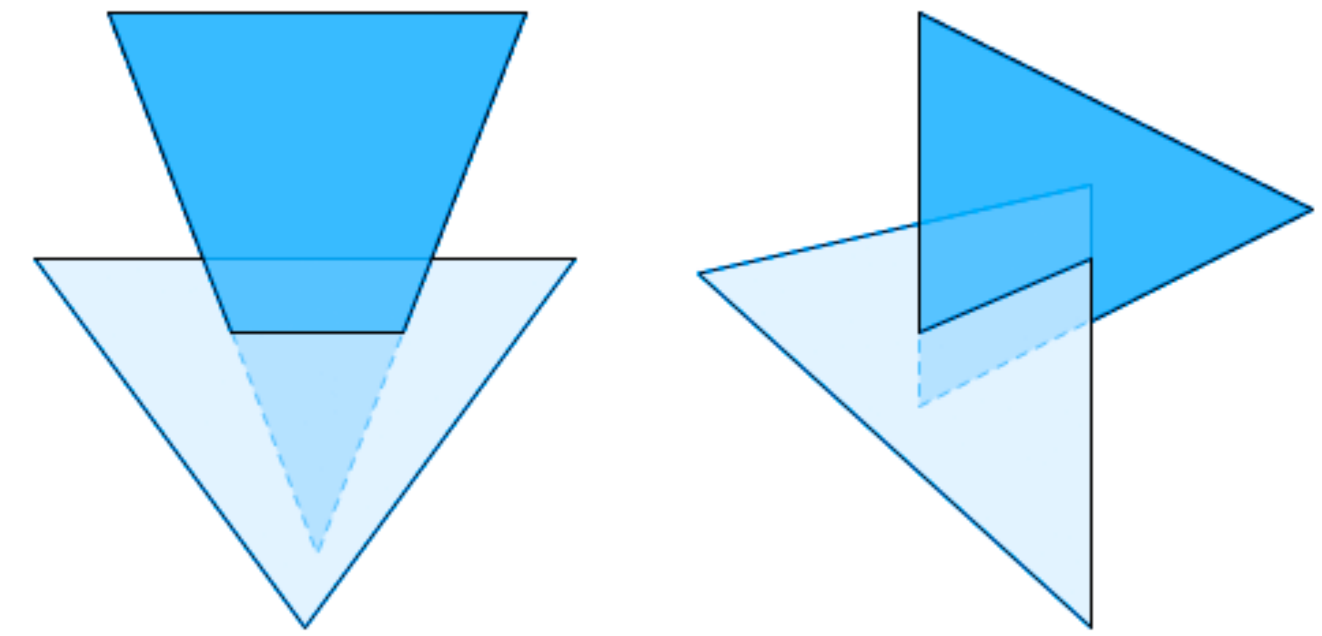
Degenerate line-line intersection?

- **What if lines are almost parallel?**
- **Small change in normal can lead to big change in intersection!**
- **Instability very common, very important with geometric predicates. Demands special care (e.g., analysis of matrix).**

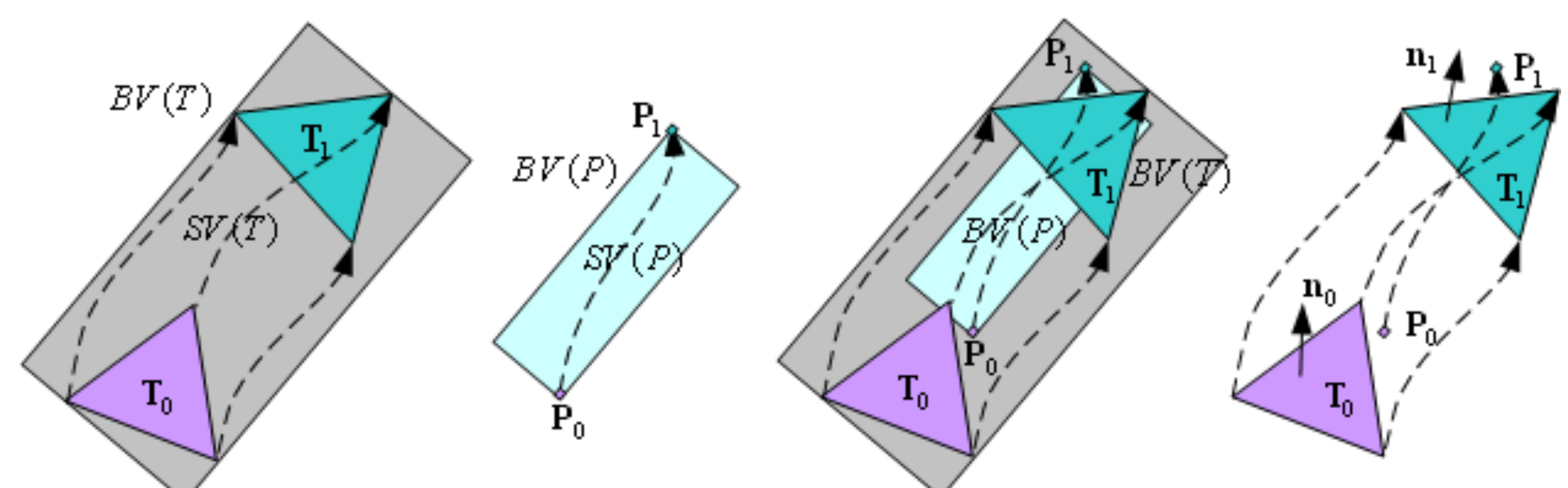


See for example Shewchuk, "*Adaptive Precision Floating-Point Arithmetic and Fast Robust Geometric Predicates*"

Triangle-Triangle Intersection?



- Lots of ways to do it
- Basic idea:
  - Q: Any ideas?
  - One way: reduce to edge-triangle intersection
  - Check if each line passes through plane
  - Then do interval test
- What if triangle is *moving*?
  - Important case for animation
  - Can think of triangles as *prisms* in time
  - Turns dynamic problem (nD + time) into purely geometric problem in (n+1)-dimensions



(a) Bounding volume of a deforming triangle (b) Bounding volume of a deforming vertex (c) Bounding volume test (d) Coplanarity test

Up Next: Spatial Acceleration Data Strucutres

- Testing every element is *slow!*
- E.g., linearly scanning through a list vs. binary search
- Can apply this same kind of thinking to geometric queries