Reparameterizing a Catmull-Rom Spline

CMU 15-462 (Fall 2015)

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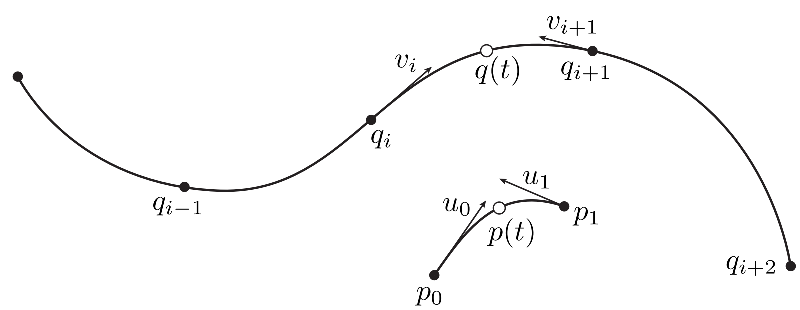

A *Catmull-Rom spline* is a piecewise cubic curve $q(t)$ that (i) interpolates a given set of data points at the endpoints of each interval (i.e., at the "knots" of the spline), and (ii) has tangents at knots parallel to the difference of adjacent knot values. More explicitly, if the knot times are $t_0 < t_1 < \cdots < t_n$ and the corresponding knot values are $q_0, q_1, \ldots, q_n$, then we have $q(t_i) = q_i$ for all i[1], and

$$\dot{q}(t_i) = \frac{q_{i+1} - q_{i-1}}{t_{i+1} - t_{i-1}} =: v_i$$

for $i \in \{1, \ldots, n-1\}$, where $\dot{q} := \frac{d}{dt}q$ denotes the time derivative of q. (Behavior of the spline near the boundary has been discussed elsewhere in the assignment writeup; the short story is that you can always "mirror" values across boundary knots in order to define the unknown data). One way to motivate this definition of the tangent values v_i is to imagine that knots are sampled from a continuous curve; in this case, m_i will approach the true tangents to the curve as the spacing between knots uniformly goes to zero (subject to sufficient regularity of the curve being sampled).

On the *unit* interval $[0,1]$, we can construct a cubic polynomial $p(t)$ that interpolates given values p_0, p_1 and given tangents u_0, u_1 at endpoints. More explicitly, p will satisfy $p(0) = p_0$, $p(1) = p_1$, $\dot{p}(0) = u_0$, and $\dot{p}(1) = u_1$, where $\dot{p} := \frac{d}{dt}p$ denotes the time derivative of p. Constructing a polynomial that interpolates these values is easily achieved using the *Hermite bases*, as discussed in the assignment writeup.

Suppose now that we want to use the function p to interpolate values within a given interval $t_i \leq t < t_{i+1}$ of our Catmull-Rom spline. In other words, suppose we've implemented the function p as a subroutine, and want to call it from our routine that implements q. Mathematically, this means we want to find values p_0, p_1, t_0, t_1 such that

$$p(t) = q(t_i + t(t_{i+1} - t_i)), \tag{1}$$

[1]A word about notation here: you'll notice that we use the same letter q to denote a time-continuous function $q(t)$ as well as discrete samples of this function q_i. This usage is unambiguous, because if q appears *without* a subscript we know it refers to the continuous quantity, and if q appears *with* a subscript, we know it refers to a sample. What do you think about this notation? Do you find it clearer—or *less* clear—than using different letters for continuous and sampled quantities? One benefit is that you don't have to ask "which quantity was sampled in order to get the discrete values?" On the other hand, it can be easy to confuse continuous and discrete quantities. Ultimately, notation is a personal style choice—but one that you should pay close attention to if you want people to understand what you write.

i.e., such that over the interval $[0,1]$, our "local" cubic polynomial p agrees with our "global" Catmull-Rom spline q. Given this relationship, what is the relationship between the tangents u_0, u_1 at the endpoints of the curve $p(t)$, and the corresponding tangents v_i, v_{i+1} at the endpoints of the current interval of $q(t)$? Well, tangents are the first time derivative of position. So, let's differentiate Equation 1 with respect to t:

$$\begin{array}{rrcl} & p(t) & = & q(t_i + t(t_{i+1} - t_i)) \\ \Rightarrow & \frac{d}{dt}p(t) & = & \frac{d}{dt}q(t_i + t(t_{i+1} - t_i)) \\ \iff & \dot{p}(t) & = & (t_{i+1} - t_i)\dot{q}(t_i + t(t_{i+1} - t_i)), \end{array}$$

and if we evaluate this last expression at the two critical times $t = 0$ and $t = 1$ we get

$$\begin{array}{rrcl} & \dot{p}(0) & = & (t_{i+1} - t_i)\dot{q}(t_i) \\ \iff & u_0 & = & (t_{i+1} - t_i)v_i \end{array}$$

and likewise

$$\begin{array}{rrcl} & \dot{p}(1) & = & (t_{i+1} - t_i)\dot{q}(t_{i+1}), \\ \iff & u_1 & = & (t_{i+1} - t_i)v_{i+1}. \end{array}$$

In short: *to get the tangents u_i for the local cubic polynomial, we must **multiply** the tangents v_i by the length of the current interval on the global Catmull-Rom spline*. This should make sense intuitively: for instance, if $t_{i+1} - t_i$ is greater than 1, then we are moving faster through space as we move along the cubic polynomial p than the Catmull-Rom spline q (hence the tangent should be bigger).

To further simplify this expression, we can recall our definition of v_i from above to get

$$u_0 = (t_{i+1} - t_i)v_i = \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}}(q_{i+1} - q_{i-1})$$

and

$$u_1 = (t_{i+1} - t_i)v_{i+1} = \frac{t_{i+1} - t_i}{t_{i+2} - t_i}(q_{i+2} - q_i).$$

In the special case where knots are uniformly spaced, all intervals t_i, t_{i+1} have the same length and these expressions can be simplified to

$$\begin{array}{rcl} u_0 & = & (q_{i+1} - q_{i-1})/2, \\ u_1 & = & (q_{i+2} - q_i)/2. \end{array}$$

However, this assumption does not hold in general, and you should **not** use these final two expressions in your assignment! (Since the user can insert knots at whatever times she pleases.)