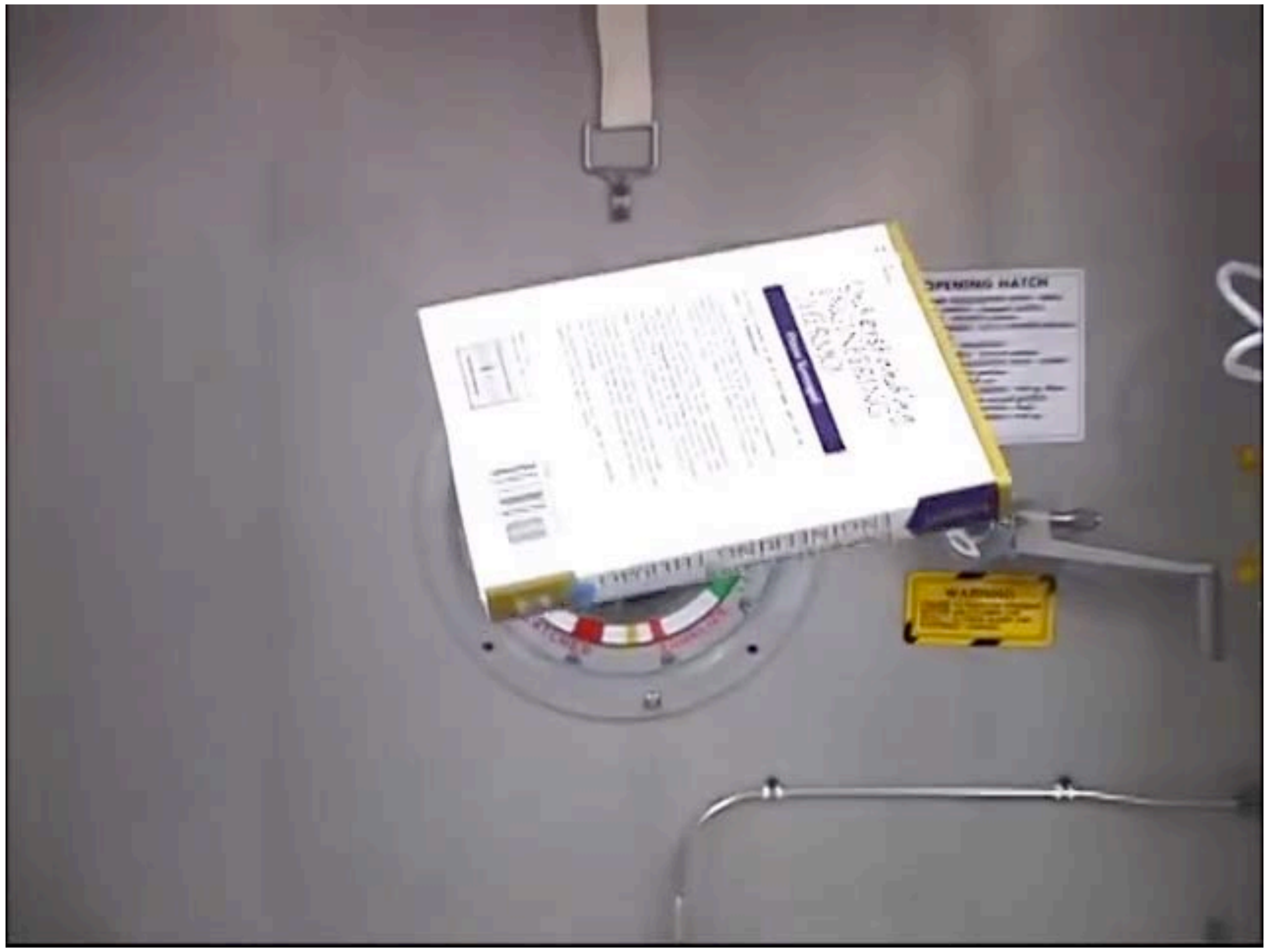
3D Rotations and Complex Representations

Computer Graphics
CMU 15-462/15-662

Rotations in 3D

- What is a rotation, intuitively?
- *How do you know a rotation when you see it?*
  - length/distance is preserved (no stretching/shearing)
  - orientation is preserved (e.g., text remains readable)
  - origin is preserved (otherwise it's a rotation + translation)



3D Rotations—Degrees of Freedom

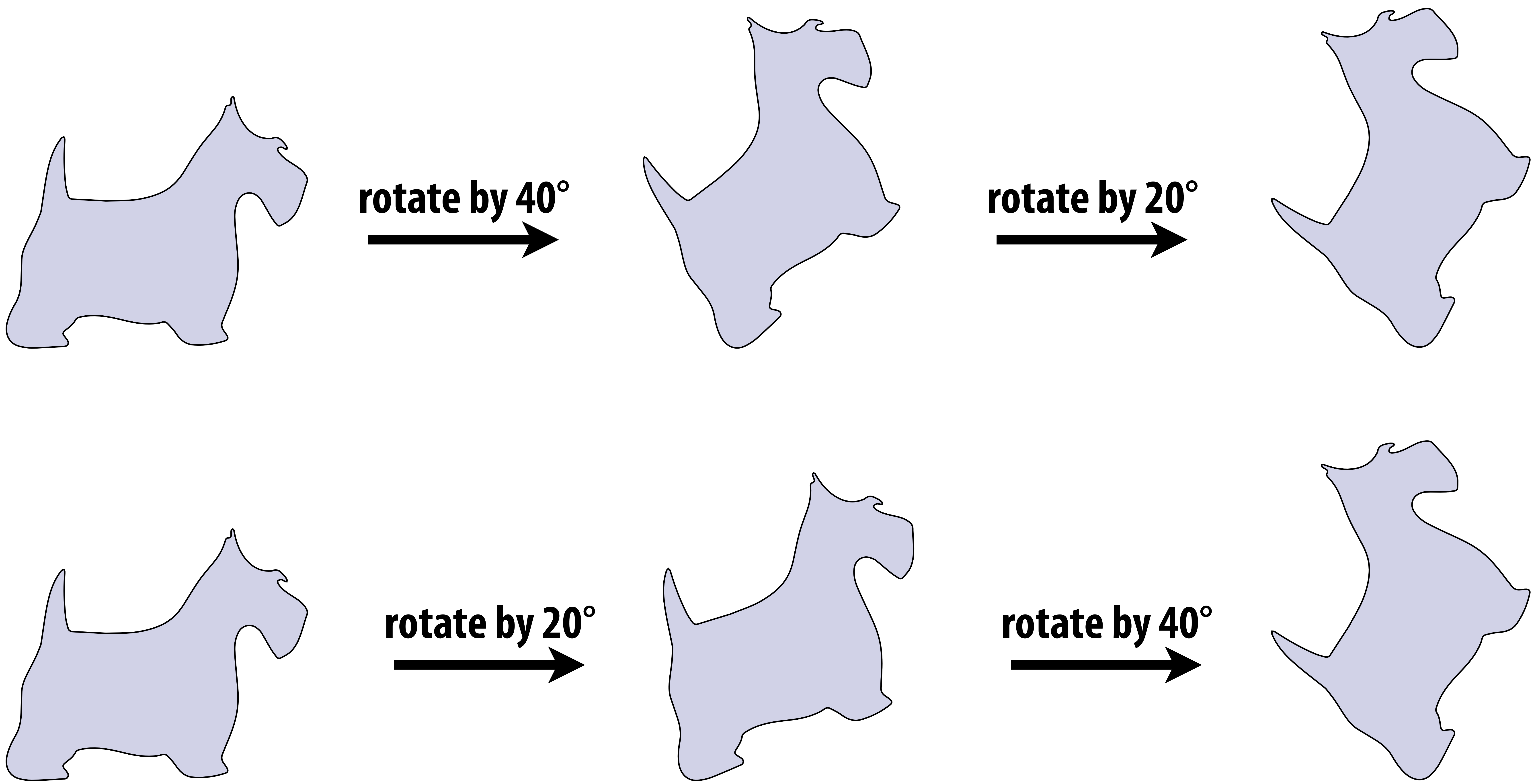
- How many numbers do we need to specify a rotation in 3D?
- For instance, we could use rotations around X, Y, Z. But do we *need* all three?
- Well, to rotate Pittsburgh to another city (say, São Paulo), we have to specify two numbers: latitude & longitude:
- Do we really need both latitude and longitude? Or will one suffice?
- Is that the *only* rotation from Pittsburgh to São Paulo? (How many more numbers do we need?)

NO: We can keep São Paulo fixed as we rotate the globe.

Hence, we MUST have three degrees of freedom.

Commutativity of Rotations—2D

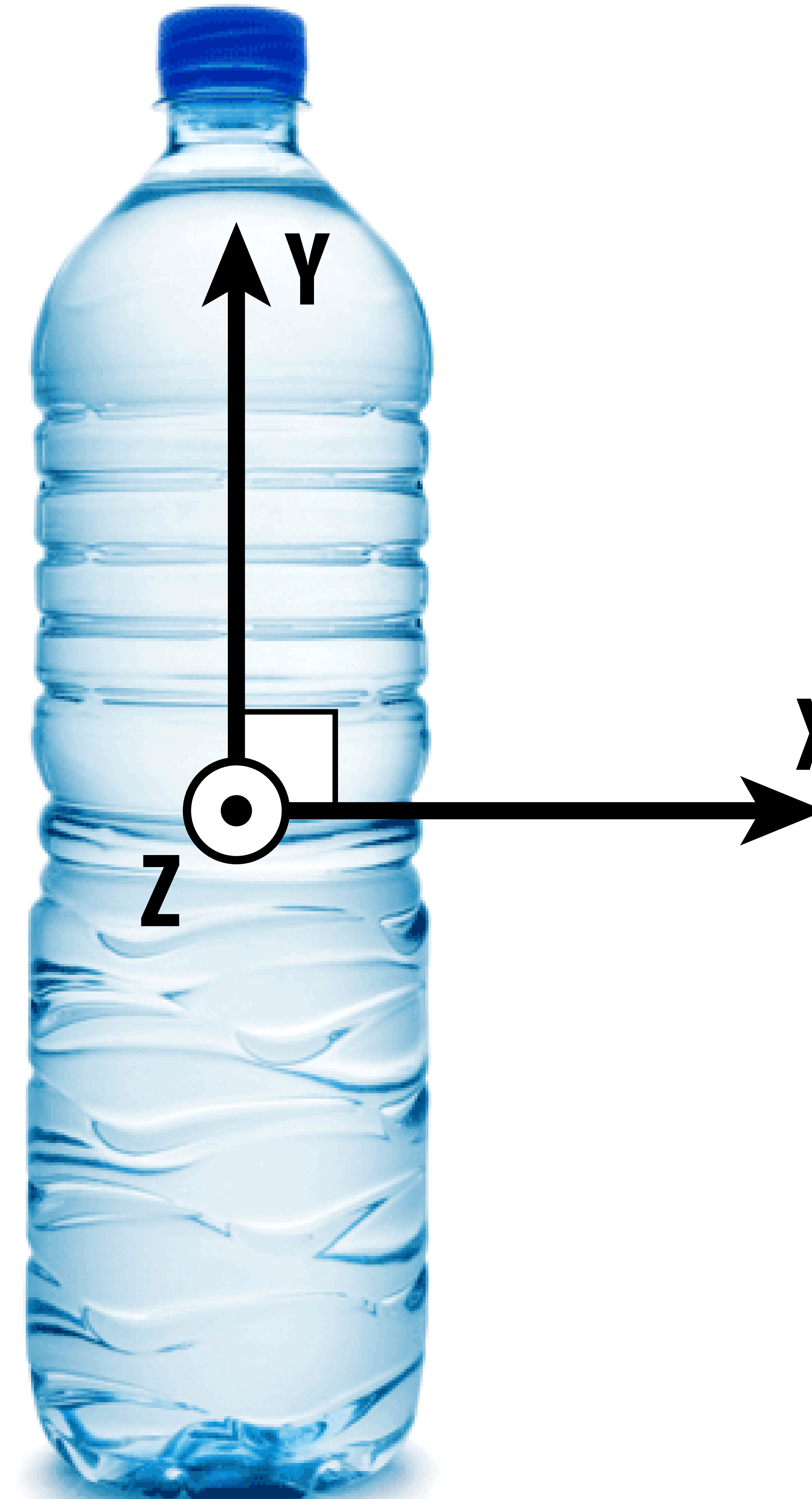
- **In 2D, order of rotations doesn't matter:**



Same result! ("2D rotations commute")

Commutativity of Rotations—3D

- What about in 3D?
- Try it at home—grab a water bottle!
  - Rotate 90° around Y, then 90° around Z, then 90° around X
  - Rotate 90° around Z, then 90° around Y, then 90° around X
  - (Was there any difference?)

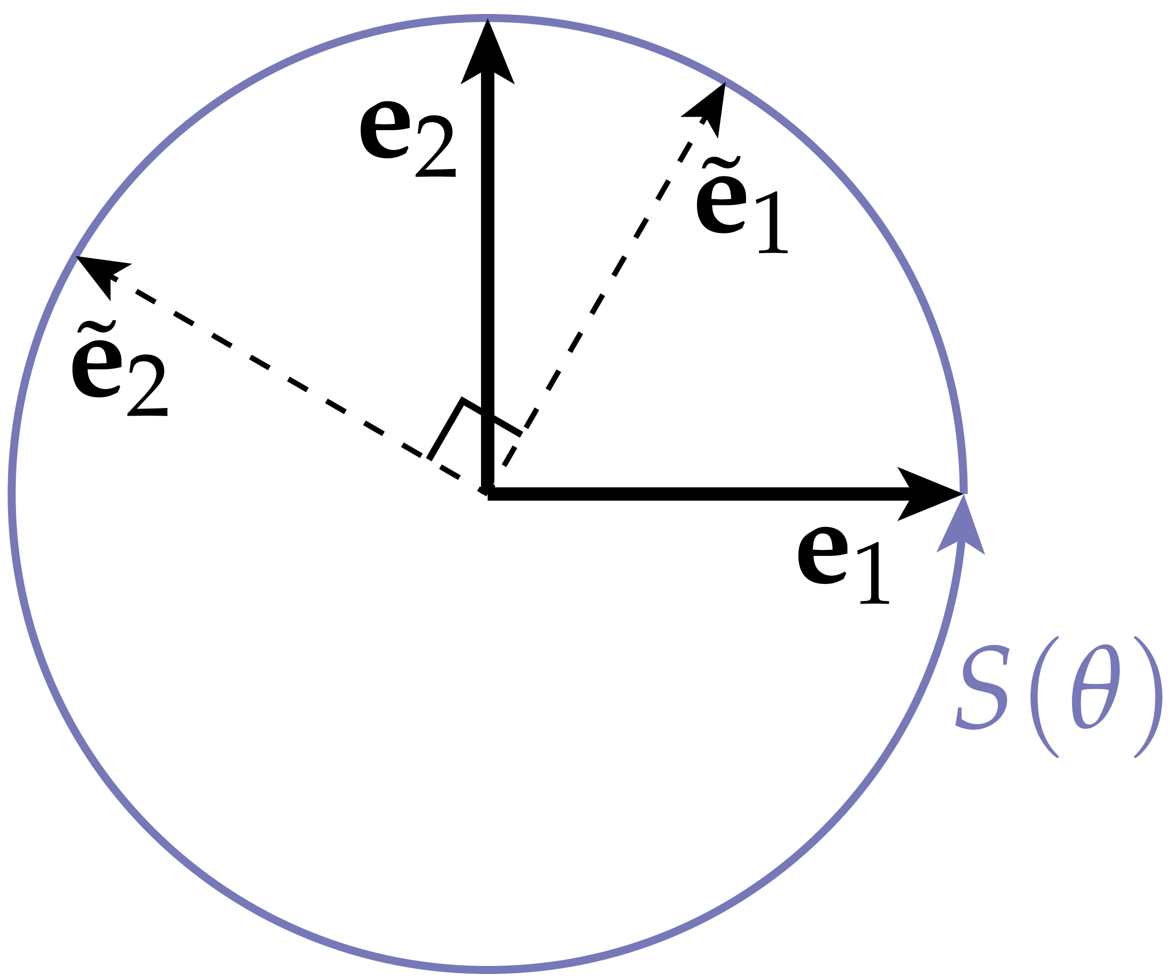


CONCLUSION: bad things can happen if we're not careful about the order in which we apply rotations!

