Dynamics and Time Integration

Computer Graphics
CMU 15-462/15-662, Fall 2016

Last Quiz: IK (and demo)

Last few classes…

- Keyframing and motion capture
  - Q: what can one NOT do with these techniques?

The Animation Equation

- We previously saw the *rendering equation*
  - Rasterization and path tracing give approximate solutions to the rendering equation
- What's the *animation equation?*
  - Leverage tools from computational physics: dynamical descriptions, numerical integration, etc.

Dynamical Description of Motion

"A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."

—Sir Isaac Newton, 1687

The Animation Equation

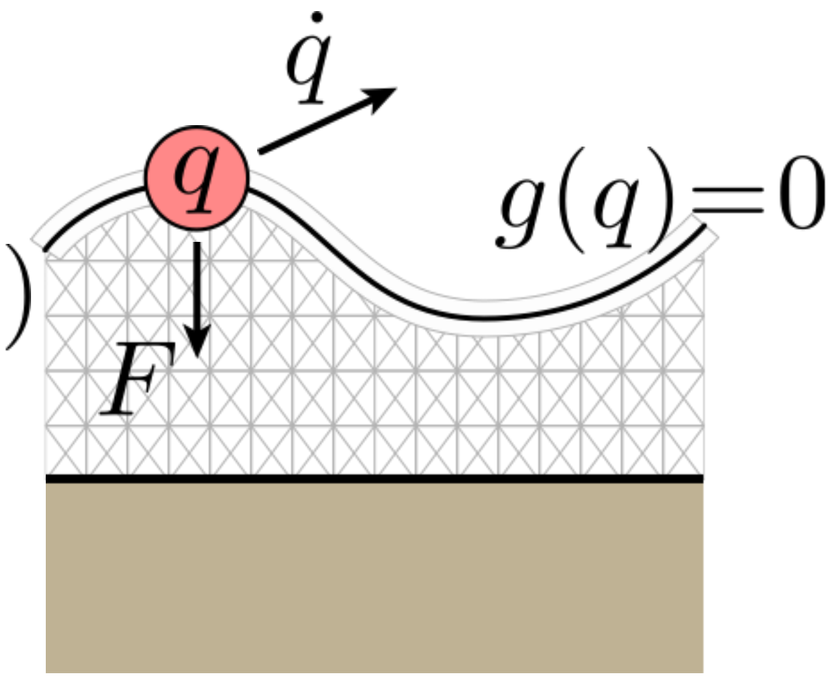
mass

acceleration

$$F = ma$$

force

The "Animation Equation"



Well there is more to be said…

- **Every system has a *configuration*** $q(t)$
- **It also has a *velocity*** $\dot{q} := \frac{d}{dt} q$
- **And some kind of *mass*** M
- **There are probably some *forces*** F
- **And also some *constraints*** $g(q, \dot{q}, t) = 0$

Can write Newton's 2nd law as $\ddot{q} = F/m$

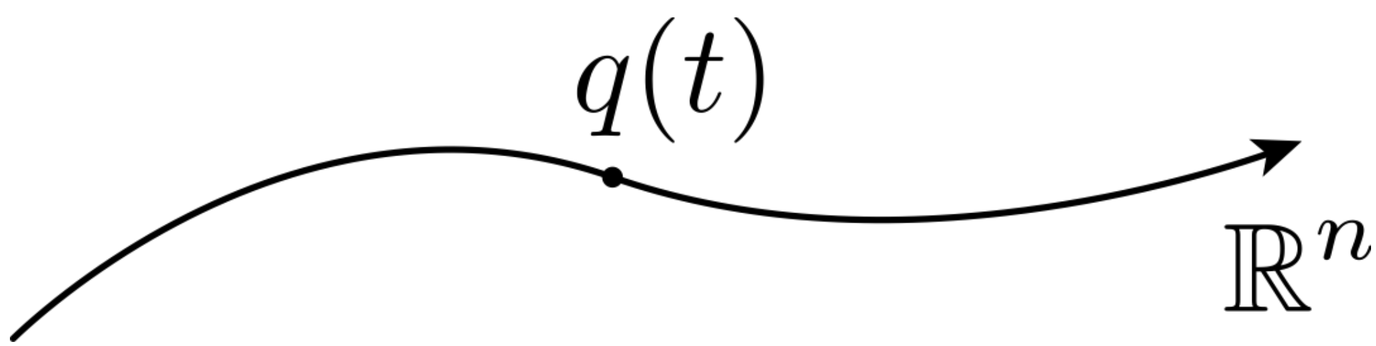
Makes two things clear:

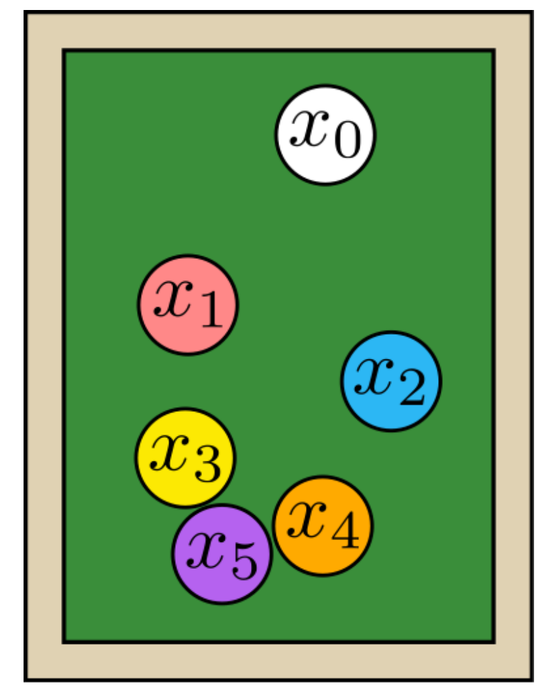
- **acceleration is 2nd time derivative of configuration**
- **ultimately, we want to solve for the configuration *q***

Generalized Coordinates

- Often describing systems with many, many moving pieces
- E.g., a collection of billiard balls, each with position x_i
- Collect them all into a single vector of *generalized coordinates:*

