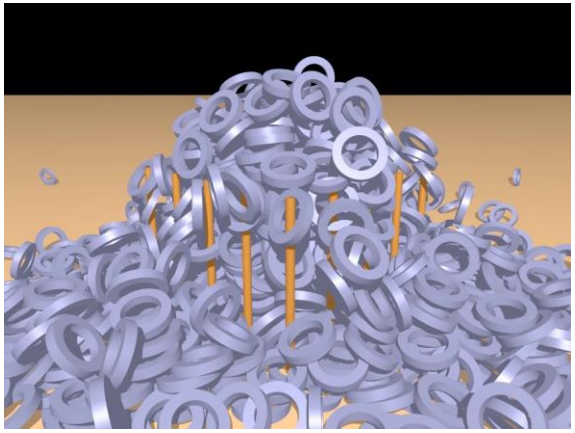
Rigid Body Dynamics and Beyond

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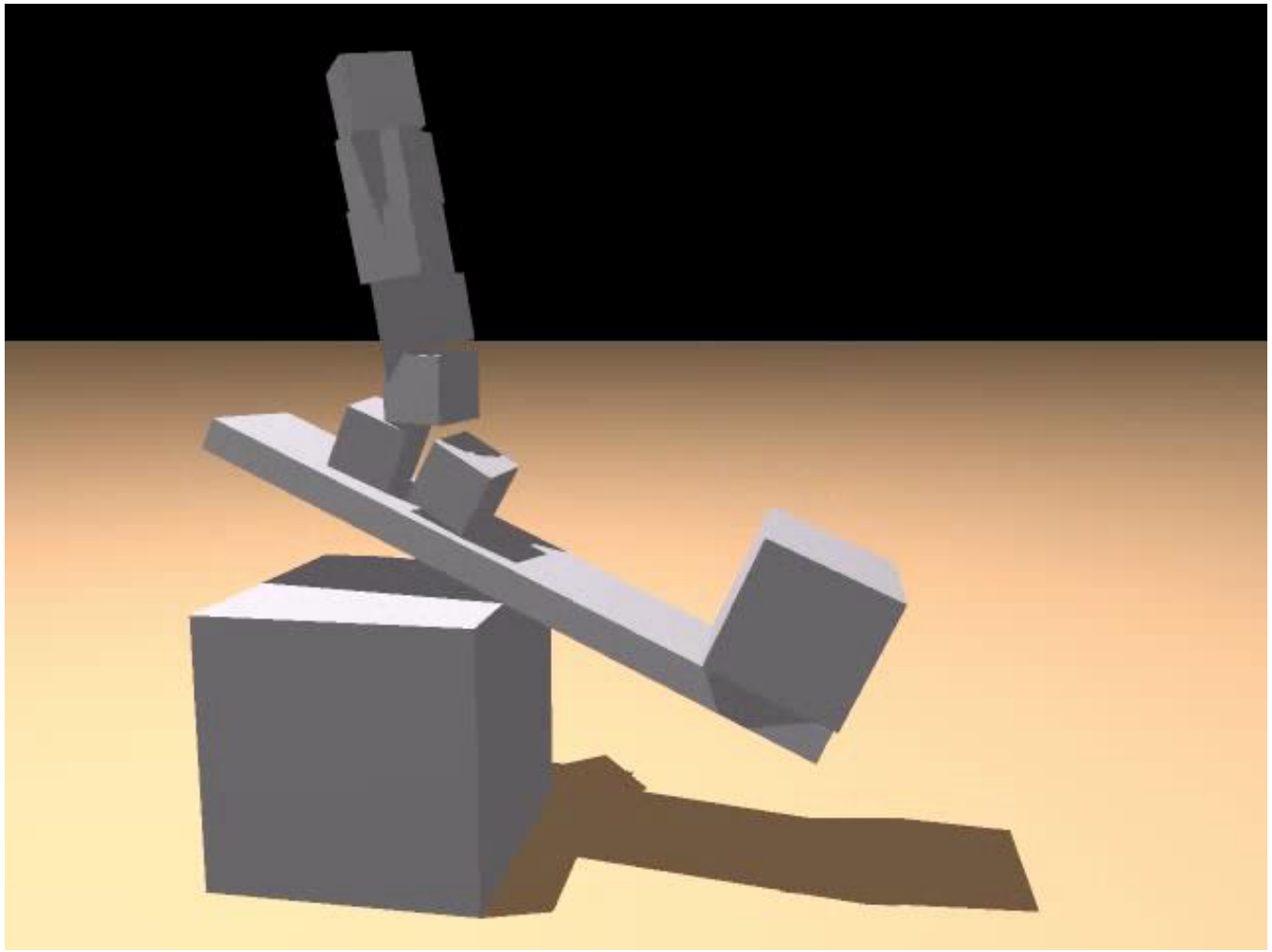
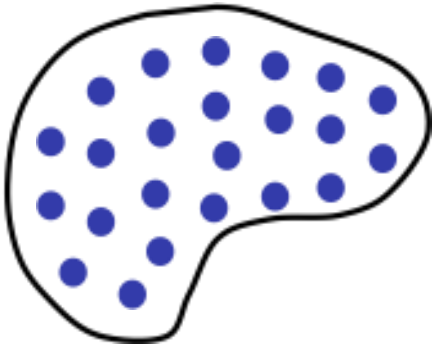
MOVIECLIPS

Rigid Bodies

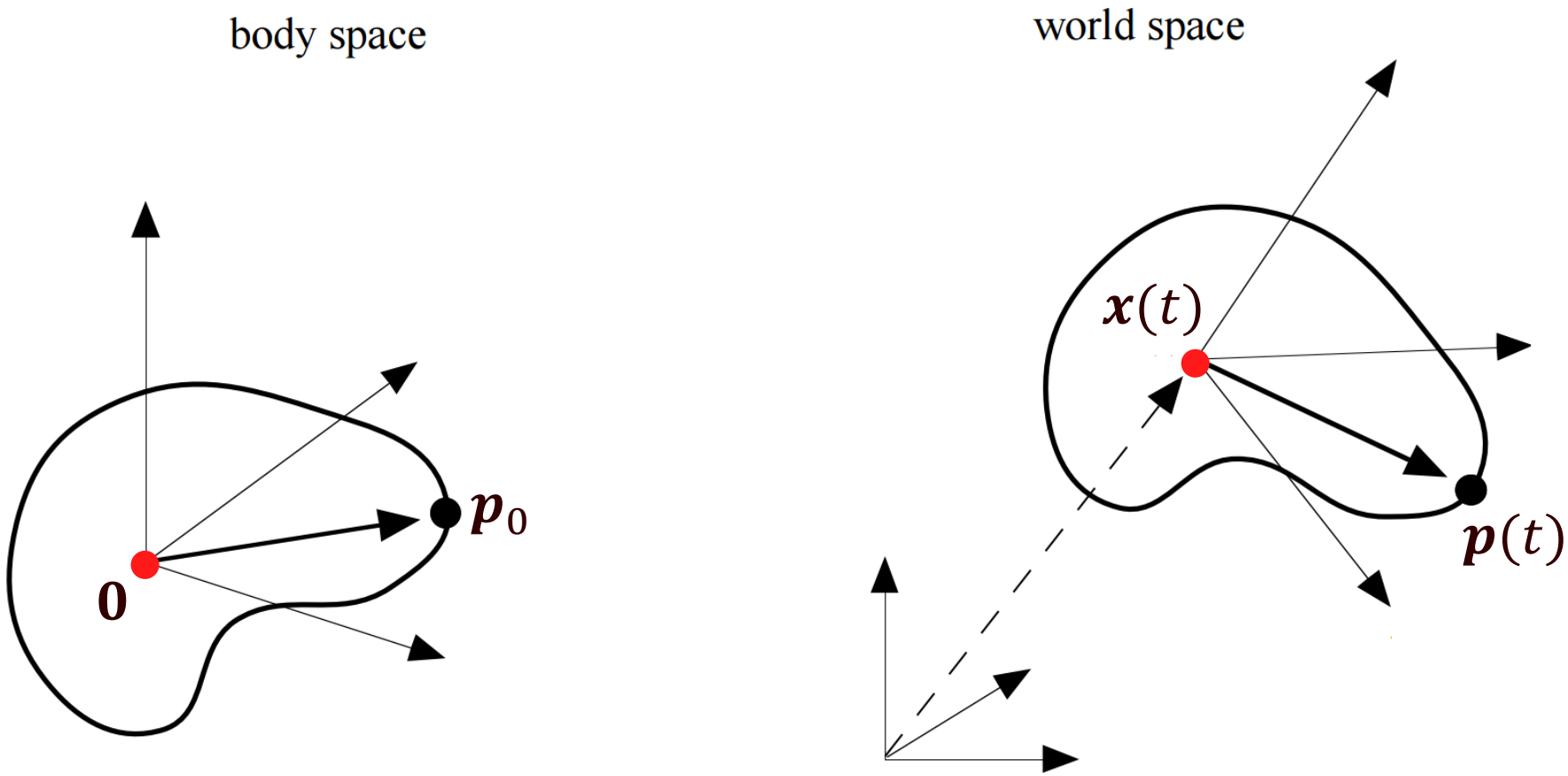


A rigid body



- Collection of particles
- Distance between any two particles is always constant
- What types of motions preserve these constraints?
  - Translation, rotation

