Full Name: ______________________________

Andrew ID: ______________________________

15-462/662, Fall 2020

Midterm Exam

October 22, 2020

Instructions:

- Answers must be submitted by 11:59:59pm Eastern time on October 22, 2020.
- There is no time limit; you may work on the exam as much as you like up until the deadline.
- This exam is **open book, open notes, open internet**, but you *must work alone*[1].
- Your answers **must** be filled out via the provided plain-text template, and submitted via Autolab. We will *not* accept any other form of submission (such scans of written exams, email, etc.).
- For answers involving code, your solution should have the same prototype as the function given in the prompt.
- Partial credit will be awarded, so please try to clearly explain what you are doing, especially if you are uncertain about the final answer. Comments in code are especially helpful.

Problem	Your Score	Possible Points
1		20
2		40
3		40
Total		100

[1] "Alone" means you cannot discuss the exam with your classmates, your friends, your dog, your cat, or any other creature containing deoxyribonucleic acid.

1 (20 points) Getting Warmed Up

Just a few assorted questions to get your brain in graphics mode!

(a) (6 points) In our lecture on spatial transformations, we saw that the choice of matrix decomposition made a big difference when interpolating between two poses—for instance, separately interpolating each component of the polar decomposition gave natural motion, whereas directly interpolating the original transformation matrix resulted in weird artifacts. Likewise, the choice of color space will make a big difference if we want to interpolate between color images. Consider two common color models:

- **RGB**—encodes the intensity of red, green, and blue emission as values in $[0, 1]$.
- **HSV**—encodes hue as an angle $\theta \in [0, 360)$, and both saturation/value as values in $[0, 1]$.

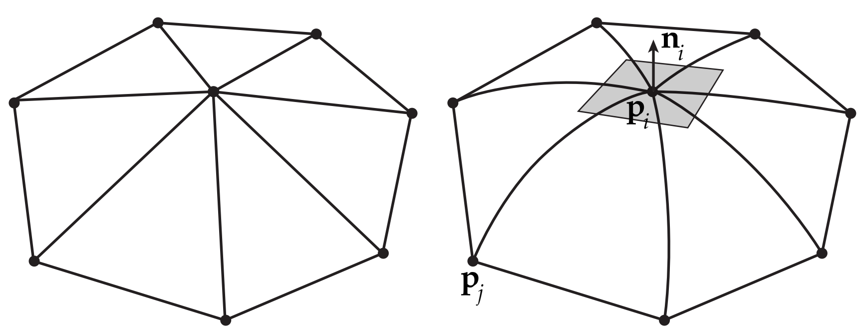
Question: Suppose we want to fade between two images by linearly interpolating either RGB or HSV values. Which choice will give a more natural fade? What artifacts might you see?

(b) (7 points) A compact way to store the connectivity of a halfedge mesh is to index the halfedges from 0 to $2E - 1$ (where E is the number of edges) and implicitly assume that the twin of each even halfedge n is $n + 1$, and likewise, the twin of each odd halfedge n is $n - 1$. So for instance, 0 and 1 are twins, 2 and 3 are twins, and so on. This way, we only have to explicitly store index of the next halfedge, as done in the table below.

Question: Does the given data describe valid manifold connectivity? If so, how many vertices, edges, and faces does the mesh have? Is it possible to draw each face as a flat polygon in 3D, without causing any polygons to intersect or become degenerate?

