CMU CS 462/662 (INTRO TO COMPUTER GRAPHICS) FALL 2016
HOMEWORK 0.5—MATH REVIEW/PREVIEW
VECTOR CALCULUS

Andrew ID: bpx

September 13, 2016

This assignment continues with our review of mathematical tools you will use throughout our course—this time covering vector calculus and some basic complex analysis that will be useful for geometric/coordinate manipulations in 2D and 3D. As with the last assignment, we do not provide a comprehensive tutorial; the assignment is **open-book, open-notes, open-internet** and you are welcome to use any resource (including your professors, TAs, and fellow classmates!) that will helps you understand and complete the exercises. As always, however, you must perform the final calculations yourself. Finally, you can (and should!) use a calculator, computer, *etc.*, to evaluate the final numerical answers—as noted above the template below, all answers should be numerical and given in decimal form to three decimal places. Unless stated otherwise, all angles should be expressed in radians (not degrees).

WARNING: The numerical values in your assignment are unique to your Andrew ID. Do *not* read these values off of someone else's assignment sheet!

To submit the homework, enter your numerical solutions into the template at the bottom of this document and email it to both `kmcrane@cs.cmu.edu` and `zshearer@andrew.cmu.edu`, with the string `462FA2016A05` in the subject line. (It is important that you use *exactly* this string, otherwise your assignment might not get graded!)

The assignment will be due no later than **11:59 PM, EDT on September 13, 2016**, after which point you will lose 10% credit per day. However, we realize (as with the first part) this is a lot of material to cover. Therefore, we will ask you to complete only 25 out of the 79 total sub-exercises (many of which are shorter/simpler than in Assignment 0.0). As usual you may choose this subset however you like—for instance, it is fine to solve problem 22a and 22c but not 22b, as long as the total number of answers in your final solution is at least 25. If you want additional practice, you may also complete additional exercises, for up to 20% extra credit (corresponding to 1% extra credit in the overall course grade). In other words, if you complete 25 exercises (correctly), you will get 100%, if you complete all 79 correctly you will get 120%, and anywhere in between the amount of extra credit will be a linear function of the number of additional exercises completed.

TLDR: Do 25 sub-exercises for full credit; do more for more credit. Don't turn it in late.

Good luck! (And remember that we are here to help!)

1 Vector Calculus

1.1 Dot and Cross Product

In our study of linear algebra, we looked *inner products* in the abstract, *i.e.*, we said that an inner product $\langle \cdot, \cdot \rangle$ was *any* operation that is symmetric, bilinear, *etc.* When working with two- and three-dimensional geometry, we typically want to work with one very special inner product called the **Euclidean inner product**, which has a concrete geometric relationship to lengths and angles. In particular, for any two Euclidean vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the Euclidean inner product is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\text{Euc}} := |\mathbf{u}||\mathbf{v}| \cos(\theta).$$

where $|\mathbf{u}|$ and $|\mathbf{v}|$ are the lengths of $\mathbf{u}$ and $\mathbf{v}$, respectively, and $\theta \geq 0$ is the (unsigned) angle between them. If the components u_i, v_i of these two vectors are expressed with respect to some orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then the Euclidean inner product can be computed via the **dot product**

$$\mathbf{u} \cdot \mathbf{v} := \sum_{i=1}^{n} u_i v_i,$$

Notice that we made two key assumptions here:

1. The vectors $\mathbf{u}, \mathbf{v}$ represent vectors in Euclidean $\mathbb{R}^n$; they are not tangent vectors on the sphere, bitmap images, or rows in a database.
2. The values $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are the components of $\mathbf{u}$ and $\mathbf{v}$ with respect to an *orthonormal basis*.

If either of these two assumptions are violated, then the dot product no longer carries the geometric meaning one might expect, *i.e.*, it is no longer true that $\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle_{\text{Euc}}$. If we want to recover length or angle in a different basis, we need to carefully account for the effect of our choice of basis on the coordinates—mismanagement of coordinate systems is common source of bugs in graphics code. The following exercises should help illuminate these ideas.

