Meshes and Manifolds
Fractal Quiz
Last time: overview of geometry

- Many types of geometry in nature
- Demand sophisticated representations
- Two major categories:
  - IMPLICIT - “tests” if a point is in shape
  - EXPLICIT - directly “lists” points
- Lots of representations for both
- Today:
  - what is a surface, anyway?
  - nuts & bolts of polygon meshes
  - geometry processing / resampling
Manifold Assumption

- Today we’re going to introduce the idea of manifold geometry
- Can be hard to understand motivation at first!
- So first, let’s revisit a more familiar example...
Bitmap Images, Revisited

To encode images, we used a *regular grid* of pixels:
But images are not fundamentally made of little squares:

Goyō Hashiguchi, *Kamisuki* (ca 1920)
So why did we choose a square grid?

...rather than dozens of possible alternatives?
Regular grids make life easy

- **One reason: SIMPLICITY / EFFICIENCY**
  - E.g., always have four neighbors
  - Easy to index, easy to filter...
  - Storage is just a list of numbers

- **Another reason: GENERALITY**
  - Can encode basically any image

- Are regular grids *always* the best choice for bitmap images?
  - No! E.g., suffer from anisotropy, don’t capture edges, ...
  - But *more often than not* are a pretty good choice

- Will see a similar story with geometry...

<table>
<thead>
<tr>
<th>i,j-1</th>
<th>(i-1,j)</th>
<th>i,j</th>
<th>(i+1,j)</th>
<th>(i,j+1)</th>
</tr>
</thead>
</table>
So, how should we encode surfaces?
Smooth Surfaces

- Intuitively, a *surface* is the boundary or “shell” of an object
- (Think about the candy shell, not the chocolate.)
- Surfaces are *manifold*:
  - If you zoom in far enough (at any point) looks like a plane*
  - E.g., the Earth from space vs. from the ground

*…or can easily be flattened into the plane, without cutting or ripping.
Isn’t every shape manifold?

- No, for instance:

Center point never looks like the plane, no matter how close we get.
More Examples of Smooth Surfaces

Which of these shapes are manifold?
A manifold polygon mesh has fans, not fins

- For polygonal surfaces just two easy conditions to check:
  1. Every edge is contained in only two polygons (no “fins”)
  2. The polygons containing each vertex make a single “fan”
What about boundary?

- The boundary is where the surface “ends.”
- E.g., waist & ankles on a pair of pants.
- Locally, looks like a half disk
- Globally, each boundary forms a loop

Polygon mesh:
- one polygon per boundary edge
- boundary vertex looks like “pacman”
Ok, but why is the manifold assumption *useful*?
Keep it Simple!

- Same motivation as for images:
  - make some assumptions about our geometry to keep data structures/algorithms simple and efficient
  - in *many common cases*, doesn’t fundamentally limit what we can do with geometry
How do we actually encode all this data?
Warm up: storing numbers

Q: What data structures can we use to store a list of numbers?

One idea: use an array (constant time lookup, coherent access)

Alternative: use a linked list (linear lookup, incoherent access)

Q: Why bother with the linked list?

A: For one, we can easily insert numbers wherever we like...
Polygon Soup

- Most basic idea:
  - For each triangle, just store three coordinates
  - No information about connectivity
  - Not much different from point cloud! ("Triangle cloud?")

- Pros:
  - Really stupidly simple

- Cons:
  - Redundant storage
  - Hard to do anything beyond simply drawing the mesh on screen

(x0,y0,z0) (x1,y1,z1)
(x1,y1,z1) (x2,y2,z2)
(x2,y2,z2) (x3,y3,z3)
(x3,y3,z3) (x0,y0,z0)
Adjacency List (Array-like)

- Store triples of coordinates \((x,y,z)\), tuples of indices
- E.g., tetrahedron:

<table>
<thead>
<tr>
<th>VERTICES</th>
<th>POLYGONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>0:</td>
<td>-1</td>
</tr>
<tr>
<td>1:</td>
<td>1</td>
</tr>
<tr>
<td>2:</td>
<td>1</td>
</tr>
<tr>
<td>3:</td>
<td>-1</td>
</tr>
</tbody>
</table>

Q: How do we find all the polygons touching vertex 2?

Ok, now consider a more complicated mesh:

Very expensive to find the neighboring triangles! (What’s the cost?)
Incidence Matrices

- If we want to answer neighborhood queries, why not simply store a list of neighbors?
- Can encode all neighbor information via *incidence matrices*
- E.g., tetrahedron: 
  
  ```
  \begin{array}{c|cccc}
  & v0 & v1 & v2 & v3 \\
  \hline 
  e0 & 1 & 1 & 0 & 0 \\
  e1 & 0 & 1 & 1 & 0 \\
  e2 & 1 & 0 & 1 & 0 \\
  e3 & 1 & 0 & 0 & 1 \\
  e4 & 0 & 0 & 1 & 1 \\
  e5 & 0 & 1 & 0 & 1 \\
  \end{array}
  \quad
  \begin{array}{c|cccccccc}
  & e0 & e1 & e2 & e3 & e4 & e5 \\
  \hline 
  f0 & 1 & 0 & 0 & 1 & 0 & 1 \\
  f1 & 0 & 1 & 0 & 0 & 1 & 1 \\
  f2 & 1 & 1 & 1 & 0 & 0 & 0 \\
  f3 & 0 & 0 & 1 & 1 & 1 & 0 \\
  \end{array}
  
  ```

