The Rendering Equation
Recap: What is radiance?

Radiance at point $p$ in direction $N$ is radiant energy ("#hits") per unit time, per solid angle, per unit area perpendicular to $N$.

$$L = \frac{\partial^2 \Phi}{\partial \Omega \partial A \cos \theta}$$

- $\Phi$ — radiant flux
- $\Omega$ — solid angle
- $A \cos \theta$ — projected area

*Confusing point: this cosine has to do w/ parameterization of sphere, not Lambert’s law
Intuition: Radiance and Irradiance

\[ E = \int_{H^2} L(\omega) \cos \theta \, d\omega \]

irradiance

radiance in direction $\omega$

angle between $\omega$ and normal
Incident vs. Exitant Radiance

In both cases: intensity of illumination is highly dependent on direction (not just location in space or moment in time).
The Rendering Equation

- Core functionality of photorealistic renderer is to estimate radiance at a given point \( p \), in a given direction \( \omega_0 \).
- Summed up by the rendering equation (Kajiya):

\[
L_o(p, \omega_0) = L_e(p, \omega_0) + \int_{H^2} f_r(p, \omega_i \rightarrow \omega_0) L_i(p, \omega_i) \cos \theta \, d\omega_i
\]

Key challenge: to evaluate incoming radiance, we have to compute yet another integral. I.e., rendering equation is recursive.
Recursive Raytracing

- Basic strategy: recursively evaluate rendering equation!

(This is why you’re writing a ray tracer—rasterizer isn’t enough!)
Renderer measures radiance along a ray

At each “bounce,” want to measure radiance traveling in the direction opposite the ray direction.
Renderer measures radiance along a ray

Radiance entering camera in direction $d =$ light from scene light sources that is reflected off surface in direction $d$. 
How does reflection of light affect the outgoing radiance?

\[ L_0(p, \omega_0) = \int_{\mathcal{H}^2} f_r(p, \omega_i, \omega_0) L_i(p, \omega_i) \cos \theta \, d\omega_i \]
Reflection models

- Reflection is the process by which light incident on a surface interacts with the surface such that it leaves on the incident (same) side without change in frequency.
- Choice of reflection function determines surface appearance.
Some basic reflection functions

- **Ideal specular**
  Perfect mirror

- **Ideal diffuse**
  Uniform reflection in all directions

- **Glossy specular**
  Majority of light distributed in reflection direction

- **Retro-reflective**
  Reflects light back toward source

Diagrams illustrate how incoming light energy from given direction is reflected in various directions.
Materials: diffuse
Materials: plastic
Materials: red semi-gloss paint
Materials: Ford mystic lacquer paint
Materials: mirror
Materials: gold
Materials
Models of Scattering

- How can we model “scattering” of light?
- Many different things that could happen to a photon:
  - bounces off surface
  - transmitted through surface
  - bounces around inside surface
  - absorbed & re-emitted
  - …

- What goes in must come out! (Total energy must be conserved)
- In general, can talk about “probability*” a particle arriving from a given direction is scattered in another direction

*Somewhat more complicated than this, because some light is absorbed!
Hemispherical incident radiance

At any point on any surface in the scene, there’s an incident radiance field that gives the directional distribution of illumination at the point.
Diffuse reflection

Exitant radiance is the same in all directions

Incident radiance

Exitant radiance
Ideal specular reflection

Incident radiance is “flipped around normal” to get exitant radiance

Incident radiance

Exitant radiance
Plastic

Incident radiance gets “flipped and blurred”

Incident radiance

Exitant radiance
Copper

More blurring, plus coloration (nonuniform absorption across frequencies)

Incident radiance

Exitant radiance
Scattering off a surface: the BRDF

- “Bidirectional reflectance distribution function”
- Encodes behavior of light that “bounces off” surface
- Given incoming direction \( \omega_i \), how much light gets scattered in any given outgoing direction \( \omega_o \)?
- Describe as distribution \( f_r(\omega_i \rightarrow \omega_o) \)

\[
f_r(\omega_i \rightarrow \omega_o) \geq 0
\]

\[
\int_{\mathcal{H}_2} f_r(\omega_i \rightarrow \omega_o) \cos \theta \, d\omega_i \leq 1
\]

why less than or equal?

where did the rest of the energy go?!

\[
f_r(\omega_i \rightarrow \omega_o) = f_r(\omega_o \rightarrow \omega_i)
\]

“Helmholtz reciprocity”

Q: Why should Helmholtz reciprocity hold? Think about little mirrors...
Radiometric description of BRDF

\[ f_r(\omega_i \rightarrow \omega_o) = \frac{dL_o(\omega_o)}{dE_i(\omega_i)} = \frac{dL_o(\omega_o)}{dL_i(\omega_i) \cos \theta_i} \left[ \frac{1}{sr} \right] \]

“For a given change in the incident irradiance, how much does the exitant radiance change?”
Example: Lambertian reflection

Assume light is equally likely to be reflected in each output direction

\[
L_o(\omega_o) = \int_{H^2} f_r L_i(\omega_i) \cos \theta_i \, d\omega_i
\]

\[
= f_r \int_{H^2} L_i(\omega_i) \cos \theta_i \, d\omega_i
\]

\[
= f_r E
\]

“albedo” (between 0 and 1)

\[
f_r = \frac{\rho}{\pi}
\]
Example: perfect specular reflection

[Zátonyi Sándor]
Geometry of specular reflection

\[ \theta = \theta_o = \theta_i \]

\[ \omega_o = -\omega_i + 2(\omega_i \cdot \hat{n})\hat{n} \]
Specular reflection BRDF

\[ L_i(\theta_i, \phi_i) \quad \Rightarrow \quad L_o(\theta_o, \phi_o) \]

\[ f_r(\theta_i, \phi_i; \theta_o, \phi_o) = \frac{\delta(\cos \theta_i - \cos \theta_o)}{\cos \theta_i} \delta(\phi_i - \phi_o \pm \pi) \]

- Strictly speaking, \( f_r \) is a distribution, not a function.
- In practice, no hope of finding reflected direction via random sampling; simply pick the reflected direction!
Transmission

In addition to reflecting off surface, light may be transmitted through surface.