Exercise 1. *Suppose we are working in $\mathbb{R}^2$ with the standard orthonormal basis $\mathbf{e}_1 := (1, 0)$, $\mathbf{e}_2 := (0, 1)$.*

*(a) Compute the Cartesian coordinates of a vector $\mathbf{u}$ with length $\ell_1 := 3$ and counter-clockwise angle $\theta_1 := 0.200$ relative to the positive $\mathbf{e}_1$-axis. [**Hint:** You may want to revisit our earlier discussion of polar coordinates.]*

(b) Compute the Cartesian coordinates of a vector $\mathbf{v}$ with length $\ell_2 := 7$ and counter-clockwise angle $\theta_2 := 0.150$ relative to the positive $\mathbf{e}_1$-axis.

(c) For the same values used above, compute the quantity $\ell_1 \ell_2 \cos(|\theta_2 - \theta_1|)$ (where here $|\cdot|$ just denotes the usual absolute value of a real number).

(d) For the same vectors computed above, evaluate the dot product $\mathbf{u} \cdot \mathbf{v}$.

Exercise 2. *Consider the non-orthonormal basis $\tilde{\mathbf{e}}_1 := (4, 0)$, $\tilde{\mathbf{e}}_2 := (0, 5)$. Let $\mathbf{u} := (4, 2)$ and $\mathbf{v} := (2, 8)$ be two vectors expressed with respect to the standard orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$, as in the previous exercise.*

*(a) Compute the length and angle of the vector $\mathbf{u}$, giving the result as a pair (ℓ_1, θ_1), with the angle θ_1 expressed relative to the $\mathbf{e}_1$ axis. [**Hint:** Remember that to recover the angle θ with the proper sign, one can use the* atan2 *function.]*

(b) Compute the length and angle of the vector $\mathbf{v}$, giving the result as a pair (ℓ_2, θ_2), with the angle θ_2 expressed relative to the $\mathbf{e}_1$ axis.

(c) Re-express $\mathbf{u}$ with respect to the basis $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$, i.e., *find coefficients $\tilde{u}_1, \tilde{u}_2$ such that $\tilde{u}_1 \tilde{\mathbf{e}}_1 + \tilde{u}_2 \tilde{\mathbf{e}}_2 = (u_1, u_2)$.*

(d) Likewise, re-express $\mathbf{v}$ with respect to $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$.

(e) Compute the dot product $\tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} = \sum_{i=1}^{2} \tilde{u}_i \tilde{v}_i$.

(f) Compute the quantity $\ell_1 \ell_2 \cos(|\theta_2 - \theta_1|)$ *using the length and angle values computed above. (Think about why this value might not match the result of the previous part.)*

Unlike the dot product, which maps two vectors to a scalar, the **cross product** maps two vectors to another vector. In particular, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the cross product can be defined as the unique vector $\mathbf{u} \times \mathbf{v} \in \mathbb{R}^3$ such that

$$\sqrt{\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v})} = |\mathbf{u}||\mathbf{v}| \sin(\theta)$$

where det denotes the determinant of a matrix, and $\theta \in [0, \pi]$ is the (unsigned) angle between $\mathbf{u}$ and $\mathbf{v}$. [1] From these three properties, one can infer that in an orthonormal coordinate system the cross product must be equal to

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$

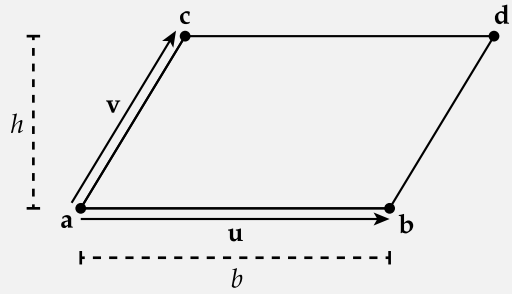
Note that this operation is only well-defined for vectors in $\mathbb{R}^3$. However, it is sometimes convenient (especially in computer graphics) to abuse notation and write

$$\mathbf{u} \times \mathbf{v} := u_1 v_2 - u_2 v_1$$

for a pair of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. This expression effectively treats $\mathbf{u}$ and $\mathbf{v}$ as 3-vectors $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$, yielding the third (nonzero) component as the result.