$$q = (x_0, x_1, \ldots, x_n)$$



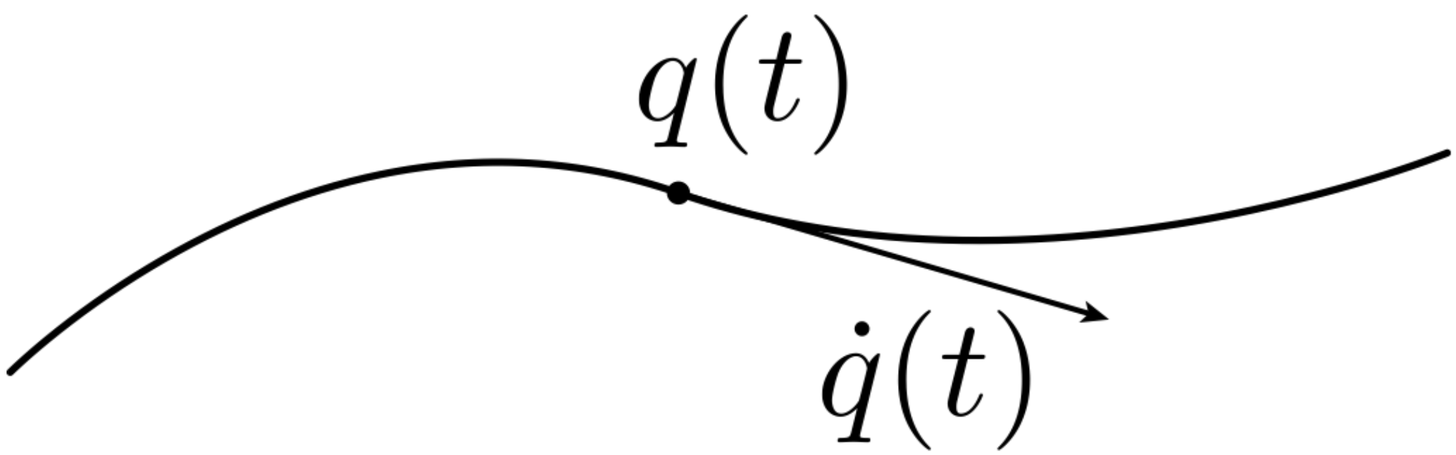


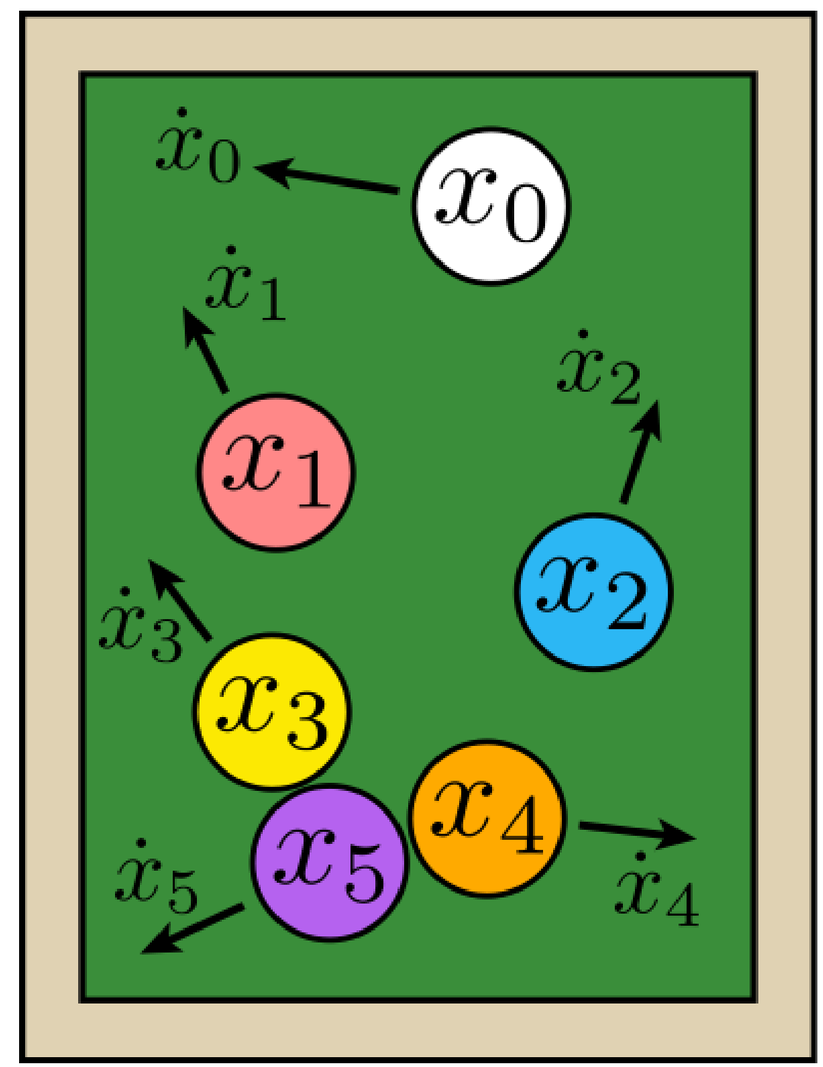
- Can think of q as a *single point* moving along a trajectory in R^n
- This way of thinking naturally maps to the way we actually solve equations on a computer: all variables are often "stacked" into a big vector and handed to a solver.

Generalized Velocity

- **Just the time derivative of the generalized coordinates!**

$$\dot{q} = (\dot{x}_0, \dot{x}_1, \ldots, \dot{x}_n)$$



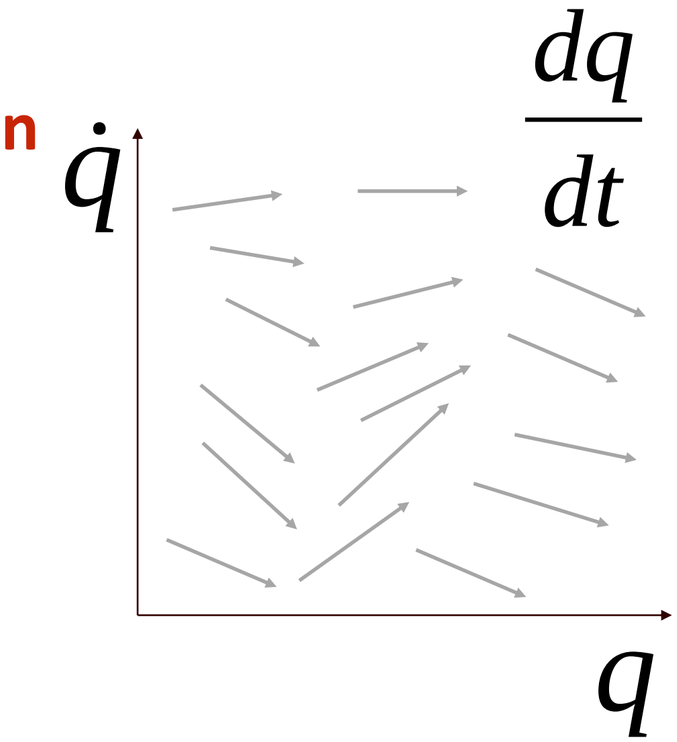


Ordinary Differential Equations

- **Many dynamical systems can be described via an *ordinary differential equation (ODE)* in generalized coordinates:**

change in configuration over time **velocity function**

$$\frac{d}{dt}q = f(q, \dot{q}, t)$$



- **Example:**

$$\frac{d}{dt}u(t) = au$$

"rate of growth is proportional to value"

- **Solution?** $u(t) = be^{at}$
- **Describes exponential decay (a < 1), or exponential growth (a > 1)**
- **"Ordinary" means "involves derivatives in time but not space"**
- **We'll talk about spatial derivatives (PDEs) in another lecture...**

Dynamics via ODEs

- **Another key example: Newton's 2nd law!**

$$\ddot{q} = F/m$$

- **"Second order" ODE since we take *two* time derivatives**
- **Can also write as a *system* of two *first order* ODEs, by introducing new "dummy" variable for velocity:**

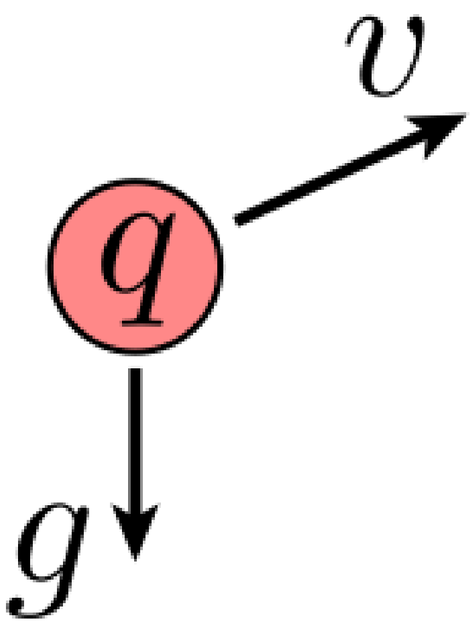
$$\dot{q} = v$$
$$\dot{v} = F/m$$

$$\frac{d\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{v} \end{bmatrix}}{\boldsymbol{dt}} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{F}/m \end{bmatrix}$$

- **Splitting things up this way will make it easy to talk about solving these equations numerically (among other things)**