Rigid Body Parameterization (reduced coordinates)



$$\boldsymbol{p}(t) = \boldsymbol{x}(t) + \boldsymbol{R}(t)\boldsymbol{p}_0$$

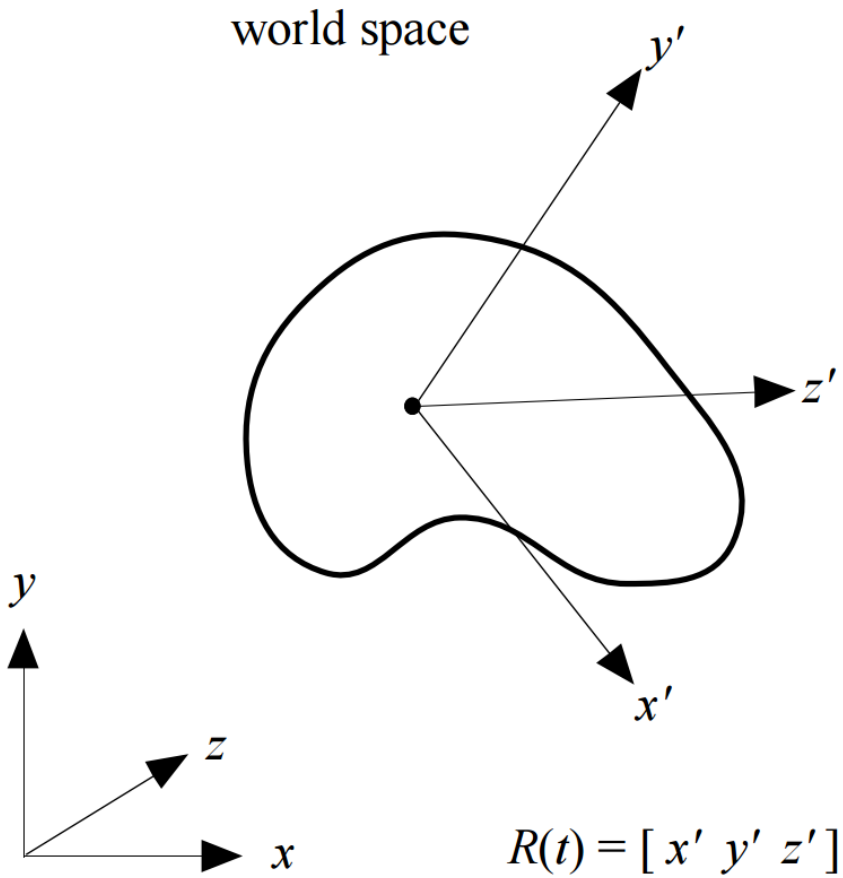
$\boldsymbol{x}(t) \in R^3$ *position*

$R(t) \in R^{3x3}$ *orientation*

Body Orientation

Rotates vectors from body to world coordinates

- Columns of $\boldsymbol{R}(t)$ encode world coordinates of body a



Center of Mass (COM)

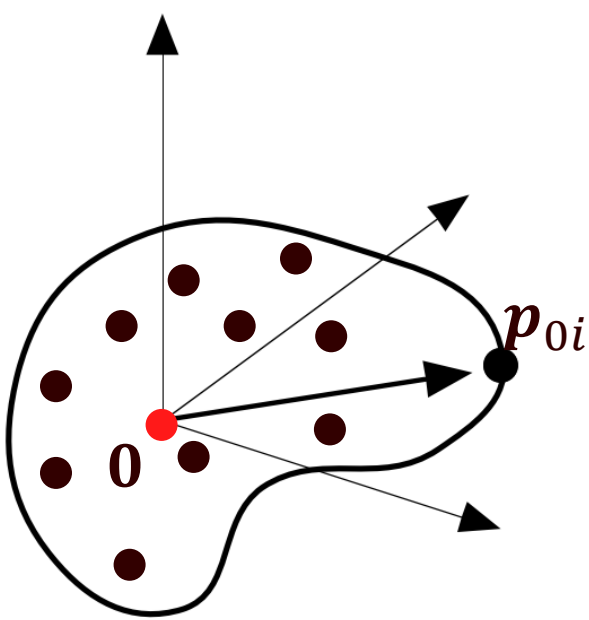
Geometric center of the body

- (0,0,0) in body coordinates
- $\boldsymbol{x}(t)$ in world coordinates

Geometric center: $\sum m_i \boldsymbol{p}_{0i} \coloneqq (0,0,0)$

$$M = \sum m_i$$

$$\boldsymbol{COM}: \frac{\sum m_i \boldsymbol{p}_i(\boldsymbol{t})}{M} = \boldsymbol{x}(\boldsymbol{t})$$



Body velocities

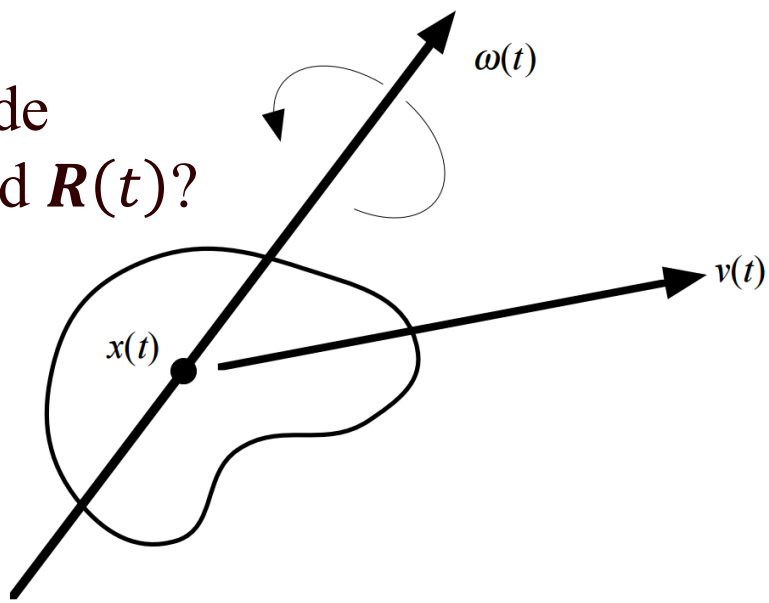
How do the COM position and orientation change with time?

Linear velocity: $\boldsymbol{v}(t) = \frac{d\boldsymbol{x}(t)}{dt} = \dot{\boldsymbol{x}}(t)$

Angular velocity: $\boldsymbol{\omega}(t) =?$

$\boldsymbol{\omega}(t)$ encodes spin direction and magnitude

What is the relationship between $\boldsymbol{\omega}(t)$ and $\boldsymbol{R}(t)$?

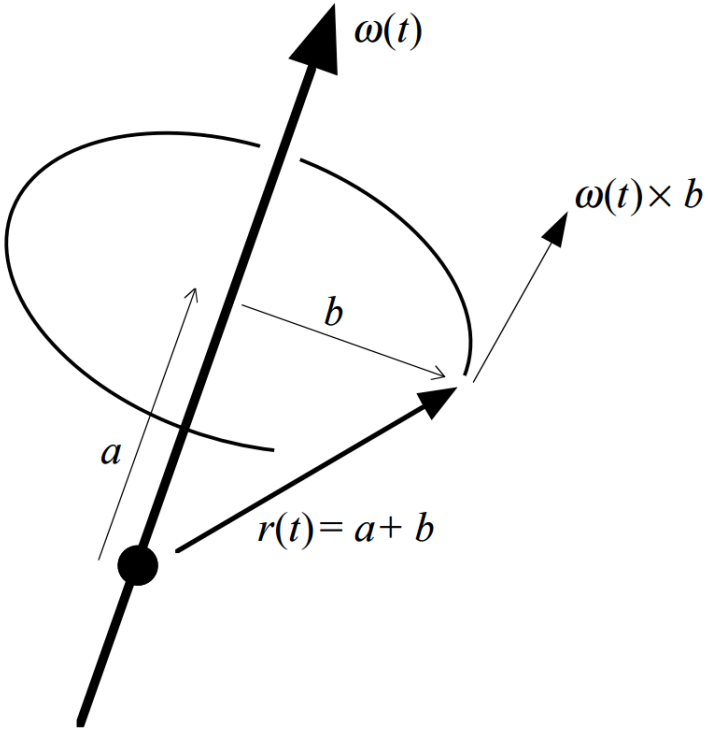


Angular velocity

- Consider vector $\boldsymbol{r}(t)$. What is $\dot{\boldsymbol{r}}(t)$?

$$\dot{\boldsymbol{r}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{r}(t)$$

- Relation to $\dot{\boldsymbol{R}}(\boldsymbol{t}) = \dfrac{d\boldsymbol{R}(t)}{dt}$?