An easy, geometric way to think about the magnitude of the cross product is provided by the next exercise.

Exercise 3. *Consider a parallelogram with vertices* $\mathbf{a} := (0,0)$, $\mathbf{b} := (2,0)$, $\mathbf{c} := (3,3)$, *and* $\mathbf{d} := (5,3)$, *as depicted below.*



(a) The area of a parallelogram is equal to its base length times its height. Compute the base length b, assuming that the "base" is along the horizontal direction $(1,0)$.

*(b) Now compute the height h of the parallelogram. [**Hint:** The height can be found by computing the vertical component of the edge vector from* $\mathbf{a}$ *to* $\mathbf{c}$.*]*

(c) Compute the area $A := bh$.

(d) Letting $\mathbf{u} := \mathbf{b} - \mathbf{a}$ *and* $\mathbf{v} := \mathbf{c} - \mathbf{a}$, *compute* $|\mathbf{u} \times \mathbf{v}|$ *(where* $|\cdot|$ *denotes the usual absolute value of a real number).*

[1] You might have seen a different definition of the cross product before; something like, *"The cross product is defined as* $\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin(\theta)\mathbf{w}$, *where* $\mathbf{w}$ *is a unit vector orthogonal to* $\mathbf{u}$ *and* $\mathbf{v}$ *with direction given by the right-hand rule."* In other words, since there are two vectors $\pm\mathbf{w}$ orthogonal to $\mathbf{u}$ and $\mathbf{v}$ with the right magnitude, we have to have some way to pick which one defines the cross product. The determinant in the definition above effectively gives a more precise meaning to the "right hand rule," *i.e.*, it provides a way to determine the direction of the cross product that also works if you are an octopus.

Exercise 4.

(a) For $\mathbf{u} := (3,6,5)$*, compute* $\mathbf{u} \times \mathbf{u}$*. (Think about why you get this result, in light of the geometric interpretation above.)*

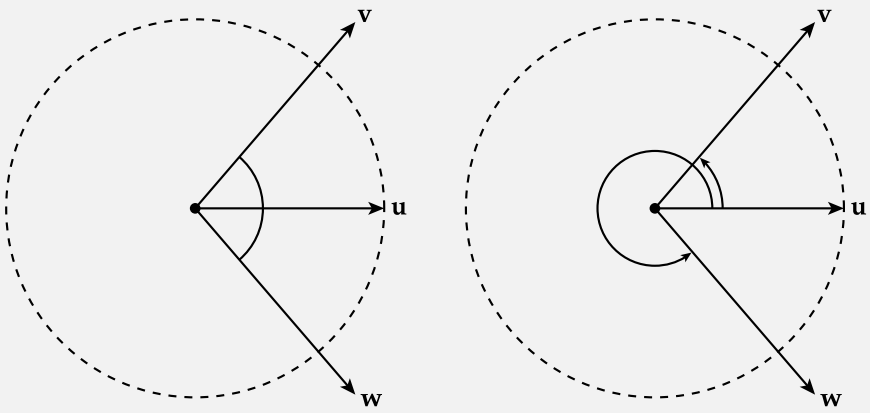
(b) For the same vector $\mathbf{u}$*, and the vector* $\mathbf{v} := (3,5,6)$*, compute* $\mathbf{u} \times \mathbf{v}$*.*

(c) For the same two vectors $\mathbf{u}, \mathbf{v}$*, compute* $\mathbf{v} \times \mathbf{u}$*. (Again, think about why you got this result, in terms of both the geometric interpretation and our original definition for the cross product.)*

(d) For the same two vectors $\mathbf{u}, \mathbf{v}$*, and a vector* $\mathbf{w} := (5,6,9)$*, compute* $\mathbf{w} \times (\mathbf{u} + \mathbf{v})$*.*

(e) For the same three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$*, compute* $\mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v}$*. (Think about the result of the last two parts—why should this be true, geometrically?)*