Simple Example: Throwing a Rock

- **Consider a rock* of mass *m* tossed under force of gravity *g***
- **Easy to write dynamical equations, since only force is gravity (constant acceleration):**



$$F = mg \qquad \text{or} \qquad \dot{q} = v$$

$$\ddot{q} = g \qquad\qquad \dot{v} = g$$

Solution:

$$v(t) = v_0 + \frac{t}{m}F$$

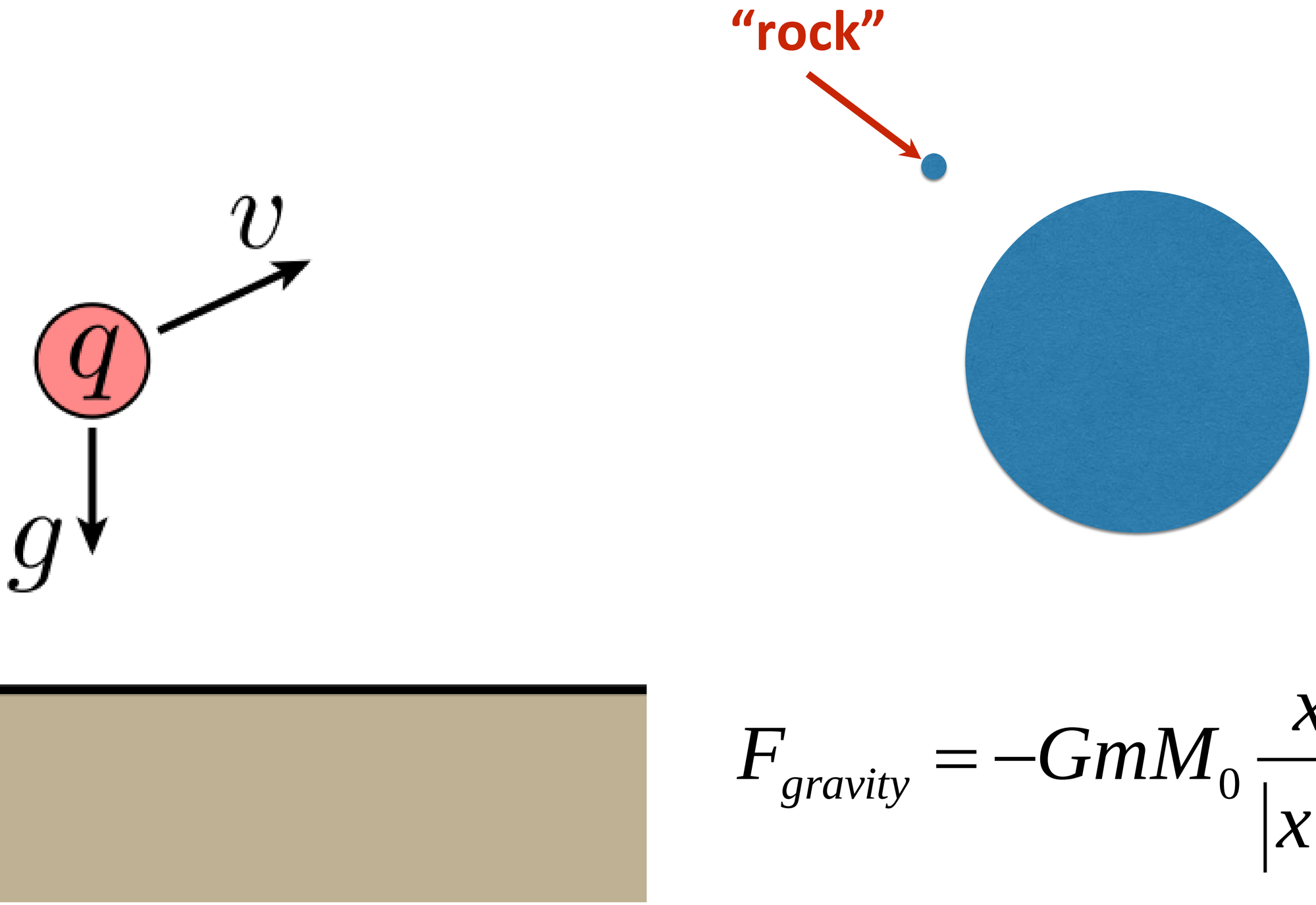
$$q(t) = q_0 + tv_0 + \frac{t^2}{2m}F$$

(What do we need a computer for?!)