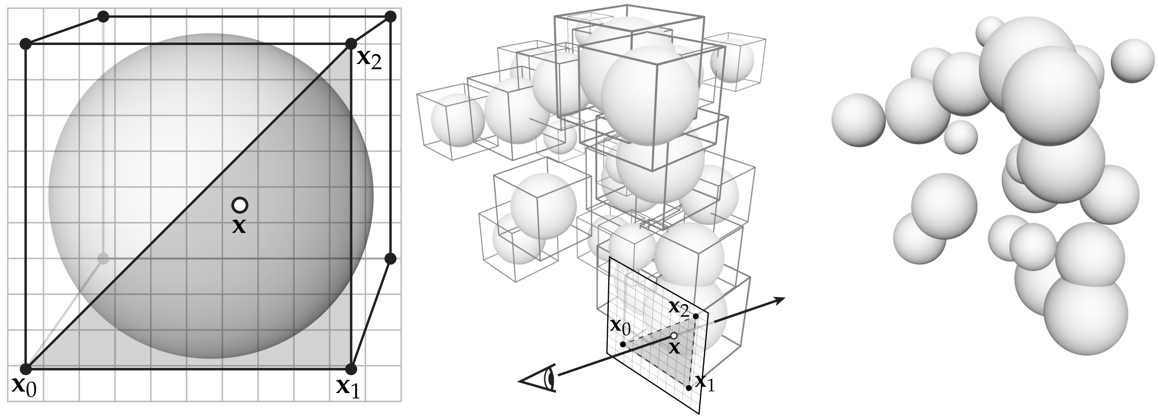
`i`	0	1	2	3	4	5
`next[i]`	2	4	1	3	0	5

(c) (7 points) Suppose you're given a triangle mesh where you know the vertex positions $\mathbf{p}$, the vertex normals $\mathbf{n}$, and the connectivity. If you imagine this data was sampled from a smooth surface, then there's really no reason you *have* to interpolate it using straight line segments and flat triangles (as depicted below, left). For instance, you could draw a wireframe of the mesh using Bézier curves where (i) the curve for each edge ij interpolates the positions $\mathbf{p}_i, \mathbf{p}_j \in \mathbb{R}^3$ at the two endpoints and where (ii) the tangents of all curves meeting at a common vertex i lie in the plane perpendicular to the normal $\mathbf{n}_i$ at this vertex (as depicted below, right).

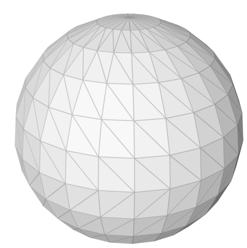


Question: What's the lowest-degree Bézier curve you can use to interpolate the point and normal data as described above? Do the given constraints *uniquely* determine the interpolating curve? If not, what might you do to pin down a unique solution? (There are many possible answers for the final question—we just want you to think creatively!)

2 (40 points) Cubing the Sphere



Often in this class we've presented rasterization and ray tracing as two "competing" ways to draw images on the screen. The reality is that these techniques are getting combined more and more to achieve the best of both worlds: beautiful effects at lightning speed. One nice example is drawing a very large number of spheres, which might be used to, say, nicely display vertices in a mesh editor, or render high-quality points in a point cloud (among many other things!).



The traditional way to draw a sphere using the rasterization pipeline is to tessellate it into a bunch of triangles (as shown at right), then rasterize these triangles as usual. An alternative route, which we'll explore here, is to rasterize a bounding box around the sphere (as illustrated above). For each pixel in the bounding box, we then trace a ray from the eye through the pixel center $\mathbf{x}$, and see if and where this ray intersects the sphere. The intersection information is then used to update the color and depth buffers. In essence, rather draw the color and depth of the triangle itself, we treat the triangle as a "portal" that looks into a box where the sphere lives. We'll build up this procedure one small piece at a time.

(a) (5 points) Write a routine to build a matrix that will transform a cube with vertices $(\pm 1, \pm 1, \pm 1)$ to the bounding box for a sphere with center $\mathbf{c} \in \mathbb{R}^3$ and radius $r > 0$. You should assume that this matrix will be applied to the homogeneous coordinates for the eight vertices. Matrices are indexed as `A[i][j]`, where `i` is the row index, and `j` is the column index, and `Matrix4x4::Zero` gives the matrix of all zeros.

```
Matrix4x4 bboxTransform( Vec3 c, // sphere center
                         double r ) // sphere radius
```

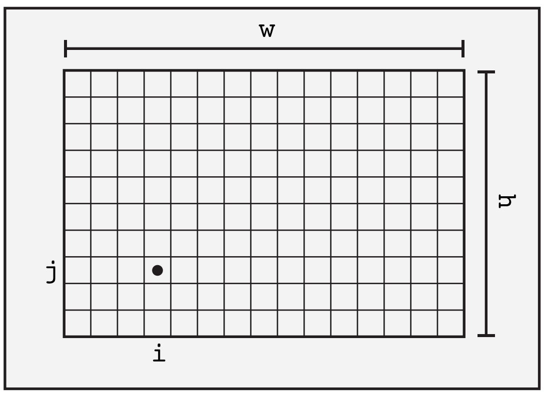
(b) (6 points) Implement a method that performs barycentric interpolation of three given vertex coordinates in world coordinates, assuming we are given the barycentric coordinates $\mathbf{b}$ of a point in the 2D projection of the triangle.

```
Vec3 interpolateWorldPosition(
      Vec3 p0, Vec3 p1, Vec3 p2, // vertex world coordinates
      Vec3 b ) // barycentric coordinates of sample point
```