Representing Rotations—2D

- **First things first: how do we get a rotation matrix in 2D? (Don't just regurgitate the formula!)**
- **Suppose I have a function S(θ) that for a given angle θ gives me the point (x,y) around a circle (CCW).**
  - **Right now, I *do not care how this function is expressed!*****
- **What's e1 rotated by θ?** $\tilde{\mathbf{e}}_1 = S(\theta)$
- **What's e2 rotated by θ?** $\tilde{\mathbf{e}}_2 = S(\theta + \pi/2)$
- **How about** $\mathbf{u} := a\mathbf{e}_1 + b\mathbf{e}_2$ **?**

$$\mathbf{u} := aS(\theta) + bS(\theta + \pi/2)$$



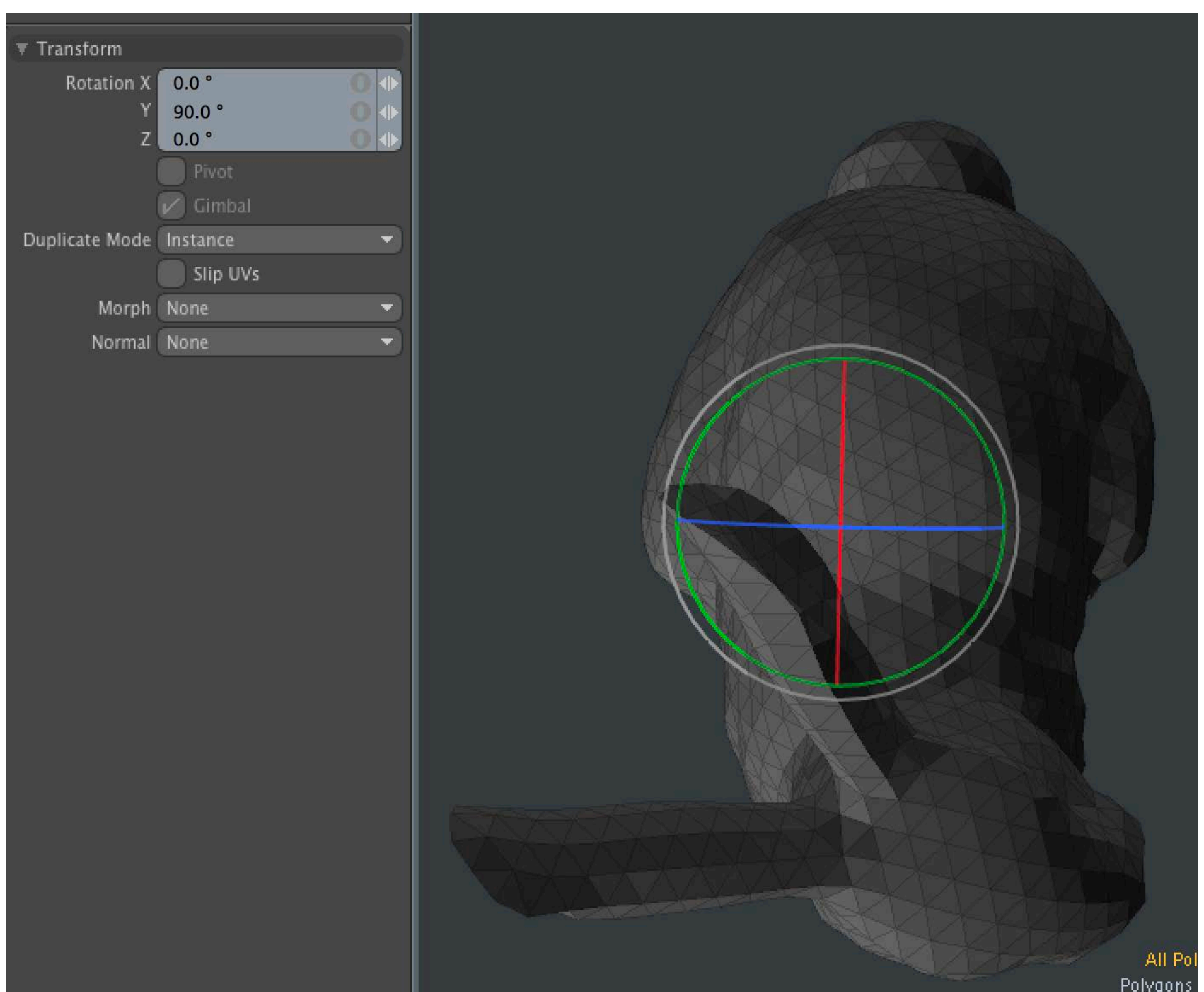
What then must the matrix look like?

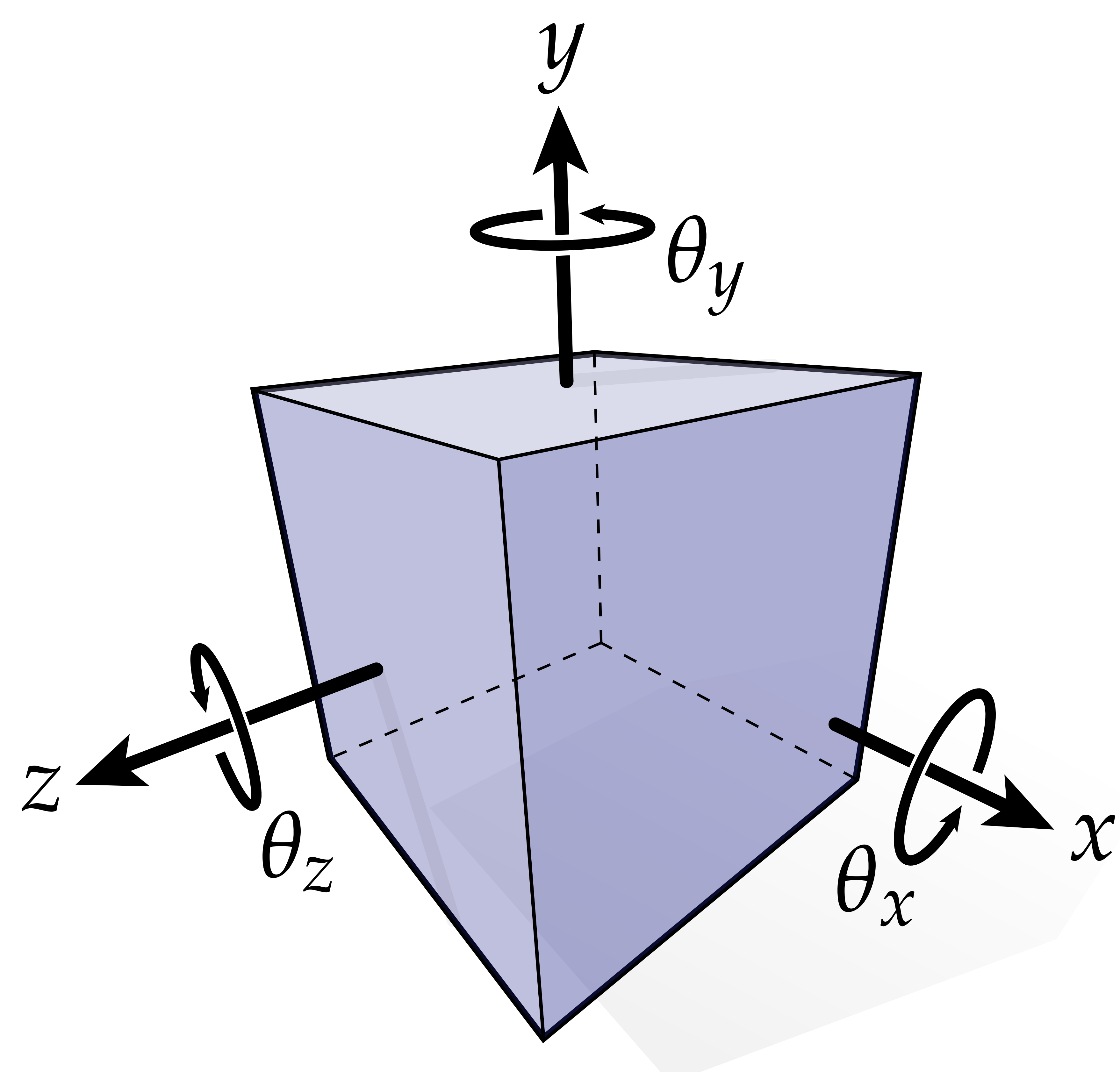
$$\left[\begin{array}{cc} S(\theta) & S(\theta+\pi/2) \end{array}\right] = \left[\begin{array}{cc} \cos(\theta) & \cos(\theta+\pi/2) \\ \sin(\theta) & \sin(\theta+\pi/2) \end{array}\right] = \left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right]$$

***I.e., I don't yet care about sines and cosines and so forth.**

Representing Rotations in 3D—Euler Angles

- How do we express rotations in 3D?
- One idea: we know how to do 2D rotations.
- Why not simply apply rotations around the three axes? (X,Y,Z)
- Scheme is called *Euler angles*
- "Gimbal Lock"





Gimbal Lock

- **When using Euler angles θ_x, θ_y, θ_z, may reach a configuration where there is *no way to rotate around one of the three axes!***
- **Recall rotation matrices around three axes:**

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \quad R_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \quad R_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Product of these matrices represents rotation by Euler angles:**

$$R_x R_y R_z = \begin{bmatrix} \cos\theta_y\cos\theta_z & -\cos\theta_y\sin\theta_z & \sin\theta_y \\ \cos\theta_z\sin\theta_x\sin\theta_y + \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z - \sin\theta_x\sin\theta_y\sin\theta_z & -\cos\theta_y\sin\theta_x \\ -\cos\theta_x\cos\theta_z\sin\theta_y + \sin\theta_x\sin\theta_z & \cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_y\sin\theta_z & \cos\theta_x\cos\theta_y \end{bmatrix}$$

- **Consider special case $\theta_y = \pi/2$ (so, $\cos\theta_y = 0$, $\sin\theta_y = 1$):**

$$\implies \begin{bmatrix} 0 & 0 & 1 \\ \cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_z & \cos\theta_x\cos\theta_z - \sin\theta_x\sin\theta_z & 0 \\ -\cos\theta_x\cos\theta_z + \sin\theta_x\sin\theta_z & \cos\theta_z\sin\theta_x + \cos\theta_x\sin\theta_z & 0 \end{bmatrix}$$

Gimbal Lock, continued

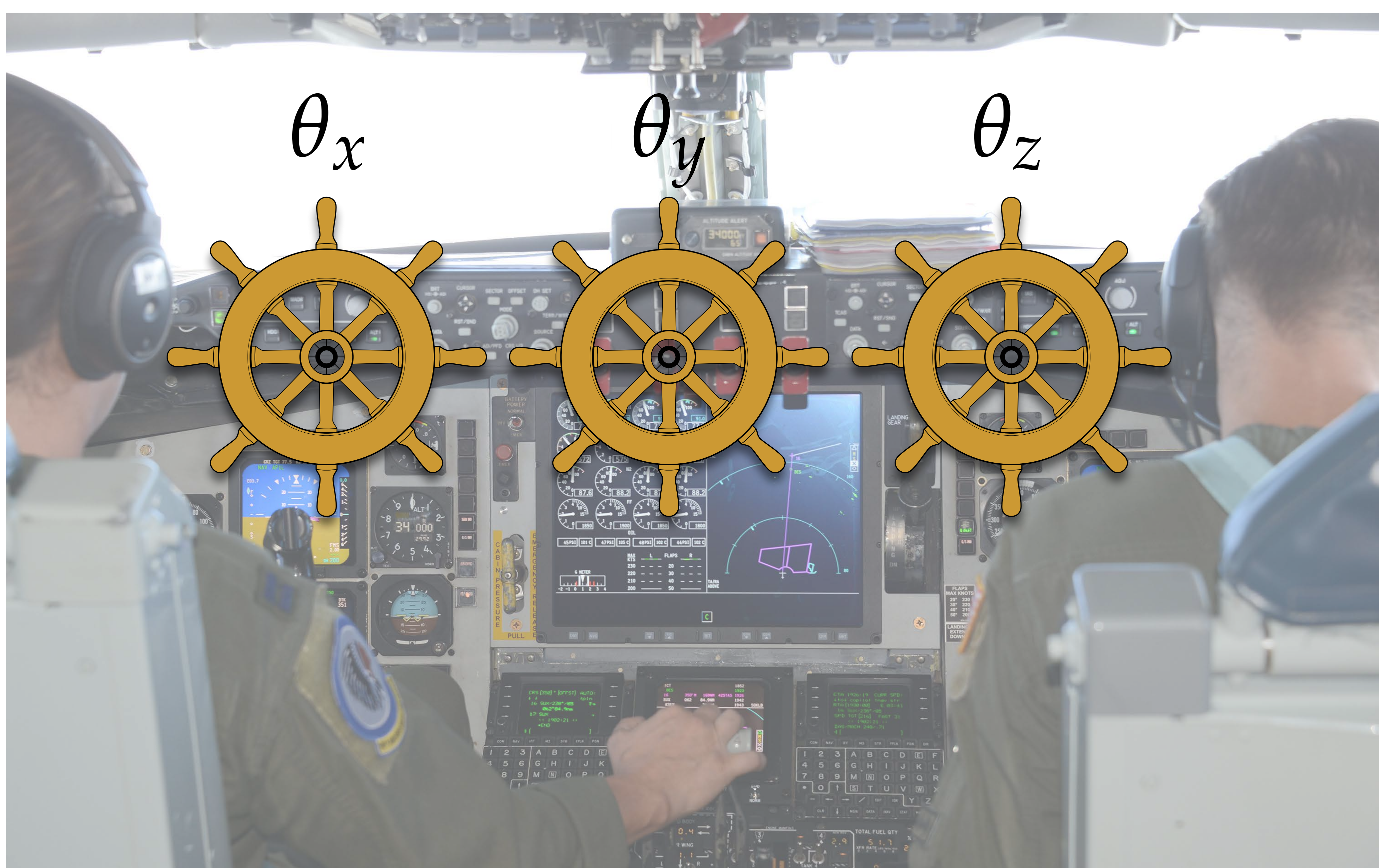
- Simplifying matrix from previous slide, we get

no matter how we adjust θ_x, θ_z, can only rotate in one plane!

$$\begin{bmatrix} 0 & 0 & 1 \\ \sin(\theta_x + \theta_z) & \cos(\theta_x + \theta_z) & 0 \\ -\cos(\theta_x + \theta_z) & \sin(\theta_x + \theta_z) & 0 \end{bmatrix}$$

Q: What does this matrix do?