***Yes, this rock is spherical and has uniform density.**
Note: this is an initial value problem!

Simple Example: the two-body problem

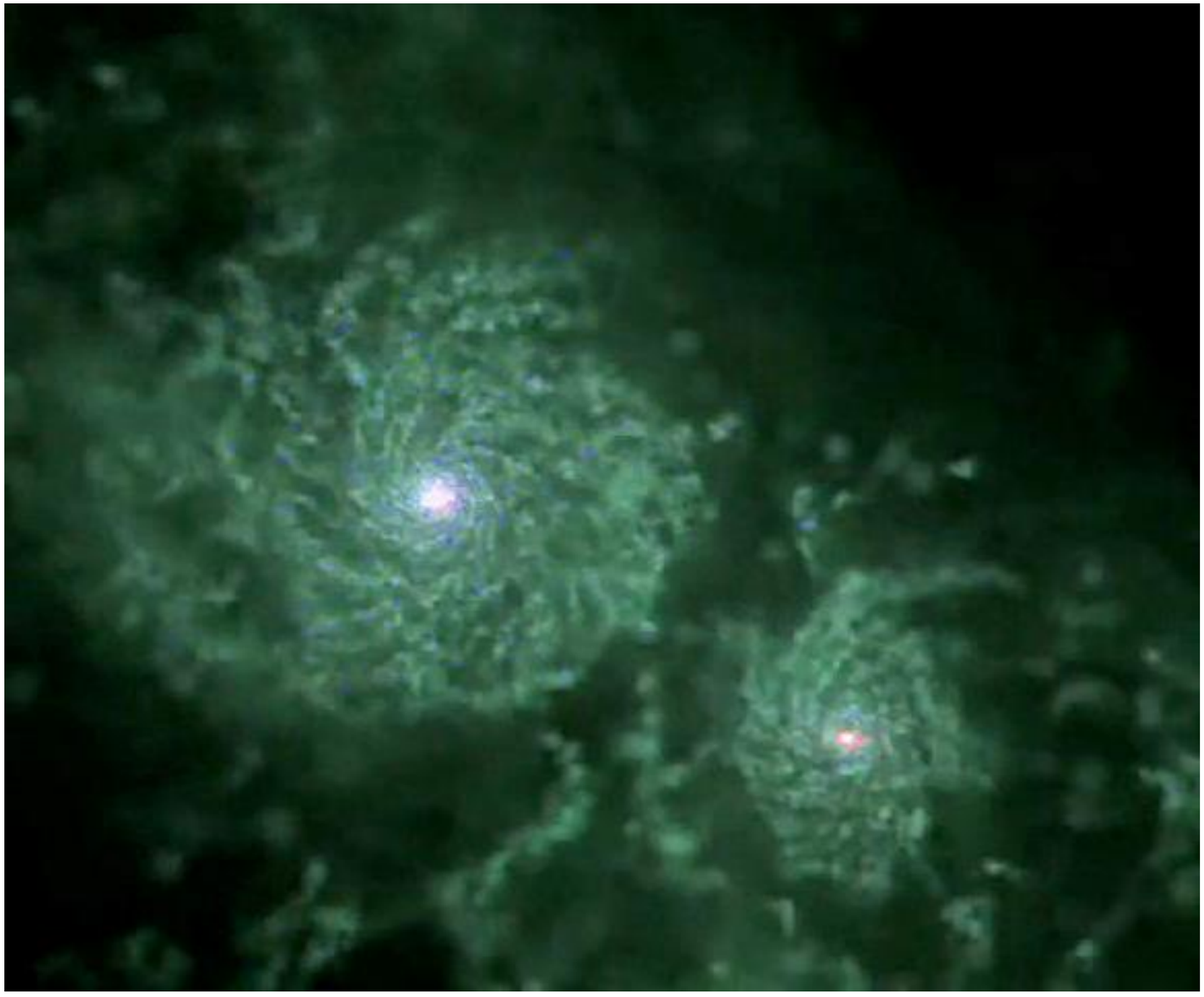
- **Let's take a closer look…**



$$F_{gravity} = -GmM_0 \frac{x - x_0}{|x - x_0|^3}$$

Not-So-Simple Example: *n*-Body Problem

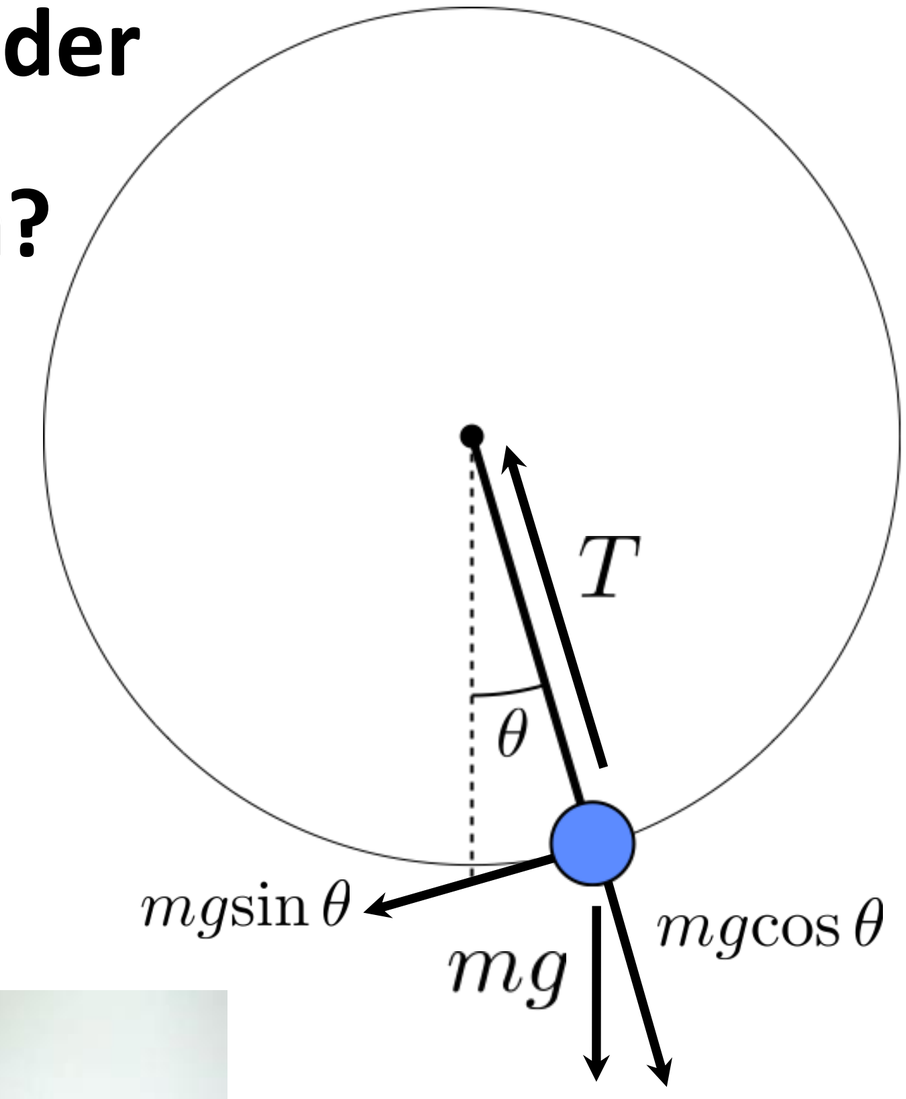
- Consider the Earth, moon, and sun—where do they go?
- As soon as n ≥ 3, no closed form (chaotic solutions)
- What if we want to simulate entire *galaxies?*



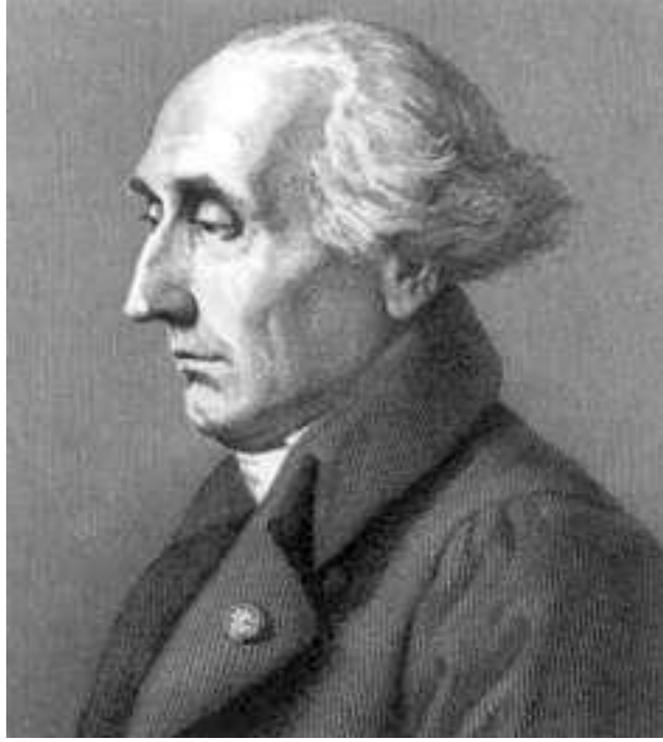
Credit: Governato et al / NASA

Slightly Harder Simple Example: Pendulum

- Mass on end of a bar, swinging under gravity
- What are the equations of motion?
- Same as "rock" problem, but *constrained*
- Could use a *"force diagram"*
  - You probably did this for many hours in high school/college
  - Let's do something different!



Lagrangian Mechanics



Joe Lagrange

- **Beautifully simple and general recipe:**
  - **Write down kinetic energy** K
  - **Write down potential energy** U
  - **Write down *Lagrangian*** $\mathcal{L} := K - U$
  - **Dynamics then given by *Euler-Lagrange equation***

becomes (generalized) "MASS TIMES ACCELERATION"

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

becomes (generalized) "FORCE"

- **Why is this useful?**
  - often easier to come up with (scalar) energies than forces
  - very general, works in any kind of generalized coordinates

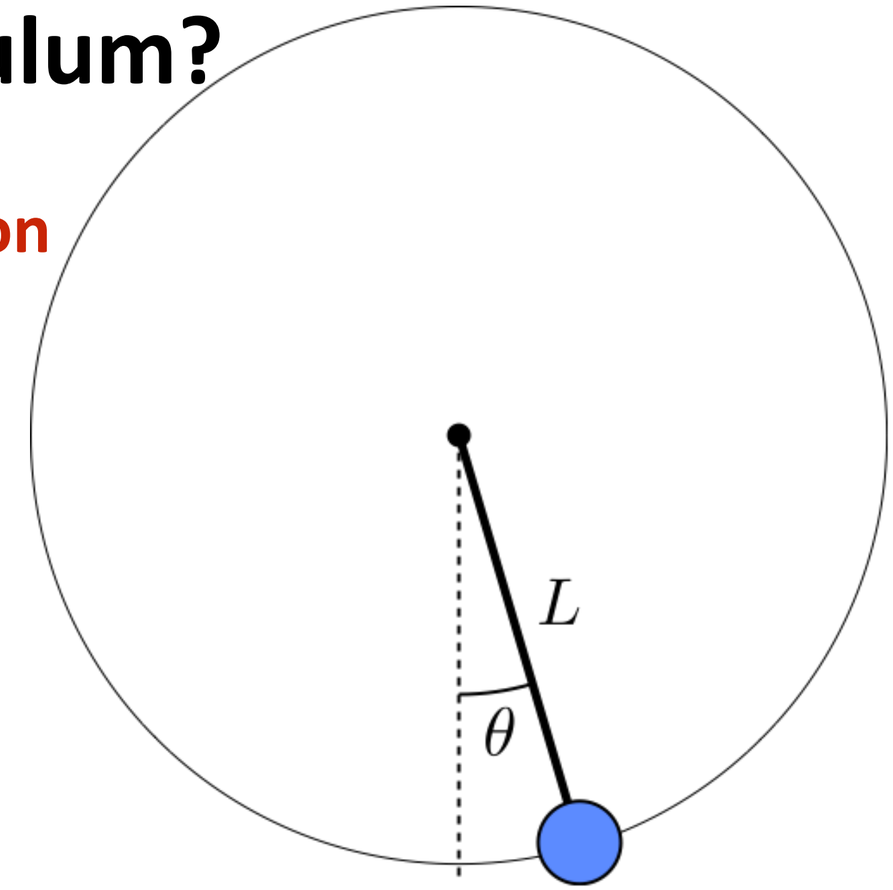
Great reference: Sussman & Wisdom, "Structure and Interpretation of Classical Mechanics"