Angular Velocity

R rotates vectors from body to world coords

- Columns of $\boldsymbol{R}(t)$: world coordinates of body axes
- Columns of $\dot{\boldsymbol{R}}(t)$: change of body axes world coordinates wrt time

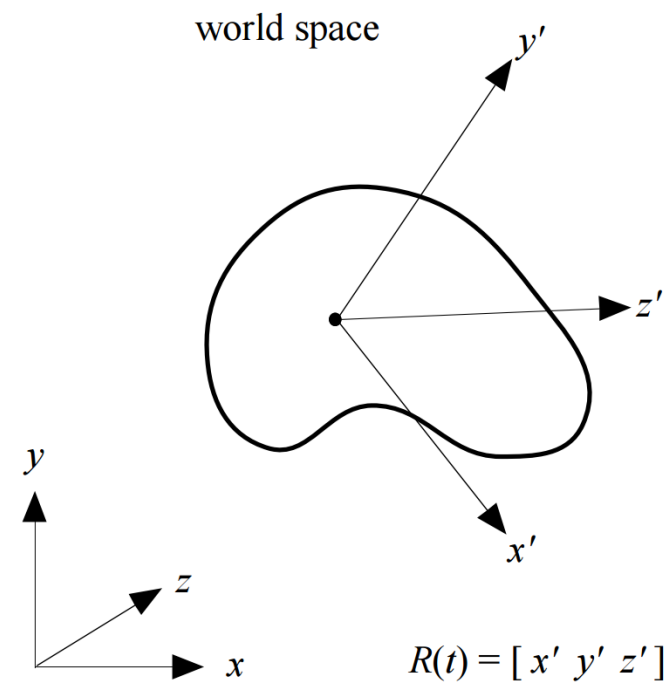
$$\dot{\boldsymbol{x}}'(t) = \boldsymbol{\omega}(t) \times \boldsymbol{x}'(t)$$

$$\dot{\boldsymbol{y}}'(t) = \boldsymbol{\omega}(t) \times \boldsymbol{y}'(t)$$

$$\dot{\boldsymbol{z}}'(t) = \boldsymbol{\omega}(t) \times \boldsymbol{z}'(t)$$

Putting these all together:

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{\omega}(t)_{\times} \boldsymbol{R}(t)$$



Recall

$$a \times b = \begin{pmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{pmatrix}$$

$$= \underbrace{\boxed{\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}}}_{\boldsymbol{a}_\times} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

Summary

- Kinematics: how does the body move?

$$\boldsymbol{p}(t), \boldsymbol{R}(t)$$

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{x}(t)}{dt} = \dot{\boldsymbol{x}}(t)$$

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{\omega}(t)_\times \boldsymbol{R}(t)$$

$$\boldsymbol{p}(t) = \boldsymbol{x}(t) + \boldsymbol{R}(t)\boldsymbol{p}_0$$

$$\dot{\boldsymbol{p}}(t) = \dot{\boldsymbol{x}}(t) + \boldsymbol{\omega}(t)_\times \boldsymbol{R}(t)\boldsymbol{p}_0$$
$$= \dot{\boldsymbol{x}}(t) + \boldsymbol{\omega}(t)_\times (\boldsymbol{p}(t) - \boldsymbol{x}(t))$$

- Dynamics: *what causes this motion?*

Forces and Torques

- External forces: acting on individual particles

$$\text{Net force on body: } \boldsymbol{F} = \sum \boldsymbol{F}_i$$

- Conservation of linear momentum (Newton's 2nd law):

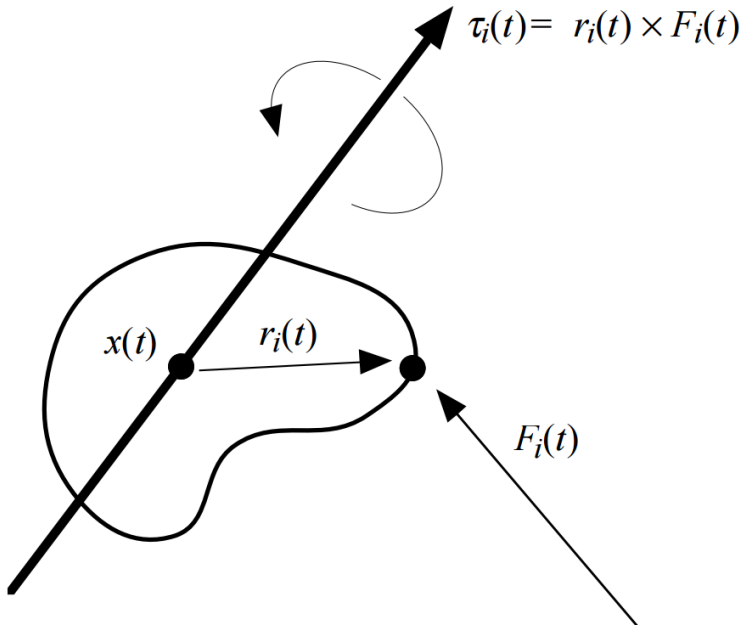
$$\boldsymbol{p} = M\boldsymbol{v}; \dot{\boldsymbol{p}} = \boldsymbol{F}$$

- Analogous concepts for angular motion

Forces and Torques

- Forces on individual particles generate torques
  - (consequence of constant inter-particle distance)

Net torque on body: $\boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \boldsymbol{r}_i \times \boldsymbol{f}_i$



Forces and Torques

- Forces on individual particles generate torques
  - (consequence of constant inter-particle distance)

$$\text{Net torque on body: } \boldsymbol{\tau} = \sum \boldsymbol{\tau}_i = \sum \boldsymbol{r}_i \times \boldsymbol{f}_i$$

- Conservation of angular momentum:

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}; \dot{\boldsymbol{L}} = \boldsymbol{\tau}$$

- What is $\boldsymbol{I}$?
  - Moment of Inertia tensor

The Inertia Tensor

- Analogous to mass, but for rotational motions
  - quantifies distribution of mass as a 2nd order tensor

$$\boldsymbol{I} = \boldsymbol{R}\boldsymbol{I}_b\boldsymbol{R}^T$$

$$\boldsymbol{I}_b = \sum_{\boldsymbol{i}} m_i(\boldsymbol{p}_{0i}{}^T\boldsymbol{p}_{0i}\mathbf{1} - \boldsymbol{p}_{0i}\boldsymbol{p}_{0i}{}^T)$$

- Body-coords MOI is constant, can be precomputed – easy to look up for common shapes!
- World-cords MOI changes with time!

Conservation of Linear and Angular Momenta

- Linear Momentum:

$$\boldsymbol{p} = M\boldsymbol{v}; \dot{\boldsymbol{p}} = \boldsymbol{F};\ \dot{\boldsymbol{v}} = \frac{1}{\boldsymbol{M}}\boldsymbol{F}$$

- Angular Momentum:

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}; \dot{\boldsymbol{L}} = \boldsymbol{\tau};\ \dot{\boldsymbol{\omega}} = \boldsymbol{I}^{-1}(\boldsymbol{\tau} - \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega})$$

- Note: they are decoupled!

Numerical Integration

- COM Acceleration → Velocity → Position
  - Easy: $\boldsymbol{v}_{t+1} = \boldsymbol{v}_t + \Delta t \dot{\boldsymbol{v}}$; $\boldsymbol{x}_{t+1} = \boldsymbol{x}_t + \Delta t \boldsymbol{v}_{t+1}$
- Angular Acceleration → Angular Velocity
  - Easy: $\boldsymbol{\omega}_{t+1} = \boldsymbol{\omega}_t + \Delta t \dot{\boldsymbol{\omega}}$
- Angular Velocity to Rotations?
  - A bit trickier: $\boldsymbol{R}_{t+1} = \boldsymbol{R}_t + \Delta t \dot{\boldsymbol{R}}_{t+1}$?