- We are now "locked" into a single axis of rotation
- Not a great design for airplane controls!



Rotation from Axis/Angle

- **Alternatively, there is a general expression for a matrix that performs a rotation around a given axis u by a given angle θ:**

$$\begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z\left(1-\cos\theta\right) + u_y\sin\theta \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z\left(1-\cos\theta\right) - u_x\sin\theta \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \cos\theta + u_z^2\left(1-\cos\theta\right) \end{bmatrix}$$

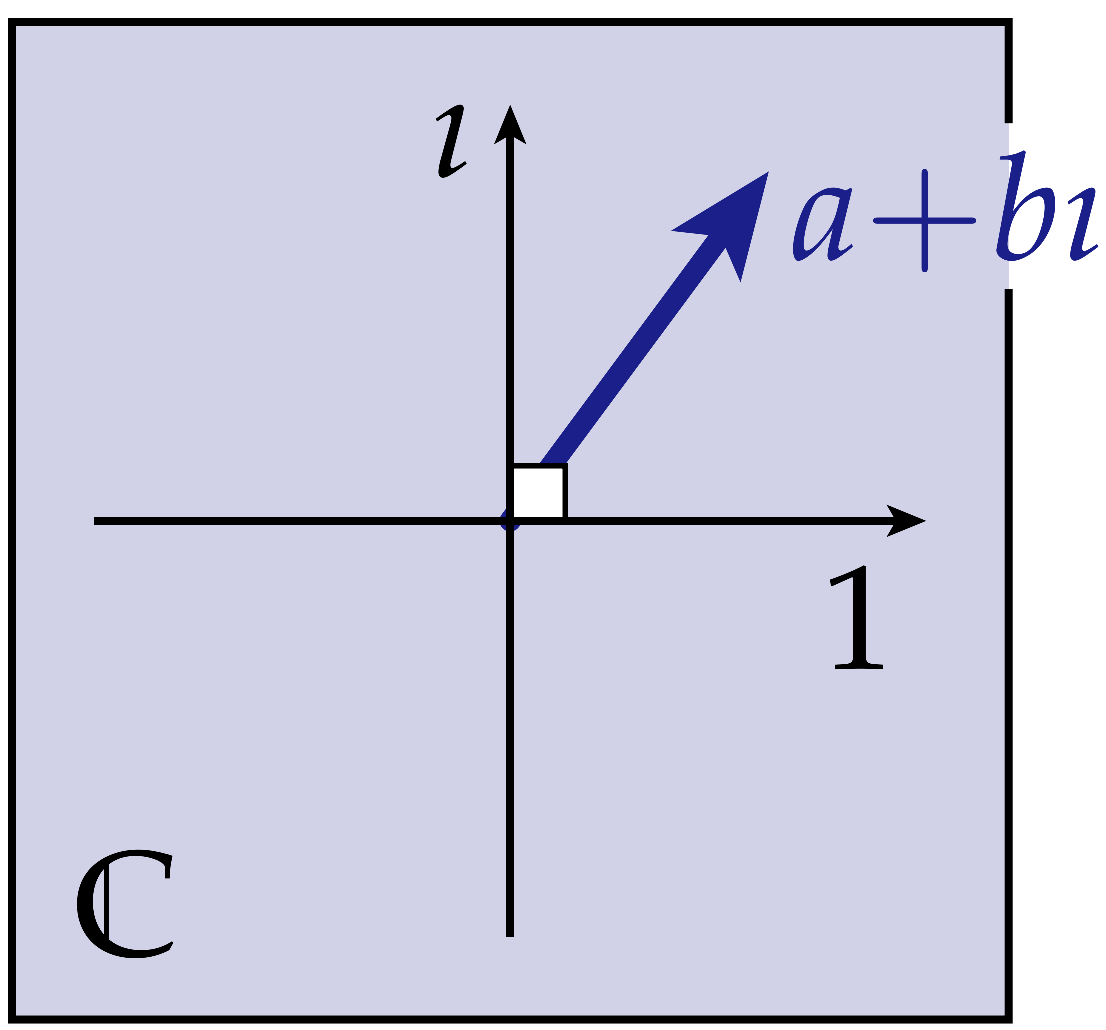
Just memorize this matrix! :-)

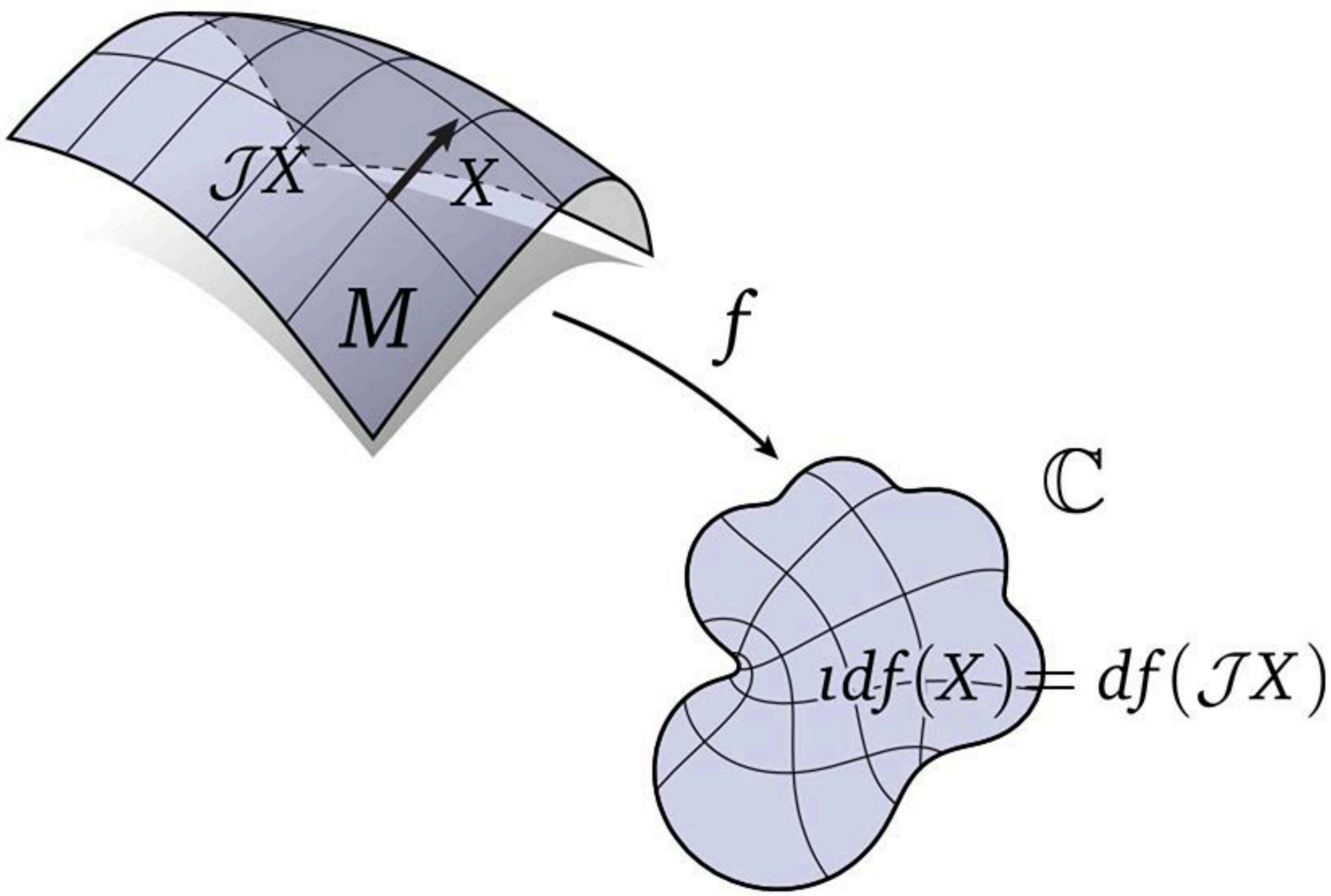
…we'll see a much easier way, later on.

Complex Analysis—Motivation

- Natural way to encode geometric transformations in 2D
- Simplifies code / notation / debugging / *thinking*
- *Moderate* reduction in computational cost/bandwidth/storage
- Fluency with complex analysis can lead into deeper/novel solutions to problems…

COMPLEX





Truly: no good reason to use 2D vectors instead of complex numbers…

DON'T: Think of these numbers as "complex."

DO: Imagine we're simply defining additional operations (like dot and cross).

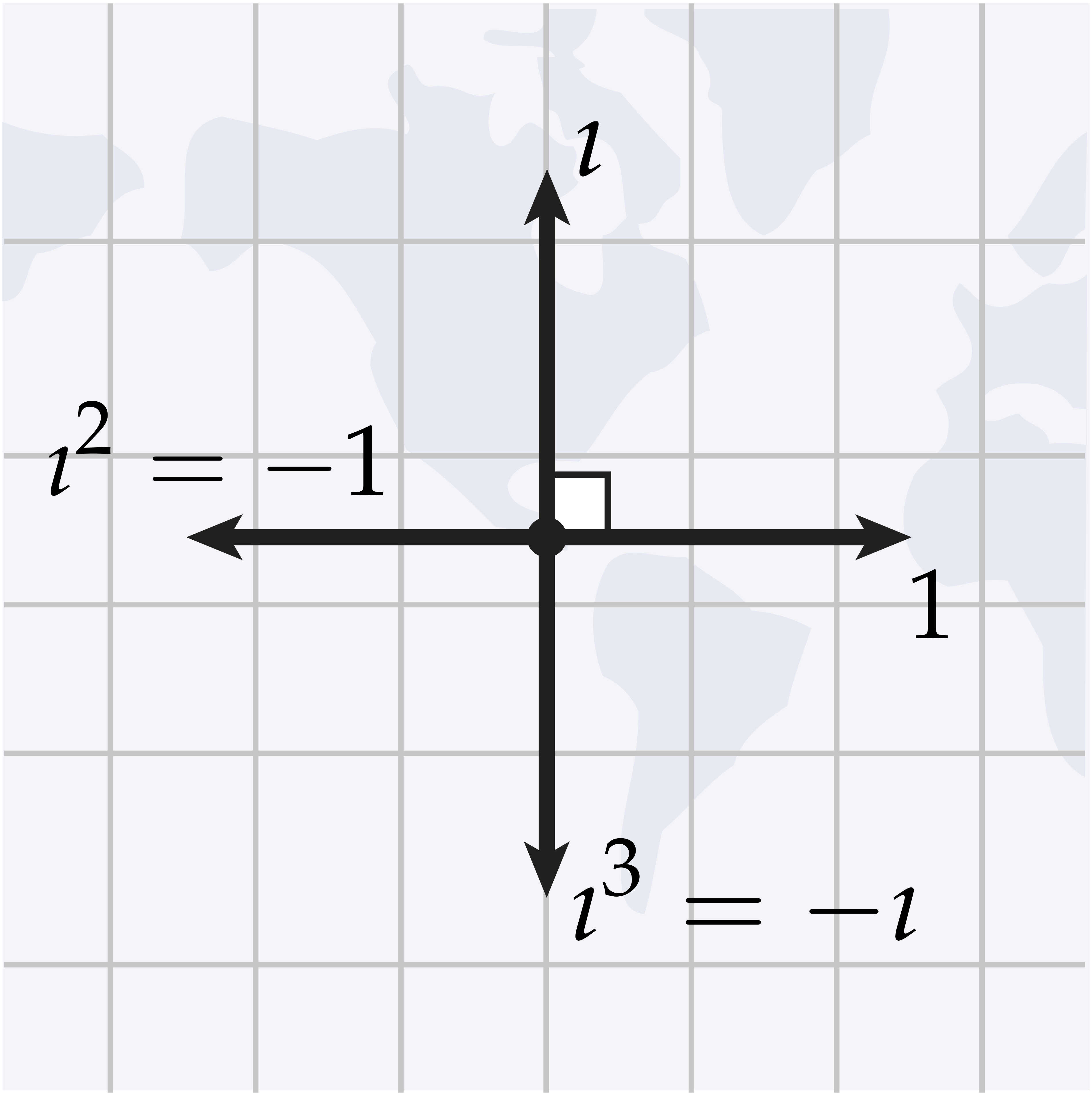
Imaginary Unit

$$\imath := \sqrt{-1}$$

nonsense!

More importantly: obscures geometric meaning.