Lagrangian Mechanics - Example

- **Generalized coordinates for pendulum?**

$$q = \theta$$

just one coordinate:
angle with the vertical direction



- **Kinetic energy (mass *m*)?**

$$K = \frac{1}{2} I \omega^2 = \frac{1}{2} m L^2 \dot{\theta}^2$$

- **Potential energy?**

$$U = mgh = -mgL\cos\theta$$

- **Euler-Lagrange equations?**

(from here, just "plug and chug"—even a computer could do it!)

$$\mathcal{L} = K - U = m\left(\frac{1}{2} L^2 \dot{\theta}^2 + gL\cos\theta\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} \qquad \frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{L}}{\partial \theta} = -mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \quad \Rightarrow \quad \boxed{\ddot{\theta} = -\frac{g}{L}\sin\theta}$$

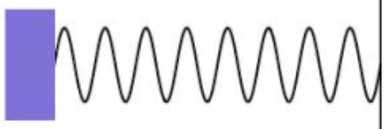
Solving the Pendulum

- **Great, now we have a nice simple equation for the pendulum:**

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

- **For small angles (e.g., clock pendulum) can approximate as**

$$\ddot{\theta} = -\frac{g}{L}\theta \qquad \Rightarrow \theta(t) = a\cos(t\sqrt{g/L} + b)$$

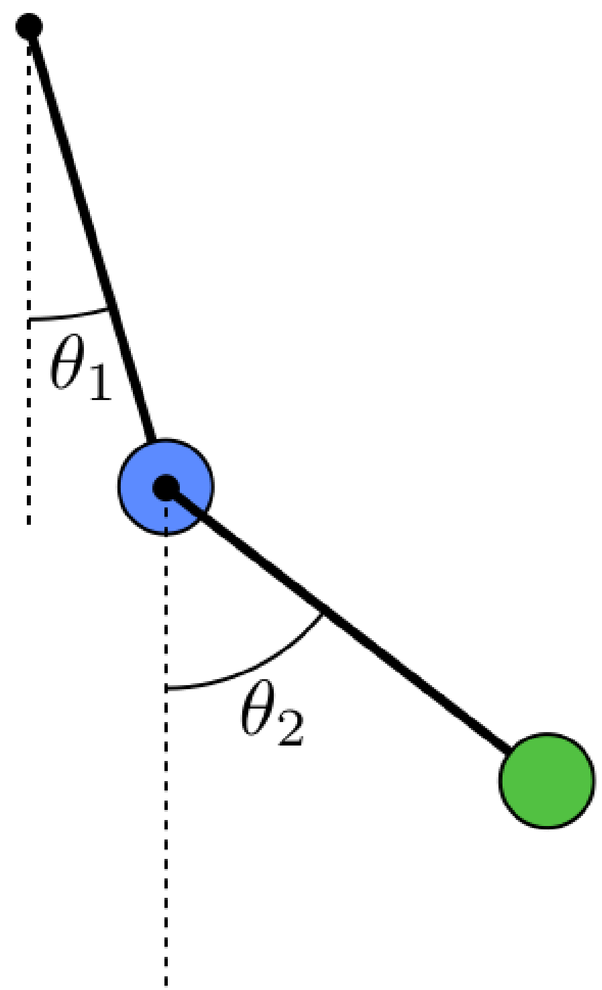


"harmonic oscillator"

- **In general, there is *no closed form solution!***
- **Hence, we *must* use a numerical approximation**
- **...And this was one of the simpler systems we can think of!**

Not-So-Simple Example: Double Pendulum

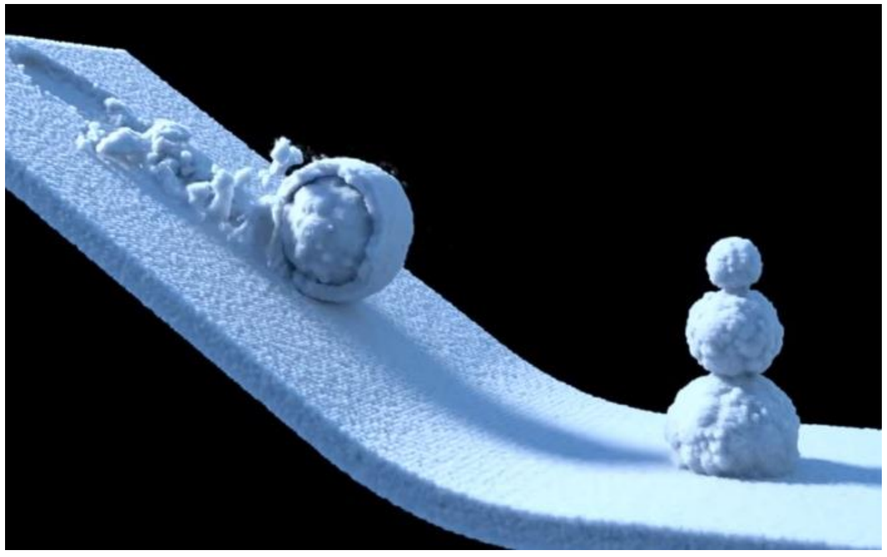
- Blue ball swings from fixed point; green ball swings from blue one
- Simple system... not-so-simple motion!
- Chaotic: small changes to input, wild changes to output
- Must use numerical approximation

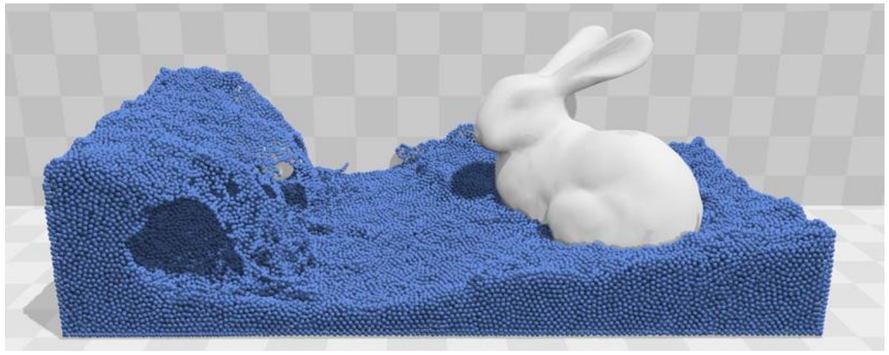


For animation, we *want* to simulate phenomena that are even more complex!

Particle Systems

- Model complex phenomena as large collection of particles
- Each particle has a behavior described by (physical or *non-*physical) forces
- Extremely common in graphics/games
  - easy to understand
  - simple equation for each particle
  - easy to scale up/down
- May need *many* particles to capture certain phenomena (e.g., fluids)
  - may require fast hierarchical data structure (kd-tree, BVH, ...)
  - sometimes better to use continuum model!