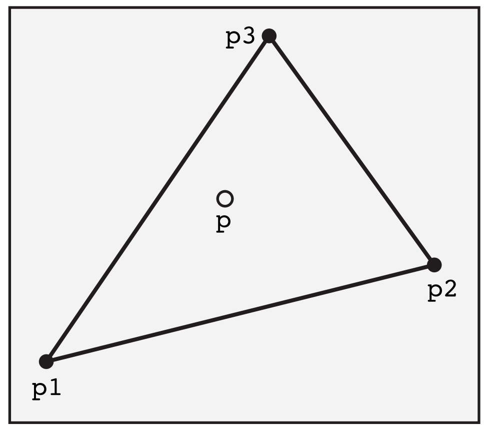
(c) (5 points) Implement a routine that intersects a ray $\mathbf{o} + t\mathbf{d}$ with a sphere of radius $r > 0$ centered at $\mathbf{c} \in \mathbb{R}^3$. This routine assumes that the ray direction $\mathbf{d}$ has unit magnitude, and should return the smallest *positive* t where the ray hits the sphere; if there is no such intersection, it should return -1.

```
double intersectSphere( Vec3 c, // center
                        double r, // radius
                        Vec3 o, // ray origin
                        Vec3 d ) // ray direction (unit)
```

(d) (6 points) Ok, let's put it all together. To draw a super high-quality sphere, we will rasterize its bounding box, which has been diced into triangles. Your job is just to implement an "unusual" triangle rasterization routine `drawBBoxTriangle` that shades each pixel of the triangle according to the closest sphere-ray intersection (as discussed at the beginning). For each pixel covered by the triangle, your routine should figure out the location $\mathbf{x} \in \mathbb{R}^3$ of this pixel in world coordinates. It should then trace a ray from the eye through $\mathbf{x}$ to see if it hits the sphere. If the hit point is the closest thing seen so far, your routine should shade the pixel using the color of the *sphere*, rather than the color of the bounding box. It should also update the depth buffer so that subsequent objects are properly occluded by the sphere.



`pixelCenter`



`baryCoords`

Implementation notes: You can (and should!) call the routines from the earlier parts of this question—and can assume these routines work correctly, independent of what answers you gave above. You may also assume you have two other basic routines (with inputs illustrated above):

- `Vec2 pixelCenter( i, j, w, h )` — returns the location of pixel (i, j) for an image of width $w \times h$.
- `Vec3 baryCoords( p, p1, p2, p3 )` — returns the barycentric coordinates of a point $\mathbf{p}$ within a triangle with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^2$.

For the main routine `drawBBoxTriangle`, the three input vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$ give the world coordinates of the triangle vertices, after the camera transformation but before being transformed into clip space. (Hence, you can assume that the camera is sitting at the origin, looking down the $-z$-axis.) The inputs $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^2$ give the same three coordinates projected into the 2D image plane, and transformed into final 2D image coordinates $[0, w] \times [0, h]$. The `depth` and `color` buffers have size `w*h` and store a single value per pixel (hence, `color` is just a greyscale value rather than an RGB color). You do not need to worry about efficiency: it is ok to test every pixel in the image to see if it's covered by the triangle.