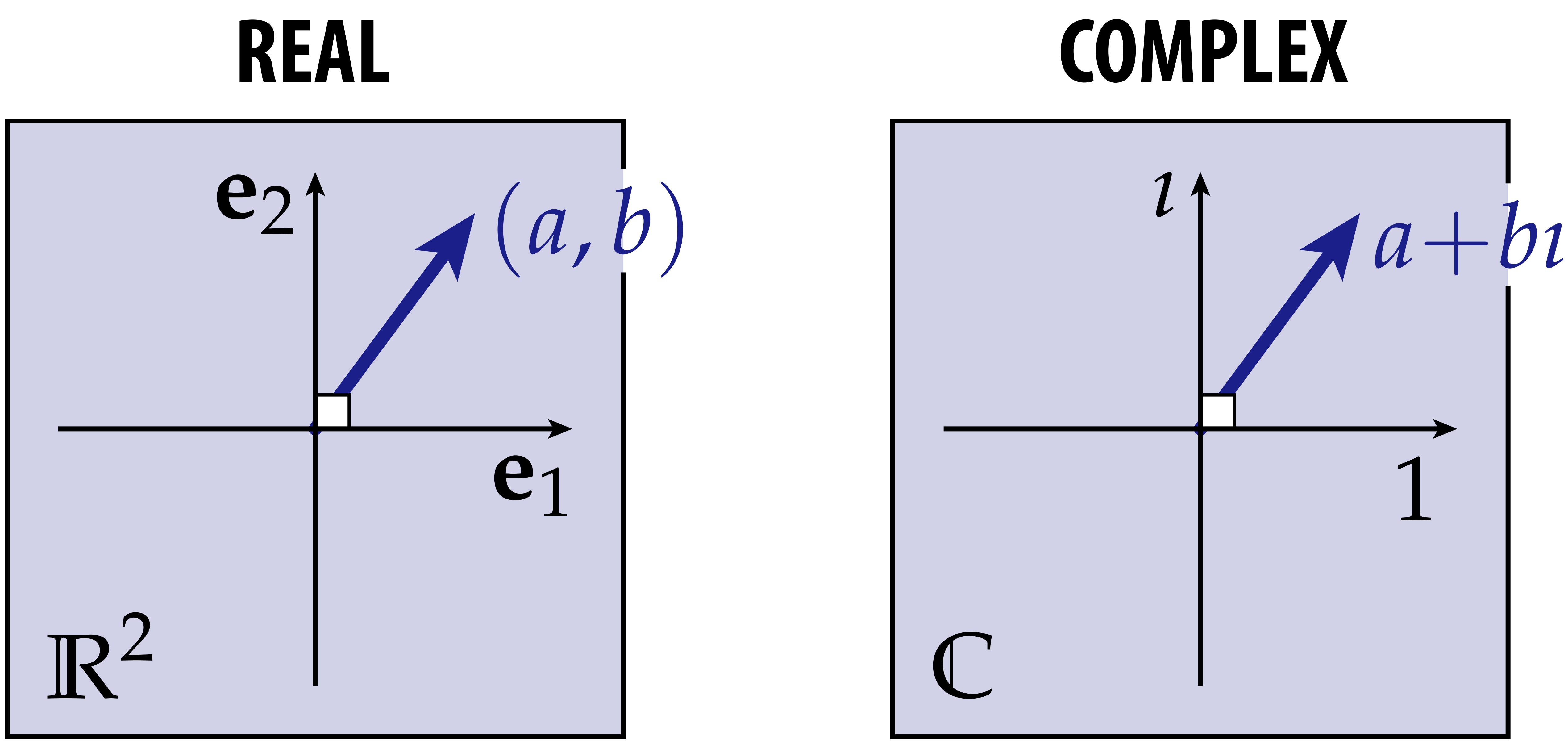
Imaginary Unit—Geometric Description



Imaginary unit is just a quarter-turn in the counter-clockwise direction.

Complex Numbers

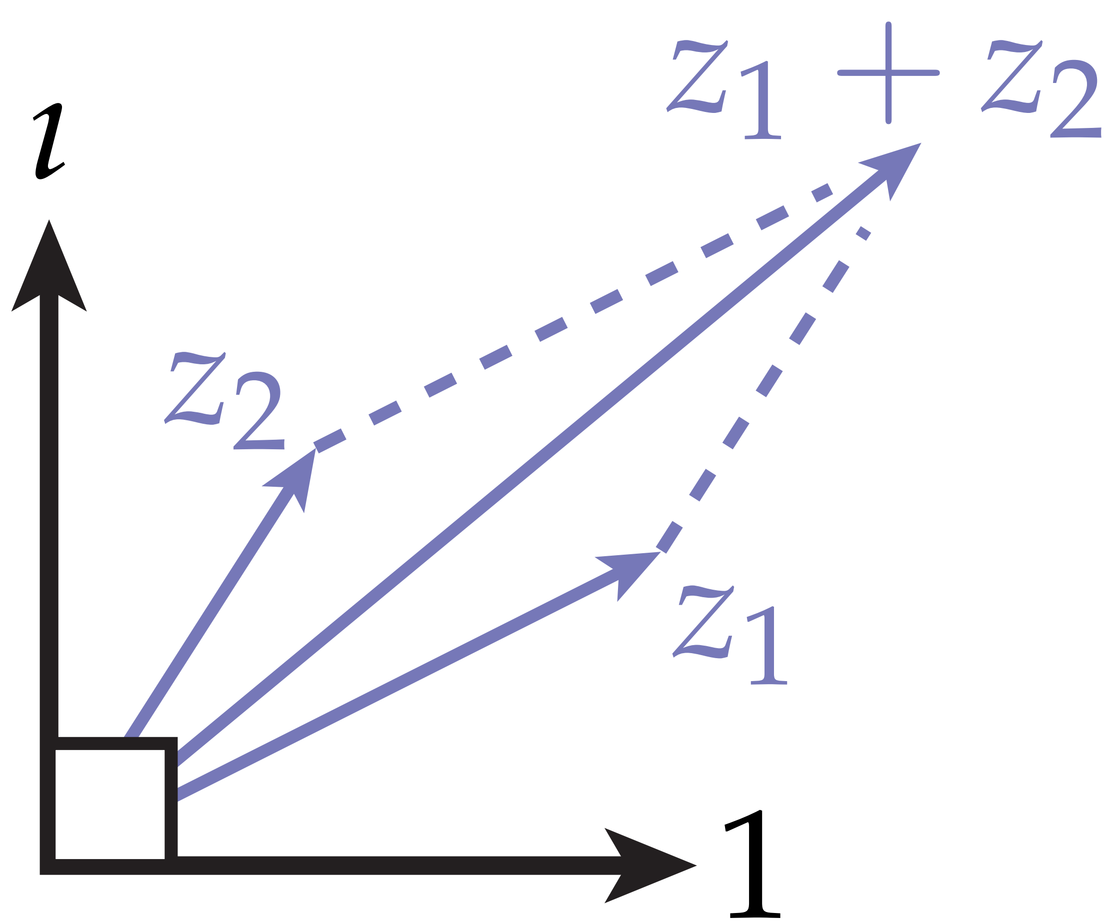
- Complex numbers are then just 2-vectors
- Instead of e_1,e_1, use "1" and "ι" to denote the two bases
- Otherwise, behaves exactly like a real 2-dimensional space



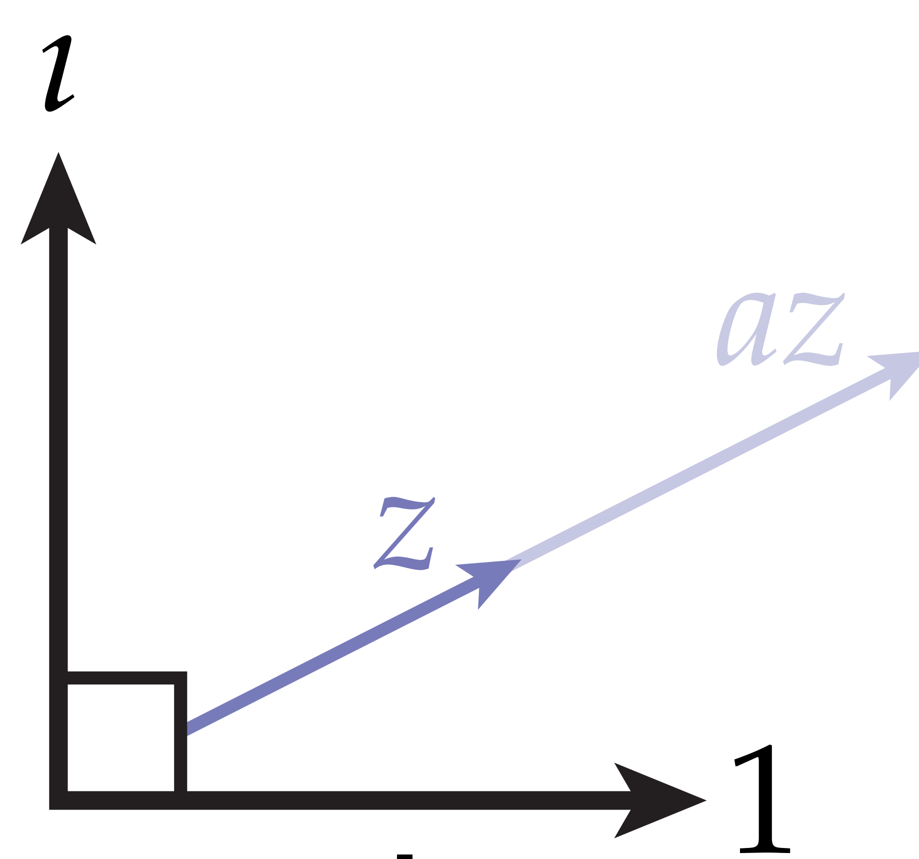
- …except that we're also going to get a very useful new notion of the *product* between two vectors.

Complex Arithmetic

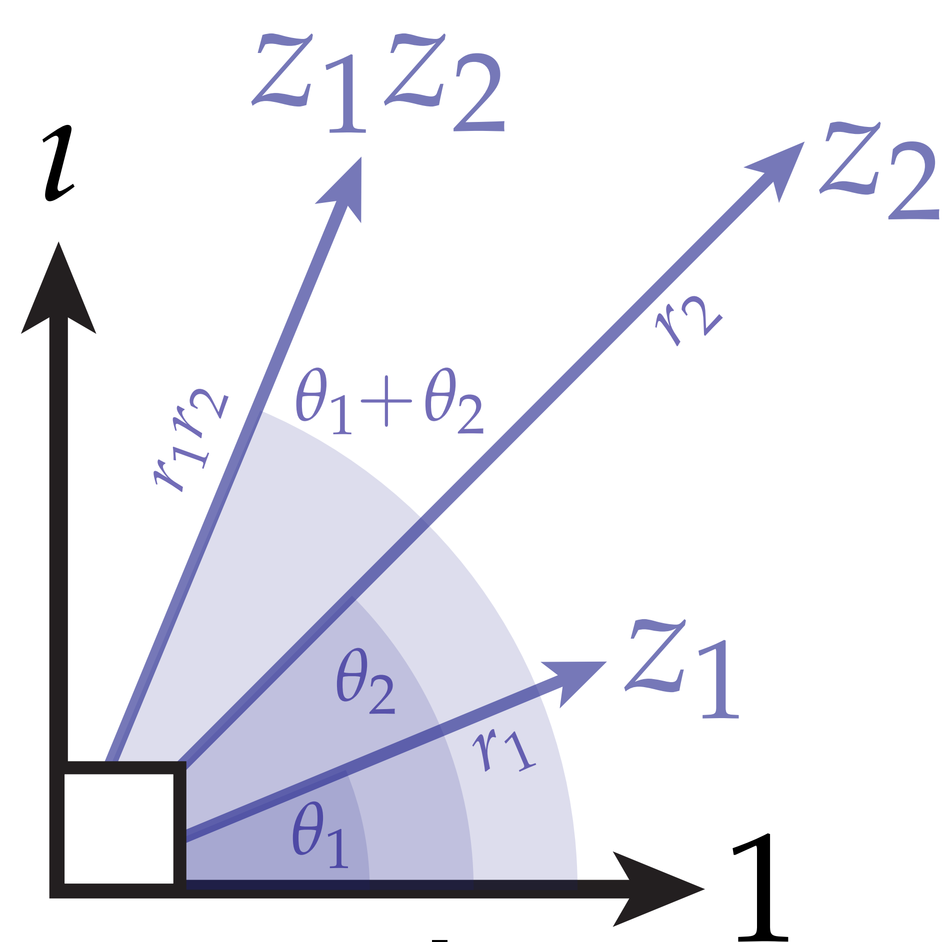
- Same operations as before, plus one more:



vector addition



scalar multiplication



complex multiplication

- Complex multiplication:
  - angles *add*
  - magnitudes *multiply*

"POLAR FORM"*:

$$z_1 := (r_1, \theta_1)$$
$$z_2 := (r_2, \theta_2)$$
$$z_1 z_2 = (r_1 r_2, \theta_1 + \theta_2)$$

have to be more careful here!

*Not *quite* how it really works, but basic idea is right.

Complex Product—Rectangular Form

- **Complex product in "rectangular" coordinates (1, ı):**

$$z_1 = (a + b\imath)$$

$$z_2 = (c + d\imath)$$

two quarter turns—
same as -1

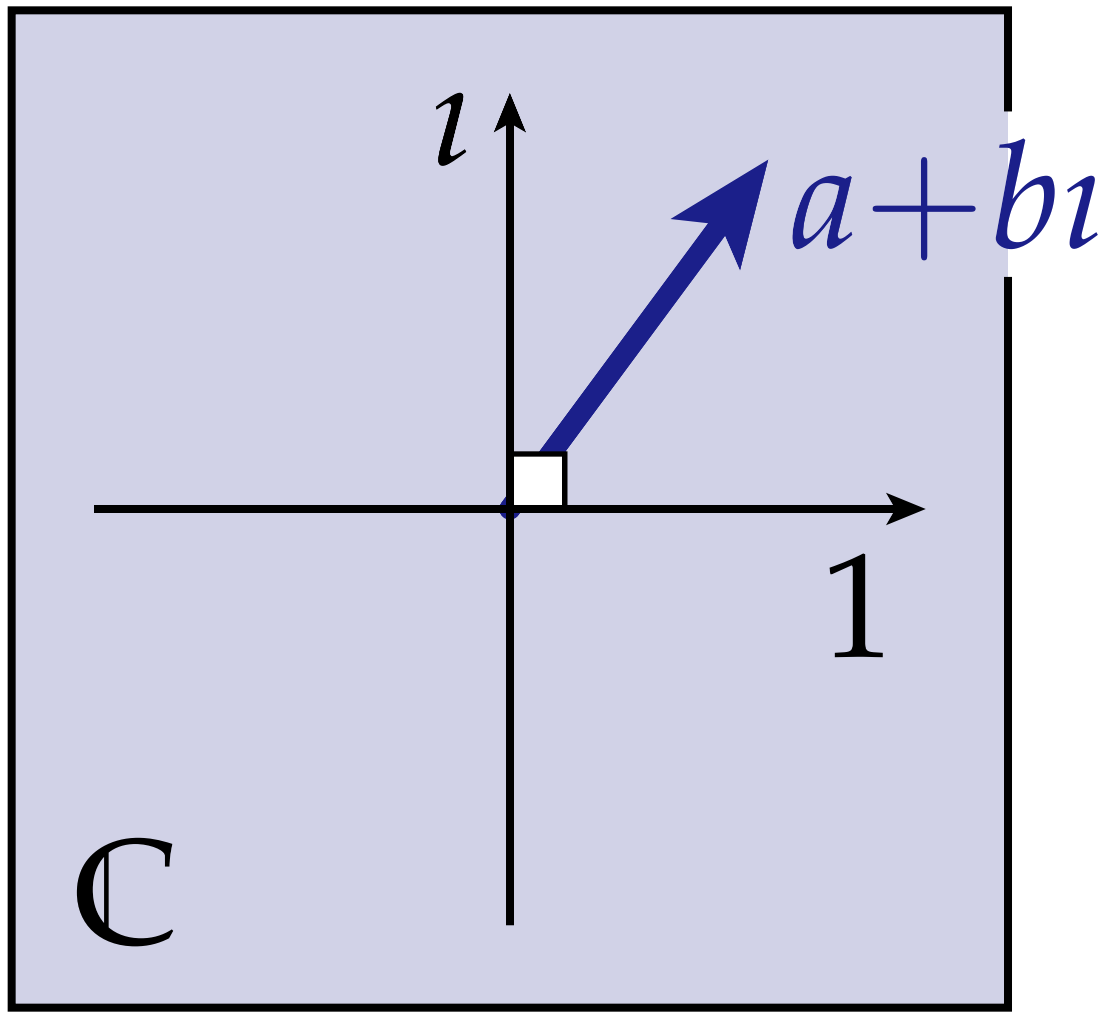
$$z_1 z_2 = ac + ad\imath + bc\imath + bd\imath^2 =$$

$$\boxed{(ac - bd) + (ad + bc)\imath.}$$

"real part" $\mathrm{Re}(z_1 z_2)$

"imaginary part" $\mathrm{Im}(z_1 z_2)$

- **We used a lot of "rules" here. Can you justify them geometrically?**
- **Does this product agree with our geometric description (last slide)?**



Complex Product—Polar Form

- **Perhaps most beautiful identity in math:**

$$e^{\imath\pi} + 1 = 0$$

- **Specialization of *Euler's formula*:**

$$e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta)$$

- **Can use to "implement" complex product:**

$$z_1 = ae^{\imath\theta}, \quad z_2 = be^{\imath\phi}$$

$$z_1 z_2 = abe^{\imath(\theta+\phi)}$$

(as with real exponentiation, exponents *add*)

Leonhard Euler
(1707–1783)

Q: How does this operation differ from our earlier, "fake" polar multiplication?