Exercise 5. *An extremely common mistake in computer graphics is incorrectly computing the angle between two vectors. Simple as this task may seem, it is easy to compute the wrong angle in the wrong situation. Sometimes, one needs only the unsigned magnitude of the smallest angle between two vectors* $\mathbf{u}, \mathbf{v}$*. At other times, it is necessary to compute a signed angle that takes* $\mathbf{u}$ *to* $\mathbf{v}$ *(or vice-versa). For the exercises below, let* $\mathbf{u} := (1,0)$*,* $\mathbf{v} := (7,3)$*, and* $\mathbf{w} := (7,-3)$ *be vectors expressed relative to the standard Euclidean basis* $\mathbf{e}_1 := (1,0)$*,* $\mathbf{e}_2 := (0,1)$*. Please express all answers in* ***radians****, not degrees.*



(a) From the relationship $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$*, it appears that the angle* θ *between two vectors* $\mathbf{u}$*,* $\mathbf{v}$ *"should be" expressed as* [a]

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right).$$

Compute the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *using the* arccos *formula.*

(b) Likewise, compute the angle between $\mathbf{u}$ *and* $\mathbf{w}$ *via the* arccos *function.*

(c) Another popular trigonometric identity is $\sin(\theta)/\cos(\theta) = \tan(\theta)$*. Interestingly enough, we've already seen two vector-vector products that essentially give us the data on the left-hand side:* $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ *and* $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\theta)$*. So it would seem that*

$$\frac{|\mathbf{u} \times \mathbf{v}|}{\mathbf{u} \cdot \mathbf{v}} = \tan(\theta).$$

A different way to recover the angle, then, would be to apply the arctan *function to the left-hand side of this expression. Actually, we can do a little better by using the function* atan2*, which properly determines the* sign *of an angle by using two separate arguments (rather than their quotient).* [b]

Compute the angle from **u** *to* **v** *using* atan2.

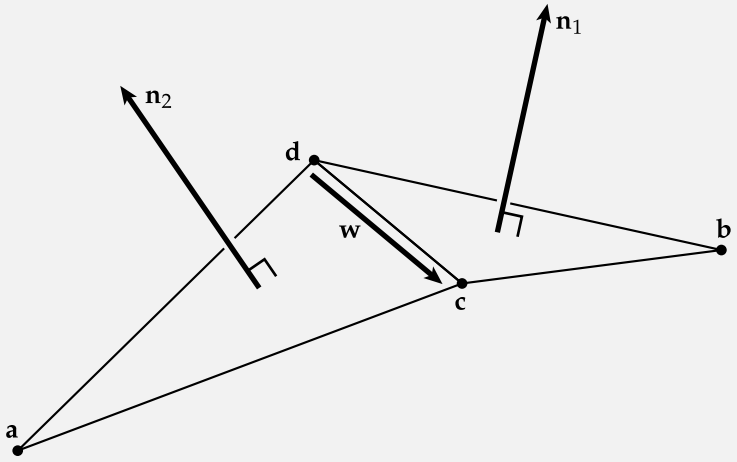
(d) Compute the angle from **u** *to* **w** *using* atan2.

(e) The last calculation yields the angle from **u** *to* **w** *via a clockwise motion. Suppose that we instead want the angle from* **u** *to* **w** *via a counter-clockwise motion. What is the smallest such angle?*

[a] For clarity, we use arccos to denote the inverse of the cosine function over the interval $[-\pi, \pi)$, and $\cos^{-1}(x)$ to mean $1/\cos(x)$. Otherwise, it is easy to erroneously interchange the two!

[b] **WARNING:** Some programming environments define $\arctan(x, y)$ whereas others swap the arguments and define $\arctan(y, x)$. If you are uncomfortable with atan2, you may wish to take a moment and read the Wikipedia page.