- 1 means “touches”; 0 means “does not touch”
- Instead of storing lots of 0’s, use *sparse matrices*
- Still large storage cost, but finding neighbors is now $O(1)$
- Hard to change connectivity, since we used fixed indices
- Bonus feature: mesh does not have to be manifold
Halfedge Data Structure (Linked-list-like)

- Store *some* information about neighbors
- Don’t need an exhaustive list; just a few key pointers
- Key idea: two *halfedges* act as “glue” between mesh elements:

Each vertex, edge, or face points to just *one* of its halfedges.
Use "twin" and "next" pointers to move around mesh

Use "vertex", "edge", and "face" pointers to grab element

Example: visit all vertices of a face:

```cpp
Halfedge* h = f->halfedge;
do {
    h = h->next;
    // do something w/ h->vertex
} while( h != f->halfedge );
```

Example: visit all neighbors of a vertex:

```cpp
Halfedge* h = v->halfedge;
do {
    h = h->twin->next;
} while( h != v->halfedge );
```

Note: only makes sense if mesh is manifold!
Halfedge meshes are *always* manifold

- Consider simplified halfedge data structure
- Require only “common-sense” conditions

```c
struct Halfedge {
    Halfedge *next, *twin;
};
```

twin->twin == this
next != this
twin != this

- Keep following `next`, and you’ll get faces.
- Keep following `twin` and you’ll get edges.
- Keep following `next->twin` and you’ll get vertices.

Q: Why, therefore, is it impossible to encode the red figures?
Halfedge meshes are easy to edit

- Remember key feature of linked list: insert/delete elements
- Same story with halfedge mesh (“linked list on steroids”)
- E.g., for triangle meshes, several atomic operations:

- **flip**

- **split**

- **collapse**

- Must be careful to preserve manifoldness!
Edge Flip (Triangles)

- Triangles \((a,b,c), (b,d,c)\) become \((a,d,c), (a,b,d)\):

- Long list of pointer reassignments \((\text{edge} \rightarrow \text{halfedge} = \ldots)\)
- However, no elements created/destroyed.
- Q: What happens if we flip twice?
- Challenge: can you implement edge flip such that pointers are unchanged after two flips?
Insert midpoint m of edge (c,b), connect to get four triangles:

This time, have to add new elements.
Lots of pointer reassignments.
Q: Can we “reverse” this operation?
Edge Collapse (Triangles)

- Replace edge \((b,c)\) with a single vertex \(m\):

Now have to **delete** elements.

Still lots of pointer assignments!

Q: How would we implement this with an adjacency list?

Any other good way to do it? (E.g., different data structure?)
# Comparison of Polygon Mesh Data Structures

**Case study: triangles.**

<table>
<thead>
<tr>
<th></th>
<th>Adjacency List</th>
<th>Incidence Matrices</th>
<th>Halfedge Mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>storage cost</strong>*</td>
<td>~6 x #vertices</td>
<td>~24 x #vertices</td>
<td>~36 x #vertices</td>
</tr>
<tr>
<td><strong>constant-time</strong></td>
<td>NO</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>neighborhood access?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>add/remove mesh</strong></td>
<td>NO</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>elements efficiently?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>nonmanifold</strong></td>
<td>YES</td>
<td>YES</td>
<td>NO**</td>
</tr>
<tr>
<td>geometry?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Conclusion: pick the right data structure for the job!**

*number of integer values and/or pointers required to encode connectivity (all data structures require same amount of storage for vertex positions)

**(but perhaps it is possible to modify this data structure to support nonmanifold geometry...)**
Alternatives to Halfedge

- Many very similar data structures:
  - winged edge
  - corner table
  - quadedge
  - ...

- Each stores local neighborhood information

- Similar tradeoffs relative to simple polygon list:
  - **CONS**: additional storage, incoherent memory access
  - **PROS**: better access time for individual elements, intuitive traversal of local neighborhoods

- (Food for thought: can you design a halfedge-like data structure with reasonably coherent data storage?)
Ok, but what can we actually *do* with our fancy new data structure?
Subdivision Modeling
Subdivision Modeling

- Common modeling paradigm in modern 3D tools:
  - Coarse “control cage”
  - Perform local operations to control/edit shape
  - Global subdivision process determines final surface
Subdivision Modeling—Local Operations

For general polygon meshes, we can dream up lots of local mesh operations that might be useful for modeling:

- EdgeBevel
- FaceBevel
- VertexBevel
- EdgeCollapse
- EdgeFlip
- EraseVertex
- EdgeVertexSplit
- EraseEdge
- FaceCollapse

...and many, many more!
Global Subdivision

- Start with coarse polygon mesh ("control cage")
- Subdivide each element
- Update vertices via local averaging
- Many possible rule:
  - Catmull-Clark (quads)
  - Loop (triangles)
  - ...
- Common issues:
  - interpolating or approximating?
  - continuity at vertices?
- Easier than splines for modeling; harder to evaluate pointwise
Next Time: Digital Geometry Processing

- Extend traditional digital signal processing (audio, video, etc.) to deal with *geometric* signals:
  - upsampling / downsampling / resampling / filtering ...
  - aliasing (reconstructed surface gives “false impression”)

- Also some new challenges:
  - over which domain is a geometric signal expressed?
  - no terrific sampling theory, no fast Fourier transform, ...

- Often need new data structures & new algorithms