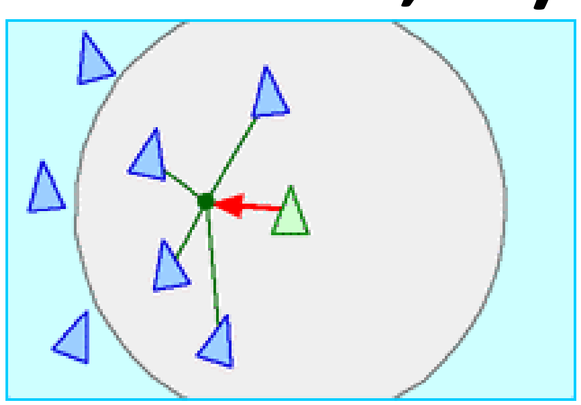




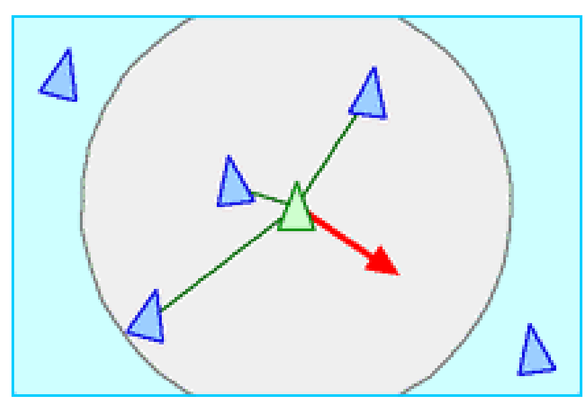
Example: Flocking

Simulated Flocking as an ODE

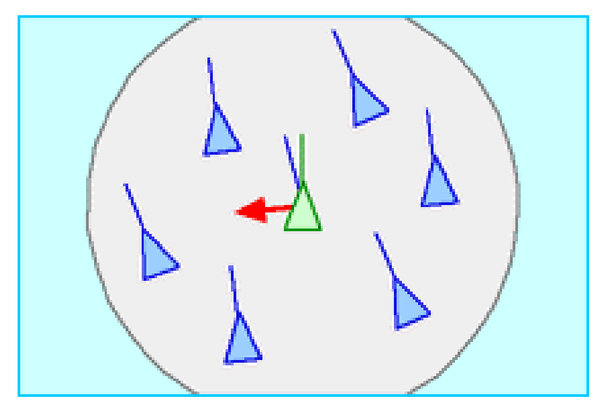
- Each bird is a particle
- Subject to very simple forces:
  - *attraction* to center of neighbors
  - *repulsion* from individual neighbors
  - *alignment* toward average trajectory of neighbors
- Solve large system of ODEs (numerically!)
- Emergent complex behavior (also seen in fish, bees, ...)



attraction



repulsion



alignment

Credit: Craig Reynolds (see http://www.red3d.com/cwr/boids/)

Example: Crowds

Where are the bottlenecks in a building plan?

Example: Granular Materials

Bell et al, "Particle-Based Simulation of Granular Materials"

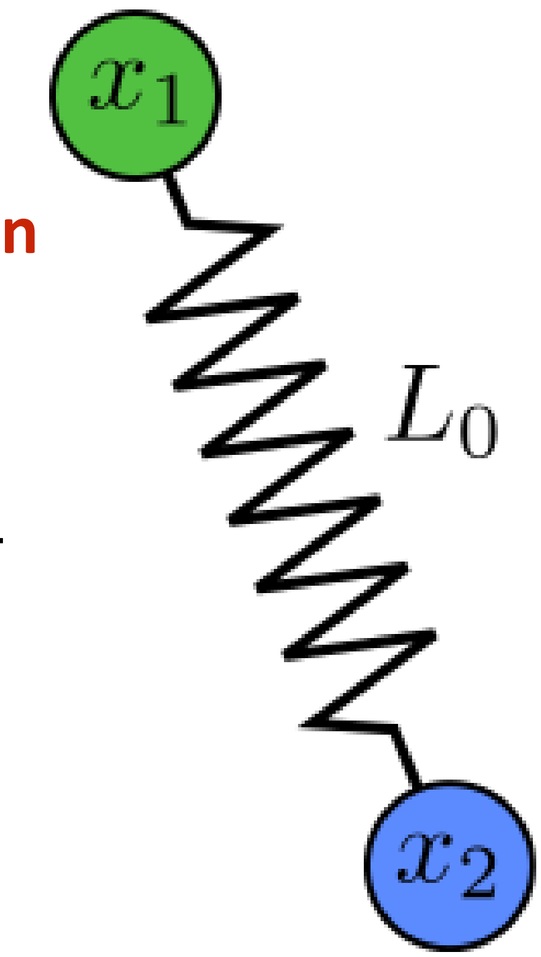
Example: Particle-Based Fluids

Movie *Battleship*

(Fluid: particles or continuum?)

Example: Mass-Spring System

- Connect particles x_1, x_2 by a spring of length L_0
- Spring force is given by Hooke's law:
  (*Ut tensio, sic vis.*)



current length

direction

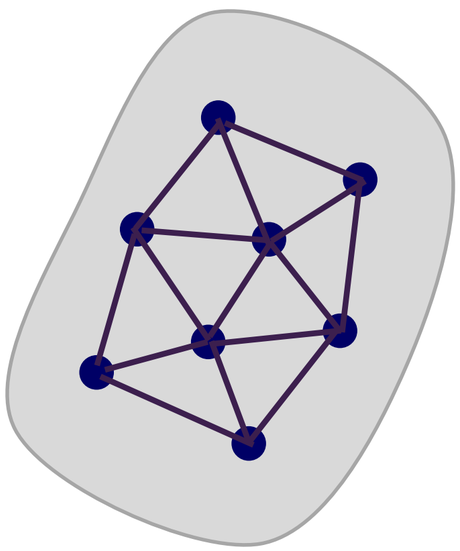
stiffness

$$F_{spring} = -K\left(\frac{|x_1 - x_2|}{L_0} - 1\right)\frac{x_1 - x_2}{|x_1 - x_2|}$$

rest length

- Extremely common in graphics/games
  - easy to understand
  - simple force equation
  - Easy to combine many springs + particles to model complex phenomena!

Example: Mass-Spring System



Spatial discretization: sample object with mass points

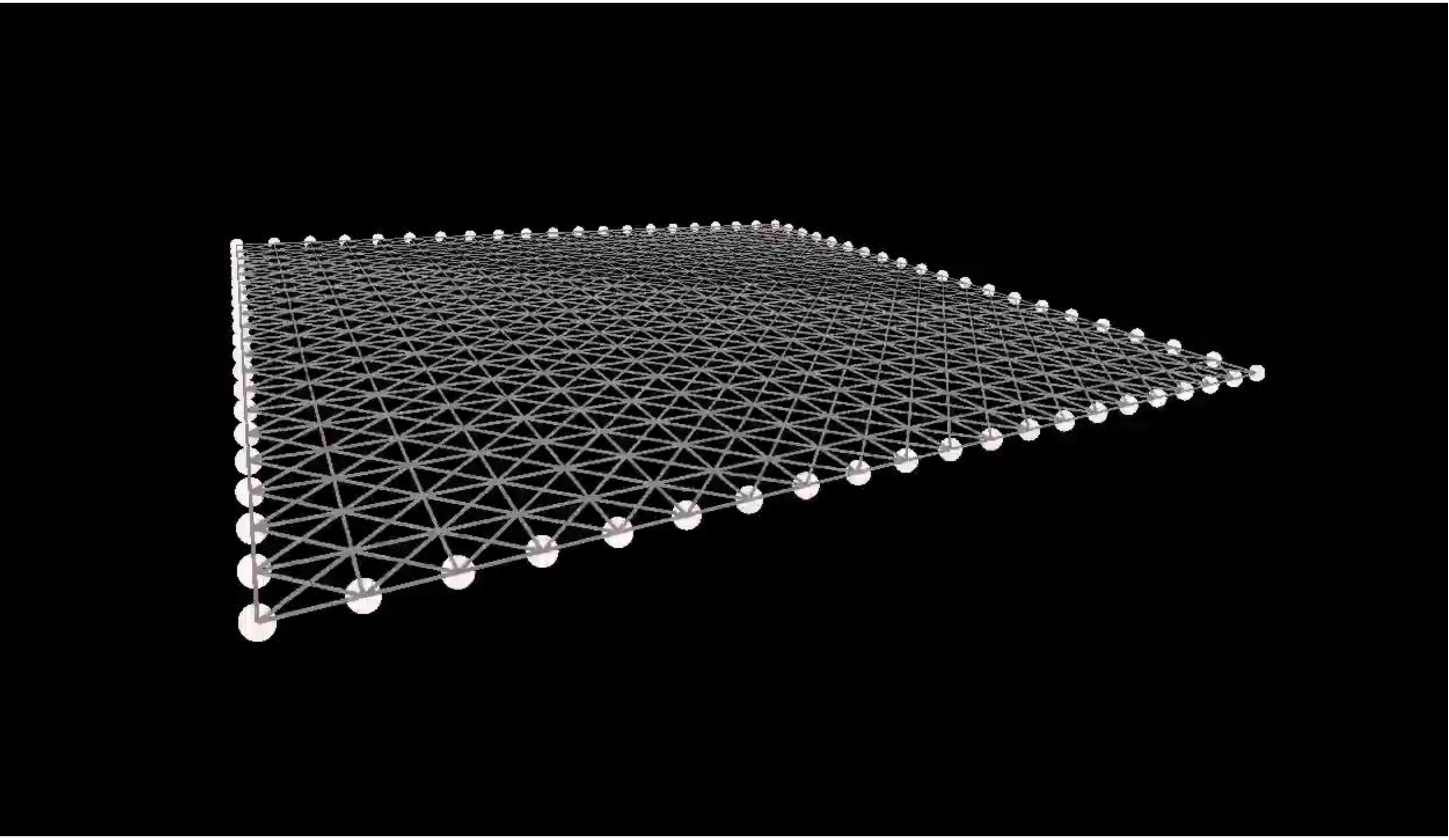
- Total mass of object: M
- Number of mass points: p
- Mass of each point: $m=M/p$ (*uniform distribution*)