Exercise 6. *Accounting for orientation is also important when computing angles in 3D. Consider the figure below, depicting two triangles with vertices* $\mathbf{a} := (7, -7, 0)$, $\mathbf{b} := (7, 7, 0)$, $\mathbf{c} := (0, 0, 4)$, *and* $\mathbf{d} := (0, 0, -8)$, *and their unit normals* $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{R}^3$. *(A* unit normal *is a unit-length vector orthogonal to the plane of a given triangle; notice that for any triangle there are two possible choices of unit normal, with opposite orientation.) Note that the following exercises involve a fair amount of computation; you may find it easier to use some kind of programming/scripting language to carry out the calculations—which will also prepare you for writing real graphics code!*



(a) Compute the unit normal $\mathbf{n}_1$ *for the triangle with vertices* $\mathbf{b}, \mathbf{d}, \mathbf{c}$, *being careful to pick the orientation that matches the picture. [**Hint:** Think about the fact that the cross product of two vectors* $\mathbf{u}, \mathbf{v}$ *yields a vector orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$. *For a triangle normal, for instance, one can compute a vector orthogonal to two of the edges. To reason about the orientation, use the so-called "right hand rule" (which is explained in any standard reference, like Wikipedia). Remember that the final result must have unit length.]*

(b) Compute the unit normal $\mathbf{n}_2$ *for the triangle with vertices* $\mathbf{a}, \mathbf{c}, \mathbf{d}$, *being careful to pick the orientation that matches the picture.*

(c) Compute the signed angle θ *representing a counter-clockwise rotation from* $\mathbf{n}_1$ *to* $\mathbf{n}_2$ *around the edge vector* $\mathbf{w} := \mathbf{c} - \mathbf{d}$. *[**Hint:** The vector* $\mathbf{w}$ *should not actually be needed for your calculations; it is used only to give a meaning to the notion of "counter-clockwise" in 3D. For instance, if we instead used the vector from* $\mathbf{c}$ *to* $\mathbf{d}$ *as the axis of rotation, our notion of counter-clockwise would be reversed. Keeping careful track of orientation in 3D helps to avoid bugs in computer graphics code. For this calculation, you should need only the dot product, cross product, and arctangent formula.]*

Exercise 7. *For performance-intensive code, trigonometric functions (and inverse trigonometric functions) can be quite expensive to evaluate numerically. For geometric calculations, one can sometimes devise suitable alternatives using only dot products, cross products,* etc., *which may involve simpler floating-point arithmetic.* [a] *In the*

following exercises, let θ denote the angle between vectors $\mathbf{u} := (2, 8, 4)$ and $\mathbf{v} := (9, 5, 0)$.

(a) Compute the quantity $\mathbf{u} \cdot \mathbf{v}$.

(b) Compute the quantity $|\mathbf{u} \times \mathbf{v}|$.

(c) Compute the angle $\theta = \arccos(\mathbf{u} \cdot \mathbf{v}/|\mathbf{u}||\mathbf{v}|)$.

(d) Evaluate the cotangent of the angle theta $\cot(\theta)$. (In many calculators/programming environments a `cot` function is not available; in this case, the cotangent can be evaluated via $1/\tan(\theta)$.)

(e) Compute the quantity $\mathbf{u} \cdot \mathbf{v}/|\mathbf{u} \times \mathbf{v}|$.

[a] I would be very interested in knowing *how much* faster, on real hardware...

Exercise 8. *It is often useful to be able to express the cross product via a matrix-vector multiplication. For any two vectors $\mathbf{u} := (u_1, u_2, u_3)$ and $\mathbf{a} := (a_1, a_2, a_3)$, the cross product $\mathbf{u} \times \mathbf{a}$ can be represented as $\hat{\mathbf{u}}\mathbf{a}$, where $\hat{\mathbf{u}} \in \mathbb{R}^{3\times 3}$ denotes the matrix*

$$\hat{\mathbf{u}} := \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

(a) Using the values $\mathbf{u} := (7, 6, 5)$ and $\mathbf{a} := (8, 5, 9)$, compute $\mathbf{u} \times \mathbf{a}$.

(b) Evaluate the matrix $\hat{\mathbf{u}}$, giving the result in row-major form.

(c) Compute the matrix-vector product $\hat{\mathbf{u}}^{\mathsf{T}}\mathbf{a}$, where ^{T} denotes the transpose. (Think about what operation this product represents, in terms of the cross product.)