2D Rotations: Matrices vs. Complex

- **Suppose we want to rotate a vector u by an angle θ, then by an angle ϕ.**

REAL / RECTANGULAR	COMPLEX / POLAR
$\mathbf{u} = (x, y)$ $\quad \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$u = re^{\imath\alpha}$
$\mathbf{B} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$	$a = e^{\imath\theta}$ $b = e^{\imath\phi}$
$\mathbf{Au} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$	$abu = re^{\imath(\alpha+\theta+\phi)}.$
$\mathbf{BAu} = \begin{bmatrix} (x\cos\theta - y\sin\theta)\cos\phi - (x\sin\theta + y\cos\theta)\sin\phi \\ (x\cos\theta - y\sin\theta)\sin\phi + (x\sin\theta + y\cos\theta)\cos\phi \end{bmatrix}$	
$= \cdots \text{some trigonometry} \cdots =$	
$\mathbf{BAu} = \begin{bmatrix} x\cos(\theta+\phi) - y\sin(\theta+\phi) \\ x\sin(\theta+\phi) + y\cos(\theta+\phi) \end{bmatrix}.$	

Pervasive theme in graphics:

Sure, there are often many "equivalent" representations.

...But why not choose the one that makes life easiest*?

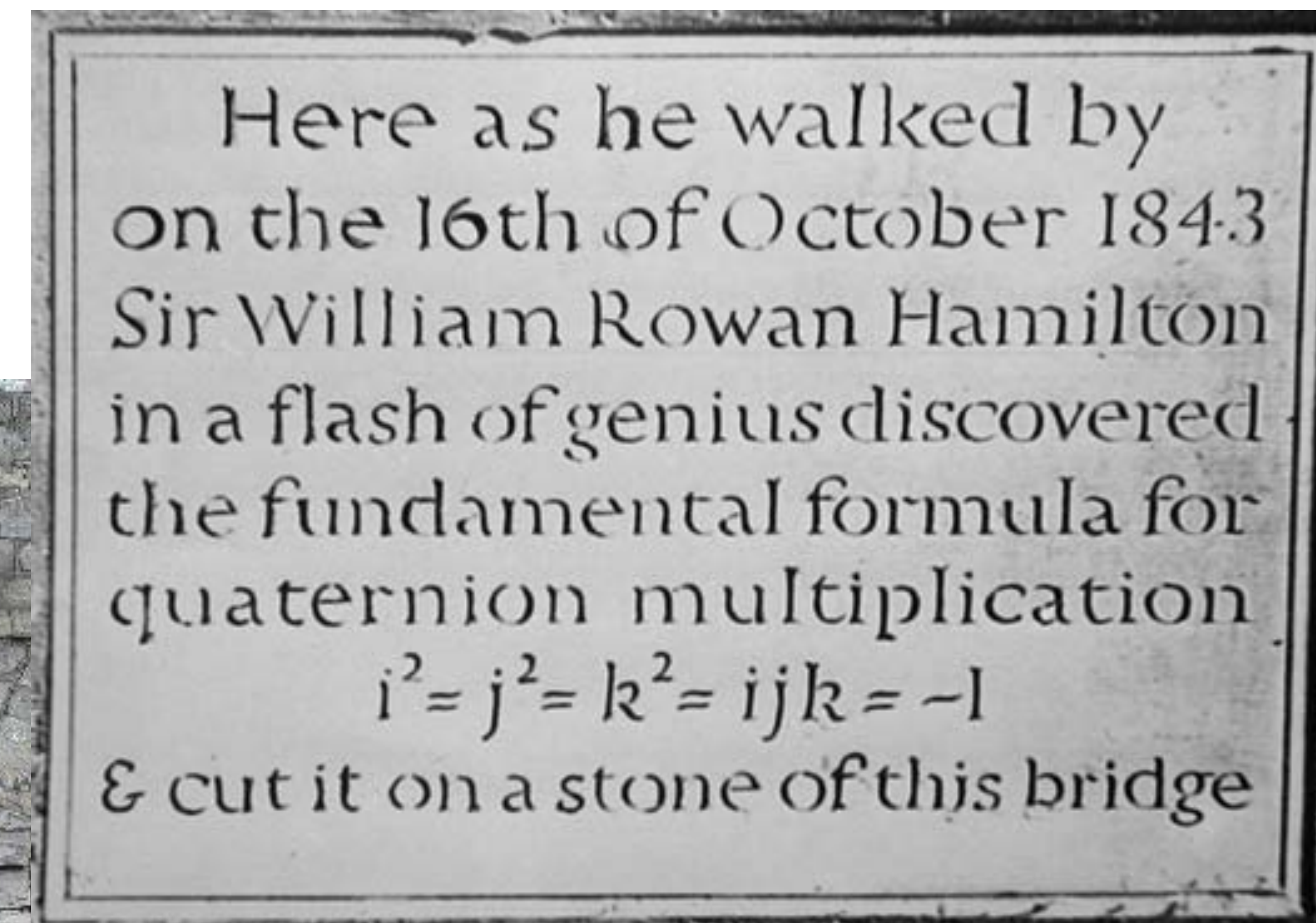
*Or most efficient, or most accurate...

Quaternions

- **TLDR: Kind of like complex numbers but for 3D rotations**
- **Weird situation: can't do 3D rotations w/ only 3 components!**

William Rowan Hamilton (1805-1865)

(Not Hamilton)



Quaternions in Coordinates

- **Hamilton's insight: in order to do 3D rotations in a way that mimics complex numbers for 2D, actually need FOUR coords.**
- **One real, *three* imaginary:**

$$\mathbb{H} := \text{span}(\{1, \imath, \jmath, k\})$$

"H" is for *Hamilton!*

$$q = a + b\imath + c\jmath + dk \in \mathbb{H}$$

- **Quaternion product determined by**

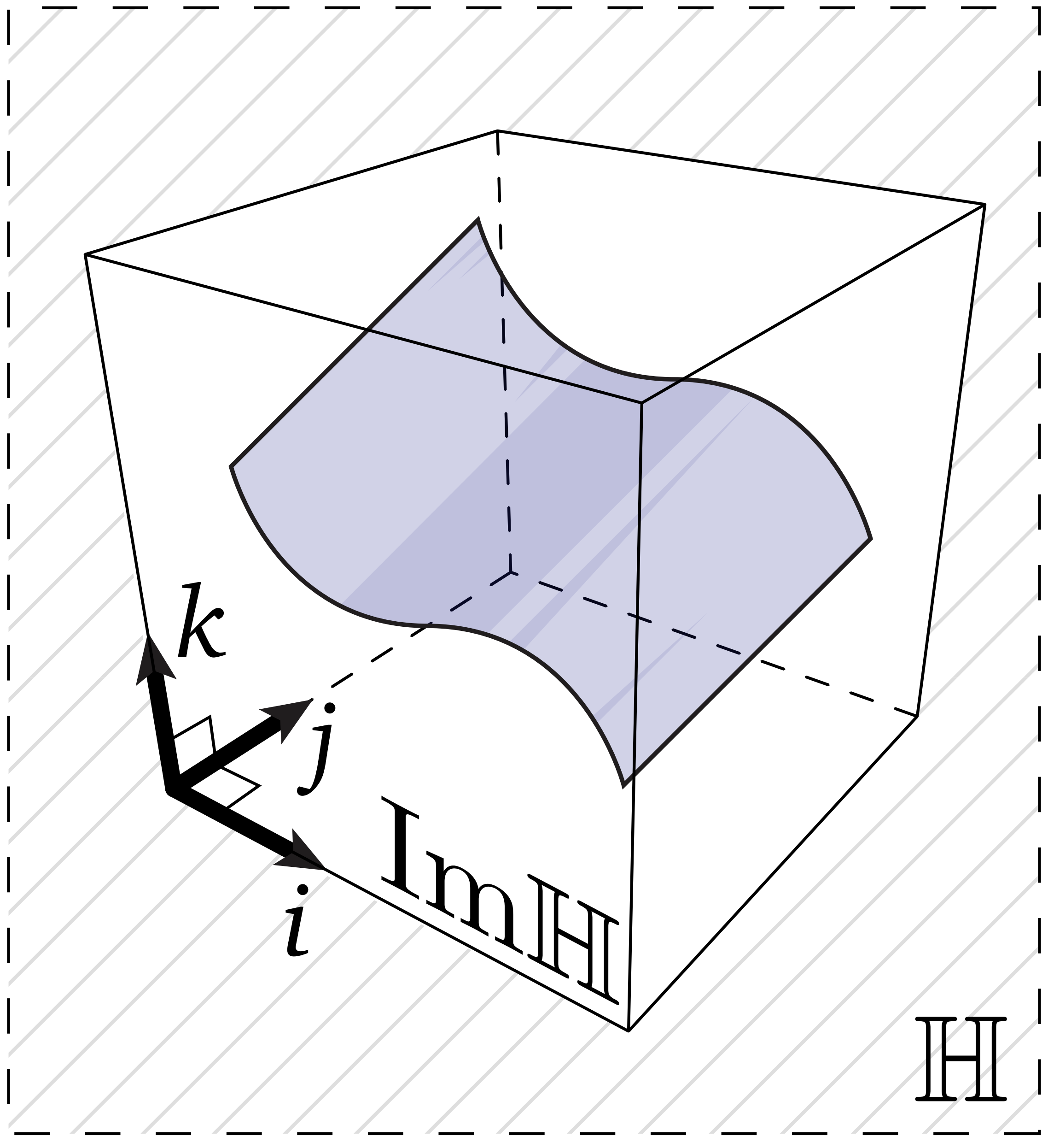
$$\imath^2 = \jmath^2 = k^2 = \imath\jmath k = -1$$

together w/ "natural" rules (distributivity, associativity, etc.)

- **WARNING: product no longer commutes!**

$$\text{For } q, p \in \mathbb{H}, \quad qp \neq pq$$

(Why might it make sense that it doesn't commute?)



Quaternion Product in Components

- **Given two quaternions**

$$q = a_1 + b_1\imath + c_1\jmath + d_1k$$
$$p = a_2 + b_2\imath + c_2\jmath + d_2k$$

- **Can express their product as**

$$\begin{aligned} qp = {} & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ & +(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\imath \\ & +(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\jmath \\ & +(a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k \end{aligned}$$

…fortunately there is a (much) nicer expression.

Quaternions—Scalar + Vector Form

- **If we have *four* components, how do we talk about pts in 3D?**
- **Natural idea: we have three imaginary parts—why not use these to encode 3D vectors?**

$$(x, y, z) \mapsto 0 + x\imath + y\jmath + zk$$

- **Alternatively, can think of a quaternion as a pair**

$$\underset{\underset{\mathbb{R}}{\cap}}{(\text{ scalar}}, \underset{\underset{\mathbb{R}^3}{\cap}}{\text{vector}} \text{)} \in \mathbb{H}$$

- **Quaternion product then has simple(r) form:**

$$(a, \mathbf{u})(b, \mathbf{v}) = (ab - \mathbf{u} \cdot \mathbf{v}, a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v})$$

- **For vectors in R3, gets even simpler:**

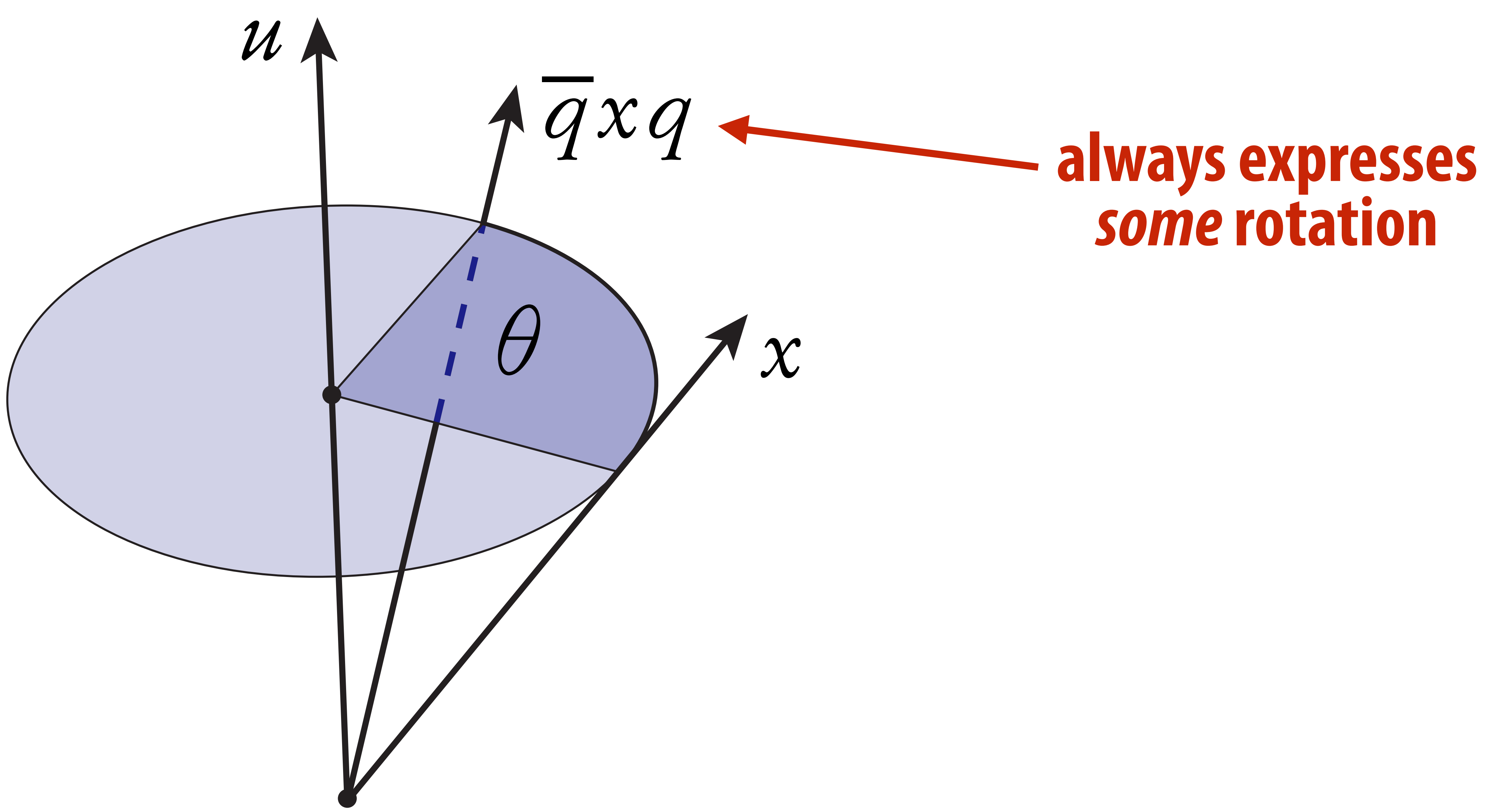
$$\mathbf{uv} = \mathbf{u} \times \mathbf{v} - \mathbf{u} \cdot \mathbf{v}$$