Updating Rotations

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t + \Delta t \dot{\boldsymbol{R}}_{t+1}$$
$$= \boldsymbol{R}_t + \Delta t \boldsymbol{\omega}_{t+1_\times} \boldsymbol{R}_t = (I + \Delta t \boldsymbol{\omega}_{t+1_\times})\, \boldsymbol{R}_t$$

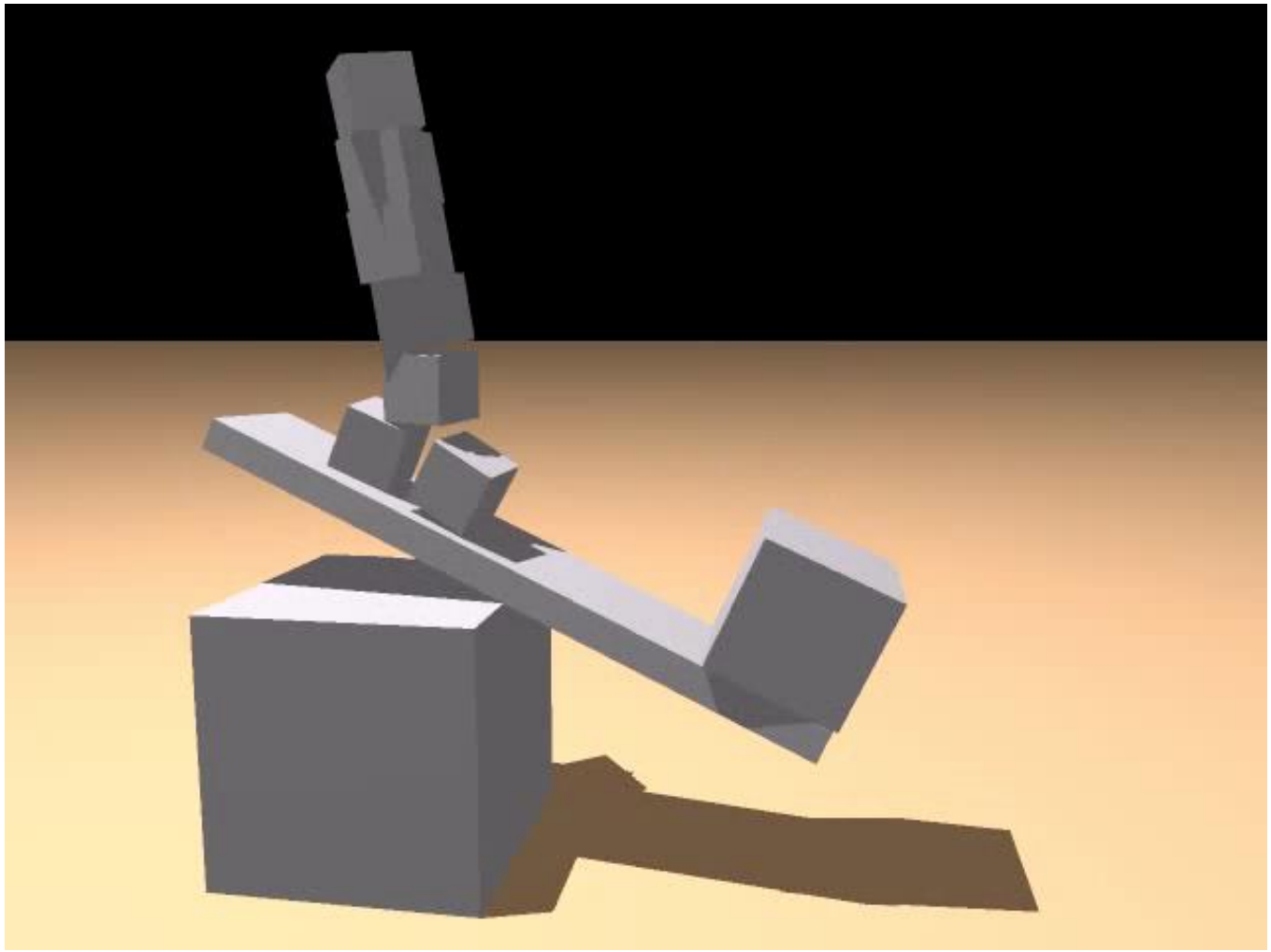
No longer a rotation matrix!

- Option 1: orthonormalize (Gram–Schmidt)
- Option 2: explicitly compute rotation $\boldsymbol{R}_{\Delta t}$ due to spinning with angular speed $\boldsymbol{\omega}_{t+1}$ for Δt seconds, apply incremental rotations: $\boldsymbol{R}_{t+1} = \boldsymbol{R}_{\Delta t} \boldsymbol{R}_t$
- NOTE: same concept applies if other rotation parameterizations (i.e. quaternions) are employed

Computing forces

- Given a set of forces, you know how to compute the motion of a rigid body
- Where do forces come from?
  - User interaction
  - Gravity

Rigid Bodies



Computing forces

- Given a set of forces, you know how to compute the motion of a rigid body
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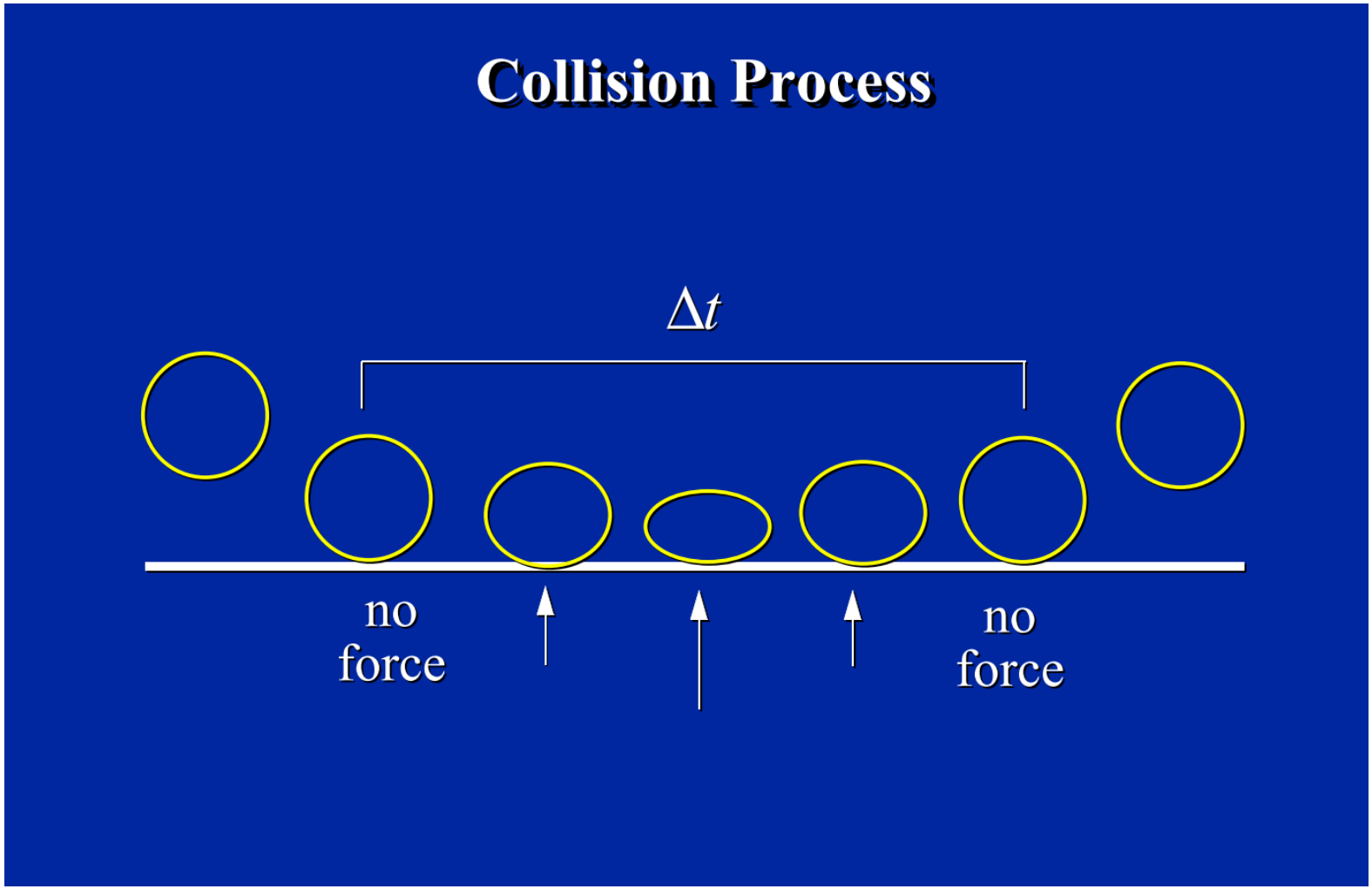
  - Gravity
  - Collisions and contacts