Light refracts when it enters a new medium.
Snell’s Law

Transmitted angle depends on relative index of refraction of material ray is leaving/entering.

\[ \eta_i \sin \theta_i = \eta_t \sin \theta_t \]

<table>
<thead>
<tr>
<th>Medium</th>
<th>( \eta^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1.0</td>
</tr>
<tr>
<td>Air (sea level)</td>
<td>1.00029</td>
</tr>
<tr>
<td>Water (20°C)</td>
<td>1.333</td>
</tr>
<tr>
<td>Glass</td>
<td>1.5-1.6</td>
</tr>
<tr>
<td>Diamond</td>
<td>2.42</td>
</tr>
</tbody>
</table>

* index of refraction is wavelength dependent (these are averages)
Law of refraction

\[ \eta_i \sin \theta_i = \eta_t \sin \theta_t \]

\[ \cos \theta_t = \sqrt{1 - \sin^2 \theta_t} \]

\[ = \sqrt{1 - \left( \frac{\eta_i}{\eta_t} \right)^2 \sin^2 \theta_i} \]

\[ = \sqrt{1 - \left( \frac{\eta_i}{\eta_t} \right)^2 (1 - \cos^2 \theta_i)} \]

Total internal reflection:
When light is moving from a more optically dense medium to a less optically dense medium: \( \frac{\eta_i}{\eta_t} > 1 \)

Light incident on boundary from large enough angle will not exit medium.
Optical manhole

Only small “cone” visible, due to total internal reflection (TIR)
Fresnel reflection

Many real materials: reflectance increases w/ viewing angle

[Lafortune et al. 1997]
Snell + Fresnel: Example

Refraction (Snell)

Reflection (Fresnel)
Without Fresnel (fixed reflectance/transmission)
Glass with Fresnel reflection/transmission
Anisotropic reflection

Reflection depends on azimuthal angle $\phi$

Results from oriented microstructure of surface e.g., brushed metal
Translucent materials: Jade
Translucent materials: skin
Translucent materials: leaves
Subsurface scattering

- Visual characteristics of many surfaces caused by light entering at different points than it exits
  - Violates a fundamental assumption of the BRDF
  - Need to generalize scattering model (BSSRDF)

[Jensen et al 2001]
[Donner et al 2008]
Scattering functions

- Generalization of BRDF; describes exitant radiance at one point due to incident differential irradiance at another point:
  \[ S(x_i, \omega_i, x_o, \omega_o) \]

- Generalization of reflection equation integrates over all points on the surface and all directions (!):
  \[ L(x_o, \omega_o) = \int_A \int_{H^2} S(x_i, \omega_i, x_o, \omega_o) L_i(x_i, \omega_i) \cos \theta_i \, d\omega_i \, dA \]
Ok, so scattering is *complicated*!

What’s a (relatively simple) algorithm that can capture all this behavior?
The reflection equation

\[ L_r(x, \omega_r) = f_r(\omega_i \rightarrow \omega_r) \int dL_i(\omega_i) \cos \theta_i \]

\[ L_r(p, \omega_r) = \int_{H^2} f_r(p, \omega_i \rightarrow \omega_r) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]
The reflection equation

- Key piece of overall rendering equation:

\[ L_r(p, \omega_r) = \int_{H^2} f_r(p, \omega_i \to \omega_r) L_i(p, \omega_i) \cos \theta_i \, d\omega_i \]

- Approximate integral via Monte Carlo integration
- Generate directions \( \omega_j \) sampled from some distribution \( p(\omega) \)
- Compute the estimator

\[ \frac{1}{N} \sum_{j=1}^{N} f_r(p, \omega_j \to \omega_r) L_i(p, \omega_j) \cos \theta_j \frac{p(\omega_j)}{p(\omega)} \]

- To reduce variance \( p(\omega) \) should match BRDF or incident radiance function
Estimating reflected light

// Assume:
//   Ray ray hits surface at point hit_p
//   Normal of surface at hit point is hit_n

Vector3D wr = -ray.d;   // outgoing direction
Spectrum Lr = 0.;
for (int i = 0; i < N; ++i) {
    Vector3D wi;        // sample incident light from this direction
    float pdf;          // p(wi)

    generate_sample(brdf, &wi, &pdf); // generate sample according to brdf

    Spectrum f = brdf->f(wr, wi);
    Spectrum Li = trace_ray(Ray(hit_p, wi)); // compute incoming Li
    Lr += f * Li * fabs(dot(wi, hit_n)) / pdf;
}
return Lr / N;
The rendering equation

Now that we know how to handle reflection, how do we solve the full rendering equation? Have to determine incident radiance...
Key idea in (efficient) rendering: take advantage of special knowledge to break up integration into “easier” components.
Path tracing: overview

- Partition the rendering equation into direct and indirect illumination
- Use *Monte Carlo* to estimate each partition separately
  - One sample for each
  - Assumption: 100s of samples per pixel
- Terminate paths with *Russian roulette*
Direct illumination + reflection + transparency

Image credit: Henrik Wann Jensen
Global illumination solution

Image credit: Henrik Wann Jensen
Next Time: Monte Carlo integration

\[ \int_{\Omega} f(p) \, dp \approx \text{vol}(\Omega) \frac{1}{N} \sum_{i=1}^{N} f(X_i) \]