3D Transformations via Quaternions

- **Main use for quaternions in graphics?** ***Rotations.***
- **Consider vector x ("pure imaginary") and *unit* quaternion q:**

$$x \in \mathrm{Im}(\mathbb{H})$$

$$q \in \mathbb{H}, \quad |q|^2 = 1$$



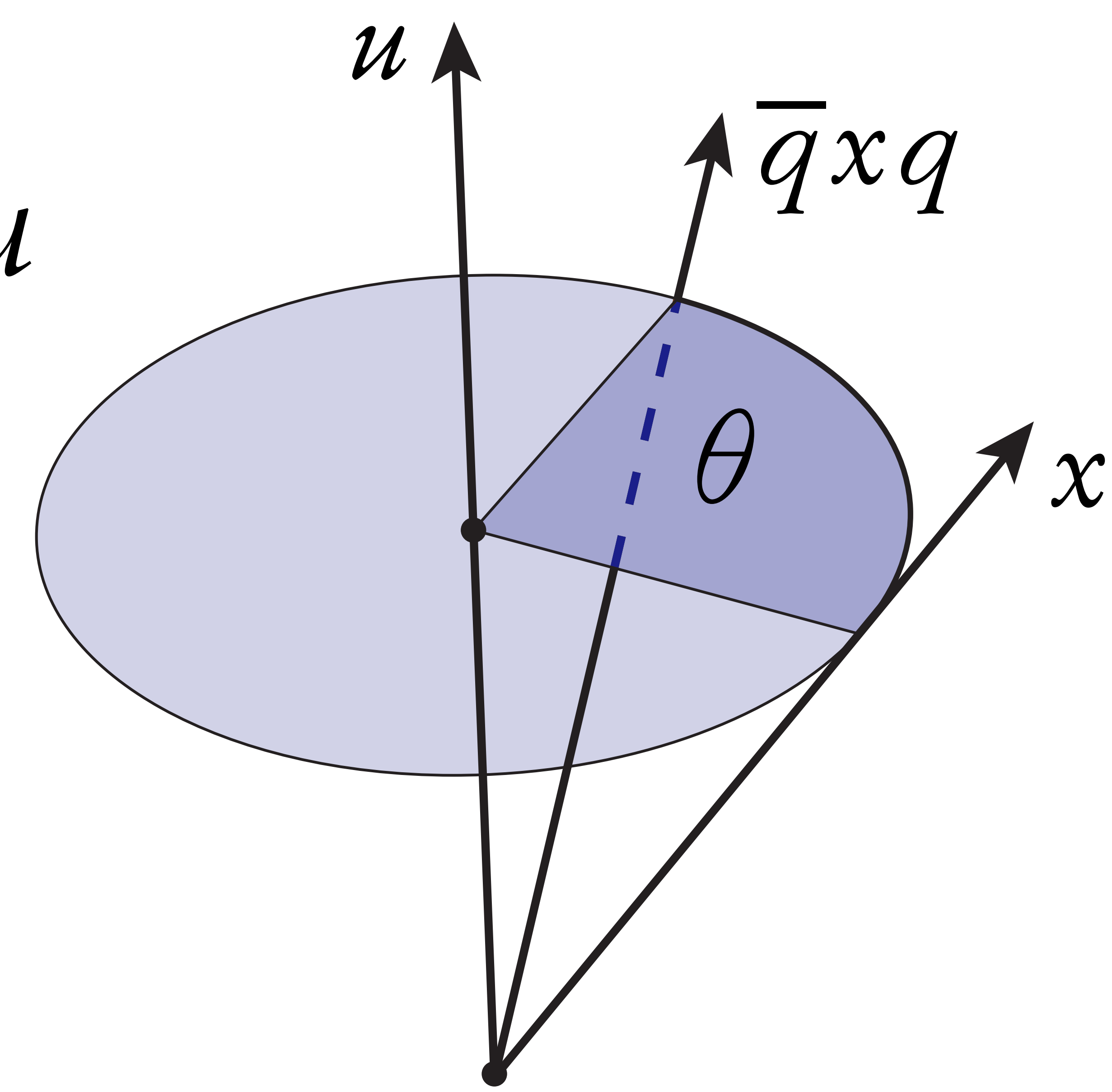
Rotation from Axis/Angle, Revisited

- **Given axis u, angle θ, quaternion q representing rotation is**

$$q = \cos(\theta/2) + \sin(\theta/2)u$$

angle

axis



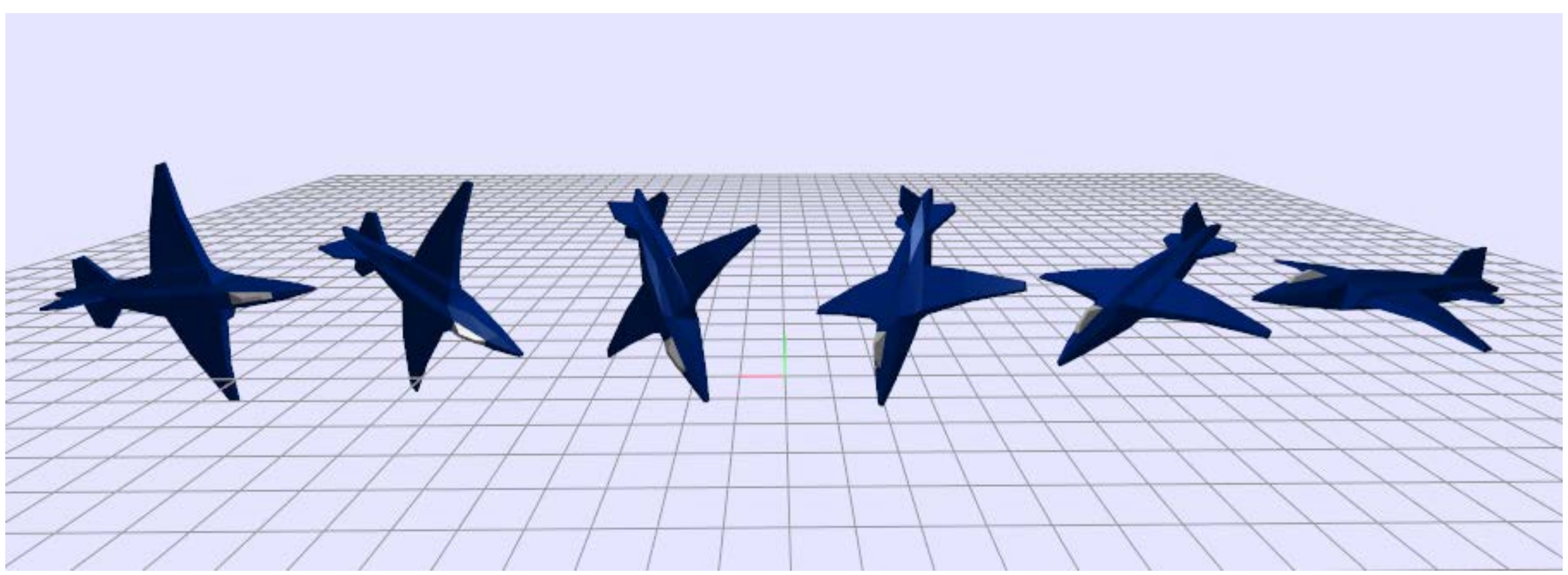
- **Much easier to remember (and manipulate) than matrix!**

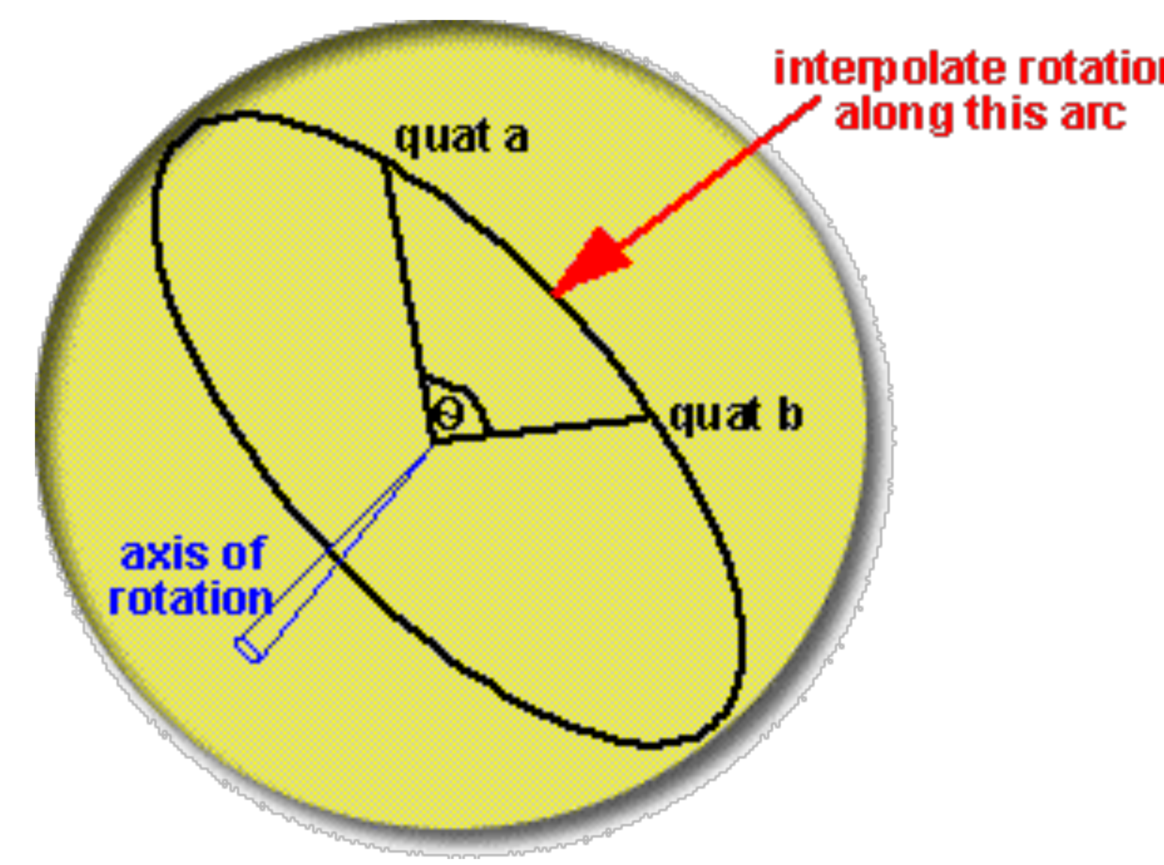
$$\begin{bmatrix} \cos\theta + u_x^2(1-\cos\theta) & u_x u_y(1-\cos\theta) - u_z\sin\theta & u_x u_z(1-\cos\theta) + u_y\sin\theta \\ u_y u_x(1-\cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1-\cos\theta) & u_y u_z(1-\cos\theta) - u_x\sin\theta \\ u_z u_x(1-\cos\theta) - u_y\sin\theta & u_z u_y(1-\cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1-\cos\theta) \end{bmatrix}$$

Interpolating Rotations

- Suppose we want to smoothly interpolate between two rotations (e.g., orientations of an airplane)
- Interpolating Euler angles can yield strange-looking paths, non-uniform rotation speed, …
- Simple solution* w/ quaternions: "SLERP" (spherical linear interpolation):