Collision Response



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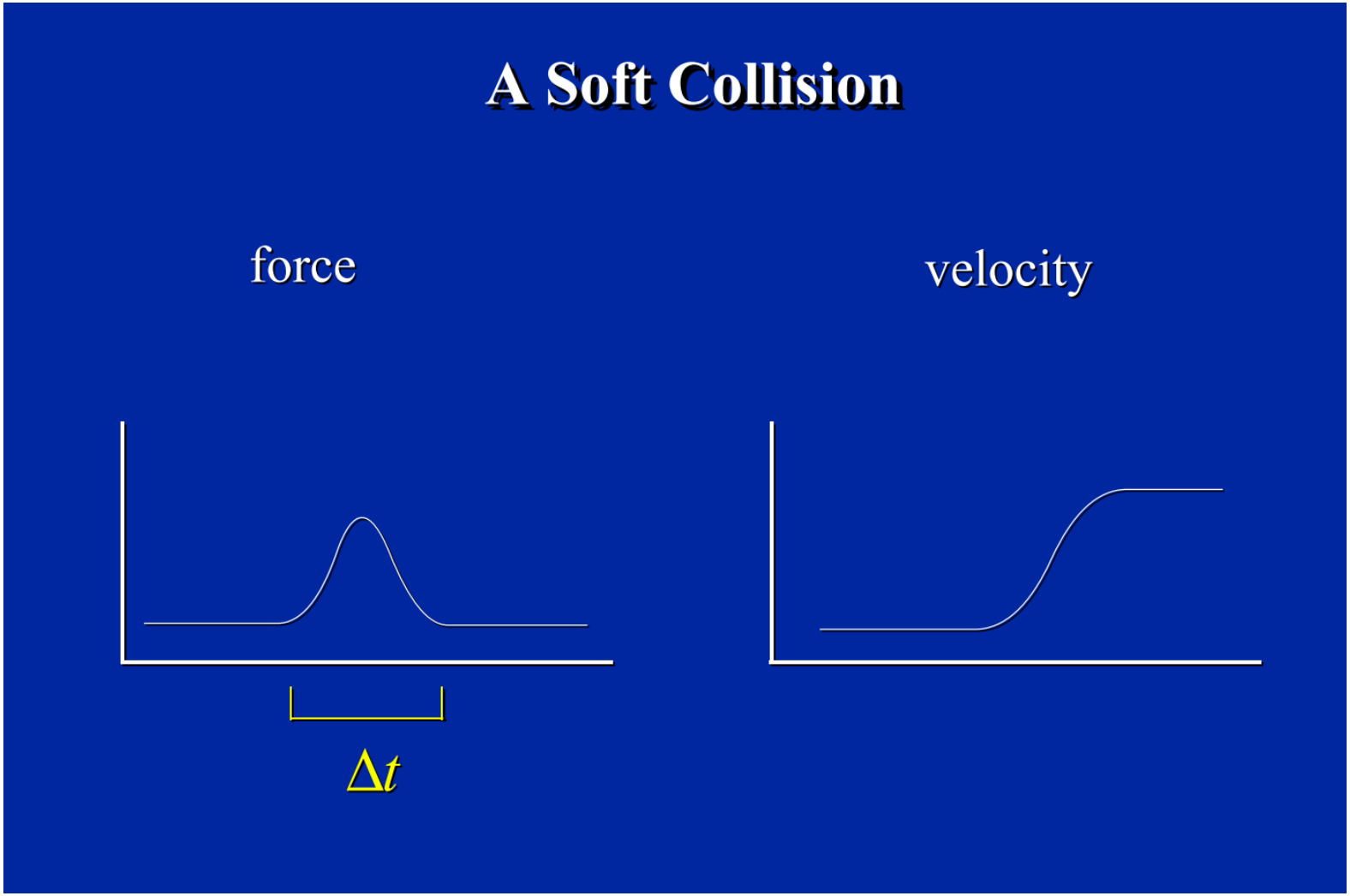

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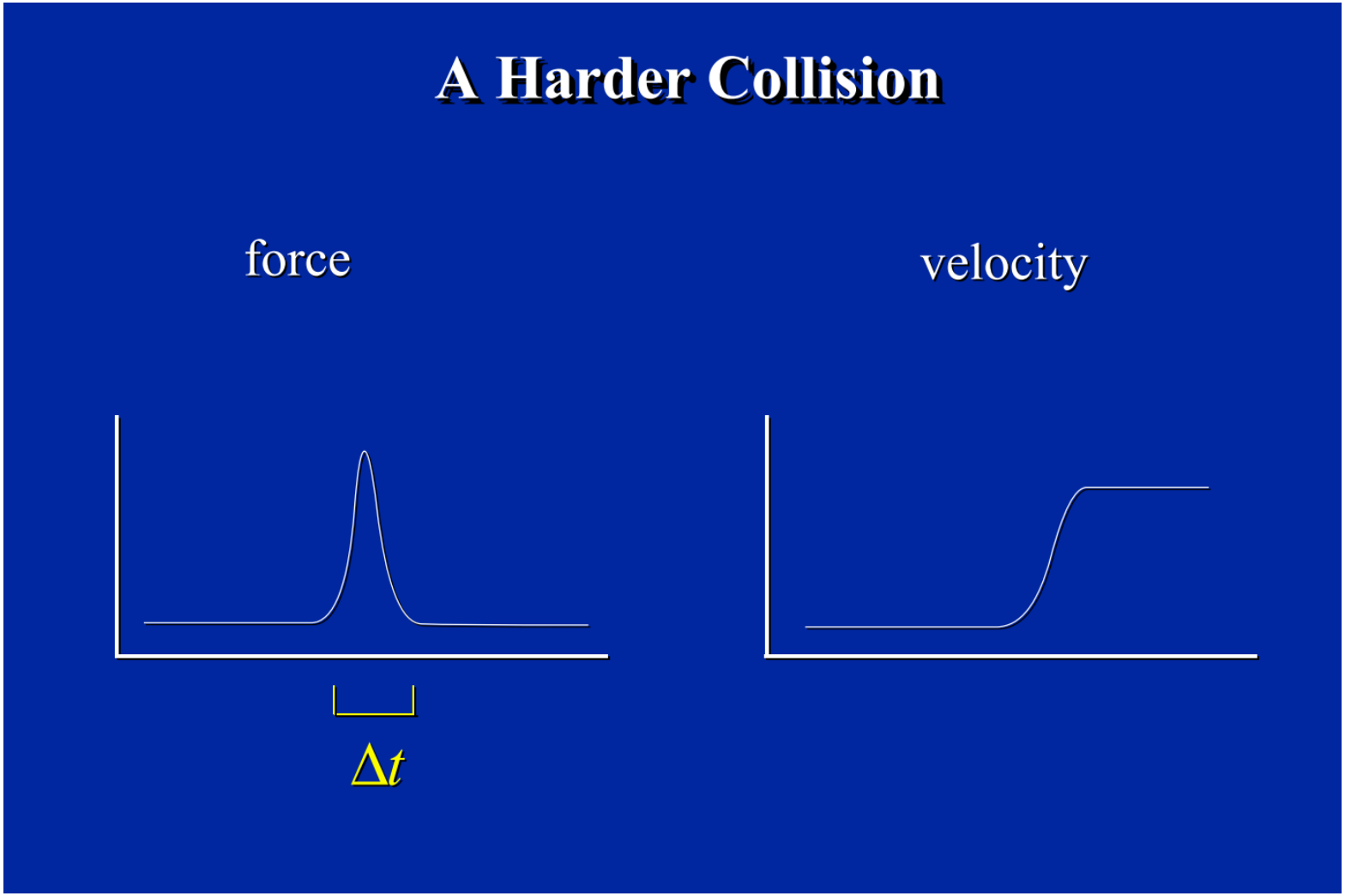