Each point is a particle, just like before. It has

- Mass
- Position
- Velocity

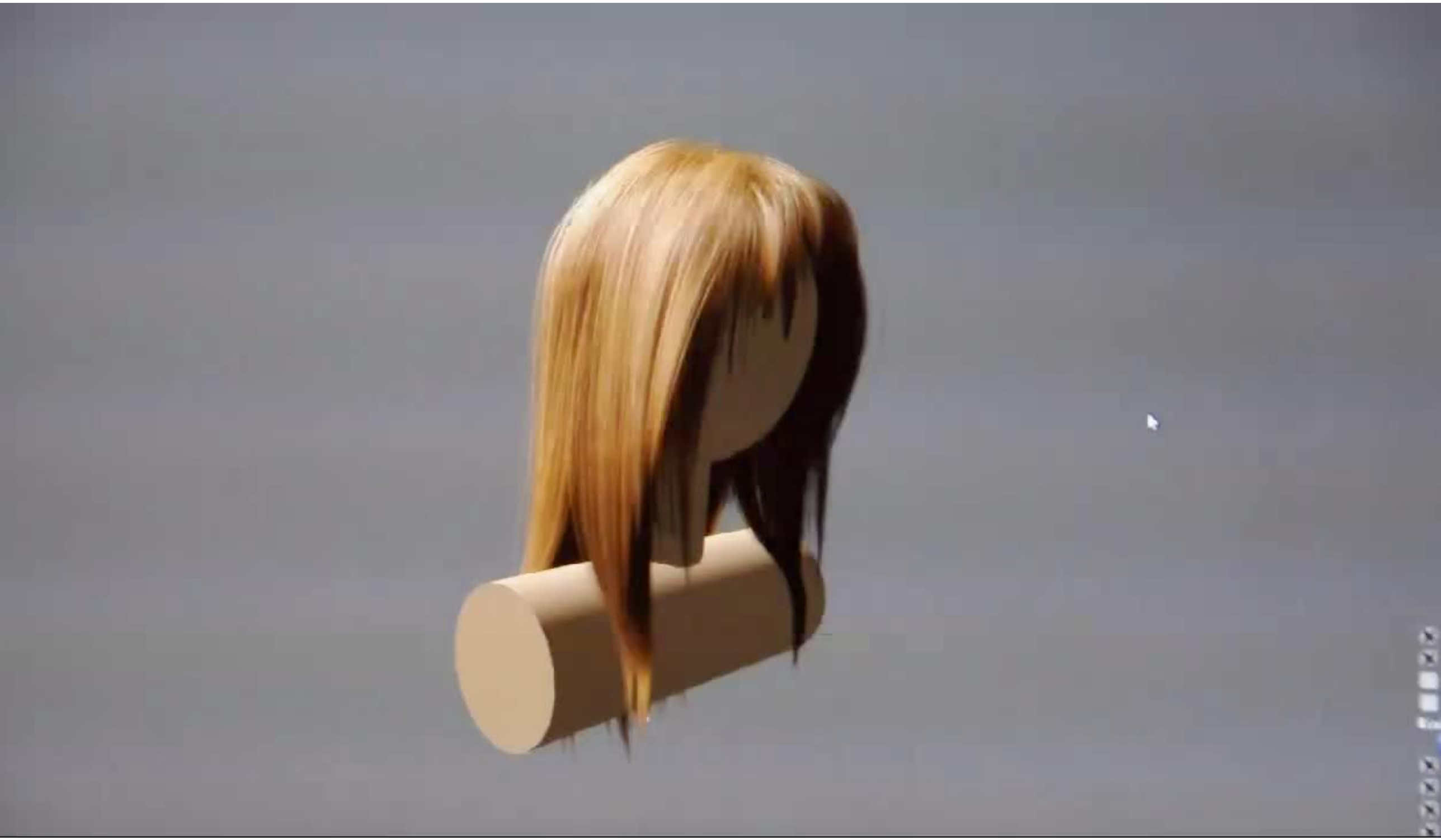
Connect particles with springs, evaluate force due to each spring, add gravity, etc and integrate.

Example: Mass Spring System



Example: Mass Spring + Character

Example: Hair

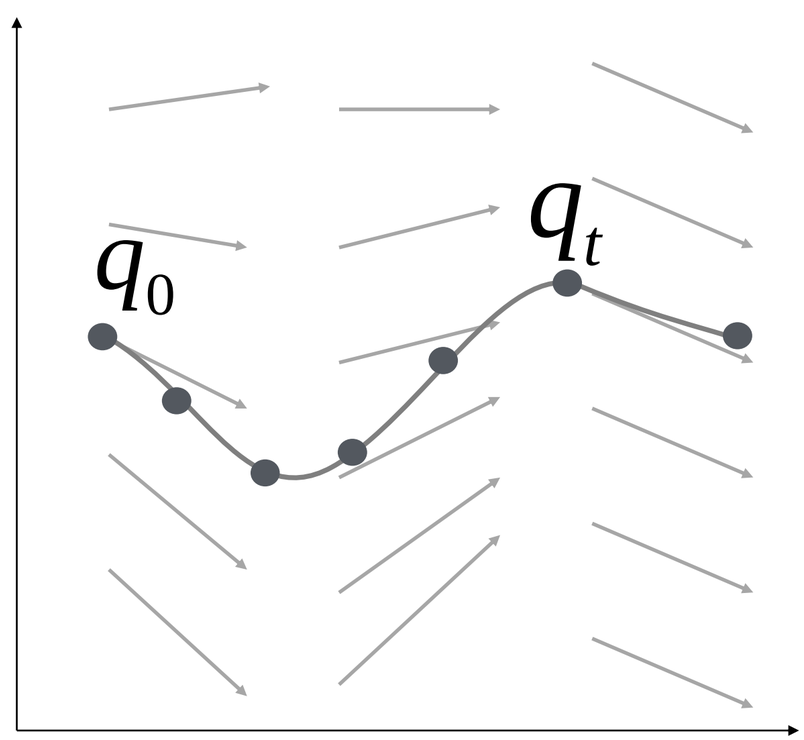


Ok, I'm convinced.
So how do we solve these things numerically?

Numerical Integration

- **Want to obtain *function* $q(t)$, given $q(0)$ and $\dot{q}(t)$**

- **Replace time-continuous function $q(t)$ with samples q_t in time**



Numerical Integration

- **How do you compute time-discretized samples?**
  - **Key idea: replace *derivatives* with *differences***

$$\frac{d}{dt}q(t) = f(q(t))$$

$$\Downarrow$$

$$\frac{q_{k+1} - q_k}{\tau} = f(q)$$

new configuration (unknown—want to solve for this!)

current configuration (known)

"time step," i.e., interval of time between q_k and q_{k+1}

Wait... where do we evaluate the velocity function? At the new or old configuration?