Exercise 9. *Often the so-called* triple product formulas *are often quite useful for evaluating or simplifying an expression involving three vectors. Evalute the expressions below for the vectors $\mathbf{u} := (4, 2, 2)$, $\mathbf{v} := (4, 9, 6)$, and $\mathbf{w} := (6, 4, 8)$. [**Hint:** For most of these, you should not need to perform brute-force calculation! Look up the triple-product formulas if you don't know them already. (If you're comfortable with linear algebra, you may might find it helpful to think about what these formulas mean / why they're true in terms of the determinant.)]*

(a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{u})$

(b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

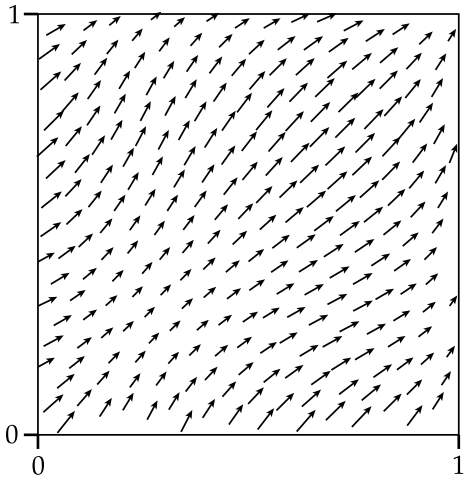
(c) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$

(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

(e) $\mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})$

(f) $\mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v})$.

1.2 Vector Fields



A *vector field* associates each point of a space with a vector. For instance, a vector field $X : [0,1]^2 \to \mathbb{R}^2$ assigns a 2-vector to each point (u, v) in the unit square. We will specify vector fields as tuples of coordinate functions; for instance

$$X(u, v) = (1, u + v)$$

is a vector field whose horizontal component is the same at each point, and whose vertical component is determined by the sum of the two coordinates at the specified point.

Exercise 10. *Just like individual vectors, vector fields can be added together or scaled by a constant by simply applying these operations to the vectors at each point. For instance, for a pair of vector fields $X, Y : \mathbb{R}^m \to \mathbb{R}^n$ and a scalar $a \in \mathbb{R}$ we have $(X + Y)(\mathbf{u}) := X(\mathbf{u}) + Y(\mathbf{u})$ and $(aX)(\mathbf{u}) := a(X(\mathbf{u}))$. Consider the two vector fields $X, Y : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$\begin{aligned} X(\mathbf{u}) &:= (2\cos(5u_2), 3u_1^2), \\ Y(\mathbf{u}) &:= (6, 7(u_1^2 + u_2^2)). \end{aligned}$$

(a) Compute an expression for the vector field $Z := X + Y$ and evaluate it at the point $\mathbf{u} = (0.800, 0.400)$.

(b) Compute an expression for the vector field $5Z$ and evaluate it at the point $\mathbf{u} := (0.800, 0.700)$.

Exercise 11. *Just like we can apply addition and scalar multiplication at each point, we can also combine two vector fields by applying the dot product and cross product pointwise. In other words, $(X \cdot Y)(\mathbf{u}) := X(\mathbf{u}) \cdot Y(\mathbf{u})$ and $(X \times Y)(\mathbf{u}) := X(\mathbf{u}) \times Y(\mathbf{u})$. Consider the two vector fields $X, Y : \mathbb{R}^3 \to \mathbb{R}^3$ given by*

$$\begin{aligned} X(\mathbf{u}) &:= (3u_1, 7u_2, 1), \\ Y(\mathbf{u}) &:= (-7u_2, 3u_1, u_3). \end{aligned}$$

(a) Compute an expression for the scalar function $f := X \cdot X$ and evaluate it at the point $\mathbf{u} := (4, 5, 7)$.

(b) Compute an expression for the scalar function $f := X \cdot Y$ and evaluate it at the point $\mathbf{u} := (9, 3, 4)$.

(c) Compute an expression for the vector field $Z := X \times Y$ and evaluate it at the point $\mathbf{u} := (7, 4, 7)$.

1.3 Gradient, Divergence and Curl

Exercise 12.

(a) *Compute the gradient of the function* $f(x,y) := 2x^2 + y\cos(6x)$ *and evaluate it at the point* $(x,y) = (5,3)$.

(b) *Compute the gradient of the function* $g(x,y,z) := 8xy + 7yz + 6zx$, *evaluating it at the point* $(x,y,z) = (2,5,8)$.