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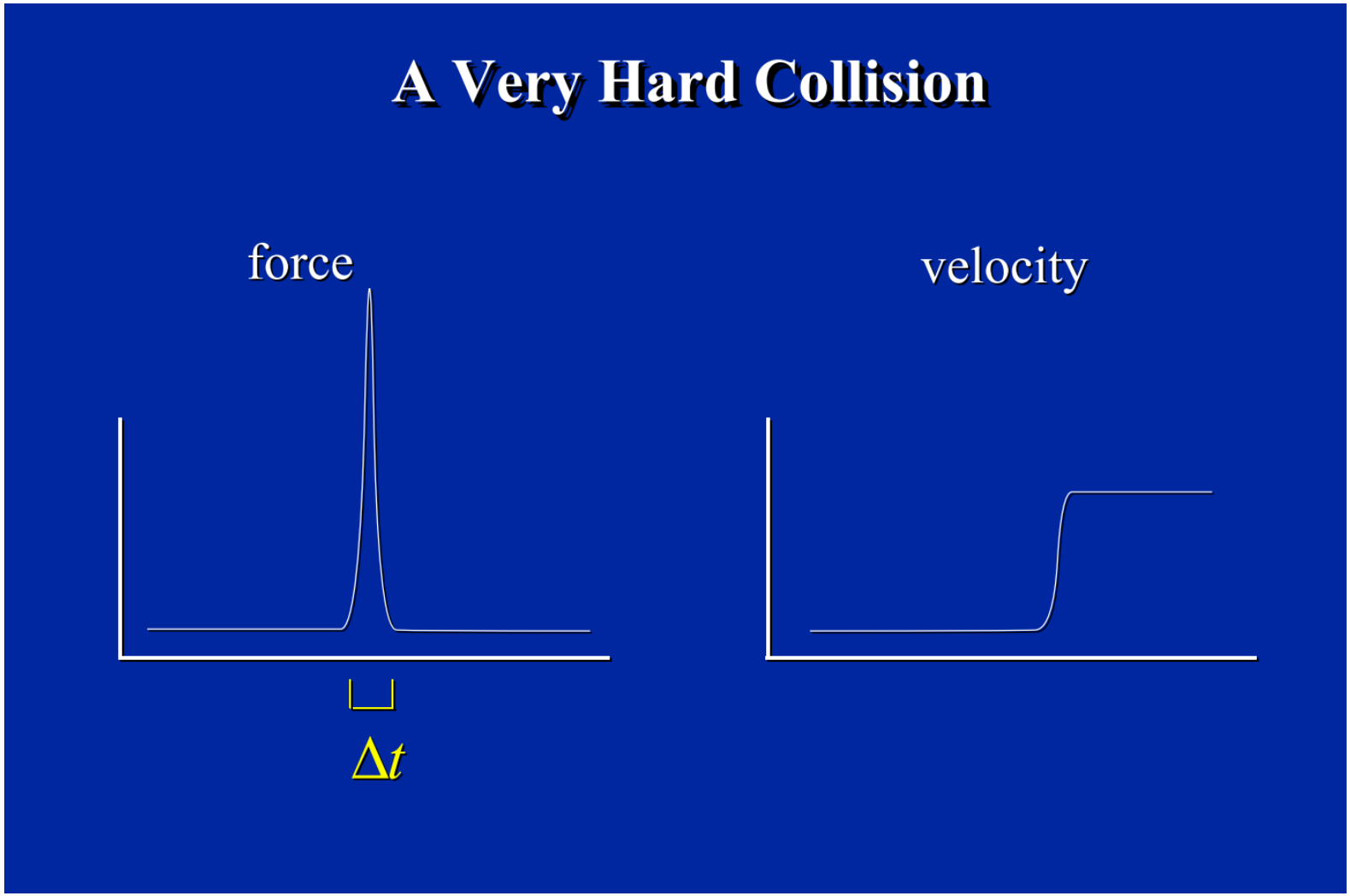

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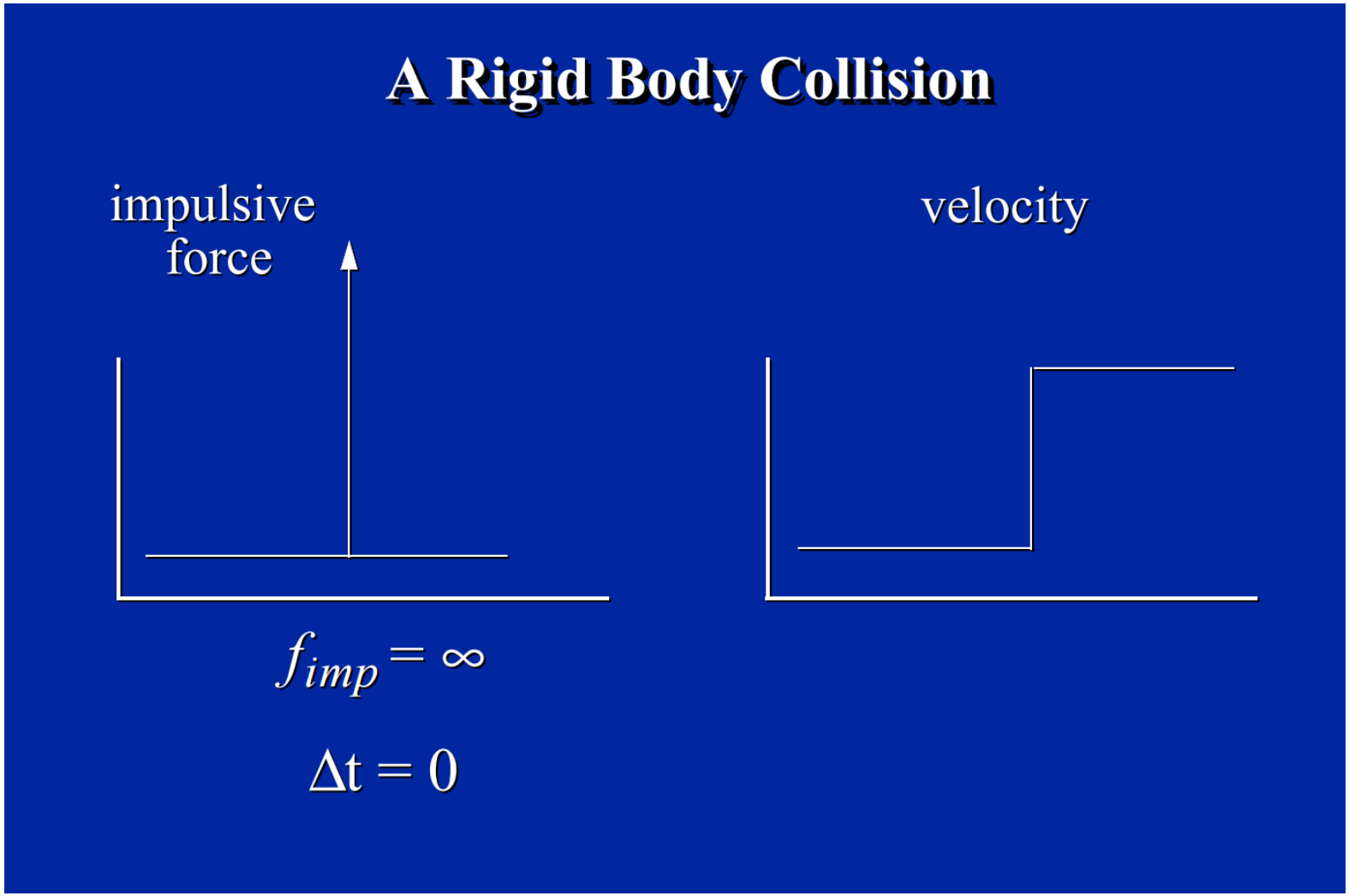

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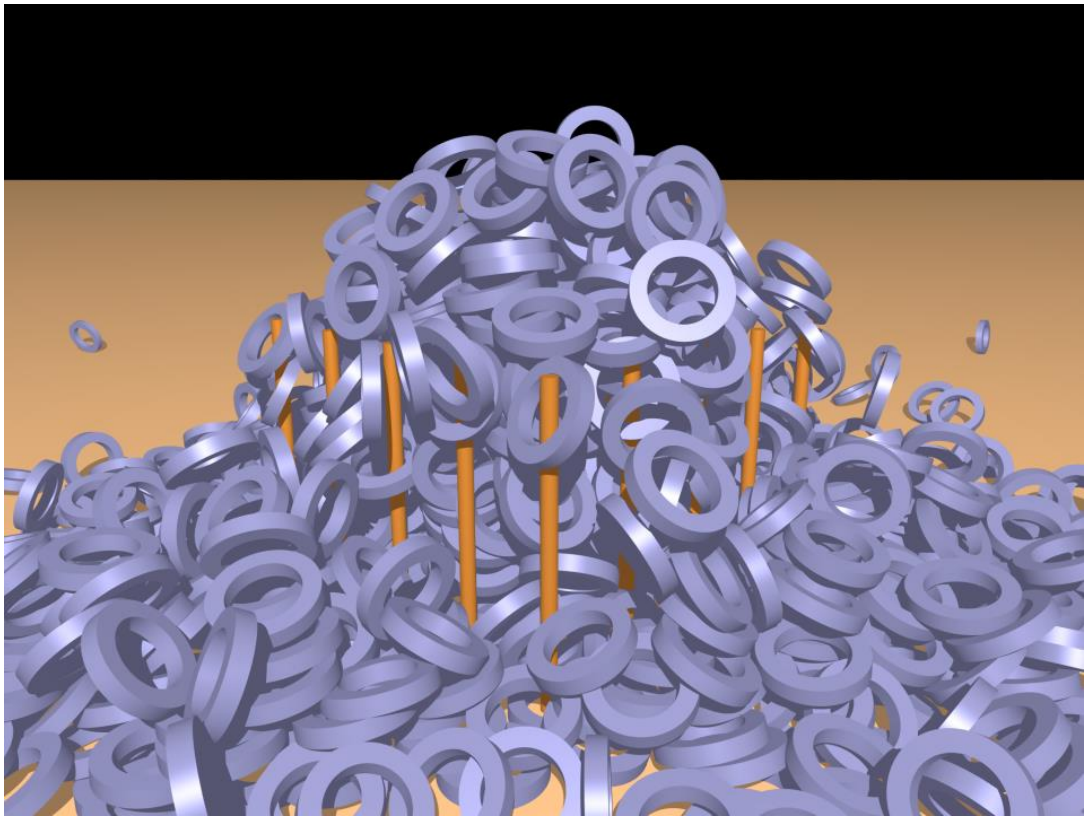