Forward Euler

- Simplest scheme: evaluate velocity at current configuration
- New configuration can then be written *explicitly* in terms of known data:

new configuration

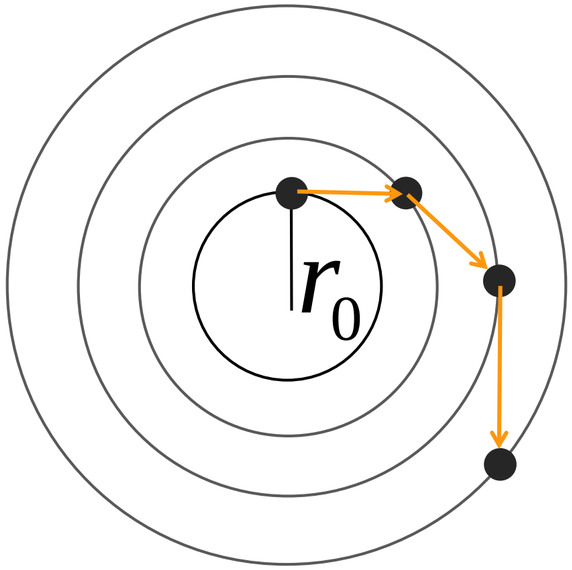
current configuration

velocity at current time

$$q_{k+1} = q_k + \tau f(q_k)$$

- Very intuitive: walk a tiny bit in the direction of the velocity
- Unfortunately, not very *stable* ☹
- Consider the following 2D, first order ODE (what does it do?):

$$\dot{q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q$$



Forward Euler - Stability Analysis

- **Let's consider behavior of forward Euler for simple linear ODE (e.g. temperature of an object):**

$$\dot{u} = -au, \qquad a > 0$$

- **Importantly: *u* should *decay* (exact solution is $u(t)=e^{-at}$)**
- **Forward Euler approximation is**

$$\begin{aligned} u_{k+1} &= u_k - \tau a u_k \\ &= (1 - \tau a) u_k \end{aligned}$$

- **Which means after n steps, we have**

$$u_n = (1 - \tau a)^n u_0$$

- **Decays only if $|1-\tau a| < 1$, or equivalently, if $\tau < 2/a$**
- **In practice: may need *very small* time steps ("stiff ODE")**

Backward Euler

- **Let's try something else: evaluate velocity at *next* configuration**
- **New configuration is then *implicit*, and we must solve for it:**

new configuration

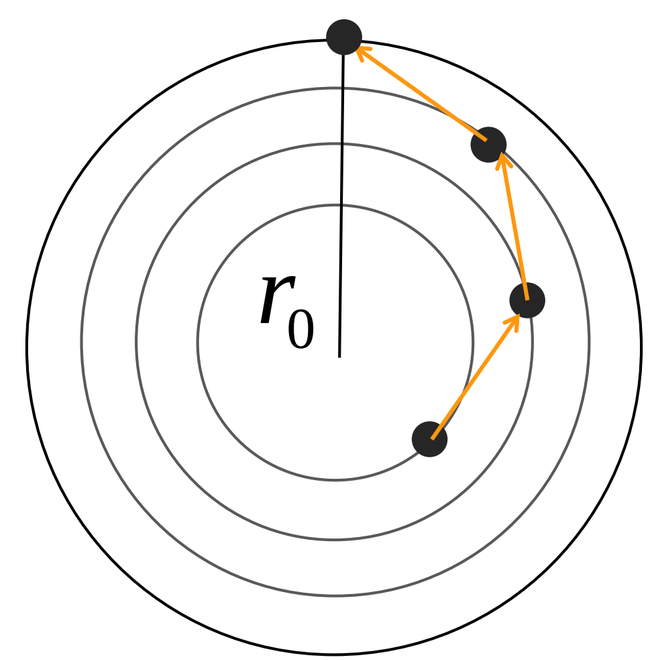
current configuration

velocity at next time

$$q_{k+1} = q_k + \tau f(q_{k+1})$$

- **Much harder to solve, since in general f can be very nonlinear!**

$$\dot{q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q$$



Backward Euler - Stability Analysis

- **Again consider a simple linear ODE:**

$$\dot{u} = -au, \quad a > 0$$

- **Remember: *u* should *decay* (exact solution is *u*(*t*)=e $^{-at}$)**
- **Backward Euler approximation is**

$$(u_{k+1} - u_k)/\tau = -au_{k+1}$$

$$\Longleftrightarrow \qquad u_{k+1} = \frac{1}{1+\tau a} u_k$$

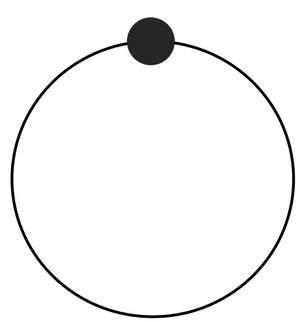
- **Which means after n steps, we have**

$$u_n = \left(\frac{1}{1+\tau a}\right)^n u_0$$

- **Decays if |1+τa| > 1, which is always true!**
- **⇒ Backward Euler is *unconditionally stable* for linear ODEs**

Symplectic Euler

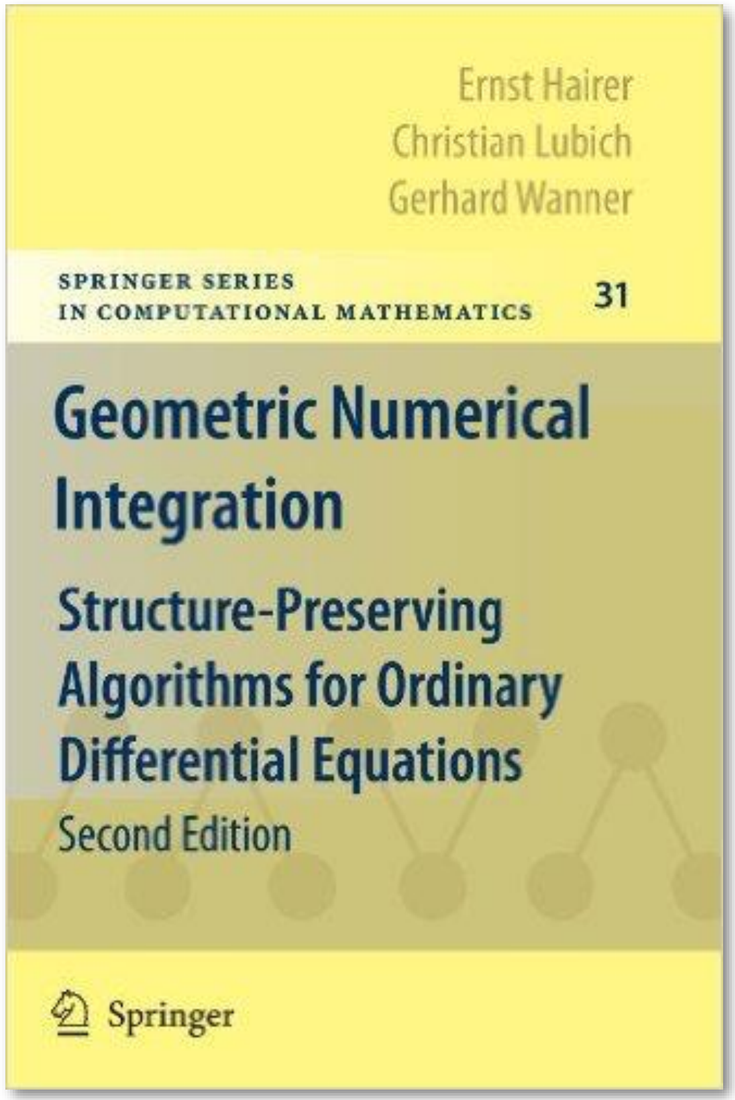
- Backward Euler was stable, but we also saw (empirically) that it exibits *numerical damping* (damping not found in original eqn.)
- Nice alternative is symplectic Euler (for 2nd order ODEs)
  - update velocity using current configuration
  - update configuration using *new* velocity
- Easy to implement; used often in practice (or leapfrog, Verlet, ...)
- Energy is conserved *almost exactly*, forever. Is that desirable?



(Proof? The analysis is not so easy...)

Numerical Integrators

- Barely scratched the surface
- *Many* different integrators
- Why? Because many notions of "good":
  - stability
  - accuracy
  - consistency/convergence
  - conservation, symmetry, ...
  - computational efficiency (!)
- No one "best" integrator—*pick the right tool for the job!*
- Could do (at least) an entire course on time integration...
- Great book: Hairer, Lubich, Wanner



Let’s look at an example…