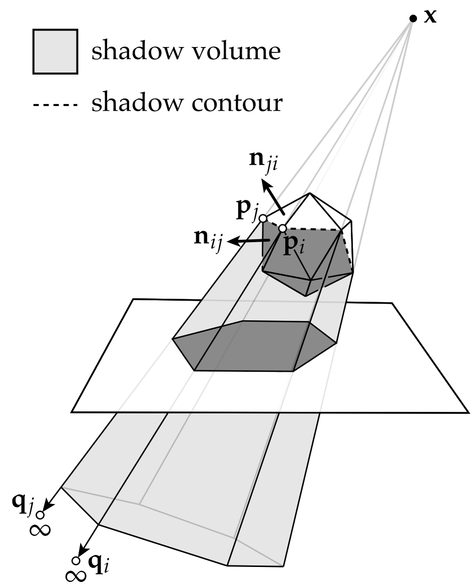
```
void drawBBoxTriangle( Vec3 x0, Vec3 x1, Vec3 x2, // world coordinates
                       Vec2 u0, Vec2 u1, Vec2 u2, // projected coordinates
                       Vec3 c, double r, // sphere center/radius
                       double sphereColor,
                       double* depth, double* color, // buffers
                       int w, int h ) // buffer width/height
```

(e) (6 points) Suppose you want to rasterize a scene that combines your beautiful, pixel-perfect spheres with ordinary triangles. Will everything work out if you just rasterize triangles in the usual way? For instance, will you correctly resolve depth for spheres that intersect triangles? Why or why not?

(f) (6 points) A completely different strategy is to just use *instancing* to draw a bunch of copies of a triangle mesh of a sphere (using standard triangle rasterization). What are some pros and cons of instancing relative to the mixed ray tracing/rasterization scheme we've devised above?

(g) (6 points) Finally, the other obvious strategy is to just ray trace everything, i.e., shoot a ray through every pixel, that gets tested against every sphere (possibly using some kind of spatial data structure). What advantage(s) does our hybrid "bounding box" strategy provide—assuming an efficient implementation that does *not* test every pixel for every triangle? What advantage(s) does pure ray tracing provide? (**Hint:** consider situations where you have either a very small or very large number of spheres.)

3 (40 points) Step Into the Shadow



Up until now, we've said that ray tracing is typically needed to render effects like reflections and shadows. But actually, there's a clever way to render exact shadows for a point light source using just a rasterizer. The basic idea is to explicitly construct a polygon mesh called the "shadow volume," corresponding to the shadowed region of space (see above). These polygons are then rasterized as usual—but with a twist: rather than rasterize directly into the image buffer, we rasterize them into a so-called *stencil buffer*. This buffer does not keep track of color values, but instead counts the *number* of shadow polygons that cover each pixel. More accurately: this number is *incremented* (+1) for shadow polygons that face the camera, and *decremented* (-1) for shadow polygons that face away from the camera. The stencil buffer can then be used in a final pass to shade only those pixels where the closest primitive is outside the shadow volume.

(a) (5 points) In a conventional rasterization pipeline[2], what is the fundamental reason why we can't just directly evaluate whether a pixel is in/out of shadow during the triangle rasterization stage?

(b) (5 points) Consider a triangle mesh that is manifold and has no boundary, and a point light source at a point $\mathbf{x} \in \mathbb{R}^3$. The *shadow contour* is the set of edges on the boundary between the shadowed and non-shadowed region. More precisely, an edge ij is part of the shadow contour if $\mathbf{x}$ is in front of one of the two triangles containing ij, and behind the other.

Give an explicit mathematical expression (**not code**) that evaluates to true if ij is part of the shadow contour, and false otherwise. Your expression can use the light position $\mathbf{x} \in \mathbb{R}^3$, the two edge endpoints $\mathbf{p}_i, \mathbf{p}_j \in \mathbb{R}^3$, and the two normal vectors $\mathbf{n}_{ij}, \mathbf{n}_{ji} \in \mathbb{R}^3$ on either side of an edge ij. If you like, you may use logical operations (*AND*, *OR*, *etc.*) in your expression, but there is a nice way to write the answer without such expressions.

[2]**Note**: RTX is *not* a conventional rasterization pipeline!

(c) (5 points) Using your expression from the previous part, write some pseudocode to decide if the given edge is on the shadow contour. You may assume a basic halfedge data structure[3], which provides methods like `face->normal()` and `vertex->position()`, but you must explicitly call these methods (rather than assuming that quantities like $\mathbf{p}_i$, $\mathbf{n}_{ij}$, *etc.*, are already given). You may also assume basic vector operations (cross product, dot product, *etc.*). **Note:** you will *not* be penalized if your expression from the previous part is wrong, as long as the rest of your code is right.

```
bool onContour( Edge e, // edge to be tested
                Vec3 x // location of point light )
```