"Nonconvex Rigid Bodies with Stacking", Guendelman et al., SIGGRAPH 2003

Computing forces

- Given a set of forces, you know how to compute the motion of a rigid body
- Where do forces come from?
  - User interaction
  - Gravity
  - Collisions and contacts
    - Easiest way to model: “spring” penalty forces
  - Articulation

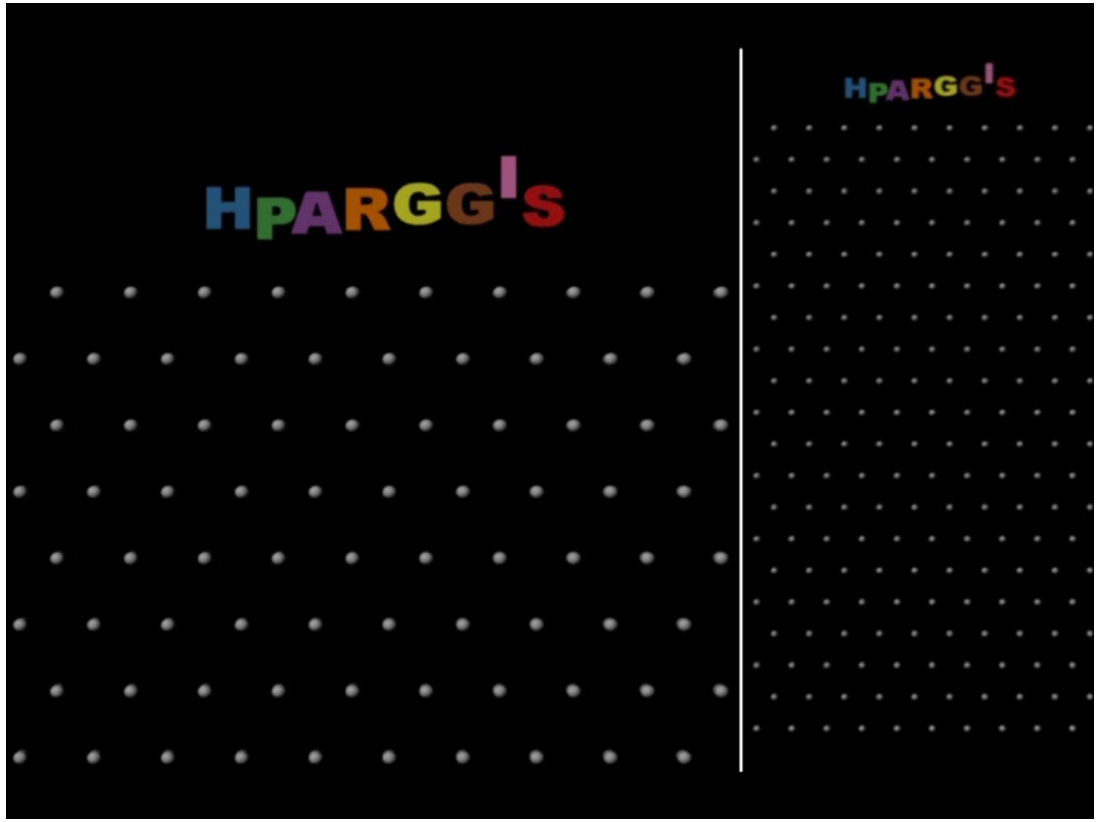
Articulated Rigid Body Dynamics

Computing forces

- Given a set of forces, you know how to compute the motion of a rigid body
- Where do forces come from?
  - User interaction
  - Gravity
  - Collisions and contacts
    - Easiest way to model: “spring” penalty forces
  - Articulation
    - Easiest way to model: “spring” penalty forces

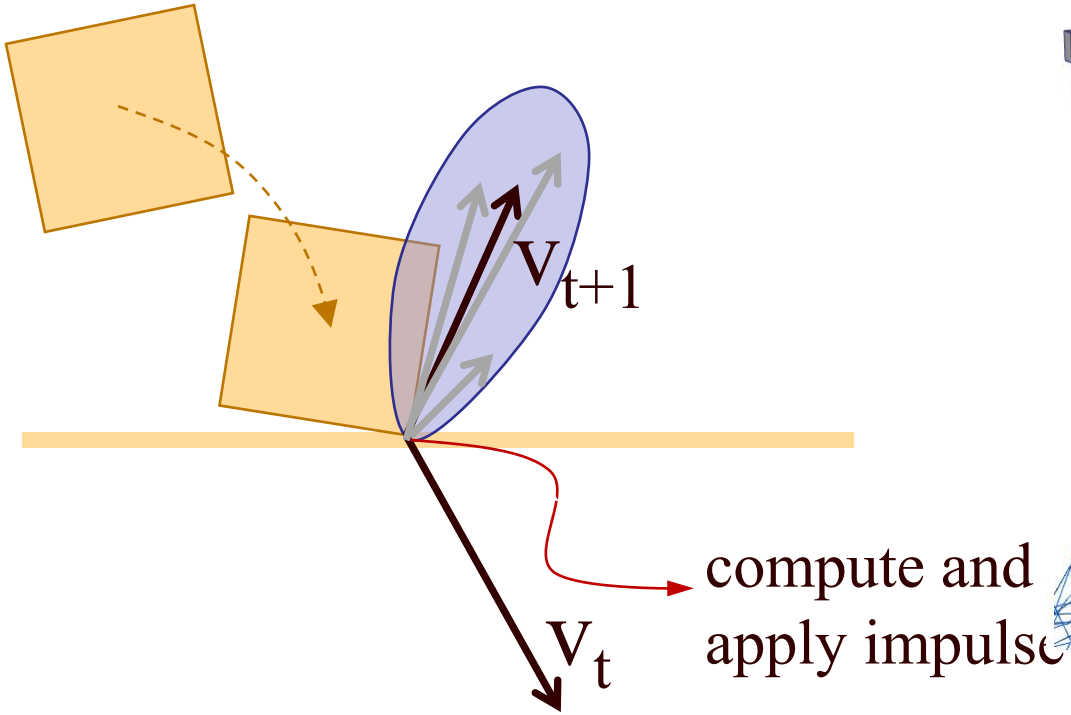
Artistic control over rigid body simulations

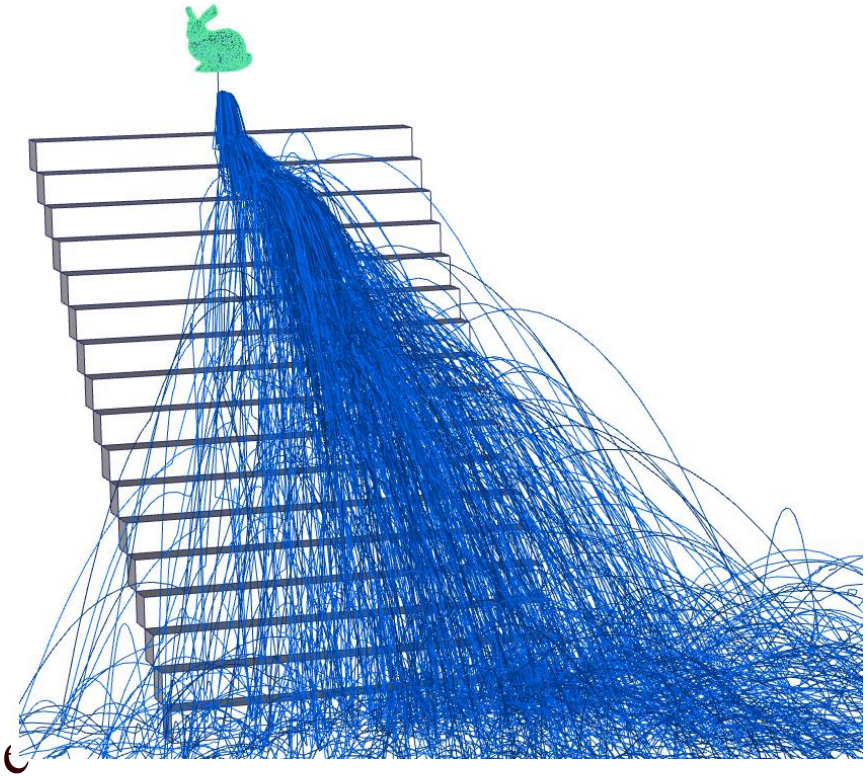
Many-Worlds Browsing for Control of Multibody Dynamics Twigg and James, 2007