$$\text{Slerp}(q_0, q_1, t) = q_0(q_0^{-1}q_1)^t, \quad t \in [0,1]$$



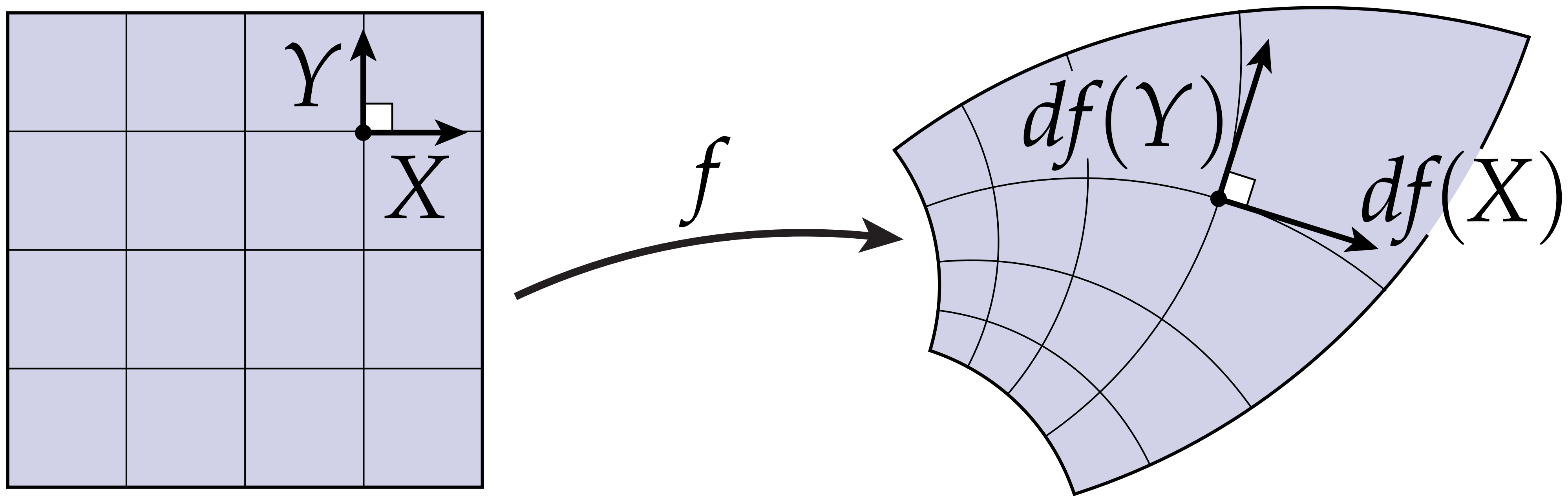


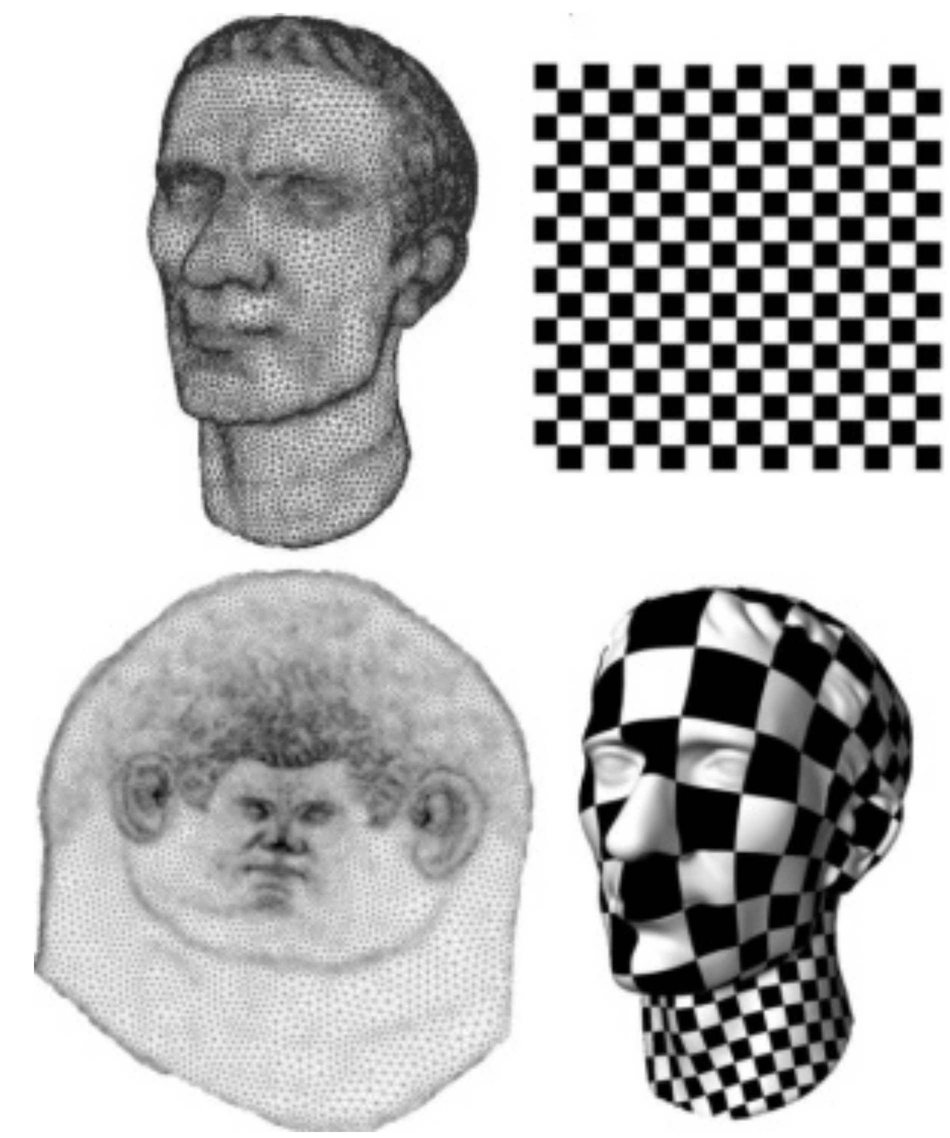
*Shoemake 1985, "Animating Rotation with Quaternion Curves"

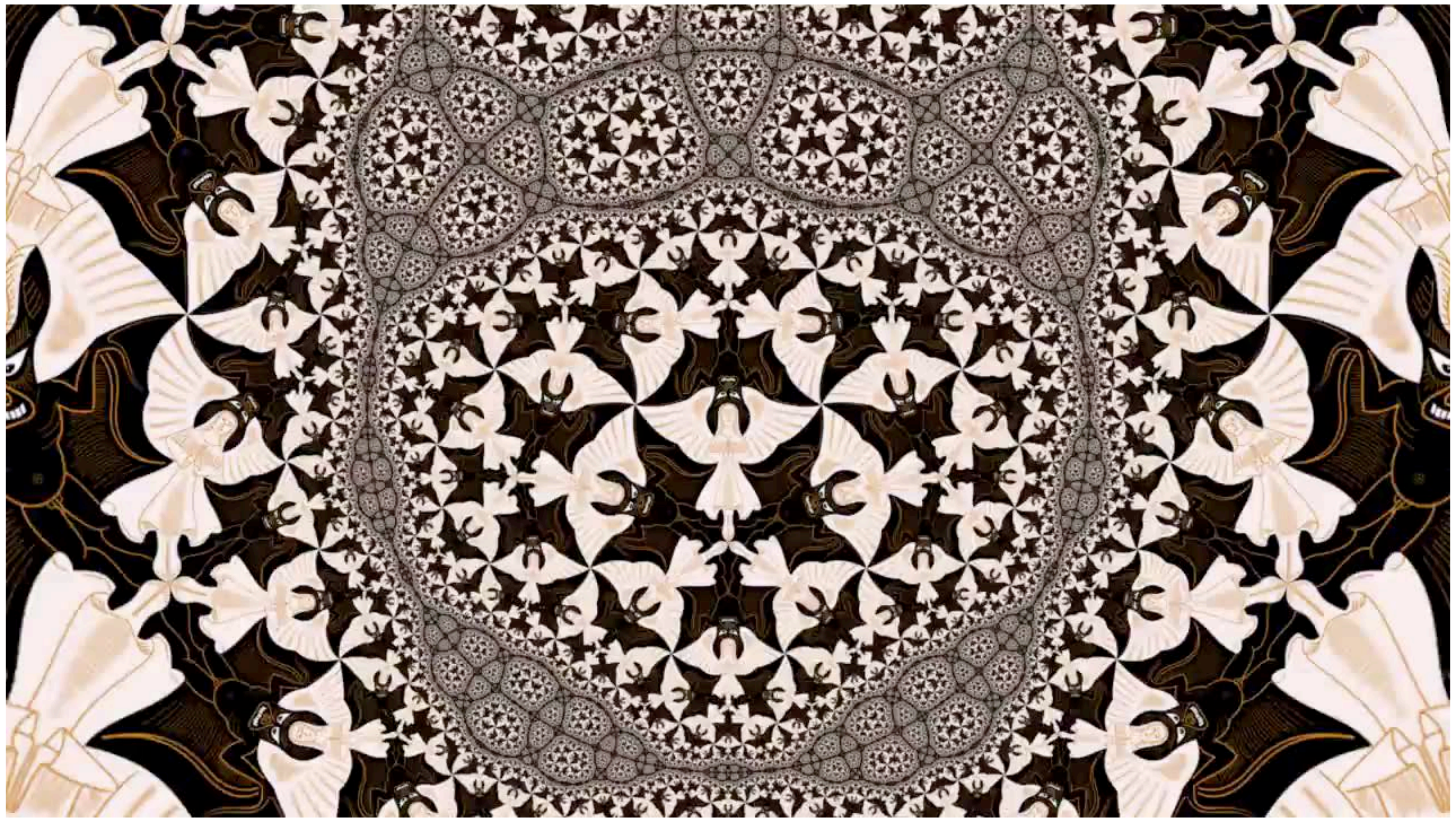
Where else are (hyper-)complex numbers useful in computer graphics?

Generating Coordinates for Texture Maps

Complex numbers are natural language for angle-preserving ("conformal") maps



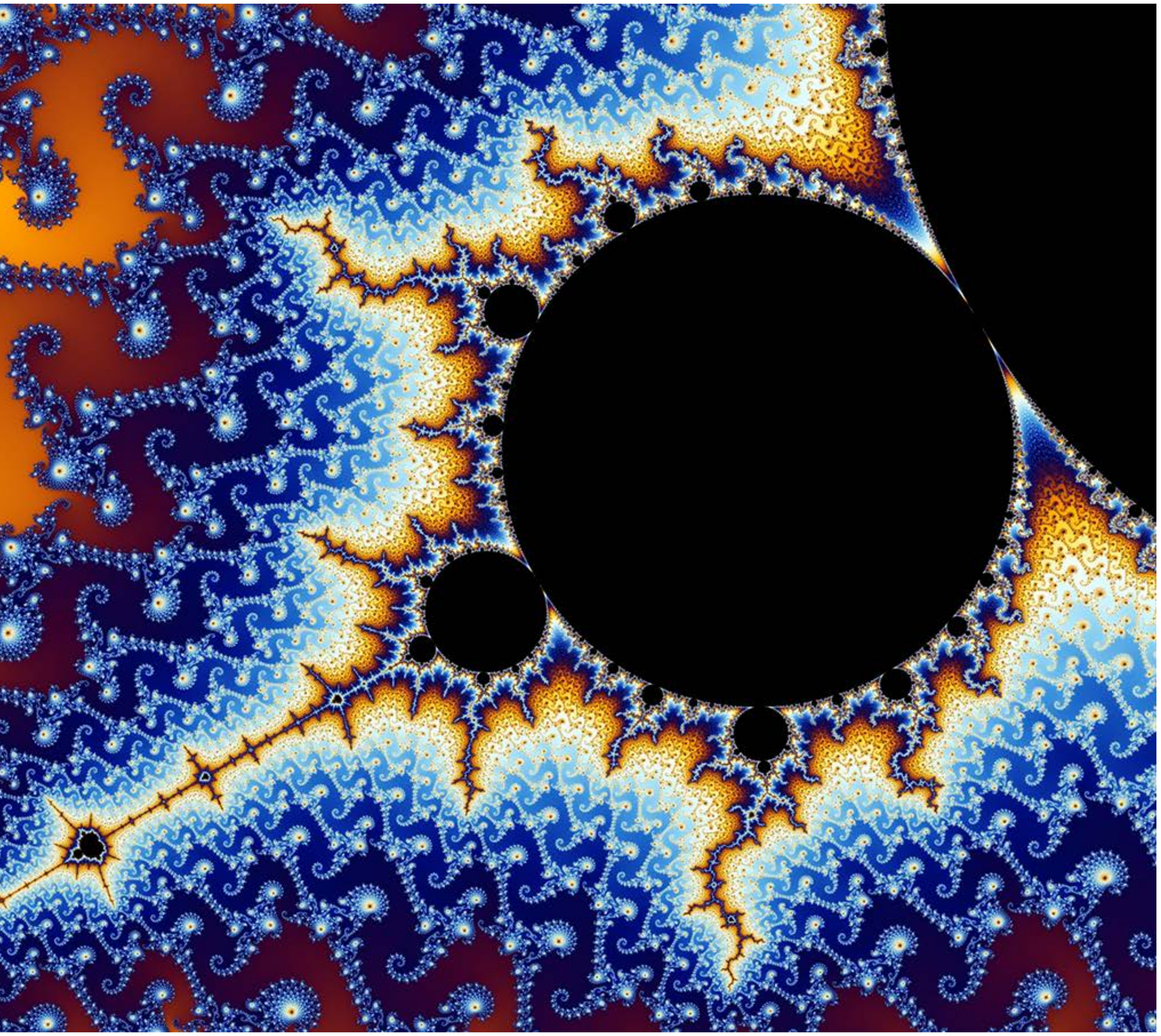


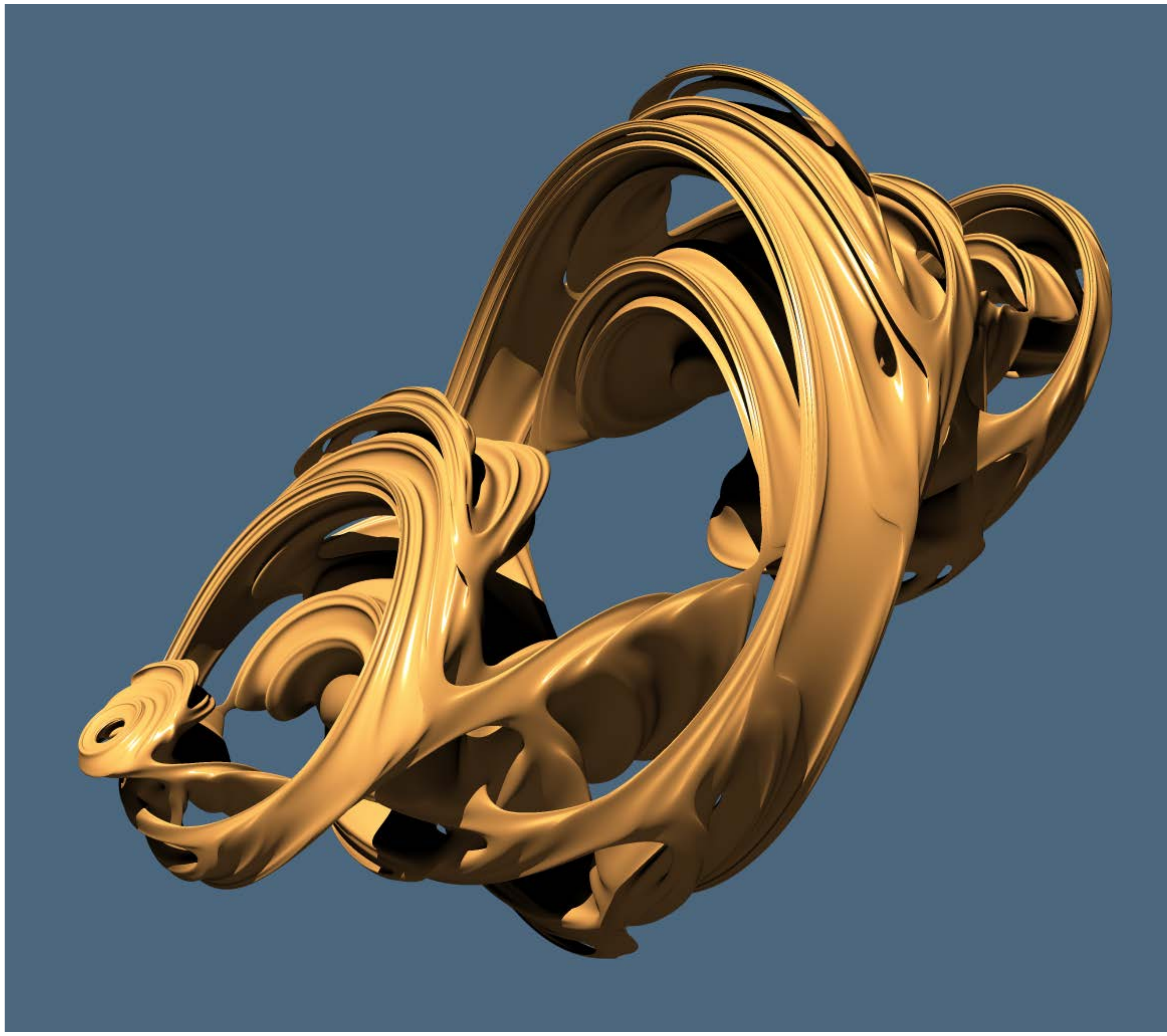


Preserving angles in texture well-tuned to human perception…

Useless-But-Beautiful Example: Fractals

- Defined in terms of iteration on (hyper)complex numbers:

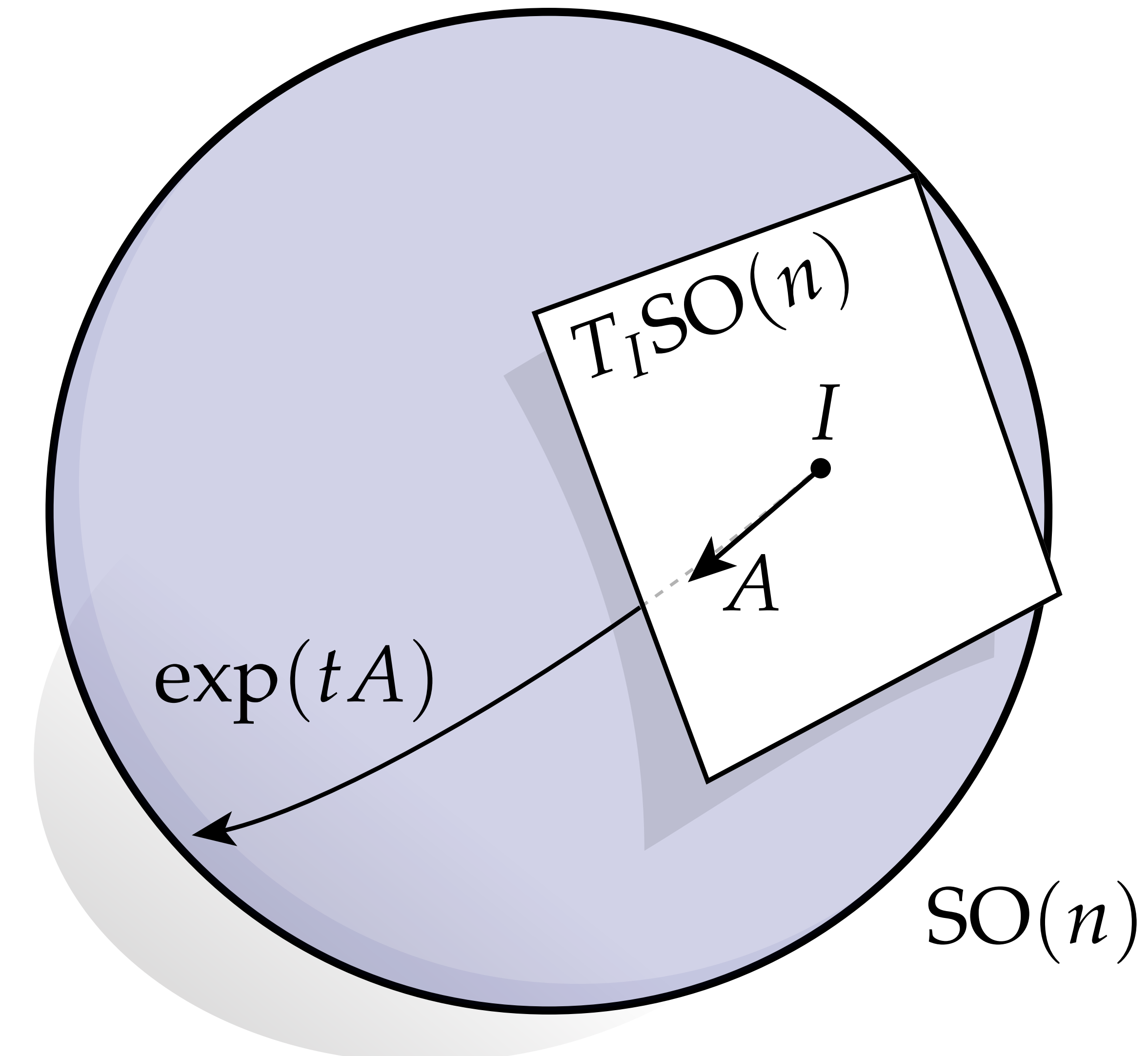


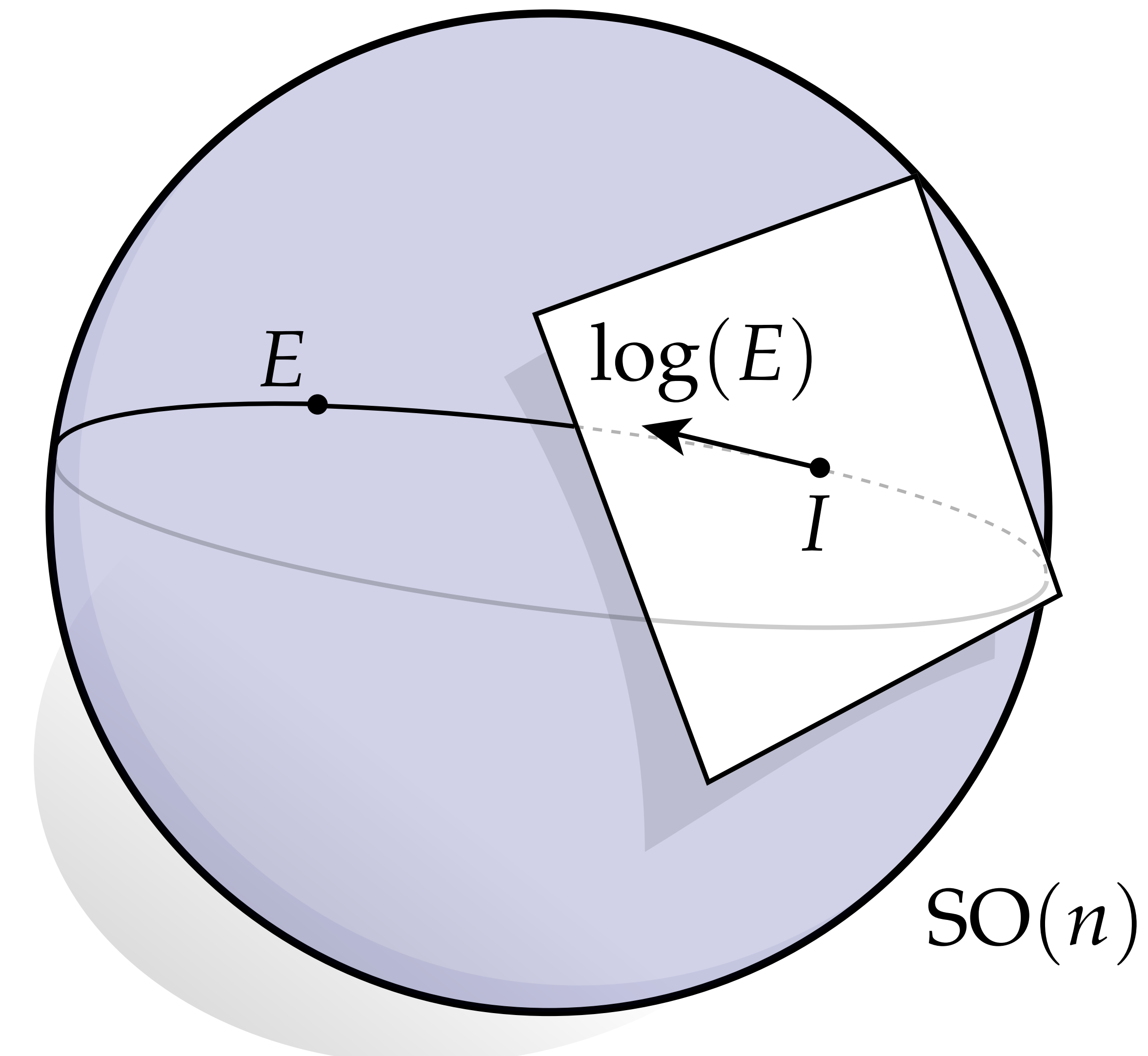


(Will see exactly how this works later in class.)

<u>Not Covered</u>: Lie algebras/Lie Groups

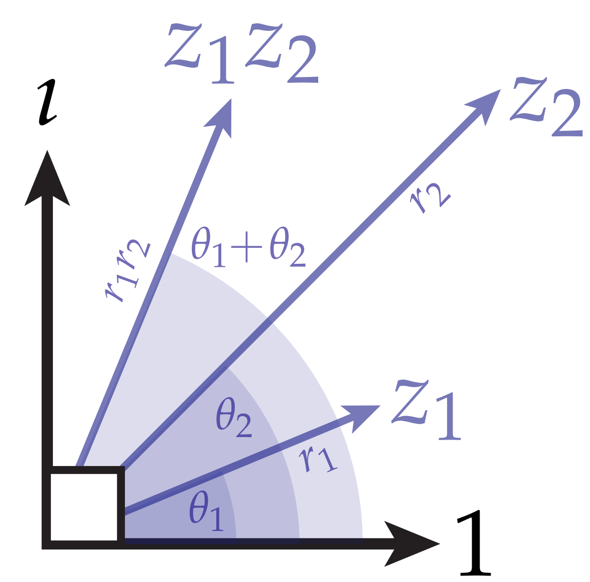
- Another <u>super</u> nice/useful perspective on rotations is via "Lie groups" and "Lie algebras"
- More than we have time to cover!
- Many benefits similar to quaternions (easy axis/angle representation, no gimbal lock, ...)
- Nice for encoding angles bigger than 2π
- Also *very* useful for taking <u>averages of rotations</u>
- (Very) short story:
  - exponential map takes you from axis/angle to rotation matrix
  - logarithmic map takes you from rotation matrix to axis/angle

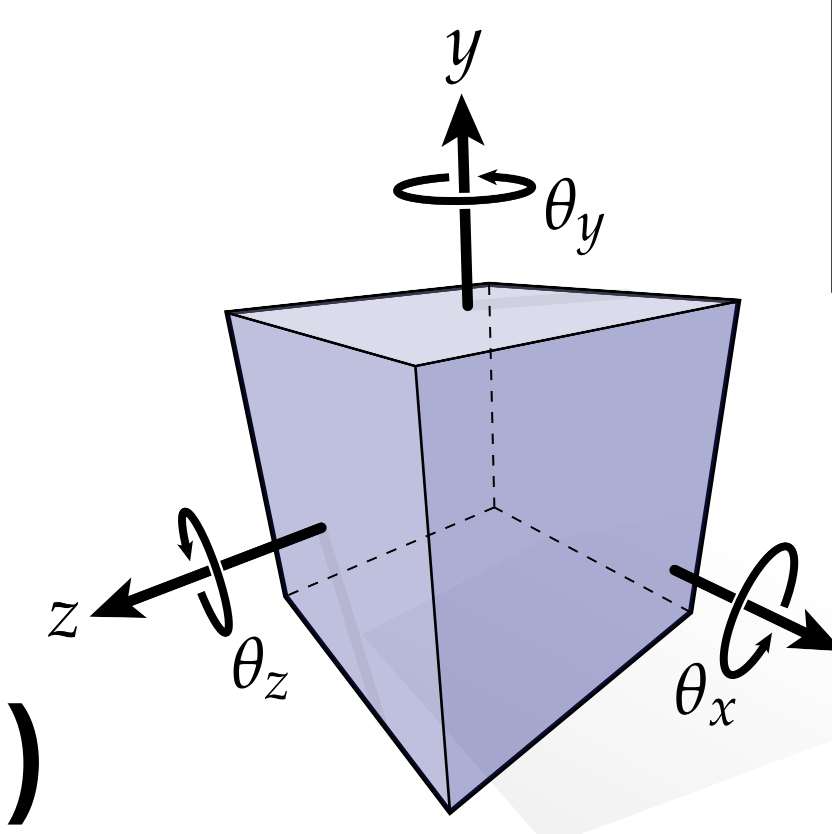


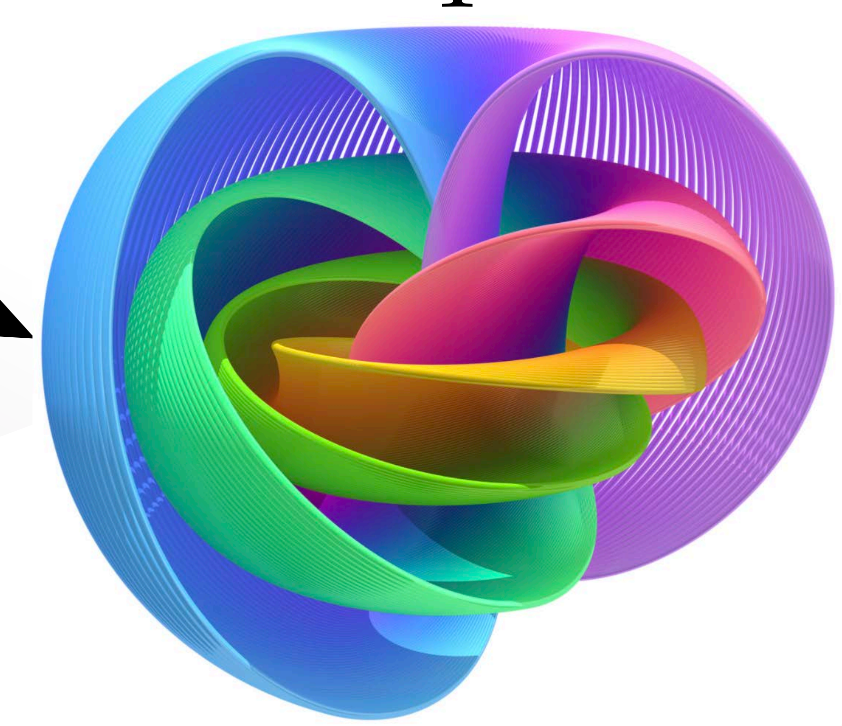


Rotations and Complex Representations—Summary

- **Rotations are surprisingly complicated in 3D!**
- **Today, looked at how <u>complex</u> representations help understand/work with rotations in 3D (& 2D)**
- **In general, many possible representations:**
  - **Euler angles**
  - **axis-angle**
  - **quaternions**
  - **Lie group/algebra (not covered)**
  - **geometric algebra (not covered)**
- **There's no "right" or "best" way—the more you know, the more you'll be able to do!**







Next time: Perspective & Texture Mapping