(c) *Compute the gradient of the function* $h(x) := \log(x) + 3/x$, *evaluating it at the point* $x = 5$.

Exercise 13. *In class we saw that there are several different approaches to computing the gradient, each of which can prove useful in different situations. For the problems below, let* $\mathbf{x} := (5,2)$ *and* $\mathbf{y} := (4,4)$.

(a) *Let* $f(\mathbf{x}) := \mathbf{x}^\mathsf{T}\mathbf{y}$ *and evaluate* $(\partial f/\partial x_1, \partial f/\partial x_2)$ *(using the data specified above).*

(b) *For the same function* f, *evaluate the directional derivative*

$$D_\mathbf{u} f(\mathbf{x}_0) := \lim_{\varepsilon \to 0} \frac{f(\mathbf{x}_0 + \varepsilon\mathbf{u}) - f(\mathbf{x}_0)}{\varepsilon}$$

for $\mathbf{x_0} := (7,4)$ *and* $\mathbf{u} := (9,6)$.

(c) *Using the result from the previous exercise, consider the relationship* $(\nabla f(\mathbf{x}_0))^\mathsf{T}\mathbf{u} = D_\mathbf{u} f(x_0)$. *What must* $\nabla f(\mathbf{x}_0)$ *be equal to in order for this relationship hold for all vectors* $\mathbf{u}$*? Once you have an answer, evaluate it using the data provided in the previous part.*

(d) *Next, consider the function* $F(p) := \langle\langle p, q\rangle\rangle = \int_0^1 p(s)q(s)\,ds$, *which maps any (square-integrable) function* $p : [0,1] \to \mathbb{R}$ *to its inner product with the function* $q : [0,1] \to \mathbb{R}; s \mapsto 9s$. *Evaluate the limit*

$$D_r F(p_0) := \lim_{\varepsilon \to 0} \frac{F(p_0 + \varepsilon r) - F(p_0)}{\varepsilon},$$

for the functions $p_0(s) := 9\sin^2(s)$ *and* $r(s) := 5/4$.

(e) *Using the result from the previous exercise, consider the relationship* $\langle\langle \nabla_p F(p_0), r\rangle\rangle = D_r F(p_0)$. *What must* $\nabla_p F(p_0)$ *be equal to in order for this relationship to hold for all functions* r*? Once you have an answer, evaluate it using the data provided in the previous part, at the point* $s = 5$.

Exercise 14. *In this exercise we consider the 2nd-order Taylor series and its generalization to multivariable functions, which plays a central role in numerical approximation and optimization used throughout computer graphics. Throughout this exercise we consider thefunction* $f(x) := \cos(x)$, *and the multivariable function* $g(x,y) := \cos(x)\sin(y)$.

(a) *Evaluate* f *as well as its first and second derivatives at the point* $x_0 := 8$, *reporting the result as a tuple* $(f(x_0), f'(x_0), f''(x_0))$.

(b) *Evaluate the 2nd-order Taylor series approximation* $\hat{f}(x) := f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$ *at the point* $x_1 := 8.060$.

(c) *Compute the approximation error* $\hat{f}(x_1) - f(x_1)$.

(d) *Evaluate the gradient* ∇g *at the point* $\mathbf{x}_0 := (8,9)$.

(e) *Evaluate the Hessian* $\nabla^2 g$ *at the same point, giving the result in row-major form.*

(f) *Evaluate the 2nd-order Taylor series approximation*

$$\hat{g}(\mathbf{x}) := g(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)^\mathsf{T}\nabla g(\mathbf{x}_0) + \tfrac{1}{2}(\mathbf{x} - \mathbf{x}_0)^\mathsf{T}\nabla^2 g(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

at the point $\mathbf{x}_1 := (8.090, 9.030)$.

(g) Evaluate the approximation error $\hat{g} - g$ *at* $\mathbf{x}_1$.

Exercise 15. *Consider the vector fields* $X(u,v) := (3+2v^2, 4uv)$, $Y(u,v) = (-4uv, 3+2v^2)$, *and* $Z(u,v) := (6,3)$. *Evaluate each of the results below at the point* $(u,v) := (5,3)$.