(d) (5 points) For each edge ij on the shadow contour, we have to build a polygon that forms one "side" of the shadow volume[4]. This polygon is formed by the two endpoints $\mathbf{p}_i$ and $\mathbf{p}_j$ of the edge, which are extended along rays from the light source $\mathbf{x}$ to points $\mathbf{q}_i$, $\mathbf{q}_j$ at infinity. Write a routine that computes the shadow polygon for a given edge, yielding four points in homogeneous coordinates. If you find it convenient, you can assume that `Vec4` has a constructor that takes a `Vec3` and a scalar as input (*e.g.*, `Vec4 u( v, 1 )`, where `v` is a `Vec3`). As in part (c), you should assume a standard halfedge data structure, and must explicitly access any data you need to perform the computation.

```
void shadowPolygon( Edge e, // contour edge
                    Vec3 x, // light position
                    Vec4& pi, Vec4& pj,  // endpoints along edge
                    Vec4& qi, Vec4& qj ) // endpoints at infinity
```

(e) (5 points) What challenge might you run into when trying to build a shadow volume for a polygon mesh that is manifold but does not have triangular faces?

(f) (5 points) For the shadow volume strategy to work, we need to make sure not to draw shadow polygons that are occluded by scene objects, and we need to make sure not to shade pixels that should be in shadow (*i.e.*, pixels where the final stencil buffer value is greater than zero). So, the general strategy is to rasterize the scene in multiple "passes." Each pass might render a different set of objects, and uses data from one buffer to decide what data should get written into another buffer.

Describe—**in words, not code**—any sequence of rasterization passes that will correctly draw shadowed pixels as black and lit pixels as white. The only three buffers you should consider are a *depth buffer* which keeps track of the closest primitive seen so far, a *stencil buffer* which counts the number of primitives drawn into each pixel (incrementing for camera-facing polygons, and decrementing otherwise), and a *color buffer* which stores the final color values (in this case just black or white). The only two sets of primitives you can rasterize are the *scene primitives* describing the objects in the scene, and the *shadow primitives* which are the polygons describing the shadow volume. All pixels that are neither lit nor in shadow should be given a background color (like grey).

(g) (5 points) What happens if the camera is *inside* the shadow volume? Does your scheme correctly draw objects that are in shadow? If so, why? If not, why not? (Note: you are *not* required to come up with a procedure that produces correct shadows in this case! You merely need to be able to correctly analyze the behavior of your algorithm in this scenario.)

[3] For instance, you may assume you have the same operations available as in the skeleton code for Scotty3D.

[4] To complete the shadow volume, you'd also have to add two "end caps," where triangles facing toward the light are drawn as usual, and triangles facing away from the light are drawn at infinity. For this exam, you do not need to worry about drawing these end caps.

(h) (5 points) Finally, let's put it all together—write a routine that rasterizes a polygon from the shadow volume, and updates the stencil buffer. Independent of how you designed your algorithm in part (f), this routine should take the four points of the polygon as input, and update the stencil buffer only if this polygon is closer than the closest object stored in the depth buffer. The four vertices of the polygon are given as points `P0`, `P1`, `Q0`, `Q1` that have already been projected into 2D image coordinates; the third coordinate of each point gives its depth value, and all vertices are at finite locations (*i.e.*, not points at infinity).

Implementation notes: You may use any method from any other part of the exam—even if you did not complete that part. As before, you may also assume that you have the methods `pixelCenter` and `baryCoords`. All buffers have size `w*h` and store a single value per pixel (`color` is just a greyscale value rather than an RGB color). You do not need to worry about efficiency—to rasterize, you can just test coverage for every pixel in the entire image.

This routine will be a bit longer/more complicated than the simple subroutines you wrote above. Some questions to think about:

- What's the standard way to draw a quad via the rasterization pipeline?
- How do you check if a sample point is inside a triangle?
- How do you determine the depth at an arbitrary location inside a triangle?
- How do you know whether to increment or decrement the stencil buffer?

(**Note:** these questions are just to help you think about how to write the routine! You do not have to answer them directly.)

```
void drawShadowPolygon(
   Vec3 P0, Vec3 P1, Vec3 Q0, Vec3 Q1, // polygon vertices
   double* depth, double* stencil, double* color, // buffers
   int w, int h ) // buffer width/height
```