Many Worlds Browsing…

Sampling Plausible Worlds

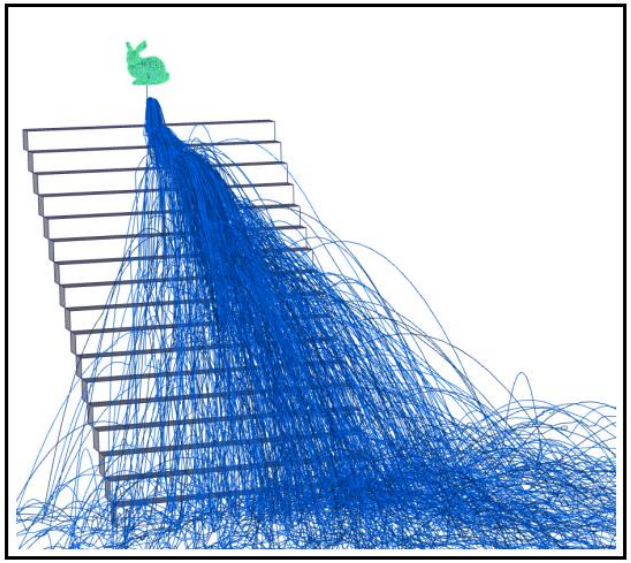




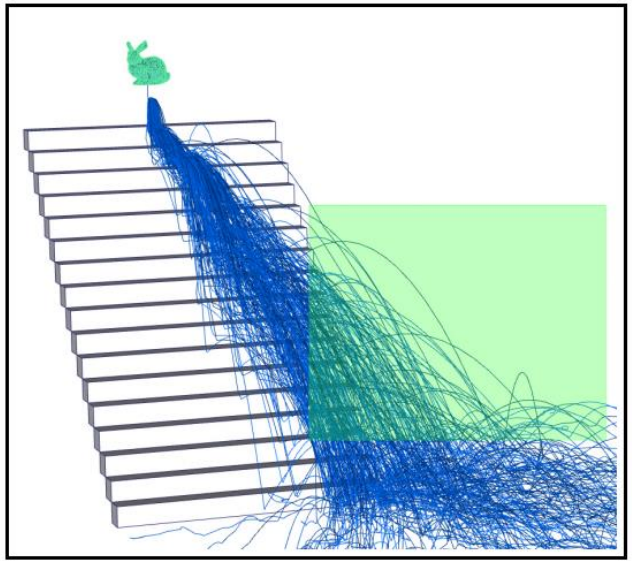
[O'Sullivan et al., 2003]

Many Worlds Browsing…

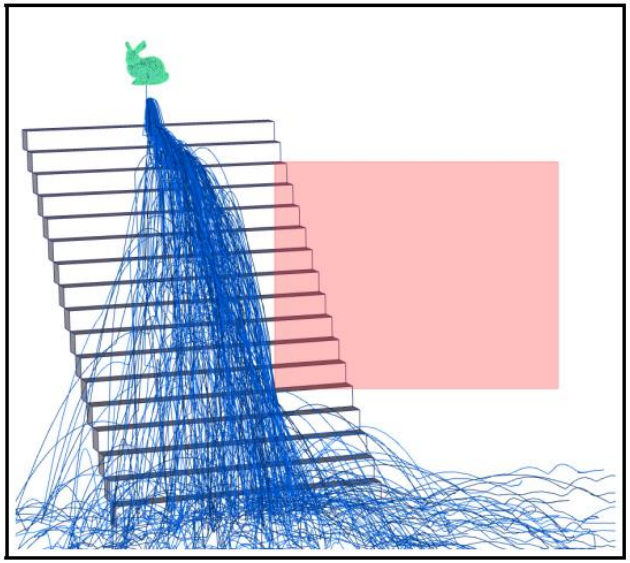
Interactive Browsing – various criteria



Input

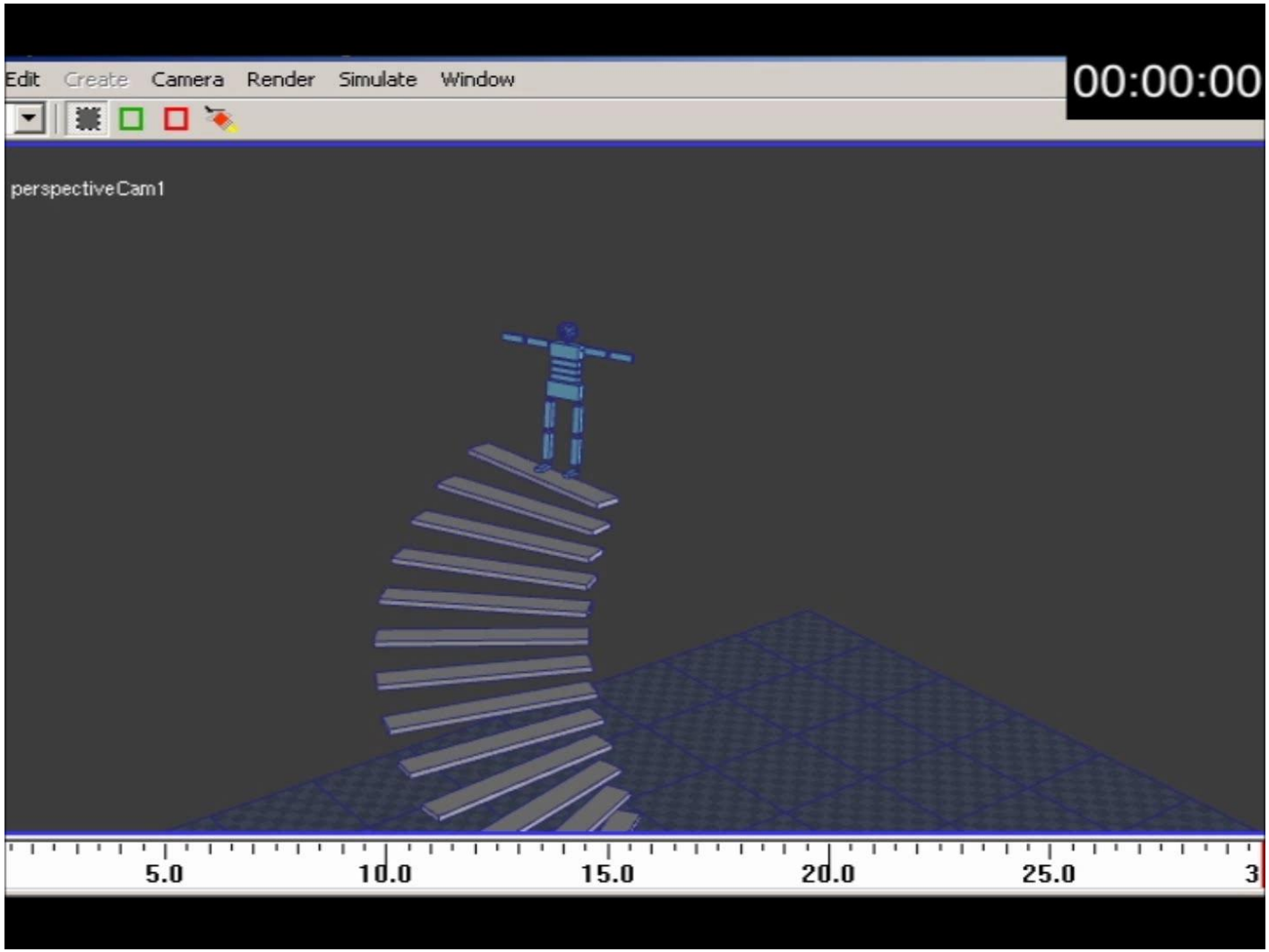


Positive



Negative

Many Worlds Browsing



For more information

An Introduction to Physically Based Modeling:
- http://www.cs.cmu.edu/~baraff/pbm/