(a) $\nabla \cdot X$

(b) $\nabla \cdot Y$

(c) $\nabla \times X$

(d) $\nabla \times Y$

(e) $\nabla \cdot Z + \nabla \times Z$

Exercise 16. *For the following exercises consider the scalar function* $f(u,v) := 6uv$ *and the vector field* $Z(u,v) := (4u^2, 2v^2)$, *and evaluate each expression at the point* $(u,v) := (8,5)$.

(a) Evaluate the gradient of f.

(b) Evaluate the divergence Z.

(c) Evaluate $(\nabla f) \cdot Z$.

(d) Evaluate $f(\nabla \cdot Z)$.

(e) Evaluate the sum $(\nabla f) \cdot Z + f(\nabla \cdot Z)$.

(f) Evaluate the product fZ.

(g) Evaluate $\nabla \cdot (fZ)$.

(Think about the values you computed above; do some of them agree? Why might that happen?)

Exercise 17. *Consider the scalar function* $g(x,y,z) := 2x^2y + 7y^2z + 9z^2x$ *and the vector field* $U(x,y,z) := (0,1,3)$.

(a) Compute the vector field $g(\nabla \times U) + (\nabla g) \times U$, *evaluating the result at the point* $(x,y,z) = (9,9,2)$.

(b) Compute the vector field $\nabla \times (gU)$, *evaluating the result at the same point as in the previous part.*

Exercise 18. *Compute the Laplacian* $\Delta f(x,y) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ *of the functions below, evaluating the result at the point* $(x,y) := (0.030, 0.030)$.

(a) $f(x,y) := e^{-2(x^2+y^2)}$

(b) $f(x,y) := 5(x^2 - y^2)$

(c) $f(x,y) := 9x^3 - 24x^2y - 27xy^2 + 8y^3$

(d) $f(x,y) := 9\cos(6x)\sin(6y)$

2 Solution Template

To submit your solutions, you should copy and paste the template below into a plain-text email. Each ? mark should be replaced with the numerical solutions you computed—no other text should appear in the answers other than numerical values. (For questions where we said something like, "Think about..." or "Why does..." in parentheses, you need not write any answers to these parenthetical remarks—they are merely for your own contemplation.) For questions you did not answer, you may simply leave the ? marks. All numbers should be expressed as decimal values with at least three decimal places. ***Please make sure to preserve the formatting exactly!*** Since the assignment will be auto-graded, missing parentheses, commas, etc., may result in lost points.

```
andrewID: ???????????????
1a. (?,?)
1b. (?,?)
1c. ?
1d. ?
2a. (?,?)
2b. (?,?)
2c. (?,?)
2d. (?,?)
2e. ?
2f. ?
3a. ?
3b. ?
3c. ?
3d. ?
4a. (?,?,?)
4b. (?,?,?)
4c. (?,?,?)
4d. (?,?,?)
4e. (?,?,?)
5a. ?
5b. ?
5c. ?
5d. ?
5e. ?
6a. (?,?,?)
6b. (?,?,?)
6c. ?
7a. ?
7b. ?
7c. ?
7d. ?
7e. ?
8a. (?,?,?)
8b. ((?,?,?),(?,?,?),(?,?,?))
8c. (?,?,?)
9a. ?
9b. ?
9c. ?
9d. (?,?,?)
9e. (?,?,?)
9f. (?,?,?)
10a. (?,?)
10b. (?,?)
11a. ?
11b. ?
11c. (?,?,?)
12a. (?,?)
12b. (?,?,?)
12c. ?
13a. (?,?)
13b. ?
13c. (?,?)
13d. ?
13e. ?
14a. (?,?,?)
14b. ?
14c. ?
14d. (?,?)
14e. ((?,?),(?,?))
14f. ?
14g. ?
15a. ?
15b. ?
15c. ?
15d. ?
15e. ?
16a. (?,?)
16b. ?
16c. ?
16d. ?
16e. ?
16f. (?,?)
16g. ?
17a. (?,?,?)
17b. (?,?,?)
18a. ?
18b. ?
18c. ?
18d. ?
```